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EXISTENCE AND GLOBAL STABILITY OF ALMOST AUTOMORPHIC SOLUTIONS FOR SHUNTING INHIBITORY CELLULAR NEURAL NETWORKS WITH TIME-VARYING DELAYS IN LEAKAGE TERMS ON TIME SCALES*

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Abstract In this paper, shunting inhibitory cellular neural networks(SICNNs) with time-varying delays in leakage terms on time scales are investigated. With the aid of the existence of the exponential dichotomy of linear dynamic equations on time scales, fixed point theorem and the theory of calculus on time scales, we establish some sufficient conditions to ensure the existence and exponential stability of almost automorphic solutions for the model. An example with its numerical simulations is given to illustrate the feasibility and effectiveness of the theoretical findings.

Keywords Shunting inhibitory cellular neural networks, almost automorphic solution, exponential stability, leakage term, time-varying delay.

MSC(2010) 34K14, 34K25, 34C25, 45G10.

1. Introduction

Since the classical research of Roska and Chua [42], cellular neural networks with delay play an important role in variety of areas such as signal processing, pattern recognition, chemical processes, nuclear reactors, biological systems, static image processing, associative memories, optimization problems and so on [14–16,56]. Thus many authors pay much attention to the dynamical properties of networks and many excellent findings have been reported. We refer the readers to [7,9,11,18,22,24–28,31,41,43,45–47,49–55].

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Recently, some scholars pointed out that in the negative feedback terms of neural networks, there often exists a leakage delay [5,20,36,57]. Generally speaking, the leakage delay has important effect on the stability of neural networks. Various studies show that it is difficult for us to handle the dynamical behavior of neural networks with leakage delay. Therefore, it is meaningful to consider neural networks with time delays in leakage terms [33].

In 1990, Hilger [21] proposed the theory of time scales. Many scholars [19, 29, 30, 34, 35, 57, 59] suggested that one could investigate continuous time and discrete time neural networks in a unity way [33] by applying the theory of time scales. Thus it is significant to investigate the dynamical behaviors of neural networks on time scales.

In real word, almost periodicity is universal than periodicity. Moreover, almost automorphic functions, which were introduced by Bochner, are much more general than almost periodic functions. Almost automorphic solutions in the context of differential equations were studied by several authors. We refer the readers to [1–4,12,13,17,20,21,38–40,48]. However, to the best of our knowledge, there are very few papers published on the almost automorphic solutions of cellular neural networks with time-varying leakage delays on time scales.

Inspired by the discussion above, in this paper, we consider the following shunting inhibitory cellular neural networks with time-varying leakage delays on time scales

$$x_{ij}^{\Delta}(t) = -a_{ij}(t)x_{ij}(t-\eta_{ij}(t)) + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)f(x_{kl}(t-\tau(t)))x_{ij}(t) + L_{ij}(t), \quad (1.1)$$

where $\mathbb{T}$ is an almost periodic time scale, C_{ij} denotes the cell at the (i,j) position of the lattice, the r-neighborhood $N_r(i,j)$ of C_{ij} is

$$N_r(i,j) = \{C_{kl} : \max(|k-i|, |l-i|) \le r, 1 \le k \le m, 1 \le l \le n\}.$$

x_{ij} is the activity of the cell C_{ij}, $L_{ij}(t)$ is the external put to C_{ij}, the function $a_{ij}(t) > 0$ stands for the passive decay rate of the cell activity, $C_{ij}^{kl} \ge 0$ is the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell C_{ij}, and the activity function $f(x_{kl})$ is a continuous function representing the output or firing rate of the cell C_{kl}, $\eta_i(t) \ge 0$ and $\tau(t) \ge 0$ denote the leakage delay and transmission delay, $t - \eta_i(t) \in \mathbb{T}, t-\tau(t) \in \mathbb{T}$ for all $t \in \mathbb{T}, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$

In this paper, we will apply the existence of the exponential dichotomy of linear dynamic equations on time scales, fixed point theorem and the theory of calculus on time scales to investigate the existence and exponential stability of almost automorphic solutions for model (1.1).

For convenience, we denote by $[a,b]_{\mathbb{T}} = \{t | t \in [a,b] \cap \mathbb{T}\}, \Lambda = \{11, 12, \cdots, 1n, 21, 22, \cdots, mn\}$. For an almost automorphic function $f : \mathbb{T} \to \mathbb{R}$, $f^+ = \sup_{t\in\mathbb{T}} |f(t)|$, $f^- = \inf_{t\in\mathbb{T}} |f(t)|$. We denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^+$ the set of positive real numbers and by $\mathbb{X}$ a real Banach space with the norm $||.||$. The initial conditions associated with system(1.1) are of the form:

$$x_{ij}(s) = \varphi_{ij}(s), s \in (-\theta, 0]_{\mathbb{T}}, \quad (1.2)$$

where $\theta = \max\{\max_{ij\in\Lambda} \eta_{ij}^+, \tau^+\}, \varphi_{ij} \in C^1([-\theta, 0]_{\mathbb{T}}, \mathbb{R})$ and $i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$.

The remainder of the paper is organized as follows. In Section 2, we introduce some lemmas and definitions on almost automorphic solutions of system (1.1). In Section 3, we present some sufficient conditions for the existence of almost automorphic solutions of (1.1). Some sufficient conditions on the global exponential stability of almost automorphic solutions of (1.1) are established in Section 4. An example is given to illustrate the effectiveness of the obtained findings in Section 5. A brief conclusion is drawn in Section 6.

2. Preliminary results

In this section, we will give some definitions and lemmas.

Definition 2.1. ([6]) Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \text{ and } \mu(t) = \sigma(t) - t.$$

Lemma 2.1. ([6]) *Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, then*
(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$*;*
(ii) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
(iii) $e_p(t, s)e_p(s, r) = e_p(t, r)$
(iv) $(e_p(t, s))^\Delta = p(t)e_p(t, s)$.

Lemma 2.2. ([6]) *Let f, g be Δ-differentiable functions on $\mathbb{T}$, then*
(i) $(\nu_1 f + \nu_2 g)^\Delta = \nu_1 f^\Delta + \nu_2 g^\Delta$, *for any constants* ν_1, ν_2;
(ii) $(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t))$.

Lemma 2.3. ([6]) *Let $p(t) \geq 0$ for $t \geq s$, then $e_p(t, s) \geq 1$.*

Definition 2.2. ([6]) A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$; $p : \mathbb{T} \to \mathbb{R}$ is called positively regressive if $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathbf{R} = \mathbf{R}(\mathbb{T}, \mathbb{R})$ and the set of all positively regressive functions and rd-continuous functions will be denoted by $\mathbf{R}^+ = \mathbf{R}^+(\mathbb{T}, \mathbb{R})$.

