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MULTIPLE SIGN-CHANGING SOLUTIONS FOR A CLASS OF SEMILINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^N$

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Abstract In this paper, we study the following semilinear elliptic equations

$$-\triangle u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N,$$

where $V \in C(\mathbb{R}^N, \mathbb{R})$ and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. Under some suitable conditions, we prove that the equation has three solutions of mountain pass type: one positive, one negative, and sign-changing. Furthermore, if f is odd with respect to its second variable, this problem has infinitely many sign-changing solutions.

Keywords Semilinear elliptic equations, critical point theorem, sign-changing solutions.

MSC(2010) 35J20, 35J70, 35P05, 35P30,34B15, 58E05, 47H04.

1. Introduction and preliminaries

In this paper, we study the following semilinear elliptic equations

$$-\triangle u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3, V \in C(\mathbb{R}^N, \mathbb{R})$ and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.

During the past years, problem of the form (1.1) has been extensively studied via the critical point theory, for example, see [?, ?, ?, ?, ?, ?, ?] and the reference therein, because such problems arise naturally in various branches of Mathematical Physics, on the other hand, they present specific mathematical difficulties that make them challenging to the researchers. Moreover, because the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is continuous, but not compact, the usual variational methods, that allow to prove the existence of infinitely many solutions to (1.1) in a bounded domain Ω.

When the nonlinearity f is of super-quadratic growth near infinity in u and is also allowed to be sign-changing, in [?] the author proved the existence of infinitely many nontrivial solutions of (1.1). If f is of subcritical growth and satisfies the global (AR) condition, Bartsch and Wang [?] obtained the existence of a sign-changing of (1.1). Recently, in [?], the author obtained the existence of positive solutions for the following class of elliptic equation

$$-\triangle u + V(x)u = f(u), \quad x \in \mathbb{R}^N, \tag{1.2}$$

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where f has a subcritical growth and V is a nonnegative potential, which can vanish at infinity. When $V(x) \equiv 1$, in [?] Weth gave a lower bound for the energy of sign changing solutions of (1.2) if f is superlinear by using the compactness lemma. By applying another version of the compactness lemma under a more general condition, Wang [?] proved that the result of Weth [?] was also true if f is asymptotically linear. When $V(x) \equiv a > 0$ is fixed, Bartsch and Weth [?] proved the existence of three nodal solutions of (1.2) provided Ω contains a large ball $B_R(0)$, where Ω is bounded domain with smooth boundary in $\mathbb{R}^N$. For the following equation:

$$-\triangle u + V(x)u = \lambda f(u), \quad \text{in} \quad \mathbb{R}^N,$$

for λ sufficiently large, the authors [?] proved the existence of positive, negative and a sign-changing solution by using Minimax methods. Liu and Chen [?] proved the existence of sign changing solutions and multiple solutions for the following the semilinear elliptic eigenvalue problem with constraint of the form:

$$\begin{cases} -\triangle u + V(x)u = \lambda f(x,u) \quad x \in \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N) \\ \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx = r^2 \\ u(x) \to 0, \quad |x| \to +\infty, \end{cases} \tag{1.3}$$

where $\lambda \in \mathbb{R}, r > 0$. The authors constructed nonempty invariant sets of the gradient flow which contained the positive and the negative solutions of the problem (1.3). Several authors have studied the following equation:

$$\begin{cases} -\triangle u + V(x)u = f(x,u) \quad x \in \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N) \\ u(x) \to 0, \quad |x| \to +\infty. \end{cases} \tag{1.4}$$

Bartsch, Liu and Weth [?] proved the existence of sign changing solutions and estimated the number of nodal domain of (1.4) under some other condition on $V(x)$ and $f(x,u)$. Moreover, if f is odd they obtained an unbounded sequence of sign changing solutions of (1.4). Qian [?] obtained infinitely many sign-changing solutions of (1.4) by using critical point theorem. Zhao and Ding [?] proved the existence and multiplicity of solutions of (1.4) by variational methods.

In this paper, we study the sign-changing solutions of problem (1.1). We need the following assumption:

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$, $V_0 := \inf_{x \in \mathbb{R}^N} V(x) > 0$, $\lim_{|x| \to +\infty} V(x) = +\infty$.

(f_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and there exist constants $C > 0$ and $2 < p < 2^*$ such that

$$|f(x,t)| \leq C(1 + |t|^{p-1}),$$

where $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent.

(f_2) $f(x,t) = o(|t|)$ uniformly in $x \in \mathbb{R}^N$, as $|t| \to 0$.

(f_3) $\lim_{|t| \to +\infty} \frac{F(x,t)}{t^2} = +\infty$ uniformly in $x \in \mathbb{R}^N$, where $F(x,t) := \int_0^t f(x,s) ds$.

(f_4) $tf(x,t) \geq 2F(x,t), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}$.

(f_5) There exist $\mu > 2$ and $\alpha > 0$ such that

$$\inf_{x \in \mathbb{R}^N, |t| = \alpha} F(x,t) > 0,$$

and

$$\mu F(x,t) \le f(x,t)t$$

for all $x \in \mathbb{R}^N$ and $|t| \ge \alpha$.

(f_6) $f(x,-t) = -f(x,t)$, $\forall (x,t) \in \mathbb{R}^N \times \mathbb{R}$.

We need the following several notations. Let

$$H^1(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$$

with the inner product and the norm

$$\langle u, v\rangle_1 = \int_{\mathbb{R}^N} \Big(\nabla u \nabla v + uv\Big)dx, \quad \|u\|_1 = \langle u, u\rangle_1^{\frac{1}{2}}.$$

Set

$$X := \Big\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty\Big\}$$

with the inner product and the norm

$$\langle u, v\rangle = \int_{\mathbb{R}^N} \Big(\nabla u \nabla v + V(x)uv\Big)dx, \quad \|u\| = \langle u, u\rangle^{\frac{1}{2}}.$$

Recall that a function $u \in X$ is called a weak solution of problem (1.1) if

$$\int_{\mathbb{R}^N} \nabla u \nabla \varphi dx + \int_{\mathbb{R}^N} V(x)u\varphi dx = \int_{\mathbb{R}^N} f(x,u)\varphi dx, \quad \forall \varphi \in X.$$

Seeking a weak solution of problem (1.1) is equivalent to finding a critical point of the functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} F(x,u)dx, \quad u \in X.$$

Under the assumption (V), it is well known that the embedding $X \hookrightarrow L^s(\mathbb{R}^N)$ is continuous for all $s \in [2, 2^*]$ and the embedding $X \hookrightarrow L^s(\mathbb{R}^N)$ is compact for all $s \in [2, 2^*)$. Under our conditions, we know that $J \in C^1(X, \mathbb{R})$ and for each $u \in X$,

$$\langle J'(u), \varphi\rangle = \int_{\mathbb{R}^N} \nabla u \nabla \varphi dx + \int_{\mathbb{R}^N} V(x)u\varphi dx - \int_{\mathbb{R}^N} f(x,u)\varphi dx,$$

for all $\varphi \in X$.

The main results of this paper are the following:

Theorem 1.1. *Suppose that (V), (f_1), (f_2) and (f_5) are satisfied. Then the equation* (1.1) *has three solutions of mountain pass type: one positive, one negative, and one sign-changing. Moreover, if f is odd with respect to its second variable, i.e. (f_6) holds, then problem (1.1) has infinitely many sign-changing solutions.*

Theorem 1.2. *If conditions (f_3) and (f_4) are used in place of (f_5), then the conclusion of Theorem 1.1 holds.*

Notice that the condition (f_4) is weaker than the condition (f_4') (see [**?**]). Hence we have the following corollary.

Corollary 1.1. *If the following* (f_4') *is used in place of* (f_4)*:*

(f_4') $\frac{f(x,t)}{t}$ *is non-increasing on* $(-\infty, 0)$ *and non-decreasing on* $(0, +\infty)$*,*
then the conclusion of Theorem 1.2 holds.

