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ON THE MAXIMUM NUMBER OF PERIODIC SOLUTIONS OF PIECEWISE SMOOTH PERIODIC EQUATIONS BY AVERAGE METHOD*

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Abstract In this paper, we prove smoothness of bifurcation function for a piecewise smooth periodic equation, and then use the bifurcation function together with its smoothness to study the maximum number of periodic solutions of the piecewise smooth periodic equation by the first order average.

Keywords Averaging method, bifurcation function, periodic solution, maximum number.

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1. Introduction and main result

As we know, the average method is an important tool to study the periodic solutions of periodic equations with a small parameter. Recently, the method has been developed from smooth differential equations to piecewise smooth differential equations, see [5,6]. The results obtained for piecewise smooth differential equations concern the existence of multiple periodic solutions, which can be used to find a lower bound for the maximum number of periodic solutions for piecewise smooth differential equations. The results can then be applied to obtain a lower bound for the maximum number of limit cycles for some piecewise smooth systems on the plane by the first order average, see [1,3,4,7,8]. One can apply theorems obtained in [2] to study the maximum number of limit cycles for piecewise smooth near-Hamiltonian systems on the plane.

We note that the averaging theory obtained in [5,6] does not tell any information on upper bound of the maximum number. Then a problem arises: can we obtain the maximum number of periodic solutions or an upper bound of it for piecewise smooth differential equations by the first order average? If the answer is positive, then it can be applied to obtain an upper bound of the maximum number of limit cycles for certain piecewise smooth systems on the plane studied in [1,4,7,8]. In this paper, we study the problem of the maximum number. We first establish the smoothness of bifurcation function for piecewise smooth periodic differential equations. Then based on the smoothness of the bifurcation function we obtain an upper bound of the maximum number of periodic solutions bifurcating from a period annular under some sufficient conditions in the scalar case. To state our main result, we first present our assumptions in the following.

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(H1) There exist an open interval J, a positive constant T and $k-1$ C^r functions $h_1(x), \ldots, h_{k-1}(x)$ defined on J, satisfying

$$0 < h_1(x) < \cdots < h_{k-1}(x) < T,\ x \in J,\quad k \geq 2,\quad r \geq 1.$$

(H2) Set $h_0(x) = 0$ and $h_k(x) = T$. Introduce k regions as follows

$$D_j = \{(t,x)|\ h_{j-1}(x) \leq t < h_j(x), x \in J\},\ j = 1, \ldots, k.$$

For all $j = 1, \ldots, k$ there exist $\varepsilon_0 > 0$, k C^r functions $F_j(t,x,\varepsilon,\delta)$ defined for all $(t,x) \in U(\bar{D}_j)$ and $|\varepsilon| < \varepsilon_0$ and $\delta \in V$ with V a compact set of R^n, where $\bar{D}_j$ denotes the closure of the set D_j, and $U(\bar{D}_j)$ an open set containing $\bar{D}_j$.

Clearly

$$[0,T) \times J = \bigcup_{j=1}^{k} D_j.$$

Now we introduce our differential equation of the form

$$\frac{dx}{dt} = \varepsilon F(t,x,\varepsilon,\delta),\quad t \in R,\ x \in J, \tag{1.1}$$

where $|\varepsilon| < \varepsilon_0$, $\delta \in V$ and the function F satisfies the following conditions:

(H3) F is periodic in t with period T, that is, $F(t+T,x,\varepsilon,\delta) = F(t,x,\varepsilon,\delta)$ for all $t \in R$ and $x \in J$, and satisfies

$$F(t,x,\varepsilon,\delta) = \begin{cases} F_1(t,x,\varepsilon,\delta), & (t,x) \in D_1, \\ F_2(t,x,\varepsilon,\delta), & (t,x) \in D_2, \\ \quad\vdots & \quad\vdots \\ F_k(t,x,\varepsilon,\delta), & (t,x) \in D_k. \end{cases}$$

We can call the equation (1.1) a k-piecewise C^r smooth periodic equation. Note that F may not be continuous on the switch lines $l_1, \cdots, l_{k-1}$, where

$$l_j = \{(t,x)|\ t = h_j(x),\ x \in J\},\ j = 0, \ldots, k.$$

Let

$$f(x,\delta) = \int_0^T F(t,x,0,\delta)dt = \sum_{j=1}^{k} \int_{h_{j-1}(x)}^{h_j(x)} F_j(t,x,0,\delta)dt,\ x \in J. \tag{1.2}$$

It is easy to see that f is a C^r function under the assumptions (H1)-(H3). The main result of the paper can be stated as follows.