Lemma 2.4. ([6]) *If $p \in \boldsymbol{R}^+$, then*
(i) $e_p(t, s) > 0$, *for all* $t, s \in \mathbb{T}$;
(ii) if $p(t) \leq q(t)$ *for all* $t \geq s$, *then* $e_p(t, s) \leq e_q(t, s)$ *for all* $t \geq s$.

Lemma 2.5. ([6]) *If $p \in \boldsymbol{R}$ and $a, b, c \in \mathbb{T}$, then*

$$[e_p(c, .)]^\Delta = -p[e_p(c, .)]^\sigma$$

and

$$\int_a^b p(t)e_p(c, \sigma(t))\Delta t = e_p(c, a) - e_p(c, b).$$

Lemma 2.6. ([6]) *Let $a \in \mathbb{T}^k, b \in \mathbb{T}$ and $f : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ is continuous at (t, t) where $t \in \mathbb{T}^k$ with $t > a$. Also assume that $f^\Delta(t)$ is rd-continuous on $[a, \sigma(t)]$. Suppose that for each $\varepsilon > 0$, there exists a neighborhood U of $\epsilon \in [a, \sigma(t)]$ such that*

$$|f(\sigma(t), \epsilon) - f(s, \epsilon) - f^\Delta(t, \epsilon)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \text{ for all } s \in U,$$

where f^{Δ} *denotes the derivative of* f *with respect to the first variable. Then*
(i) $g(t) := \int_a^t f(t,\epsilon)\Delta\epsilon$ *implies* $g^{\Delta}(t) := \int_a^t f^{\Delta}(t,\epsilon)\Delta\epsilon + f(\sigma(t),t)$;
(ii) $h(t) := \int_t^b f(t,\epsilon)\Delta\epsilon$ *implies* $h^{\Delta}(t) := \int_t^b f^{\Delta}(t,\epsilon)\Delta\epsilon - f(\sigma(t),t)$.

Definition 2.3. ([23,32,38]) A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi := \{\epsilon \in \mathbb{R} : t \pm \epsilon \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}.$$

Definition 2.4. ([38]) Let $\mathbb{T}$ be an almost periodic time scale.
(i) A function $f(t) : \mathbb{T} \to \mathbb{X}$ is said to be almost automorphic, if for any sequence $\{s_n\}_{n=1}^{\infty} \subset \Pi$, there is a subsequence $\{\epsilon_n\}_{n=1}^{\infty} \subset \{s_n\}_{n=1}^{\infty}$ such that $g(t) = \lim_{n\to\infty} f(t+\epsilon_n)$ is well defined for each $t \in \mathbb{T}$ and $\lim_{n\to\infty} g(t-\epsilon_n) = f(t)$ for each $t \in \mathbb{T}$. Denote by $AA(\mathbb{T},\mathbb{X})$ the set of all such functions;
(ii) A continuous function $f : \mathbb{T}\times\mathbb{X} \to \mathbb{X}$ is said to be almost automorphic, if $f(t,x)$ is almost automorphic in $t \in \mathbb{T}$ uniformly in $x \in \mathbb{B}$, where $\mathbb{B}$ is any bounded subset of $\mathbb{X}$. Denote by $AA(\mathbb{T}\times\mathbb{X},\mathbb{X})$ the set of all such functions.

Lemma 2.7. ([38]) *Let* $f, g \in AA(\mathbb{T},\mathbb{X})$. *Then we have the following*
(i) $f + g \in AA(\mathbb{T},\mathbb{X})$;
(ii) $\alpha \in AA(\mathbb{T},\mathbb{X})$ *for any constant* $\alpha \in \mathbb{R}$;
(iii) if $\varphi : \mathbb{X} \to \mathbb{Y}$ *is a continuous function, then the composite function* $\varphi\circ f : \mathbb{T} \to \mathbb{Y}$ *is almost automorphic.*

Lemma 2.8. ([33,38]) *Let* $f \in AA(\mathbb{T}\times\mathbb{X},\mathbb{X})$ *and* f *satisfies the Lipschitz condition in* $x \in \mathbb{X}$ *uniformly in* $t \in \mathbb{T}$. *If* $\varphi \in AA(\mathbb{T},\mathbb{X})$*, then* $f(t,\varphi(t))$ *is almost automorphic.*

Definition 2.5. ([32,58]) Let $x \in \mathbb{R}^n$ and $A(t)$ be a $n\times n$ matrix-valued function on $\mathbb{T}$, the linear system

$$x^{\Delta}(t) = A(t)x(t), t \in \mathbb{T} \tag{2.1}$$

is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constants $k_i, \alpha_i, i = 1,2$, projection P and the fundamental solution matrix $X(t)$ of (2.1) satisfying

$$|X(t)PX^{-1}(s)| \leq k_1 e_{\ominus\alpha_1}(t,s), s,t \in \mathbb{T}, t \geq s$$

and

$$|X(t)(I-P)X^{-1}(s)| \leq k_2 e_{\ominus\alpha_2}(t,s), s,t \in \mathbb{T}, t \leq s,$$

where $|.|$ is a matrix norm on $\mathbb{T}$, that is, if $A = (a_{ij})_{n\times n}$, then we can take $|A| = (\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2)^{\frac{1}{2}}$.

Lemma 2.9. ([38]) *If* $A(t) \in AA(\mathbb{T},\mathbb{R}^{n\times n})$ *such that* $\{A^{-1}(t)\}_{t\in\mathbb{T}}$ *and* $\{((I+\mu(t))A(t))^{-1}\}_{t\in\mathbb{T}}$ *are bounded. Moreover, if* $g \in AA(\mathbb{T},\mathbb{R}^n)$ *and (2.1) admits an exponential dichotomy, then the following system*

$$x^{\Delta}(t) = A(t)x(t) + g(t) \tag{2.2}$$

has a solution $x(t) \in AA(\mathbb{T},\mathbb{R}^n)$ *and* $x(t)$ *is expressed as follows*

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))g(s)\Delta s - \int_t^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))g(s)\Delta s,$$

where $X(t)$ *is the fundamental solution matrix of (2.1),* I *denotes the* $n\times n$*-identity matrix.*

Lemma 2.10. ([32]) *Let $c_i > 0$ and $-c_i(t) \in \boldsymbol{R}^+, \forall t \in \mathbb{T}$. If $\min_{1\le i\le n}\{\inf_{t\in\mathbb{T}} c_i(t)\} = m > 0$, then the linear system*

$$x^{\Delta}(t) = diag(-c_1(t), -c_2(t), \cdots, -c_n(t))x(t) \tag{2.3}$$

admits an exponential dichotomy on $\mathbb{T}$.