Throughout the paper, $\to$ and $\rightharpoonup$ denote the strong and weak convergence, respectively. C, c, C_i and c_i express distinct constants. For $1 \le q < \infty$, the usual Lebesgue space is endowed with the norm

$$\|u\|_q := \Big(\int_{\mathbb{R}^N} |u|^q dx\Big)^{\frac{1}{q}}.$$

The paper is organized as follows. In Section 2, we introduce some notions and results of some critical point theorems. In Section 3, we complete the proof of the main conclusions.

2. Some critical point theorems

Let us begin by recalling some notions and results of critical point theory (see [**?**]).

In the following, $P \subset X$ is a closed convex cone. For $\varepsilon > 0$, we denote by $V_\varepsilon(S)$ the ε-neighborhood of $S \subset X$, that is

$$V_\varepsilon(S) := \{u \in X : dist(u, S) := \inf_{v \in S} \|u - v\|_X < \varepsilon\}.$$

Define

$$+P := \{u \in X : u \ge 0\}, \quad -P := \{u \in X : u \le 0\},$$

$$P_\varepsilon^\pm := V_\varepsilon(\pm P) = \{u \in X : dist(u, \pm P) < \varepsilon\}.$$

Let $J \in C^1(X, \mathbb{R})$. We denote by K the set of critical point of J and $E = X \backslash K$.

For $\varepsilon_0 > 0$, we consider the following situation:

(A_{ε_0}) There exists a locally Lipschitz continuous vector field $B : E \to X$ (B odd if J is even) such that

(i) $B(P_\varepsilon^\pm \bigcap E) \subset P_\varepsilon^\pm, \quad \forall \varepsilon \in (0, \varepsilon_0)$;

(ii) there exists a constant $\alpha_1 > 0$ such that

$$\langle J'(u), u - B(u)\rangle \ge \alpha_1 \|u - B(u)\|_X^2, \quad \forall u \in E;$$

(iii) for $\rho_1 < \rho_2$ and $\alpha < 0$, there exists $\beta > 0$ such that $\|u - B(u)\|_X \ge \beta$ if $u \in X$ is such that $J(u) \in [\rho_1, \rho_2]$ and $\|J'(u)\|_{X^*} \ge \alpha$.

Definition 2.1. Let $J \in C^1(X, \mathbb{R}), c \in \mathbb{R}$. We say that J satisfies the $(PS)_c$ condition if each sequence $\{u_n\} \subset X$ with $J(u_n) \to c$ and $J'(u_n) \to 0$ in X^* possesses a convergent subsequence.

Theorem 2.1 ([**?**])**.** *Let* $J \in C^1(X, \mathbb{R})$ *with* $J(0) = 0$*. Assume there exists* $\varepsilon_0 > 0$ *such that* (A_{ε_0}) *is satisfied. Assume also that there exist* $e_\pm \in \pm P$ *and* $r > 0$ *such that*

(A_1) $\|e_\pm\|_X > r$ *and* $\rho := \inf_{\substack{u \in X \\ \|u\|_X = r}} J(u) > \delta := \max\{J(0), J(e_\pm)\}$.

Then there exist sequences $\{u_\pm^n\} \subset \overline{P_\varepsilon^\pm}$ *such that*

$$J'(u^n_\pm) \to 0 \textit{ in } X^* \textit{ and } J(u^n_\pm) \to c_\pm := \inf_{\gamma\in\Gamma_\pm} \sup_{u\in\gamma([0,1])} J(u) \geq \rho, \quad \forall \varepsilon \in (0, \varepsilon_0),$$

where

$$\Gamma_\pm := \Big\{\gamma \in C([0,1], \overline{P^\pm_\varepsilon}) : \gamma(0) = 0, \gamma(1) = e_\pm\Big\}.$$

If in addition J satisfies the $(PS)_c$ condition for any $c > 0$, then J has critical point $u_\pm \in \pm P\backslash\{0\}$.

Theorem 2.2 ([?]). *Let $J \in C^1(X, \mathbb{R})$. Assume there exists $\varepsilon_0 > 0$ such that (A_{ε_0}) is satisfied. Assume also that there exists a continuous map $\varphi_0 : \triangle \to X$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, the following conditions are satisfied:*

(1) $\varphi_0(\partial_1 \triangle) \subset P^+_\varepsilon$ and $\varphi_0(\partial_2 \triangle) \subset P^-_\varepsilon$,

(2) $\varphi_0(\partial_0 \triangle) \bigcap P^+_\varepsilon \bigcap P^-_\varepsilon =$,

(3) $c_0 := \sup_{u\in\varphi_0(\partial_0\triangle)} J(u) < c^* := \inf_{u\in\partial P^+_\varepsilon \bigcap \partial P^-_\varepsilon} J(u)$,

where

$$\triangle = \{(s,t) \in \mathbb{R}^2 : s, t \geq 0, s + t \leq 1\},$$

$$\partial_1 \triangle = \{0\} \times [0,1], \ \partial_2 \triangle = [0,1] \times \{0\} \textit{ and } \partial_0 \triangle = \{(s,t) \in \triangle : s + t = 1\}.$$

Then there exists a sequence $\{u_n\} \subset \overline{V_{\frac{\varepsilon}{2}}(X \setminus (P^+_\varepsilon \cup P^-_\varepsilon))}$ such that

$$J'(u_n) \to 0 \textit{ in } X^*$$

and

$$J(u_n) \to c := \inf_{\varphi\in\Gamma} \sup_{u\in\varphi(\triangle)\bigcap(X\setminus(P^+_\varepsilon\cup P^-_\varepsilon))} J(u) \geq c_0, \ \forall \varepsilon \in (0, \varepsilon_0),$$

where

$$\Gamma := \{\varphi \in C(\triangle, X) : \varphi(\partial_1 \triangle) \subset P^+_\varepsilon, \ \varphi(\partial_2 \triangle) \subset P^-_\varepsilon \ \textit{ and } \ \varphi|_{\partial_0\triangle} = \varphi_0\}.$$

If in addition J satisfies the $(PS)_c$ condition for any $c > 0$, then J has a sign-changing critical point.

In this following, we assume that X is of the form

$$X := \overline{\oplus^\infty_{j=1} X_j}, \qquad \textit{with} \qquad dim X_j < \infty,$$

and that there is another norm $\|\cdot\|_*$ on X such that $(X, \|\cdot\|_X)$ embeds continuously into $(X, \|\cdot\|_*)$.

In this following, we assume that X is of the form

$$X := \overline{\oplus^\infty_{j=1} X_j}, \qquad \text{with} \qquad dim X_j < \infty,$$

and that there is another norm $\|\cdot\|_*$ on X such that $(X, \|\cdot\|_X)$ embeds continuously into $(X, \|\cdot\|_*)$.

We introduce the following notations:

$$Y_k := \oplus^k_{j=1} X_j \qquad \text{and} \qquad Z_k := \overline{\oplus^\infty_{j=k} X_j}, \qquad \text{for} \quad k \geq 2,$$

$$J^\alpha := \{u \in X : J(u) \leq \alpha\}, \qquad \text{for} \quad \alpha \in \mathbb{R}.$$

Notice that

$$(X, \|\cdot\|_X) \hookrightarrow (X, \|\cdot\|_*) \Rightarrow \exists\, C_* > 0, \quad s.t. \ \ \|u\|_* \leq C_* \|u\|_X, \forall u \in X,$$

$$dim Y_k < \infty \Rightarrow \exists\ \theta_k > 0, \quad s.t.\ \ \|u\|_X \leq \theta_k \|u\|_*, \forall u \in Y_k.$$

Assume there exist constants $\rho > 0$ and $q > 2$, and numbers $\rho_k, d_k > 0$ such that

$$\frac{(\rho_k/\theta_k)^q}{\rho_k^2} + \frac{\rho_k(\rho_k/\theta_k)}{\rho_k + C_* d_k \rho_k} > \rho, \tag{2.1}$$

and define

$$B_k := \{u \in Y_k : \|u\| \leq \rho_k\} \ \text{ and } \ N_k := \{u \in Z_k : \frac{\|u\|_*^q}{\|u\|_X^2} + \frac{\|u\|_X \cdot \|u\|_*}{\|u\|_X + d_k \cdot \|u\|_*} = \rho\}.$$

In this following, we introduce a sign-changing critical point theorem.