Theorem 1.1. *Consider the periodic equation* (1.1). *Suppose it satisfies the assumptions (H1), (H2) and (H3). If there exists an integer m, $1 \leq m \leq r$, such that the function f defined in* (1.2) *has at most m zeros in $x \in J$ for all $\delta \in V$, multiplicity taken into account, then for any closed interval $I \subset J$, there exists $\varepsilon_1 = \varepsilon_1(I) > 0$, such that for $0 < |\varepsilon| < \varepsilon_1$, $\delta \in V$ the periodic equation* (1.1) *has at most m T-periodic solutions with the property that the range of each of them is a subset of I.*

The conclusion of Theorem 1.1 can be restated simply that the period annular of the unperturbed system

$$\frac{dx}{dt} = 0, \ x \in J$$

generates at most m periodic solutions by the first order average.

We present a proof to the theorem above in the next section.

2. Proof of the main result

First of all, let us define a unique solution of the equation (1.1) with the initial condition $x(0) = x_0$ for $x_0 \in J$ in a natural way.

Taking a point $(0, x_0) \in l_0$, consider the solution of the equation

$$\frac{dx}{dt} = \varepsilon F_1(t, x, \varepsilon, \delta)$$

satisfying $x(0) = x_0$, denoted by $x_1(t, 0, x_0, \varepsilon, \delta)$. Since $F_1 \in C^r$, we have $x_1 \in C^r$. Also, one can see that

$$x_1(t, 0, x_0, \varepsilon, \delta) = x_0 + \varepsilon \bar{x}_1(t, x_0, \varepsilon, \delta), \ t \in [0, T], \tag{2.1}$$

where $\bar{x}_1$ satisfies

$$\frac{d\bar{x}_1}{dt} = F_1(t, x_0 + \varepsilon \bar{x}_1, \varepsilon, \delta), \quad \bar{x}_1|_{t=0} = 0.$$

This implies further $\bar{x}_1 \in C^r$, and

$$\bar{x}_1(t, x_0, 0, \delta) = \int_0^t F_1(t, x_0, 0, \delta) dt. \tag{2.2}$$

By (2.1), the solution $x_1(t, 0, x_0, \varepsilon, \delta)$ must meet the line l_1 at some point (t_1, x_{10}). Obviously, the intersection point depends on $(x_0, \varepsilon, \delta)$ and it goes to $(h(x_0), x_0)$ as $\varepsilon \to 0$. To see the smoothness of (t_1, x_{10}) on $(x_0, \varepsilon, \delta)$, introduce a function G below

$$G(t, x_0, \varepsilon, \delta) = t - h_1(x_1(t, 0, x_0, \varepsilon, \delta)).$$

By our assumption and (2.1), the function G satisfies

$$G \in C^r, \ G(h_1(x_0), x_0, 0, \delta) = 0, \ \frac{\partial G}{\partial t}|_{\varepsilon=0} = 1.$$

Hence, by the implicity function theorem there exists a unique C^r function $t = \tau_1(x_0, \varepsilon, \delta) = h_1(x_0) + O(\varepsilon)$ such that

$$G(\tau_1(x_0, \varepsilon, \delta), x_0, \varepsilon, \delta) = 0, \ x_0 \in J,$$

or

$$\tau_1(x_0, \varepsilon, \delta) = h_1(x_1(\tau_1(x_0, \varepsilon, \delta), 0, x_0, \varepsilon, \delta), \ x_0 \in J.$$