Definition 2.6. ([33]) Let $x^*(t) = (x_1^*(t), x_2^*(t), \cdots, x_n^*t))^T$ be an almost automorphic solution of (1.1) with initial value $\varphi^*(t) = (\varphi_1^*(t), \varphi_2^*(t), \cdots, \varphi_n^*(t))^T$. If there exist positive constants λ with $\ominus\lambda \in \mathbf{R}^+$ and $M > 1$ such that an arbitrary solution $x(t) = (x_1(t), x_2(t), \cdots, x_nt))^T$ of (1.1) with initial value $\varphi(t) = (\varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t))^T$ satisfies

$$||x - x^*|| \le M||\varphi - \varphi^*||e_{\ominus\lambda}(t, t_0), t_0 \in [-\tau, \infty)_{\mathbb{T}}, t \ge t_0.$$

Then the solution $x^*(t)$ is said to be globally exponentially stable.

3. Existence of almost automorphic solutions

In this section, we will establish sufficient conditions on the existence of almost automorphic solutions of (1.1). Let $\mathbb{X}^* = \{f \in C^1(\mathbb{T}, \mathbb{R}) | f \in AA(\mathbb{T}, \mathbb{R}^{m+n})\}$ with the norm $||f||_{\mathbb{X}^*} = f^+$. Then $\mathbb{X}^*$ is a Banach space. Let $\varphi^0(t) = (\varphi_{11}^0(t), \varphi_{12}^0(t), \cdots, \varphi_{mn}^0(t))^T$, where $\varphi_{ij}^0(t) = \int_{-\infty}^t e_{-a_{ij}}(t, \sigma(s)L_{ij}(s)\Delta s, ij \in \Lambda$ and L is a constant satisfying

$$L \ge \max\{||\varphi^0||_{\mathbb{X}^*}, |f(0)|\}.$$

Throughout this paper, we assume that

(H1) $a_{ij} \in C(\mathbb{T}, \mathbb{R}^+)$ with $-a_{ij} \in \mathbf{R}^+$ and $\inf_{t\in\mathbb{T}}\{1 - \mu(t)a_{ij}(t)\} = \bar{a} > 0, C_{ij}^{kl}, L_{ij} \in C(\mathbb{T}, \mathbb{R})$, $\tau \in C(\mathbb{T}, \mathbb{R}^+)$ are almost automorphic, where $ij \in \Lambda$.

(H2) $f \in C(\mathbb{R}, \mathbb{R})$ and there exists a constant $L > 0$ such that for any $u, v \in \mathbb{R}$,

$$|f(u) - f(v)| \le L_f|u - v|.$$

(H3) For $ij \in \Lambda$,

$$\max_{ij\in\Lambda}\left\{\frac{\varrho_{ij}}{a_{ij}^-}\right\} \le \frac{1}{2}, \max_{ij\in\Lambda}\left\{\frac{\varsigma_{ij}}{a_{ij}^-}\right\} \le 1,$$

where

$$\varrho_{ij} = a_{ij}^+\eta_{ij}^+ + \sum_{C_{kl}\in N_r(i,j)} {C_{ij}^{kl}}^+(2L_fL + |f(0)|),$$
$$\varsigma_{ij} = a_{ij}^+\eta_{ij}^+ + \sum_{C_{kl}\in N_r(i,j)} {C_{ij}^{kl}}^+ 2L_fL.$$

Theorem 3.1. *If (H1)–(H3) are satisfied. Then there exists a unique almost automorphic solution of (1.1) in $\mathbb{X}_0 = \{\varphi \in \mathbb{X}^* | ||\varphi - \varphi^0||_{\mathbb{X}^*} \le L\}$.*

Proof. Let $\varphi \in \mathbb{X}^*$. Consider the system as follows:

$$x_{ij}^{\Delta}(t) = -a_{ij}(t)x_{ij}(t) + \Gamma_{ij}(t, \varphi) + L_{ij}(t), ij \in \Lambda, \tag{3.1}$$

where

$$\Gamma_{ij}(t,\varphi)=a_{ij}(t)\int_{t-\eta_{ij}(t)}^{t}\varphi_{ij}^{\Delta}(s)\Delta s+\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}(t)f(\varphi_{kl}(t-\tau(t)))\varphi_{ij}(t), ij\in\Lambda. \tag{3.2}$$

It follows from Lemma 2.10 that the linear system

$$x_{ij}^{\Delta}(t)=-a_{ij}(t)x_{ij}(t), ij\in\Lambda, \tag{3.3}$$

admits an exponential dichotomy on $\mathbb{T}$. Thus, in view of Lemma 2.9, we derive that system (3.1) has exactly one almost automorphic solution as follows

$$x_{ij}^{\varphi}(t)=\int_{-\infty}^{t}e_{-a_{ij}}(t,\sigma(s))[\Gamma_{ij}(s,\varphi)+L_{ij}(s)]\Delta s, ij\in\Lambda. \tag{3.4}$$

For $\varphi\in\mathbb{X}^*$, then

$$||\varphi||_{\mathbb{X}^*}\le||\varphi-\varphi_0||_{\mathbb{X}^*}+||\varphi_0||_{\mathbb{X}^*}\le 2L. \tag{3.5}$$

Define an operator as follows

$$\Phi:\mathbb{X}^*\to\mathbb{X}^*,(\varphi_{11},\varphi_{12},\cdots,\varphi_{mn})^T\to(x_{11}^{\varphi},x_{12}^{\varphi},\cdots,x_{mn}^{\varphi})^T. \tag{3.6}$$

First we show that for any $\varphi\in\mathbb{X}^*$, we have $\Phi\varphi\in\mathbb{X}^*$. Note that, for $ij\in\Lambda$, we have

$$\begin{aligned}|\Gamma_{ij}(s,\varphi)|&=\Big|a_{ij}(s)\int_{s-\eta_{ij}(s)}^{s}\varphi_{ij}^{\Delta}(\vartheta)\Delta\vartheta+\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}(s)f(\varphi_{kl}(s-\tau(s)))\varphi_{ij}(s)\Big|\\&\le a_{ij}^{+}\eta_{ij}^{+}||\varphi||_{\mathbb{X}^*}+\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl+}(|f(\varphi_{kl}(s-\tau(s)))-f(0)|+|f(0)|)|\varphi_{ij}(s)|\\&\le a_{ij}^{+}\eta_{ij}^{+}||\varphi||_{\mathbb{X}^*}+\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl+}(L_f||\varphi||_{\mathbb{X}^*}+|f(0)|)||\varphi||_{\mathbb{X}^*}\\&=\Bigg[a_{ij}^{+}\eta_{ij}^{+}+\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl+}(L_f||\varphi||_{\mathbb{X}^*}+|f(0)|)\Bigg]||\varphi||_{\mathbb{X}^*}\\&\le\Bigg[a_{ij}^{+}\eta_{ij}^{+}+\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl+}(2L_fL+|f(0)|)\Bigg]2L.\end{aligned} \tag{3.7}$$