Theorem 2.3 ([?]). *Let $J \in C^1(X, \mathbb{R})$ be an even functional. Assume that there exist $\rho, \rho_k, d_k > 0$ and $q > 2$ such that (2.1) holds. Assume also that there exists $\varepsilon_0 > 0$ such that (A_{ε_0}) and the following conditions are satisfied:*

(B_1) $a_k := \sup_{u \in \partial B_k} J(u) \leq 0$ *and* $b_k := \inf_{u \in N_k \cap J^{a_0}} J(u) \to +\infty$*, as* $k \to \infty$.

(B_2) $N_k \bigcap J^{a_0} \subset X \setminus (P_\varepsilon^+ \bigcup P_\varepsilon^-)$, $\forall \varepsilon \in (0, \varepsilon_0)$*, where* $a_0 := \max_{u \in B_k} J(u)$.

Then, for k large enough there exists a sequence $\{u_k^n\}_n \subset \overline{V_{\frac{\varepsilon}{2}}(X \setminus (P_\varepsilon^+ \cup P_\varepsilon^-))}$ *such that*

$$J'(u_k^n) \to 0 \qquad in \qquad X^* \qquad as \qquad n \to \infty,$$

and

$$J(u_k^n) \to c_k := \inf_{\gamma \in \Gamma_k} \max_{u \in \gamma(B_k) \bigcap (X \setminus (P_\varepsilon^+ \bigcup P_\varepsilon^-))} J(u) \geq b_k, \quad \forall \varepsilon \in (0, \varepsilon_0),$$

where

$$\Gamma_k := \Big\{\gamma \in C(B_k, X) : \gamma \ is \ odd, \gamma|_{\partial B_k} = id, \sup_{u \in B_k} J(\gamma(u)) \leq a_0$$
$$and \ \ \gamma(P_\varepsilon^+ \cup P_\varepsilon^-) \subset (P_\varepsilon^+ \cup P_\varepsilon^-)\Big\}.$$

If in addition J satisfies the $(PS)_c$ condition for any $c > 0$, then it possesses a sequence $\{u_k\}$ of sigh-changing critical points such that $J(u_k) \to \infty$, as $k \to \infty$.

3. Proof of the main theorems

Now $X = \Big\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) u^2 dx < \infty\Big\}$, for $u \in X$ fixed, we consider the functional

$$\tilde{I}_u(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x) v^2 dx - \int_{\mathbb{R}^N} f(x, u) v dx, \ v \in X. \tag{3.1}$$

By (f_1) and (f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x,t)| \leq \varepsilon|t| + C_\varepsilon |t|^{p-1}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{3.2}$$

and

$$|F(x,t)| \leq \varepsilon|t|^2 + C_\varepsilon |t|^p, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.3}$$

It is easy to prove that $\tilde{I}_u$ is of class C^1, coercive, bounded below, weakly lower semicontinuous and strictly convex in X. Therefore, by Theorem 1.1 in [?], $\tilde{I}_u$ admits a unique global minimizer in X which is the unique solution to the problem

$$-\triangle v + V(x)v = f(x,u), \qquad u \in X. \tag{3.4}$$

Now, we may introduce an auxiliary operator $A : X \to X$: for $u \in X, Au \in X$ is the unique solution of (3.4). Then the set of fixed points of A coincide with the set K of critical points of J.

Furthermore, the operator A has the following important properties.

Lemma 3.1. *Under the assumption (f_1) and (f_2),*

(1) A is continuous and maps bounded sets to bounded sets.

(2) For any $u \in X$, $J'(u) = u - Au$.

(3) There exists $\varepsilon_0 > 0$ such that $A(P_\varepsilon^\pm) \subset P_\varepsilon^\pm$, $\forall \varepsilon \in (0, \varepsilon_0)$.

Proof. (1) Let $\{u_n\} \subset X$ be such that $u_n \to u$ in X. For any $w \in X$, by the definition of A, we have

$$\int_{\mathbb{R}^N} \nabla(Au_n)\nabla w dx + \int_{\mathbb{R}^N} V(x)(Au_n)w dx = \int_{\mathbb{R}^N} f(x,u_n)w dx, \tag{3.5}$$

and

$$\int_{\mathbb{R}^N} \nabla(Au)\nabla w dx + \int_{\mathbb{R}^N} V(x)(Au)w dx = \int_{\mathbb{R}^N} f(x,u)w dx. \tag{3.6}$$

Let $v_n = Au_n$ and $v = Au$. Taking $w = v_n - v \in X$ in (3.5) and (3.6), we obtain

$$\|v_n - v\|^2 = \int_{\mathbb{R}^N} \Big[f(x,u_n) - f(x,u)\Big] \cdot (v_n - v) dx. \tag{3.7}$$

Furthermore, because $v_n = Au_n$ is the solution of (3.4), one has

$$-\triangle v_n + V(x)v_n = f(x,u_n).$$

Thus

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} V(x)v_n^2 dx = \int_{\mathbb{R}^N} f(x,u_n)v_n dx,$$

i.e.

$$\|v_n\|^2 = \int_{\mathbb{R}^N} f(x,u_n)v_n dx.$$

By (3.2) with $\varepsilon = 1$, the Hölder inequality and the Sobolev embedding theorem, one has

$$\begin{aligned} \|v_n\|^2 &\le \|u_n\|_2 \cdot \|v_n\|_2 + C\|u_n\|_p^{p-1} \cdot \|v_n\|_p \\ &\le C_1\|u_n\|_2 \cdot \|v_n\| + C_2\|u_n\|_p^{p-1} \cdot \|v_n\|. \end{aligned}$$

Therefore,

$$\|v_n\| \le C_1\|u_n\|_2 + C_2\|u_n\|_p^{p-1}.$$

By $u_n \to u$ in X, we obtain that $\{v_n - v\}$ is bounded in X. Consider any subsequence of $\{v_n\}$, we still denotes as $\{v_n\}$. Since the embedding $X \hookrightarrow L^q(\mathbb{R}^N)$ is compact for all $q \in [2, 2^*)$, we can assume that

$$v_n - v \rightharpoonup u_0 \quad \text{in} \quad X,$$

$$v_n(x) - v(x) \to u_0(x) \quad a.e. \quad x \in \mathbb{R}^N,$$

$$v_n - v \to u_0 \quad \text{in} \quad L^2(\mathbb{R}^N),$$

and

$$v_n - v \to u_0 \quad \text{in} \quad L^p(\mathbb{R}^N).$$

By the Lemma $A.1$ in [**?**], up to a subsequence, there exists $g_2, h_2 \in L^2(\mathbb{R}^N), g_p, h_p \in L^p(\mathbb{R}^N)$ such that

$$|v_n(x) - v(x)| \leq h_2(x), \; a.e. \; x \in \mathbb{R}^N,$$

$$|u_n(x)| \leq g_2(x), \; a.e. \; x \in \mathbb{R}^N,$$

$$|v_n(x) - v(x)| \leq h_p(x), \; a.e. \; x \in \mathbb{R}^N,$$

$$|u_n(x)| \leq g_p(x), \; a.e. \; x \in \mathbb{R}^N$$

and

$$u_n(x) \to u(x), \; a.e. \; x \in \mathbb{R}^N.$$

By (3.2) with $\varepsilon = 1$, we have

$$\begin{aligned}
&\Big|\Big[f(x, u_n) - f(x, u)\Big] \cdot (v_n - v)\Big| \\
\leq\ & |u_n(x)| \cdot |v_n(x) - v(x)| + C|u_n(x)|^{p-1} \cdot |v_n(x) - v(x)| + |u(x)| \cdot |v_n(x) - v(x)| \\
&+ C|u(x)|^{p-1} \cdot |v_n(x) - v(x)| \\
\leq\ & g_2(x)h_2(x) + Cg_p^{p-1}(x)h_p(x) + |u(x)| \cdot h_2(x) + C|u(x)|^{p-1}h_p(x) \in L^1(\mathbb{R}^N).
\end{aligned}$$

Hence, by the Lebesgue domained convergence theorem, we get

$$\int_{\mathbb{R}^N} \Big[f(x, u_n) - f(x, u)\Big] \cdot (v_n - v)dx \to 0.$$

Consequently, $\|Au_n - Au\| \to 0$. This shows that A is continuous on X.