It is clear that the intersection point (t_1, x_{10}) is given by

$$t_1 = \tau_1(x_0, \varepsilon, \delta), \quad x_{10} = x_1(\tau_1(x_0, \varepsilon, \delta), 0, x_0, \varepsilon, \delta).$$

Let $g_1(x_0,\varepsilon,\delta) = \bar{x}_1(\tau_1, x_0, 0, \delta)$. Then $g_1 \in C^r$ in (x_0,ε,δ), and by (2.1) and (2.2) we can write that

$$\begin{aligned} &x_{10} = x_0 + \varepsilon g_1(x_0,\varepsilon,\delta),\ g_1 \in C^r, \\ &g_1(x_0,0,\delta) = \int_0^{t_{10}} F_1(t,x_0,0,\delta)dt,\ t_{10} = h_1(x_0). \end{aligned} \tag{2.3}$$

In order to continue the definition of x_1, consider the solution of the equation

$$\frac{dx}{dt} = \varepsilon F_2(t,x,\varepsilon,\delta)$$

satisfying $x(t_1) = x_{10}$, denoted by $x_2(t,t_1,x_{10},\varepsilon,\delta)$. Similar to (2.1) and (2.2), we have

$$x_2(t,t_1,x_{10},\varepsilon,\delta) = x_{10} + \varepsilon\bar{x}_2(t,t_1,x_{10},\varepsilon,\delta),\ t \in [0,T],$$

where $\bar{x}_2 \in C^r$, and

$$\bar{x}_2(t,t_1,x_{10},0,\delta) = \int_{t_1}^{t} F_2(t,x_{10},0,\delta)dt.$$

As before, there exists a C^r function $g_2(x_{10},\varepsilon,\delta)$, such that

$$x_{20} = x_2(t_2,t_1,x_{10},\varepsilon,\delta) = x_{10} + \varepsilon g_2(x_{10},\varepsilon,\delta),\ t_2 = h_2(x_{20}),$$

and

$$g_2(x_{10},\varepsilon,\delta) = \int_{t_1}^{t_2} F_2(t,x_{10},0,\delta)dt + O(\varepsilon).$$

It then follows from (2.3) that

$$\begin{aligned} &x_{20} = x_0 + \varepsilon[g_1(x_0,\varepsilon,\delta) + g_2(x_{10},\varepsilon,\delta)] = x_0 + \varepsilon\bar{g}_2(x_0,\varepsilon,\delta),\ \bar{g}_2 \in C^r, \\ &\bar{g}_2(x_0,0,\delta) = \int_0^{t_{10}} F_1(t,x_0,0,\delta)dt + \int_{t_{10}}^{t_{20}} F_2(t,x_0,0,\delta)dt = \int_0^{t_{20}} F(t,x_0,0,\delta)dt, \end{aligned} \tag{2.4}$$

where $t_{20} = h_2(x_0)$.

In the same way, we have by induction

$$x_j(t,t_{j-1},x_{j-1,0},\varepsilon,\delta) = x_{j-1,0} + \varepsilon\bar{x}_j(t,t_{j-1},x_{j-1,0},\varepsilon,\delta),\ t \in [0,T], \tag{2.5}$$

where $\bar{x}_j$ are C^r functions with

$$\bar{x}_j(t,t_{j-1},x_{j-1,0},0,\delta) = \int_{t_{j-1}}^{t} F_j(t,x_{j-1,0},0,\delta)dt.$$

And, we have also

$$x_{j0} = x_j(t_j,t_{j-1},x_{j-1,0},\varepsilon,\delta) = x_{j-1,0} + \varepsilon g_j(x_{j-1,0},\varepsilon,\delta),\ t_j = h_j(x_{j0}), \tag{2.6}$$

$$g_j(x_{j-1,0},\varepsilon,\delta) = \int_{t_{j-1}}^{t_j} F_j(t,x_{j-1,0},0,\delta)dt + O(\varepsilon), \tag{2.7}$$

where all g_j are C^r functions for $j = 2,\dots,k$. Evidently, we have $t_k = T$.