Thus we get

$$\begin{aligned}|(\Phi(\varphi-\varphi^0))_{ij}(t)|&=\Big|\int_{-\infty}^{t}e_{-a_{ij}}(t,\sigma(s))\Gamma_{ij}(s,\varphi)\Delta s\Big|\\&\le\int_{-\infty}^{t}e_{-a_{ij}}(t,\sigma(s))|\Gamma_{ij}(s,\varphi)|\Delta s\\&\le 2L\int_{-\infty}^{t}e_{-a_{ij}^{-}}(t,\sigma(s))\Bigg[a_{ij}^{+}\eta_{ij}^{+}\\&\quad+\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl+}(2L_fL+|f(0)|)\Bigg]\Delta s\\&\le\frac{2L\varrho_{ij}}{a_{ij}^{-}}, ij\in\Lambda.\end{aligned} \tag{3.8}$$

It follows from (H3) that

$$||\Phi(\varphi-\varphi^0)||_{\mathbb{X}^*} \le \max_{ij\in\Lambda}\left\{\frac{\varrho_{ij}}{a_{ij}^-}\right\} \le L, \tag{3.9}$$

which implies that $\Phi(\varphi) \in \mathbb{X}^*$. Next, we show that Φ is a contraction. For any $\varphi = (\varphi_{11}, \varphi_{12}, \cdots, \varphi_n)^T, \psi = (\psi_{11}, \psi_{12}, \cdots, \psi_{mn})^T \in \mathbb{X}^*$, for $ij \in \Lambda$, we denote

$$\begin{aligned}\Upsilon_{ij}(s,\varphi,\psi) = a_{ij}(s)\int_{s-\eta_{ij}(s)}^{s}(\varphi_{ij}^{\Delta}(s)-\psi_{ij}^{\Delta}(s))\Delta s + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s)\\ \times[f(\varphi_{kl}(s-\tau(s)))\varphi_{ij}(s) - f(\psi_{kl}(s-\tau(s)))\psi_{ij}(s)]. \end{aligned}\tag{3.10}$$

Then

$$\begin{aligned}|(\Phi\varphi-\Phi\psi)_{ij}(t)| &= \Big|\int_{-\infty}^{t} e_{-a_{ij}}(t,\sigma(s))\Upsilon_{ij}(s,\varphi,\psi)\Delta s\Big|\\ &\le \int_{-\infty}^{t} e_{-a_{ij}}(t,\sigma(s))|\Upsilon_{ij}(s,\varphi,\psi)|\Delta s\\ &\le \int_{-\infty}^{t} e_{-a_{ij}}(t,\sigma(s))\left[a_{ij}^+\eta_{ij}^+ + \sum_{C_{kl}\in N_r(i,j)} {C_{ij}^{kl}}^+ 2L_f L\right]\Delta s||\varphi-\psi||_{\mathbb{X}^*}\\ &\le \frac{\varsigma_{ij}}{a_{ij}^-}||\varphi-\psi||_{\mathbb{X}^*}, ij\in\Lambda. \end{aligned}\tag{3.11}$$

In view of (H3), we get that $||\Phi\varphi - \Phi\varphi|| < ||\varphi - \psi||$. Then Φ is a contraction. Thus Φ has a fixed point in $\mathbb{X}_0$, i.e., (1.1) has a unique almost automorphic solution in $\mathbb{X}_0$. The proof of Theorem 3.1 is completed. □

4. Exponential stability of almost automorphic solutions

In this section, we will obtain the exponential stability of the almost automorphic solutions of system (1.1).

Theorem 4.1. *If (H1)–(H3) are fulfilled. Then the almost automorphic solution of system (1.1) is globally exponentially stable.*

Proof. By Theorem 3.1, we know that (1.1) has an almost automorphic solution $x(t) = (x_{11}(t), x_{12}(t), \cdots, x_{mn}(t))^T$ with initial condition $\varphi(t) = (\varphi_{11}(t), \varphi_{12}(t), \cdots, \varphi_{mn}(t))^T$. Suppose that $y(t) = (y_{11}(t), y_{12}(t), \cdots, y_{mn}(t))^T$ is an arbitrary solution of (1.1) with initial condition $\psi(t) = (\psi_{11}(t), \psi_{12}(t), \cdots, \psi_{mn}(t))^T$. Denote $u(t) = (u_{11}(t), u_{12}(t), \cdots, u_{mn}(t))^T$, where $u_{ij}(t) = y_{ij}(t) - x_{ij}(t), ij \in \Lambda$. Then it follows from (1.1) that

$$\begin{aligned}u_{ij}^{\Delta}(t) = -a_{ij}(t)u_{ij}(t-\eta_{ij}(t)) + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)[f(y_{kl}(t-\tau(t)))y_{ij}(t)\\ -f(x_{kl}(t-\tau(t)))x_{ij}(t)], ij\in\Lambda. \end{aligned}\tag{4.1}$$

The initial condition of (4.1) is

$$\phi_{ij}(s) = \varphi_{ij}(s) - \psi_{ij}(s), s\in[-\theta,0]_{\mathbb{T}}, ij\in\Lambda. \tag{4.2}$$

Rewrite (4.1) as the form

$$
\begin{aligned}
u_{ij}^{\Delta}(t) = & -a_{ij}(t)u_{ij}(t) + a_{ij}(t)\int_{t-\eta_{ij}(t)}^{t} u_{ij}^{\Delta}(s)\Delta s \\
& + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)[f(y_{kl}(t-\tau(t)))y_{ij}(t) \\
& - f(x_{kl}(t-\tau(t)))x_{ij}(t)], ij \in \Lambda. \qquad (4.3)
\end{aligned}
$$