Moreover, for any $u \in X$, taking $w = Au \in X$ in (3.6), we obtain

$$\int_{\mathbb{R}^N} |\nabla(Au)|^2 dx + \int_{\mathbb{R}^N} V(x)|Au|^2 dx = \int_{\mathbb{R}^N} f(x, u) \cdot Audx.$$

By (3.2), using the Hölder inequality, the Sobolev embedding theorem, we obtain

$$\|Au\| \leq C(1 + \|u\|^{p-2})\|u\|,$$

where $C > 0$ is constant. This shows that Au is bounded in X whenever u is bounded in X.

(2) For any $u, w \in X$, using again (3.6), we have

$$\begin{aligned}
\langle J'(u), w\rangle &= \int_{\mathbb{R}^N} \nabla u \nabla w dx + \int_{\mathbb{R}^N} V(x)uwdx - \int_{\mathbb{R}^N} f(x, u)wdx \\
&= \int_{\mathbb{R}^N} \nabla(u - Au)\nabla w dx + \int_{\mathbb{R}^N} V(x)(u - Au)wdx \\
&= \langle u - Au, w\rangle.
\end{aligned}$$

Hence

$$J'(u) = u - Au.$$

(3) Set $u \in X$ and $v = Au \in X$. We denote $w^+ = \max\{0, w\}$ and $w^- = \min\{0, w\}$, for any $w \in X$. Taking $w = v^+$ in (3.6) and using the Hölder inequality, we obtain

$$\|v^+\|^2 \le \varepsilon \|u^+\|_2 \cdot \|v^+\|_2 + C_\varepsilon \|u^+\|_p^{p-1} \cdot \|v^+\|_p. \tag{3.8}$$

Since $\|z^+\|_q \le \|z - w\|_q$, for all $z \in X, w \in -P$ and $2 \le q \le 2^*$, it follows from the Sobolev embedding theorem that there is a constant $C_1 = C_1(q) > 0$ such that

$$\|u^+\|_q \le C_1 dist(u, -P).$$

Moreover, one can easily verify that

$$dist(v, -P) \le \|v^+\|.$$

Consequently, by (3.8) and the Sobolev embedding theorem, we have

$$dist(v, -P)\|v^+\| \le \|v^+\|^2 \le C_2 \Big[\varepsilon dist(u, -P) + C_\varepsilon dist(u, -P)^{p-1}\Big] \|v^+\|,$$

where $C_2 > 0$. Therefore,

$$dist(v, -P) \le C_2 \Big[\varepsilon dist(u, -P) + C_\varepsilon dist(u, -P)^{p-1}\Big].$$

Similarly, we can prove that

$$dist(v, +P) \le C_3 \Big[\varepsilon dist(u, +P) + C_\varepsilon dist(u, +P)^{p-1}\Big].$$

Hence

$$dist(v, \pm P) \le C_4 \Big[\varepsilon dist(u, \pm P) + C_\varepsilon dist(u, \pm P)^{p-1}\Big],$$

where $C_4 = \max\{C_2, C_3\}$. Now, we can choose $\varepsilon_0 > 0$ small enough such that, for all $\varepsilon \in (0, \varepsilon_0)$,

$$dist(v, \pm P) \le \tfrac{1}{2} dist(u, \pm P) \quad \text{whenever} \quad dist(u, \pm P) < \varepsilon.$$

It then follows that $A(P_\varepsilon^\pm) \subset P_\varepsilon^\pm, \quad \forall \varepsilon \in (0, \varepsilon_0)$. □

Lemma 3.2 (see [**?**], Lemma 3.4)**.** *There exists a locally Lipschitz continuous operator $B : E \triangleq X \backslash K \to X$ (B odd when J is even) such that*

(1) $\langle J'(u), u - Bu\rangle \ge \frac{1}{2}\|u - Au\|^2$, for any $u \in E$.

(2) $\frac{1}{2}\|u - Bu\| \le \|u - Au\| \le 2\|u - Bu\|$, for any $u \in E$.

(3) $B(P_\varepsilon^\pm \bigcap E) \subset P_\varepsilon^\pm$ for any $\varepsilon \in (0, \varepsilon_0)$, where ε_0 is obtained in Lemma 3.1(3).

Remark 3.1. Lemma 3.1 and Lemma 3.2 imply that

$$\langle J'(u), u - Bu\rangle \ge \frac{1}{8}\|u - Bu\|^2,$$

and

$$\|J'(u)\| \le 2\|u - Bu\|.$$

Lemma 3.3. *Let $\rho_1 < \rho_2$ and $\alpha > 0$. Then there exists $\beta > 0$ such that $\|u - Bu\| \ge \beta$ if $u \in X$ is such that $J(u) \in [\rho_1, \rho_2]$ and $\|J'(u)\| \ge \alpha$.*

Proof. Otherwise, there exists a sequence $\{u_n\} \subset X$ such that $J(u_n) \in [\rho_1, \rho_2]$, $\|J'(u_n)\| \geq \alpha$ and $\|u_n - Bu_n\| \to 0$. By remark 3.1, we conclude $\|J'(u_n)\| \to 0$. This is a contradiction. □

Lemma 3.4. *Assuming that* (V), (f_1), (f_2) *and* (f_5) *hold, then the functional* J *satisfies the* $(PS)_c$ *condition at any positive level* c.

Proof. Let $\{u_n\} \subset X$ be a sequence such that $\sup_n |J(u_n)| < \infty$ and $J'(u_n) \to 0$ in X^* as $n \to \infty$. By (f_2), for $0 < \varepsilon_1 < \frac{1}{4}\Big(\frac{1}{2} - \frac{1}{\mu}\Big)V_0$, there exists $\delta_0 > 0$ such that

$$\Big|\frac{1}{\mu}tf(x,t) - F(x,t)\Big| \leq \varepsilon_1 t^2, \quad \forall |t| \leq \delta_0.$$

By (f_1), for $\delta_0 \leq |t| \leq \alpha$ (α is the constant appearing in condition (f_5)), one has

$$\Big|\frac{1}{\mu}tf(x,t) - F(x,t)\Big| \leq C\Big(\frac{1}{\delta_0} + \alpha^{p-2}\Big)t^2.$$

Thus,

$$\Big|\frac{1}{\mu}tf(x,t) - F(x,t)\Big| \leq \varepsilon_1 t^2 + C\Big(\frac{1}{\delta_0} + \alpha^{p-2}\Big)t^2, \quad \forall |t| \leq \alpha.$$

Since $\lim_{|x|\to+\infty} V(x) = +\infty$, there exists $\rho_0 > 0$ such that

$$\frac{1}{4}\Big(\frac{1}{2} - \frac{1}{\mu}\Big)V(x) \geq C\Big(\frac{1}{\delta_0} + \alpha^{p-2}\Big), \quad \forall |x| > \rho_0.$$