Similar to (2.4), one can see from (2.4)-(2.7) that

$$\begin{aligned} &x_{j0} = x_0 + \varepsilon \bar{g}_j(x_0, \varepsilon, \delta),\ \bar{g}_j \in C^r, \\ &\bar{g}_j(x_0, 0, \delta) = \sum_{i=1}^{j} \int_{t_{i-1,0}}^{t_{i0}} F_i(t, x_0, 0, \delta)dt = \int_0^{t_{j0}} F(t, x_0, 0, \delta)dt, \end{aligned} \tag{2.8}$$

where

$$t_{00} = 0,\ t_{j0} = h_j(x_0),\ j = 2, \ldots, k.$$

Then for $x_0 \in J$, we define the solution of equation (1.1) satisfying $x(0) = x_0$ for $t \in [0, T]$ as

$$x(t, 0, x_0, \varepsilon, \delta) = \begin{cases} x_1(t, 0, x_0, \varepsilon, \delta), & t \in [0, t_1), \\ x_2(t, t_1, x_{10}, \varepsilon, \delta), & t \in [t_1, t_2), \\ \quad\vdots & \quad\vdots \\ x_k(t, t_{k-1}, x_{k-1,0}, \varepsilon, \delta), & t \in [t_{k-1}, T]. \end{cases} \tag{2.9}$$

Since (1.1) is T-periodic, we can use the same way to define $x(t, 0, x_0, \varepsilon, \delta)$ for t outside the interval $[0, T]$. As in the smooth case, we can define a Poincaré map and a bifurcation function of (1.1) by

$$P(x_0, \varepsilon, \delta) = x(T, 0, x_0, \varepsilon, \delta)$$

and

$$d(x_0, \varepsilon, \delta) = P(x_0, \varepsilon, \delta) - x_0,$$

respectively. By (2.8) and (2.9), we obtain

$$\begin{aligned} P(x_0, \varepsilon, \delta) &= x_k(T, t_{k-1}, x_{k-1,0}, \varepsilon, \delta) \\ &= x_{k0} \\ &= x_0 + \varepsilon \bar{g}_k(x_0, \varepsilon, \delta), \\ \bar{g}_k(x_0, 0, \delta) &= \int_0^T F(t, x_0, 0, \delta)dt = f(x_0, \delta), \end{aligned} \tag{2.10}$$

and

$$d(x_0, \varepsilon, \delta) = \varepsilon \bar{g}_k(x_0, \varepsilon, \delta), \quad \bar{g}_k \in C^r,$$

where $f(x_0, \delta)$ is defined by (1.2).

For the property of the bifurcation function, we have the following from the above deduction.

Lemma 2.1. *Suppose the assumptions (H1), (H2) and (H3) are satisfied. Then*

1. *For any given closed interval $I \subset J$, there exists $\epsilon^* > 0$ such that the function $\bar{g}_k(x_0, \varepsilon, \delta)$ is well defined and of C^r in $(x_0, \varepsilon, \delta)$ for all $x_0 \in I$, $|\varepsilon| < \varepsilon^*$ and $\delta \in V$.*
2. *The equation (1.1) has a T-periodic solution $x(t)$ with $x(0) = x_0 \in J$ for sufficiently small $\varepsilon \neq 0$ if and only if the initial value x_0 satisfies $\bar{g}_k(x_0, \varepsilon, \delta) = 0$.*

One can see easily that we have an analogous conclusion to the above lemma for the general case of higher dimensional equations of the form (1.1).