It follows from (4.3) that for $ij \in \Lambda$ and $t \geq t_0, t_0 \in [-\theta, 0]_{\mathbb{T}}$,

$$
\begin{aligned}
u_{ij}(t) = & u_{ij}(t_0)e_{-a_{ij}}(t,t_0) + \int_{t_0}^{t} e_{-a_{ij}}(t,\sigma(s))\Bigg\{ a_{ij}(s)\int_{s-\eta_{ij}(s)}^{s} u_{ij}^{\Delta}(\vartheta)\Delta\vartheta \\
& + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s)[f(y_{kl}(s-\tau(s)))y_{ij}(s) - f(x_{kl}(s-\tau(s)))x_{ij}(s)]\Bigg\}\Delta s, \quad (4.4)
\end{aligned}
$$

where $ij \in \Lambda$. Define $\Pi_{ij}(\omega)$ and $\Gamma_{ij}(\omega)$ as follows

$$
\Pi_{ij}(\omega) = a_{ij}^{-} - \omega - e^{\omega \sup_{s\in\mathbb{T}}\mu(s)}\left[a_{ij}^{+}\eta_{ij}^{+}e^{\omega\eta_{ij}^{+}} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl^{+}} L_f e^{\omega\tau^{+}}\right], \quad (4.5)
$$

$$
\begin{aligned}
\Gamma_{ij}(\omega) = & a_{ij}^{-} - \omega - \left(a_{ij}^{+}e^{\omega \sup_{s\in\mathbb{T}}\mu(s)} + a_{ij}^{-} - \omega\right) \\
& \times\left[a_{ij}^{+}\eta_{ij}^{+}e^{\omega\eta_{ij}^{+}} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl^{+}} L_f e^{\omega\tau^{+}}\right], \qquad (4.6)
\end{aligned}
$$

where $ij \in \Lambda$. By (H3), we get

$$
\Pi_{ij}(0) = a_{ij}^{-} - \left[a_{ij}^{+}\eta_{ij}^{+} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl^{+}} L_f\right] > 0, \qquad (4.7)
$$

$$
\Gamma_{ij}(0) = a_{ij}^{-} - \left(a_{ij}^{+} + a_{ij}^{-}\right)\left[a_{ij}^{+}\eta_{ij}^{+} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl^{+}} L_f\right] > 0. \qquad (4.8)
$$

Since $\Pi_{ij}(\omega)$ and $\Gamma_{ij}(\omega)$ are continuous on $[0,+\infty)$ and $\lim_{\omega\to+\infty}\Pi_{ij}(\omega) = -\infty$, $\lim_{\omega\to+\infty}\Gamma_{ij}(\omega) = -\infty$, then there exist $\omega_{ij}, \omega_{ij}^{*} > 0$ such that $\Pi_{ij}(\omega_{ij}) = 0$, $\Gamma_{ij}(\omega_{ij}^{*}) = 0$ and $\Pi_{ij}(\omega) > 0$ for $\omega \in (0,\omega_{ij})$, $\Gamma_{ij}(\omega) > 0$ for $\omega \in (0,\omega_{i}^{*}), ij \in \Lambda$. By choosing a positive constant $\omega_0 = \min\{\omega_{11},\omega_{12},\cdots,\omega_{mn},\omega_{11}^{*},\omega_{12}^{*},\cdots,\omega_{mn}^{*}\}$, we get $\Pi_{ij}(\omega_0) \geq 0$ and $\Gamma_{ij}(\omega_0) \geq 0, ij \in \Lambda$. Thus we can choose a positive constant $0 < \xi < \min\{\omega_0, \min_{ij\in\Lambda}\{a_{ij}^{-}\}\}$ such that

$$
\Pi_{ij}(\xi) > 0, \Gamma_{ij}(\xi) > 0, ij \in \Lambda,
$$

which implies that

$$
\frac{e^{\xi \sup_{s\in\mathbb{T}}\mu(s)}}{a_{ij}^{-} - \xi}\left[a_{ij}^{+}\eta_{ij}^{+}e^{\omega\eta_{ij}^{+}} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl^{+}} L_f e^{\omega\tau^{+}}\right] < 1 \qquad (4.9)
$$

and

$$\left[1+\frac{a_{ij}^{+}e^{\xi\sup_{s\in\mathbb{T}}\mu(s)}}{a_{ij}^{-}-\xi}\right]\left[a_{ij}^{+}\eta_{ij}^{+}e^{\omega\eta_{ij}^{+}}+\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl^{+}}L_f e^{\omega\tau^{+}}\right]<1, \tag{4.10}$$

where $ij\in\Lambda$. Let

$$M=\max_{ij\in\Lambda}\left\{\frac{a_{ij}^{-}}{a_{ij}^{+}\eta_{ij}^{+}+\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl^{+}}L_f}\right\}. \tag{4.11}$$

By (H3), we know that $M>1$. Then we get

$$\frac{1}{M}<\frac{e^{\xi\sup_{s\in\mathbb{T}}\mu(s)}}{a_{ij}^{-}-\xi}\left[a_{ij}^{+}\eta_{ij}^{+}+\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl^{+}}L_f\right]. \tag{4.12}$$

Moreover, we have that $e_{\ominus\xi}(t,t_0)>1$, where $t\in[-\theta,t_0]_{\mathbb{T}}$. Then

$$||u||_{\mathbb{X}^*}\le Me_{\ominus\xi}(t,t_0)||\varphi-\psi||_{\mathbb{X}^*},\ \text{for all } t\in[-\theta,t_0]_{\mathbb{T}}. \tag{4.13}$$

We claim that

$$||u||_{\mathbb{X}^*}\le Me_{\ominus\xi}(t,t_0)||\varphi-\psi||_{\mathbb{X}^*},\ \text{for all } t\in[-t_0,+\infty]_{\mathbb{T}}. \tag{4.14}$$

To prove this (4.14), we show that for any $p>1$, the following inequality holds

$$||u||_{\mathbb{X}^*}\le pMe_{\ominus\xi}(t,t_0)||\varphi-\psi||_{\mathbb{X}^*},\ \text{for all } t\in[-t_0,+\infty]_{\mathbb{T}}, \tag{4.15}$$

which implies that for $ij\in\Lambda$,

$$|u_{ij}(t)|\le pMe_{\ominus\xi}(t,t_0)||\varphi-\psi||_{\mathbb{X}^*},\ \text{for all } t\in[-t_0,+\infty]_{\mathbb{T}}. \tag{4.16}$$

By way of contradiction, assume that (4.15) does not hold.