Combining with $0 < \varepsilon_1 < \frac{1}{4}\Big(\frac{1}{2} - \frac{1}{\mu}\Big)V_0$, one has

$$\begin{aligned}
&\Big(\frac{1}{2} - \frac{1}{\mu}\Big)\int_{\mathbb{R}^N} V(x)u_n^2dx + \int_{|u_n(x)|\leq\alpha}\Big[\frac{1}{\mu}u_nf(x,u_n) - F(x,u_n)\Big]dx \\
\geq&\Big(\frac{1}{2} - \frac{1}{\mu}\Big)\int_{\mathbb{R}^N} V(x)u_n^2dx - \int_{|u_n(x)|\leq\alpha}\Big[\varepsilon_1 u_n^2 + C\Big(\frac{1}{\delta_0} + \alpha^{p-2}\Big)u_n^2\Big]dx \\
=&\frac{1}{2}\Big(\frac{1}{2} - \frac{1}{\mu}\Big)\int_{\mathbb{R}^N} V(x)u_n^2dx - \int_{|u_n(x)|\leq\alpha}\varepsilon_1 u_n^2dx + \frac{1}{2}\Big(\frac{1}{2} - \frac{1}{\mu}\Big)\int_{\mathbb{R}^N} V(x)u_n^2dx \\
&- \int_{|u_n(x)|\leq\alpha} C\Big(\frac{1}{\delta_0} + \alpha^{p-2}\Big)u_n^2dx \\
\geq&\int_{|u_n(x)|\leq\alpha}\Big[\frac{1}{2}\Big(\frac{1}{2} - \frac{1}{\mu}\Big)V_0 - \varepsilon_1\Big]u_n^2dx + \frac{1}{4}\Big(\frac{1}{2} - \frac{1}{\mu}\Big)\int_{\mathbb{R}^N} V(x)u_n^2dx \\
&- C\Big(\frac{1}{\delta_0} + \alpha^{p-2}\Big)\alpha^2|B_{\rho_0}| \\
\geq&\frac{1}{4}\Big(\frac{1}{2} - \frac{1}{\mu}\Big)\int_{\mathbb{R}^N} V(x)u_n^2dx - C\Big(\frac{1}{\delta_0} + \alpha^{p-2}\Big)\alpha^2|B_{\rho_0}|,
\end{aligned} \tag{3.9}$$

where $B_{\rho_0} := \{x \in \mathbb{R}^N : |x| < \rho_0\}$, $|B_{\rho_0}| := meas(B_{\rho_0})$.

Consequently, by (f_5) and (3.9), we have

$$\begin{aligned}
&J(u_n)-\frac{1}{\mu}\langle J'(u_n),u_n\rangle\\
=&\Big(\frac{1}{2}-\frac{1}{\mu}\Big)\int_{\mathbb{R}^N}|\nabla u_n|^2dx+\Big(\frac{1}{2}-\frac{1}{\mu}\Big)\int_{\mathbb{R}^N}V(x)u_n^2dx+\int_{\mathbb{R}^N}\Big[\frac{1}{\mu}f(x,u_n)u_n-F(x,u_n)\Big]dx\\
\geq&\Big(\frac{1}{2}-\frac{1}{\mu}\Big)\int_{\mathbb{R}^N}|\nabla u_n|^2dx+\Big(\frac{1}{2}-\frac{1}{\mu}\Big)\int_{\mathbb{R}^N}V(x)u_n^2dx\\
&+\int_{|u_n(x)|\leq\alpha}\Big[\frac{1}{\mu}f(x,u_n)u_n-F(x,u_n)\Big]dx\\
\geq&\Big(\frac{1}{2}-\frac{1}{\mu}\Big)\int_{\mathbb{R}^N}|\nabla u_n|^2dx+\frac{1}{4}\Big(\frac{1}{2}-\frac{1}{\mu}\Big)\int_{\mathbb{R}^N}V(x)u_n^2dx-C\Big(\frac{1}{\delta_0}+\alpha^{p-2}\Big)\alpha^2|B_{\rho_0}|\\
\geq&\frac{1}{4}\Big(\frac{1}{2}-\frac{1}{\mu}\Big)\|u_n\|^2-C\Big(\frac{1}{\delta_0}+\alpha^{p-2}\Big)\alpha^2|B_{\rho_0}|.
\end{aligned}$$

Hence, $\{u_n\}$ is bounded in X. Consequently, up to a subsequence, we have $u_n\rightharpoonup u$ in X, $u_n\to u$ in $L^q(\mathbb{R}^N)$ for $2\leq q<2^*$ and $u_n(x)\to u(x)$ for almost all $x\in\mathbb{R}^N$. Using a standard argument, one has $J'(u)=0$. Notice that

$$\begin{aligned}
o_n(1)&=\langle J'(u_n)-J'(u),u_n-u\rangle\\
&=\|u_n-u\|^2-\int_{\mathbb{R}^N}\Big[f(x,u_n)-f(x,u)\Big]\cdot(u_n-u)dx.
\end{aligned}$$

Consequently, by (3.2) and the Hölder inequality, we have

$$\begin{aligned}
\|u_n-u\|^2\leq\ &\varepsilon\|u_n\|_2\cdot\|u_n-u\|_2+C_\varepsilon\|u_n\|_p^{p-1}\cdot\|u_n-u\|_p\\
&+\varepsilon\|u\|_2\cdot\|u_n-u\|_2+C_\varepsilon\|u\|_p^{p-1}\cdot\|u_n-u\|_p+o_n(1)\to 0.
\end{aligned}$$

Hence $u_n\to u$ in X. □

Lemma 3.5. *Assuming that (V) and (f_1)-(f_4) hold. Then the functional J satisfies the $(PS)_c$ condition at any positive level c.*

Proof. Let $\{u_n\}\subset X$ be $(PS)_c$ sequence of J, i.e., $\{J(u_n)\}$ is bounded and $J'(u_n)\to 0$ as $n\to\infty$. If $\{u_n\}$ is unbounded in X, then, without loss of the generality, we may assume that $\|u_n\|\to+\infty$. Consequently, we can assume that $u_n\neq 0$ for all n.

Set $w_n=\frac{u_n}{\|u_n\|}$. Then, up to a subsequence, there exists $w\in X$ such that

$$\begin{aligned}
&w_n\rightharpoonup w \text{ in } X,\\
&w_n\to w \text{ in } L^q(\mathbb{R}^N) \text{ for } 2\leq q<2^*,\\
&w_n(x)\to w(x) \ a.e. \text{ on } \mathbb{R}^N.
\end{aligned}$$