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. Let the conclusion of the theorem be not true. Then there exist a closed interval $I^* \subset J$ and a series $(\varepsilon_n, \delta_n)$ with ε_n tending to zero and $\delta_n \in V$ such that for $(\varepsilon, \delta) = (\varepsilon_n, \delta_n)$ the equation (1.1) has $m+1$ different periodic solutions $x_{jn}(t)$, $j = 1, \dots, m+1$, satisfying $x_{jn}(t) \in I^*$. Then

$$\bar{g}_k(x_{jn}(0), \varepsilon_n, \delta_n) = 0, \; x_{jn}(0) \in I^*.$$

For definiteness, we can assume

$$x_{1n}(0) < x_{2n}(0) < \cdots < x_{m+1,n}(0).$$

Note that both I^* and V are compact. We can suppose, without loss of generality, that as $n \to \infty$

$$x_{jn}(0) \to \bar{x}_j \in I^*, \quad \delta_n \to \delta_0 \in V.$$

Clearly, we have by (2.10)

$$\bar{x}_1 \le \bar{x}_2 \le \cdots \le \bar{x}_{m+1}, \; f(\bar{x}_j, \delta_0) = 0, \; j = 1, \dots, m+1.$$

If $\bar{x}_1 < \bar{x}_2 < \cdots < \bar{x}_{m+1}$, then $f(x, \delta_0) = 0$ has $m+1$ zeros on J. This is a contradiction. If $\bar{x}_1 = \cdots = \bar{x}_j$ for some $2 \le j \le m+1$, then, using Role's theorem repeatedly we know that $\frac{\partial \bar{g}_k}{\partial x}(x, \varepsilon_n, \delta_n)$ has $j-1$ different zeros in x on $(x_{1n}(0), x_{jn}(0))$. In the same way, for $i = 2, \dots, j-1$, $\frac{\partial^i \bar{g}_k}{\partial x^i}(x, \varepsilon_n, \delta_n)$ has $j-1-i$ different zeros in x on $(x_{1n}(0), x_{jn}(0))$. Note that all of the zeros have the same limit $\bar{x}_1$. Therefore, we have

$$\frac{\partial^i f}{\partial x^i}(\bar{x}_1, \delta_0) = 0, \; i = 0, 1, \dots, j-1.$$

This means that $\bar{x}_1$ is a zero of $f(x, \delta_0)$ with a multiplicity at least j.

In general, we can write

$$\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{m+1}\} = \{x^*_{r_1}, x^*_{r_2}, \dots, x^*_{r_p}\},$$

where $1 \le r_j \le m+1$, $j = 1, \dots, p$, $r_1 + r_2 + \cdots + r_p = m+1$, and

$$x^*_{r_1} < x^*_{r_2} < \cdots < x^*_{r_p}.$$

Then as before, we can prove that for $j = 1, \dots, p$, $x^*_{r_j}$ is a zero of $f(x, \delta_0)$ with a multiplicity at least r_j. This implies that $f(x, \delta_0)$ has at least $m+1$ zeros in all on J, taking multiplicity into account. This contradicts to our assumption. Then the proof is finished. □

We now have two remarks in order.

Remark 2.1. The C^r smoothness of the function $\bar{g}_k$ is essential in the proof. The smoothness of $\bar{g}_k(x_0, 0, \delta) = f(x_0, \delta)$ is not enough to control the number of zeros of $\bar{g}_k$. For example, take $g(x, \varepsilon) = x^2 + \varepsilon f_1(x, \varepsilon)$. If we require $f_1 \in C^2$ in x, then it is easy to see that for all small ε the function g has at most two zeros in x. However, if we only require $f_1 \in C^1$ in x, then for any m, we can find a suitable f_1, such that g has more than m zeros. In fact, for any odd number $n > m$, we can choose a C^1 function f_1 such that

$$x^2 + \varepsilon f_1(x, \varepsilon) = x(x^{1/n} - \varepsilon)(x^{1/n} - 2\varepsilon) \cdots (x^{1/n} - n\varepsilon).$$

Remark 2.2. By the implicity function theorem, if for some $\delta_0 \in V$, the function $f(x, \delta_0)$ has m simple zeros, then there exists $\varepsilon_0 > 0$ such that for $0 < |\varepsilon| < \varepsilon_0$, $|\delta - \delta_0| < \varepsilon_0$ the equation (1.1) has m T-periodic solutions.

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