We assume that (4.16) is not true. Then there exist $t_1\in(t_0,+\infty)_{\mathbb{T}}$ and $ij_1\in\Lambda$ such that

$$|u_{ij_1}(t_1)|\ge pMe_{\ominus\xi}(t_1,t_0)||\varphi-\psi||_{\mathbb{X}^*},|u_{ij_1}(t)|<pMe_{\ominus\xi}(t,t_0)||\varphi-\psi||_{\mathbb{X}^*},\forall\, t\in[-t_0,t_1]_{\mathbb{T}},$$

$$|u_k(t)|<pMe_{\ominus\xi}(t,t_0)||\varphi-\psi||_{\mathbb{X}^*},\ \text{for } k\ne ij_1,k\in\Lambda,t\in[-t_0,t_1]_{\mathbb{T}},$$

Therefore, there exists a constant $\gamma_1\ge 1$ such that

$$|u_{ij_1}(t_1)|=\gamma_1pMe_{\ominus\xi}(t_1,t_0)||\varphi-\psi||_{\mathbb{X}^*},|u_{ij_1}(t)|<\gamma_1pMe_{\ominus\xi}(t,t_0)||\varphi-\psi||_{\mathbb{X}^*},\forall t\in[-t_0,t_1]_{\mathbb{T}}.$$

$$|u_k(t)|<\gamma_1pMe_{\ominus\xi}(t,t_0)||\varphi-\psi||_{\mathbb{X}^*},\ \text{for } k\in\Lambda,k\ne ij_1,t\in[-t_0,t_1]_{\mathbb{T}},$$

By (4.4), for $ij \in \Lambda$ we get

$$
\begin{aligned}
|u_{ij_1}(t_1)| = & \Bigg| u_{ij_1}(t_0)e_{-a_{ij_1}}(t_1,t_0) + \int_{t_0}^{t_1} e_{-a_{ij_1}}(t_1,\sigma(s))\Bigg\{ a_{ij_1}(s)\int_{s-\eta_{ij_1}(s)}^{s} u_{ij_1}^{\Delta}(\vartheta)\Delta\vartheta \\
& + \sum_{C_{kl}\in N_r(i,j)} C_{ij_1}^{kl}(s)[f(y_{kl}(s-\tau(s)))y_{ij_1}(s) - f(x_{kl}(s-\tau(s)))x_{ij_1}(s)]\Bigg\}\Delta s\Bigg| \\
\le & \, e_{-a_{ij_1}}(t_1,t_0)||\varphi-\psi||_{\mathbb{X}^*} + \gamma_1 pMe_{\ominus\xi}(t_1,t_0)||\varphi-\psi||_{\mathbb{X}^*} \\
& \times\Bigg|\int_{t_0}^{t_1} e_{-a_{ij_1}}(t_1,\sigma(s))e_{\xi}(t_1,\sigma(s))\Bigg\{ a_{ij_1}^{+}\int_{s-\eta_{ij_1}(s)}^{s} e_{\xi}(\sigma(s),\vartheta)\Delta\vartheta \\
& + \sum_{C_{kl}\in N_r(i,j)} C_{ij_1}^{kl\,+} L_f e_{\xi}(\sigma(s), s-\tau_{ij_1}(s))\Bigg\}\Delta s\Bigg| \\
\le & \, e_{-a_{ij_1}}(t_1,t_0)||\varphi-\psi||_{\mathbb{X}^*} + \gamma_1 pMe_{\ominus\xi}(t_1,t_0)||\varphi-\psi||_{\mathbb{X}^*} \\
& \times\Bigg|\int_{t_0}^{t_1} e_{-a_{ij_1}\oplus\xi}(t_1,\sigma(s))\Bigg\{ a_{ij_1}^{+}\eta_{ij_1}^{+} e_{\xi}(\sigma(s), s-\eta_{ij_1}(s)) \\
& + \sum_{C_{kl}\in N_r(i,j)} C_{ij_1}^{kl\,+} L_f e_{\xi}(\sigma(s), s-\tau_{ij_1}(s))\Bigg\}\Delta s\Bigg| \\
\le & \, e_{-a_{ij_1}}(t_1,t_0)||\varphi-\psi||_{\mathbb{X}^*} + \gamma_1 pMe_{\ominus\xi}(t_1,t_0)||\varphi-\psi||_{\mathbb{X}^*} \\
& \times\Bigg|\int_{t_0}^{t_1} e_{-a_{ij_1}\oplus\xi}(t_1,\sigma(s))\Bigg\{ a_{ij_1}^{+}\eta_{ij_1}^{+} e^{\xi(\eta_{ij_1}^{+}+\sup_{t\in\mathbb{T}}\mu(s))} \\
& + \sum_{C_{kl}\in N_r(i,j)} C_{ij_1}^{kl\,+} L_f e^{\xi(\tau_{ij_1}^{+}+\sup_{t\in\mathbb{T}}\mu(s))}\Bigg\}\Delta s\Bigg| \\
= & \, \gamma_1 pMe_{\ominus\xi}(t_1,t_0)||\varphi-\psi||_{\mathbb{X}^*}\Bigg\{\frac{1}{\gamma_1 pM} e_{-a_{ij_1}\oplus\xi}(t_1,t_0) + e^{\xi\sup_{t\in\mathbb{T}}\mu(s)} \\
& \times\Bigg[a_{ij_1}^{+}\eta_{ij_1}^{+} e^{\xi\eta_{ij_1}^{+}} + \sum_{C_{kl}\in N_r(i,j)} C_{ij_1}^{kl\,+} L_f e^{\xi\tau_{ij_1}^{+}}\Bigg] \\
& \times\int_{t_0}^{t_1} e_{-a_{ij_1}\oplus\xi}(t_1,\sigma(s))\Delta s\Bigg\} \\
\le & \, \gamma_1 pMe_{\ominus\xi}(t_1,t_0)||\varphi-\psi||_{\mathbb{X}^*}\Bigg\{\frac{1}{M} e_{-(a_{ij_1}-\xi)}(t_1,t_0) + e^{\xi\sup_{t\in\mathbb{T}}\mu(s)} \\
& \times\Bigg[\Bigg(a_{ij_1}^{+}\eta_{ij_1}^{+} e^{\xi\eta_{ij_1}^{+}} + \sum_{C_{kl}\in N_r(i,j)} C_{ij_1}^{kl\,+} L_f e^{\xi\tau_{ij_1}^{+}}\Bigg) \\
& \times\frac{1}{-(a_{ij_1}-\xi)}\int_{t_0}^{t_1}(-(a_{ij_1}-\xi))e_{-(a_{ij_1}-\xi)}(t_1,\sigma(s))\Delta s\Bigg]\Bigg\} \\
= & \, \gamma_1 pMe_{\ominus\xi}(t_1,t_0)||\varphi-\psi||_{\mathbb{X}^*}\Bigg\{\Bigg[\frac{1}{M} - \frac{e^{\xi\sup_{t\in\mathbb{T}}\mu(s)}}{a_{ij_1}^{-}-\xi}\Bigg(a_{ij_1}^{+}\eta_{ij_1}^{+} e^{\xi\eta_{ij_1}^{+}}
\end{aligned}
$$