Case 1: $w=0$. By (3.3) and the Sobolev embedding theorem, one has

$$\begin{aligned}
\beta(t):=J(tw_n)&=\frac{1}{2}t^2\|w_n\|^2-\int_{\mathbb{R}^N}F(x,tw_n)dx\\
&\geq\frac{1}{2}t^2-\varepsilon t^2\int_{\mathbb{R}^N}|w_n|^2dx-C_\varepsilon t^p\int_{\mathbb{R}^N}|w_n|^pdx\\
&\geq\frac{1}{2}t^2-C_1\varepsilon t^2-C_2C_\varepsilon t^p\\
&>0
\end{aligned}$$

for small $\varepsilon > 0$ and $t > 0$. Moreover, by (f_3) and Fatou Lemma imply that

$$\begin{aligned}\beta(t) = J(tw_n) &= \frac{1}{2}t^2 - \int_{\mathbb{R}^N} F(x, tw_n)dx \\ &= t^2\Big[\frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(x, tw_n)}{t^2 w_n^2} w_n^2 dx\Big] \\ &\to -\infty\end{aligned}$$

as $t \to +\infty$. Hence $\beta(\cdot)$ has a positive maximum. Let $t_n \in [0, 1]$ be such that

$$J(t_n w_n) = \max_{t\in[0,1]} J(tw_n).$$

By (f_4) we know that $\frac{F(x,t)}{t^2}$ is increasing on $(0, +\infty)$ and decreasing on $(-\infty, 0)$ in t. Hence, for any $t \in (0, 1]$, one has

$$\begin{aligned}J(tw_n) &= \frac{1}{2}t^2 - \int_{\mathbb{R}^N} F(x, tw_n)dx \\ &= t^2\Big[\frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(x, tw_n)}{(tw_n)^2} w_n^2 dx\Big] \\ &\geq t^2\Big[\frac{1}{2} - \int_{\mathbb{R}^N} F(x, w_n)dx\Big] \\ &= t^2 J(w_n),\end{aligned}$$

and hence

$$J(t_n w_n) \geq \max_{t\in[0,1]} t^2 J(w_n).$$

Notice that $\lim_{n\to\infty} J(w_n) = \frac{1}{2}$, one has $J(w_n) > 0$ for large n. Hence, for large n,

$$J(t_n w_n) \geq \max_{t\in[0,1]} t^2 J(w_n) = J(w_n) = \frac{1}{2} + o_n(1).$$

But, on the other hand,

$$\begin{aligned}J(t_n w_n) &= J(t_n w_n) - \frac{1}{2}\langle J'(t_n w_n), t_n w_n\rangle \\ &= \int_{\mathbb{R}^N} \Big[\frac{1}{2}t_n w_n f(x, t_n w_n) - F(x, t_n w_n)\Big] = o_n(1),\end{aligned}$$

a contradiction.

Case 2: $w \neq 0$. By (f_3), there exists $\rho_1 > 0$ such that

$$F(x, t) \geq 0 \ \text{ for all } \ x \in \mathbb{R}^N \text{ and } |t| \geq \rho_1.$$

By (3.3), one has

$$|F(x, t)| \leq C_1 t^2 \ \text{ for all } \ x \in \mathbb{R}^N \text{ and } |t| \leq \rho_1.$$

Consequently,

$$F(x, t) \geq -C_1 t^2, \ \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

Hence

$$\int_{\{x\in\mathbb{R}^N:w(x)=0\}}\frac{F(x,u_n)}{\|u_n\|^2}dx \geq \frac{-C_1\int_{\{x\in\mathbb{R}^N:w(x)=0\}}u_n^2dx}{\|u_n\|^2}$$
$$\geq \frac{-C_1\|u_n\|_2^2}{\|u_n\|^2} \geq \frac{-C_1C_2\|u_n\|^2}{\|u_n\|^2} = -C_1C_2,$$

which implies that

$$\liminf_{n\to\infty}\int_{\{x\in\mathbb{R}^N:w(x)=0\}}\frac{F(x,u_n)}{\|u_n\|^2}dx \geq -C_1C_2 > -\infty.$$

Moreover, set $\Omega := \{x\in\mathbb{R}^N : w(x)\neq 0\}$. Then $|u_n(x)|\to+\infty$ for each $x\in\Omega$. By (f_3) and Fatou Lemma, one has

$$\int_\Omega\frac{F(x,u_n)}{\|u_n\|^2}dx = \int_\Omega\frac{F(x,u_n)}{u_n^2(x)}w_n^2(x)dx \to +\infty$$

as $n\to\infty$. Therefore, the boundedness of $\{J(u_n)\}$ implies that

$$o_n(1)+\frac{1}{2} \geq \int_{\mathbb{R}^N}\frac{F(x,u_n)}{\|u_n\|^2}dx$$
$$= \int_{\{x\in\mathbb{R}^N:w(x)=0\}}\frac{F(x,u_n)}{\|u_n\|^2}dx + \int_{\{x\in\mathbb{R}^N:w(x)\neq 0\}}\frac{F(x,u_n)}{\|u_n\|^2}dx.$$

Let $n\to\infty$, we obtain $\frac{1}{2}\geq+\infty$, a contradiction.

Summing up the above arguments we know that $\{u_n\}\subset X$ is bounded.

Now, we prove that the sequence $\{u_n\}$ possesses a convergent subsequence in X. Indeed, since $\{u_n\}$ is bounded in X, up to a subsequence, there exists $u\in X$ such that

$$u_n \rightharpoonup u \text{ in } X,$$
$$u_n \to u \text{ in } L^q(\mathbb{R}^N) \text{ for } 2\leq q<2^*,$$
$$u_n(x)\to u(x) \ a.e. \text{ on } \mathbb{R}^N.$$

Using a standard argument, one has $J'(u)=0$. Notice that

$$o_n(1) = \langle J'(u_n)-J'(u), u_n-u\rangle$$
$$= \|u_n-u\|^2 - \int_{\mathbb{R}^N}\Big[f(x,u_n)-f(x,u)\Big]\cdot(u_n-u)dx$$
$$= \|u_n-u\|^2 + o_n(1).$$

Hence $u_n\to u$ in X. □

Lemma 3.6. *For $q\in[2,2^*]$, there exists $k_q>0$ such that for any $\varepsilon>0$*

$$\|u\|_q \leq k_q\varepsilon, \qquad \forall u\in P_\varepsilon^+\bigcap P_\varepsilon^-.$$

Proof. For any $u\in X$, this follows from the fact that

$$\|u^\pm\|_q = \inf_{w\in\mp P}\|u-w\|_q \leq C_q\inf_{w\in\mp P}\|u-w\| = C_q dist(u,\mp P),$$

where $C_q>0$ is the Sobolev constant in the continuous embedding $X\hookrightarrow L^q(\mathbb{R}^N)$ for all $q\in[2,2^*]$. □

Lemma 3.7. *Under the condition* (f_1) *and* (f_2), *for* $\varepsilon > 0$ *small enough, we have*

$$J(u) \geq \frac{1}{4}\varepsilon^2, \quad \forall u \in \partial P_\varepsilon^+ \bigcap \partial P_\varepsilon^-.$$

Proof. Let $u \in \partial P_\varepsilon^+ \bigcap \partial P_\varepsilon^-$. It is clear that

$$\|u^\pm\| \geq dist(u, \mp P) = \varepsilon, \quad \forall \varepsilon > 0.$$

Using Lemma 3.6, by (f_1) and (f_2), for small $\delta > 0$, there exists $C_\delta > 0$ such that

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|u\|^2 - \delta\|u\|_2^2 - C_\delta\|u\|_p^p \\ &\geq \frac{1}{4}\varepsilon^2, \end{aligned}$$

for $\varepsilon > 0$ small enough. □

Consider the eigenvalue problem

$$\int_{\mathbb{R}^N} \Big(\nabla u \nabla \varphi + V(x) u\varphi\Big) dx = \lambda \int_{\mathbb{R}^N} u\varphi dx, \quad \forall \varphi \in X.$$

For real number λ, if there exists $0 \neq u \in X$ such that

$$\int_{\mathbb{R}^N} \Big(\nabla u \nabla \varphi + V(x) u\varphi\Big) dx = \lambda \int_{\mathbb{R}^N} u\varphi dx, \quad \forall \varphi \in X,$$

then λ is called an eigenvalue of the operator $L = -\triangle + V$. By the assumption (V) and the compactness of the embedding $X \hookrightarrow L^2(\mathbb{R}^N)$, we know that the spectrum $\sigma(L) = \{\lambda_1, \lambda_2, \cdots, \lambda_j, \cdots\}$ of L with

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots$$

and $\lambda_j \to \infty$ (see page 62 in [**?**]). It is well known that each $\lambda_j (j \geq 2)$ has finite multiplicity, the principle eigenvalue λ_1 is simple with positive eigenfunction e_1, and the eigenfunctions e_j corresponding to $\lambda_j (j \geq 2)$ are sign-changing. Let X_j be the eigenspace associated to λ_j. We set $k \geq 2$

$$Y_k := \oplus_{j=1}^k X_j \qquad \text{and} \qquad Z_k := \overline{\oplus_{j=k}^\infty X_j}.$$

Note that any element of $Z_k \backslash \{0\}$ is sign-changing.