$$+ \sum_{C_{kl}\in N_r(i,j)} C_{ij_1}^{kl\ +} L_f e^{\xi\tau_{ij_1}^+}\Bigg)\Bigg] e_{-(a_{ij_1}-\xi)}(t_1,t_0)$$
$$+\frac{e^{\xi\sup_{t\in\mathbb{T}}\mu(s)}}{a_{ij_1}^- - \xi}\Bigg(a_{ij_1}^+\eta_{ij_1}^+ e^{\xi\eta_{ij_1}^+} + \sum_{C_{kl}\in N_r(i,j)} C_{ij_1}^{kl\ +} L_f e^{\xi\tau_{ij_1}^+}\Bigg)\Bigg\}$$
$$< \gamma_1 pMe_{\ominus\xi}(t_1,t_0)||\varphi-\psi||_{\mathbb{X}^*}, \tag{4.17}$$

which is a contradiction. Based on the discussion above, we can conclude that (4.15) holds. Let $p \to 1$, then (4.14) holds. We can take $\ominus\lambda = \ominus\xi$, then $\lambda > 0$ and $\ominus\lambda \in \mathbf{R}^+$. Then we derive

$$||u||_{\mathbb{X}^*} \le M||\varphi-\psi||_{\mathbb{X}^*} e_{\ominus\lambda}(t,t_0), t \in [-\tau,\infty)_{\mathbb{T}}, t \ge t_0, \tag{4.18}$$

which means that the almost automorphic solution of (1.1) is globally exponentially stable. The proof of Theorem 4.2 is completed. □

Remark 4.1. In 2007 and 2009, Cai and Xiong [8], Shao et al. [44] studied the almost periodic solutions of system (1.1) with the leakage delay $\eta_{ij}(t) = 0$. In 2007, Xia et al. [51] investigated the existence and exponential stability of almost periodic solutions for model (1.1) with the leakage delay $\eta_{ij}(t) = 0$ and the transmission delay $\tau(t) = 0$. In 2009, Liu [37] considered the stability of model (1.1). In this paper, we consider the almost automorphic solutions of (1.1), which is more general than those models in [8,37,44,51]. Moreover, the almost automorphy has been widely applied in the theory of ordinary differential equations(ODEs) and partial differential equations(PDEs), the theory of neural networks, physics, mechanics and mathematical biology. In this sense, our results complement some previous ones in [8,37,44,51].

5. Numerical example

In this section, we will give an example to illustrate the feasibility and effectiveness of our main results obtained in previous sections. Considering the following shunting inhibitory cellular neural networks with time-varying delays in leakage terms on time scales

$$\begin{cases} x_{11}^{\Delta}(t) = -a_{11}(t)x_{11}(t-\eta_{11}(t)) + \sum\limits_{C_{kl}\in N_r(1,1)} C_{11}^{kl}(t)f(x_{kl}(t-\tau(t)))x_{11}(t) + L_{11}(t), \\ x_{12}^{\Delta}(t) = -a_{12}(t)x_{ij}(t-\eta_{12}(t)) + \sum\limits_{C_{kl}\in N_r(1,2)} C_{12}^{kl}(t)f(x_{kl}(t-\tau(t)))x_{12}(t) + L_{12}(t), \\ x_{21}^{\Delta}(t) = -a_{21}(t)x_{21}(t-\eta_{21}(t)) + \sum\limits_{C_{kl}\in N_r(2,1)} C_{21}^{kl}(t)f(x_{kl}(t-\tau(t)))x_{21}(t) + L_{21}(t), \\ x_{22}^{\Delta}(t) = -a_{22}(t)x_{22}(t-\eta_{22}(t)) + \sum\limits_{C_{kl}\in N_r(2,2)} C_{22}^{kl}(t)f(x_{kl}(t-\tau(t)))x_{22}(t) + L_{22}(t), \end{cases} \tag{5.1}$$

where $f(u) = \sin 0.3u$ and

$$\begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} = \begin{bmatrix} 0.002+0.001|\cos\sqrt{2}t| & 0.002+0.001|\cos\sqrt{3}t| \\ 0.003+0.002|\cos\sqrt{5}t| & 0.002+0.001|\cos\sqrt{3}t| \end{bmatrix},$$

$$\begin{bmatrix} \eta_{11}(t) \ \eta_{12}(t) \\ \eta_{21}(t) \ \eta_{22}(t) \end{bmatrix} = \begin{bmatrix} 0.02\sin^2 t \ 0.02\sin^2 t \\ 0.02\sin^2 t \ 0.02\sin^2 t \end{bmatrix},$$

$$\begin{bmatrix} C_{11}(t) \ C_{12}(t) \\ C_{21}(t) \ C_{22}(t) \end{bmatrix} = \begin{bmatrix} 0.00003 + 0.00001\sin\sqrt{5}t \ 0.00003 + 0.00001\sin\sqrt{5}t \\ 0.00002 + 0.00001\sin\sqrt{5}t \ 0.00004 + 0.00002\sin\sqrt{3}t \end{bmatrix},$$

$$\begin{bmatrix} I_{11}(t) \ I_{12}(t) \\ I_{21}(t) \ I_{22}(t) \end{bmatrix} = \begin{bmatrix} 0.02 + 0.02\cos\sqrt{3}t \ 0.03 + 0.02\cos\sqrt{7}t \\ 0.02 + 0.02\cos\sqrt{7}t \ 0.01 + 0.02\cos\sqrt{3}t \end{bmatrix}.$$

Let $r = 1, L = 1$. Then we get $L_f = 0.3$ and

$$\begin{bmatrix} a_{11}^+ \ a_{12}^+ \\ a_{21}^+ \ a_{22}^+ \end{bmatrix} = \begin{bmatrix} 0.003 \ 0.003 \\ 0.005 \ 0.003 \end{bmatrix}, \begin{bmatrix} a_{11}^- \ a_{12}^- \\ a_{21}^- \ a_{22}^- \end{bmatrix} = \begin{bmatrix} 0.002 \ 0.002 \\ 0.003 \ 0.002 \end{bmatrix},$$

$$\begin{bmatrix} \eta_{11}^+ \ \eta_{12}^+ \\ \eta_{21}^+ \ \eta_{22}^+ \end{bmatrix} = \begin{bmatrix} 0.02 \ 0.02 \\ 0.02 \ 0.02 \end{bmatrix}, \begin{bmatrix} C_{11}^+ \ C_{12}^+ \\ C_{21}^+ \ C_{22}^+ \end{bmatrix} = \begin{bmatrix} 0.00004 \ 0.00004 \\ 0.00005 \ 0.00006 \end{bmatrix}.$$

Hence

$$\begin{cases} \varrho_{11} = a_{11}^+\eta_{11}^+ + \sum_{C_{kl}\in N_1(1,1)} C_{11}^{kl^+}(2L_fL + |f(0)|) \approx 0.000156, \\ \varrho_{12} = a_{12}^+\eta_{12}^+ + \sum_{C_{kl}\in N_1(1,2)} C_{12}^{kl^+}(2L_fL + |f(0)|) \approx 0.000156, \\ \varrho_{21} = a_{21}^+\eta_{21}^+ + \sum_{C_{kl}\in N_1(2,1)} C_{21}^{kl^+}(2L_fL + |f(0)|) \approx 0.00022, \\ \varrho_{22} = a_{22}^+\eta_{22}^+ + \sum_{C_{kl}\in N_1(2,2)} C_{22}^{kl^+}(2L_fL + |f(0)|) \approx 0.000204, \\ \varsigma_{11} = a_{11}^+\eta_{11}^+ + \sum_{C_{kl}\in N_1(1,1)} C_{11}^{kl^+}2L_fL \approx 0.000156, \\ \varsigma_{12} = a_{12}^+\eta_{12}^+ + \sum_{C_{kl}\in N_1(1,2)} C_{12}^{kl^+}2L_fL \approx 0.000156, \\ \varsigma_{21} = a_{21}^+\eta_{21}^+ + \sum_{C_{kl}\in N_1(2,1)} C_{21}^{kl^+}2L_fL \approx 0.00022, \\ \varsigma_{22} = a_{22}^+\eta_{22}^+ + \sum_{C_{kl}\in N_1(2,2)} C_{22}^{kl^+}2L_fL \approx 0.000204, \end{cases}$$

$$\begin{cases} \frac{\varrho_{11}}{a_{11}^-} = \frac{0.000156}{0.002} = 0.0780, \\ \frac{\varrho_{12}}{a_{12}^-} = \frac{0.000156}{0.002} = 0.0780, \\ \frac{\varrho_{21}}{a_{21}^-} = \frac{0.00022}{0.003} = 0.0070, \\ \frac{\varrho_{22}}{a_{22}^-} = \frac{0.000204}{0.002} = 0.1020, \end{cases} \begin{cases} \frac{\varsigma_{11}}{a_{11}^-} = \frac{0.000156}{0.002} = 0.0780, \\ \frac{\varsigma_{12}}{a_{12}^-} = \frac{0.000156}{0.002} = 0.0780, \\ \frac{\varsigma_{21}}{a_{21}^-} = \frac{0.00022}{0.003} = 0.0070, \\ \frac{\varsigma_{22}}{a_{22}^-} = \frac{0.000204}{0.002} = 0.1020. \end{cases}$$

Then it is easy to check that

$$\max_{1\le i,j\le 2}\left\{\frac{\varrho_{ij}}{a_{ij}^-}\right\}=0.1020<\frac{1}{2},\ \max_{1\le i,j\le 2}\left\{\frac{\varsigma_{ij}}{a_{ij}^-}\right\}=0.1020<1.$$

Thus all assumptions in Theorems 4.1 and 4.2 are fulfilled. Thus we can conclude that (5.1) has an almost automorphic solution, which is globally exponentially stable. The results are verified by the numerical simulations in Figures 1-4 ($\mathbb{T}=\mathbb{R}$). Figures 1-4 stand for the time history plots of t-x_{11}, t-x_{12}, t-x_{21} and t-x_{22}, respectively.

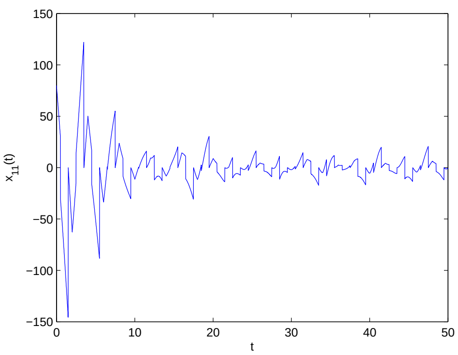

Figure 1. Time response of state variable x_{11}.

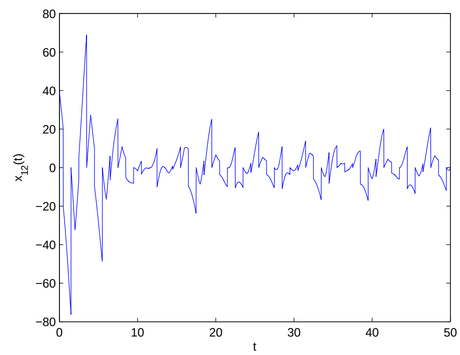

Figure 2. Time response of state variable x_{12}.

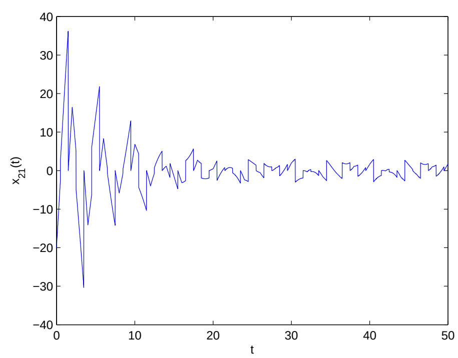

Figure 3. Time response of state variable x_{21}.

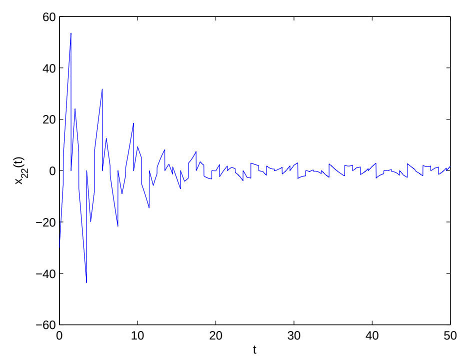

Figure 4. Time response of state variable x_{22}.

6. Conclusions

In this paper, we study a class of shunting inhibitory cellular neural networks with time-varying delays in leakage terms on time scales. Applying the existence of the exponential dichotomy of linear dynamic equations on time scales, fixed point theorem and the theory of calculus on time scales, we establish some sufficient conditions for the existence and exponential stability of almost automorphic solutions for the shunting inhibitory cellular neural networks with time-varying delays in leakage terms on time scales. It is shown that the existence and global exponential stability of almost automorphic solutions for system (1.1) only depend on time delays $\eta_{ij}(t)(ij \in \Lambda)$ (the delays in the leakage term) and do not depend on time delays $\tau(t)$ which implies that the delays in the leakage term have important effect on the existence and global exponential stability of almost automorphic solutions. To the best of our knowledge, it is the first time to deal with the almost automorphic solution for the shunting inhibitory cellular neural networks with time-varying delays in leakage terms on time scales. The method of this paper can be applied directly to many other related neural networks.

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