We define

$$N_k := \Big\{ u \in Z_k : \frac{\|u\|_p^p}{\|u\|^2} + \frac{\|u\| \cdot \|u\|_p}{\|u\| + \beta_k^{-\sigma} \cdot \|u\|_p} = \rho \Big\},$$

where

$$\rho \!: = \tfrac{1}{8C_\varepsilon}, C_\varepsilon \text{ is obtained in (3.3).} \tag{3.10}$$

$$\beta_k := \sup_{\substack{u \in Z_k \\ \|u\|=1}} \|u\|_p \qquad \text{and} \qquad \sigma > 0. \tag{3.11}$$

Lemma 3.8. *Under the conditions* (f_1) *and* (f_2),

$$\lim_{k \to \infty} \inf_{u \in N_k} J(u) = +\infty.$$

Proof. By the definition of N_k, we have

$$\frac{\|u\|_p^p}{\|u\|^2} \leq \rho, \quad \forall u \in N_k.$$

For any $u \in N_k$, for $\varepsilon > 0$ small enough, by (3.3) and (3.10), the Sobolev embedding theorem, we have

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|u\|^2 - \varepsilon C\|u\|^2 - C_\varepsilon \|u\|_p^p \\ &\geq \frac{1}{4}\|u\|^2 - C_\varepsilon \|u\|_p^p \\ &= \|u\|^2\Big(\frac{1}{4} - C_\varepsilon \frac{\|u\|_p^p}{\|u\|^2}\Big) \\ &\geq \frac{1}{8}\|u\|^2. \end{aligned} \tag{3.12}$$

Furthermore, for any $u \in N_k$, one has

$$\begin{aligned} 0 < \rho &= \frac{\|u\|_p^p}{\|u\|^2} + \frac{\|u\| \cdot \|u\|_p}{\|u\| + \beta_k^{-\sigma}\|u\|_p} \\ &\leq \frac{\|u\|_p^p}{\|u\|^2} + \|u\|_p \\ &\leq \beta_k^p \|u\|^{p-2} + \beta_k \|u\| \\ &\leq \Big(\beta_k^p + \beta_k\Big)\Big(\|u\|^{p-2} + \|u\|\Big). \end{aligned}$$

Hence,

$$\|u\|^{p-2} + \|u\| \geq \frac{\rho}{\beta_k^p + \beta_k} > 0. \tag{3.13}$$

From Lemma 3.8 in [**?**], we know that $\beta_k \to 0$, as $k \to \infty$. By (3.13), we have $\lim_{k\to\infty} \|u\| = +\infty$. By (3.12), one has

$$\lim_{k\to\infty} \inf_{u\in N_k} J(u) = +\infty.$$

This completes the proof of Lemma 3.8. □

Lemma 3.9. *For any $\alpha > 0$, we have*

$$\delta_0(\alpha) := dist(N_k \bigcap J^\alpha, P) > 0,$$

where $P = (+P)\bigcup(-P)$.

Proof. The proof is similar to the proof of Lemma 5.4 in [**?**]. □

Proof of Theorem 1.1.

Step 1: The existence of a positive and a negative solution.

By (3.3) and the Sobolev embedding theorem, there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$J(u) \geq \frac{1}{2}\|u\|^2 - \varepsilon C_1\|u\|^2 - C_\varepsilon C_2 \|u\|^p.$$

Consequently, there exists $r > 0$ (small enough) such that

$$\inf_{\|u\|=r} J(u) \geq \frac{1}{4} r^2 > 0.$$

For any $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, by (f_5), similarly [?], one has

$$F(x,t) \geq C_1 |t|^\mu, \quad \forall |t| \geq \alpha,$$

where $C_1 := \frac{1}{\alpha^\mu} \inf_{x\in\mathbb{R}^N, |t|=\alpha} F(x,t) > 0$. Combining with (3.3), we have

$$F(x,t) \geq C_1 |t|^\mu - C_2 t^2, \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{3.14}$$

for some positive constant C_1 and C_2. Thus, by (3.14) and the Sobolev embedding theorem, one has

$$J(u) \leq \Big(\frac{1}{2} + C_2\Big) \|u\|^2 - C_1 \int_{\mathbb{R}^N} |u|^\mu dx. \tag{3.15}$$

Hence, for fixed $e \in X\backslash\{0\}$, it is easy to prove that

$$J(te) \to -\infty \text{ as } t \to +\infty.$$

Therefore, we can find $e_\pm \in \pm P$ such that

$$\|e_\pm\| > r \text{ and } J(e_\pm) < 0.$$

This shows that the condition (A_1) of Theorem 2.1 is satisfied. By Lemma 3.1, Lemma 3.2 and Lemma 3.3,the condition (A_{ε_0}) is satisfied for $\varepsilon_0 > 0$ small enough. By Lemma 3.4, J satisfies the $(PS)_c$ condition at any positive level c. Hence, by Theorem 2.1, J has critical point $u_\pm \in \pm P\backslash\{0\}$. The strong maximum principle implies that $u_+(x)$ is positive and $u_-(x)$ is negative. Thus, the equation (1.1) has a positive and a negative solutions.

In the following proof, we adopt the notations of Theorem 2.2.

Step 2: The existence of a sign changing solution.

Using the the main idea of [?], we will verify the assumptions of Theorem 2.2.

Let $v_1, v_2 \in C_0^\infty(\mathbb{R}^N)\backslash\{0\}$ be such that $supp(v_1) \bigcap supp(v_2) = , v_1 \leq 0$ and $v_2 \geq 0$. we define the continuous map $\varphi_0 : \triangle \to X$ by $\varphi_0(s,t) = R(sv_1 + tv_2)$ for all $(s,t) \in \triangle$, where $R > 0$ is a constant to be determined later. Obviously, $\varphi_0(0,t) \in P_\varepsilon^+$ and $\varphi_0(s,0) \in P_\varepsilon^-$ for all $\varepsilon > 0$. This implies that $\varphi_0(\partial_1 \triangle) \subset P_\varepsilon^+$ and $\varphi_0(\partial_2 \triangle) \subset P_\varepsilon^-$, i.e. Theorem 2.2(1) holds. Now a simple computation that

$$\delta := \min \Big\{ \|(1-t)v_1 + tv_2\|_2 : t \in [0,1] \Big\} > 0.$$

Then $\|u\|_2 \geq \delta R$ for $u \in \varphi_0(\partial_0 \triangle)$ and it follows from Lemma 3.6 that

$$\varphi_0(\partial_0 \triangle) \bigcap P_\varepsilon^+ \bigcap P_\varepsilon^- = ,$$

for R large enough and for any $\varepsilon > 0$.

By (3.15), combining with Lemma 3.7, for R large enough and $\varepsilon > 0$ small enough, we obtain

$$c_0 = \sup_{u\in\varphi_0(\partial_0\triangle)} J(u) < 0 < c^* := \inf_{u\in\partial P_\varepsilon^+ \bigcap \partial P_\varepsilon^-} J(u).$$

By Theorem 2.2, J has a sign-changing critical point.

Step 3: The existence of infinitely many high-energy solutions.

Because $\dim Y_k < +\infty$, there exists $\theta_k > 0$ such that $\|u\| \le \theta_k \|u\|_\mu$ for any $u \in Y_k$. By (3.15), one has

$$J(u) \le \Big(\frac{1}{2} + C_2\Big)\|u\|^2 - C_1\theta_k^{-\mu}\|u\|^\mu, \qquad \forall u \in Y_k.$$

Hence, we have $J(u) \to -\infty$ on Y_k as $\|u\| \to +\infty$.

We can then choose $\rho_k > 0$ large enough so that

$$N_k \bigcap J^{a_0} \neq \emptyset,$$

$$\frac{(\rho_k/\theta_k)^p}{\rho_k^2} + \frac{\rho_k(\rho_k/\theta_k)}{\rho_k + C_p\beta_k^{-\sigma}\rho_k} > \rho$$

and

$$a_k := \max_{\substack{u\in Y_k \\ \|u\|=\rho_k}} J(u) < 0,$$

where $a_0 := \max\limits_{u\in B_k} J(u) > 0$, σ is given by (3.11) and C_p is the Sobolev constant. Combining with Lemma 3.8, the condition (B_1) of Theorem 2.3 is satisfied. By Lemma 3.9,

$$\delta_0(a_0) := dist(N_k \bigcap J^{a_0}, P) > 0.$$

For any $u \in N_k \bigcap J^{a_0}, \quad v \in P, \quad w \in P_\varepsilon^+ \bigcap P_\varepsilon^-$, one has

$$0 < \delta_0(a_0) = \inf_{\substack{u\in N_k\bigcap J^{a_0} \\ v\in P}} \|u - v\| \le \|u - w\| + \|w^+ - v^+\| + \|w^- - v^-\|,$$

where $v^+ = \max\{v, 0\}, v^- = \min\{v, 0\}, w^+ \in P_\varepsilon^+, w^- \in P_\varepsilon^-$. Hence

$$0 < \delta_0(a_0) = dist(u, P_\varepsilon^+ \bigcup P_\varepsilon^-) + 2\varepsilon.$$

Set $\varepsilon_0 \in \Big(0, \frac{1}{2}\delta_0(a_0)\Big)$, one has

$$dist(u, P_\varepsilon^+ \bigcup P_\varepsilon^-) \ge \delta_0(a_0) - 2\varepsilon > 0, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

This implies that the condition (B_2) of Theorem 2.3 holds. Thus, by Theorem 2.3, we obtain that J possesses a sequence $\{u_k\}$ of sign-changing critical points such that $J(u_k) \to \infty$ as $k \to \infty$. □

Proof of Theorem 1.2.

Step 1: The existence of a positive and a negative solution.

By (f_3) and Fatou Lemma, for any $u \in X\backslash\{0\}$, one has

$$\begin{aligned} J(tu) &= \frac{1}{2}t^2\|u\|^2 - \int_{\mathbb{R}^N} F(x, tu)dx \\ &= t^2\Big[\frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} \frac{F(x, tu)}{t^2u^2} \cdot u^2 dx\Big] \\ &\to -\infty \end{aligned}$$

as $t \to +\infty$. Therefore, we can find $e_\pm \in \pm P$ such that

$$\|e_\pm\| > r \text{ and } J(e_\pm) < 0.$$

Similarly to the step 1 of the proof in Theorem 1.1, there exist a positive and a negative of the problem (1.1).

Step 2: The existence of a sign changing solution.

The proof of step 2 is same to the proof of step 2 of Theorem 1.1.

By (f_3), combining with Lemma 3.7, for R large enough and $\varepsilon > 0$ small enough, we obtain

$$c_0 = \sup_{u\in\varphi_0(\partial_0\triangle)} J(u) < 0 < c^* := \inf_{u\in\partial P_\varepsilon^+ \bigcap \partial P_\varepsilon^-} J(u).$$

By Theorem 2.2, J has a sign-changing critical point.

Step 3: The existence of infinitely many high-energy solutions.

Notice that, let $S_{\tilde\rho} = \{u \in X : \|u\| = \tilde\rho\}$. For $u \in S_{\tilde\rho}$, by (3.3) and the Sobolev embedding theorem, one has

$$\begin{aligned}
J(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)dx \\
&\geq \frac{1}{2}\|u\|^2 - \varepsilon\int_{\mathbb{R}^N} u^2dx - C_\varepsilon\int_{\mathbb{R}^N} |u|^pdx \\
&\geq \frac{1}{2}\|u\|^2 - C_1\varepsilon\|u\|^2 - C_2C_\varepsilon\|u\|^p \\
&\geq \frac{1}{4}\tilde\rho^2 > 0
\end{aligned}$$

for small $\varepsilon > 0$ and $\tilde\rho > 0$. For the finite dimensional subspace Y_k, we claim that there exists a constant $\tilde R > \tilde\rho$ such that $J < 0$ on $Y_k \backslash B_{\tilde R}$. In fact, if the conclusion is false, then there exists a sequence $\{u_n\} \subset Y_k$ with $\|u_n\| \to +\infty$ such that $J(u_n) \geq 0$. Consequently,

$$\frac{1}{2} \geq \int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|^2}dx.$$

Let $w_n = \frac{u_n}{\|u_n\|}$, then up to a subsequence, there exists $w \in X$ such that

$$\begin{aligned}
&w_n \rightharpoonup w \text{ in } X, \\
&w_n \to w \text{ in } L^q(\mathbb{R}^N) \text{ for } 2 \leq q < 2^*, \\
&w_n(x) \to w(x) \ a.e. \text{ on } \mathbb{R}^N.
\end{aligned}$$

Case 1: $w = 0$. By the equivalency of all norms in Y_k, there exists a constant $\theta_k > 0$ such that

$$\|u\|_2^2 \geq \theta_k\|u\|^2, \quad \forall u \in Y_k.$$

Hence

$$0 = \lim_{n\to\infty} \|w_n\|_2^2 \geq \theta_k \lim_{n\to\infty} \|w_n\|^2 = \theta_k,$$

a contradiction.

Case 2: $w \neq 0$. In this case, by case 2 in Lemma 3.5, we have

$$\liminf_{n\to\infty} \int_{\{x\in\mathbb{R}^N : w(x)=0\}} \frac{F(x,u_n)}{\|u_n\|^2}dx \geq -C_1C_2 > -\infty,$$

and

$$\int_{\{x\in\mathbb{R}^N:w(x)\neq 0\}}\frac{F(x,u_n)}{\|u_n\|^2}dx\to+\infty,$$

as $n\to\infty$. Note that

$$\frac{1}{2}\geq\int_{\mathbb{R}^N}\frac{F(x,u_n)}{\|u_n\|^2}dx=\int_{\{x\in\mathbb{R}^N:w(x)=0\}}\frac{F(x,u_n)}{\|u_n\|^2}dx+\int_{\{x\in\mathbb{R}^N:w(x)\neq 0\}}\frac{F(x,u_n)}{\|u_n\|^2}dx.$$

Let $n\to\infty$, by (f_3), one has

$$\frac{1}{2}\geq\liminf_{n\to\infty}\int_{\{x\in\mathbb{R}^N:w(x)=0\}}\frac{F(x,u_n)}{\|u_n\|^2}dx+\liminf_{n\to\infty}\int_{\{x\in\mathbb{R}^N:w(x)\neq 0\}}\frac{F(x,u_n)}{\|u_n\|^2}dx=+\infty,$$

a contradiction. Hence there exists a constant $\tilde{R}>\tilde{\rho}$ such that $J<0$ on $Y_k\setminus B_{\tilde{R}}$.

We can then choose $\rho_k>0$ large enough so that

$$\frac{(\rho_k/\theta_k)^p}{\rho_k^2}+\frac{\rho_k(\rho_k/\theta_k)}{\rho_k+C_p\beta_k^{-\sigma}\rho_k}>\rho$$

and

$$a_k:=\max_{\substack{u\in Y_k\\ \|u\|=\rho_k}}J(u)<0,$$

where σ is given by (3.11) and C_p is the Sobolev constant. The remanent proof is similar to the step 3 of Theorem 1.1. □

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