A UNIFIED ANALYSIS OF LINEAR QUATERNION DYNAMIC EQUATIONS ON TIME SCALES

Dong Cheng[1], Kit Ian Kou[1,†] and Yong Hui Xia[2]

Abstract Over the last years, considerable attention has been paid to the role of the quaternion differential equations (QDEs) which extend the ordinary differential equations. The theory of QDEs was recently well established and has wide applications in physics and life science. This paper establishes a systematic frame work for the theory of linear quaternion dynamic equations on time scales (QDETS), which can be applied to wave phenomena modeling, fluid dynamics and filter design. The algebraic structure of the solutions to the QDETS is actually a left- or right- module, not a linear vector space. On the non-commutativity of the quaternion algebra, many concepts and properties of the classical dynamic equations on time scales (DETS) can not be applied. They should be redefined accordingly. Using q-determinant, a novel definition of Wronskian is introduced under the framework of quaternions which is different from the standard one in DETS. Liouville formula for QDETS is also analyzed. Upon these, the solutions to the linear QDETS are established. The Putzer's algorithms to evaluate the fundamental solution matrix for homogeneous QDETS are presented. Furthermore, the variation of constants formula to solve the nonhomogeneous QDETs is given. Some concrete examples are provided to illustrate the feasibility of the proposed algorithms.

Keywords Dynamic systems on time scales, difference equations, fundamental solution matrix, quaternions.

MSC(2010) 34N05, 34A30, 39A06, 20G20.

1. Introduction

The theory of dynamic equations on time scales (DETS) has enormous applications [7, 29]. It is applicable to many fields in which physical phenomena can be described by continuous or discrete dynamical models. For instance, both continuous and discrete models are used in 3D tracking of shape, motion estimation [30] and DNA dynamics [26]. An unify framework for the theory of DETS was introduced in 1988 by Hilger [22, 23]. It unifies continuous and discrete dynamic equations. The classical differential and difference equations are special cases of DETS. Many dynamical processes are either exclusively continuous or exclusively discrete.

[†]The corresponding author. Email address:kikou@umac.mo(K. I. Kou)

[1]Department of Mathematics, Faculty of Science and Technology, University of Macau, Macao, China

[2]School of Mathematical Sciences, Huaqiao University, 362021, Quanzhou, Fujian, China

Over the years, the theory of quaternion differential equations (QDEs) has received a lot of attention [12,17,39,40,43]. The QDEs have numerous applications in physics and engineering, such as spatial kinematic modelling and attitude dynamics [14,21], fluid mechanics [19,20], quantum mechanics [1,5] and so on. Recently, the basic theory and fundamental results of linear QDEs was established [27,28,41]. It is interesting and necessary to extend the theory of QDEs to quaternion dynamic equations on arbitrary time scales (QDETS), so that the theory of quaternion dynamic equations can be widely applied to physical and engineering problems. On the one hand, both discrete and mixture of continuous and discrete dynamical models are subsumed within the QDETS. On the other hand, some differential equations need to be integrated into difference equations for computations or simulations. For example, the kinematics system on discrete time was studied in [36].

The main purpose of this paper is to study the basic theory of linear QDETS. The Hilger quaternion numbers on time scales was studied in [18]. The researchers further gave the definition of the quaternion exponential function on time scales [18]. However, we will show that the quaternion exponential function, in general, is not the solution of one-dimensional homogeneous linear QDETS:

$$y^{\Delta} = p(t)y,$$

unless $p(t)$ is either a real-valued function or a quaternion constant function. This is a striking difference between DETS and QDETS. Owing to the non-commutativity of multiplication of quaternions, there are many concepts of DETS are not effective for QDETS. Besides, in consideration of the differences between QDETS and QDEs in nature, lots of results concerning QDEs can not be easily carried out to the corresponding results of QDETS. The product rule of delta derivative on time scales is more tedious than traditional derivative. Therefore the Wronskian defined in [27] is inconvenient to be applied to the QDETS since it contains many product operations. Thanks to the systematic exposition of quaternion linear algebra (refer to [33,38,42]) in recent years, there are quite a few accessible and significant results can be applied not only by mathematicians, but also by scientists and engineers. In particular, complex adjoint matrix representation of quaternion matrix plays a critical role in the current study. The definition of determinant by complex adjoint matrix, so-called q-determinant, is crucial to define Wronskian of QDETS and to derive the Liouville's formula for QDETS. Employing the newly established Wronskian and Liouville's formula for QDETS, we obtain the algebraic structure of general solutions of n dimensional linear QDETS. It is a right quaternion module.

Explicit formulations of the fundamental solution matrices (in particular, e^{At}) for the linear QDEs with quaternion constant coefficient matrix were derived in [27]. According to the discussion in [8,16], the eigenvalue problem of quaternion matrix is complicated. A quaternion matrix usually has infinite number of eigenvalues. Moreover, the set of all eigenvectors corresponding to a non-real eigenvalue is not a submodule. If the $n \times n$ coefficient matrix A has n right linearly independent eigenvectors. Then the fundamental solution matrix e^{At} can be written in terms of the eigenvalues and eigenvectors. Otherwise, more efforts need to be exerted. In [27], the authors constructed e^{At} by means of series expansion and root subspace decomposition of quaternion matrix .

For linear QDETS with quaternion constant coefficient matrix, it is not difficult to find its fundamental solution matrix if its coefficient matrix has enough right linearly independent eigenvectors. Otherwise, we cannot use the method of combining

series expansion and root subspace decomposition to construct the fundamental solution matrices of linear QDETS. This is because that the generalized exponential function on time scales does not possess simple series expansion in contrast to e^{At}. In order to overcome this difficulty, we propose a modified Putzer's algorithm to find the fundamental solution matrices of linear QDETS. The Putzer's algorithm is particularly useful for quaternion coefficient matrices that do not have enough right linearly independent eigenvectors since it avoids the computing of eigenvectors. To authors' best knowledge, the Vieta's formulas of quaternion polynomials and the theory of annihilating polynomial of quaternion matrices have not been well studied yet. Thus the operability of Putzer's algorithm for QDETS may be confronted with some challenges. Still and all, further discussion in a later section indicates that the Putzer's algorithm for QDETS may after all be accepted as a good choice.

The rest of the paper is organized as follow. In Section 2, some basic concepts of quaternion algebra and the calculus of time scales are reviewed. In Section 3, the first order linear homogeneous QDETS are studied and the properties of generalized exponential function for QDETS are investigated. In Section 4, the structure of general solutions of higher order linear QDETS are analyzed. Specifically, a novel Wronskian determinant for QDETS is defined and the Liouville's formula and variation of constants formula are given. In Section 5, explicit formulations of the fundamental solution matrices for linear QDETS with constant coefficient matrix are presented. Some examples are given to illustrate the feasibility of the established Putzer's algorithm. Finally, some conclusions are drawn at the end of the paper.

2. Preliminaries

2.1. Quaternion algebra

The quaternions were invented in 1843 by Hamilton [37]. The skew field of quaternions is denoted by

$$\mathbb{H} := \{q = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}\},$$

where q_0, q_1, q_2, q_3 are real numbers and the elements $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ obey the Hamilton's multiplication rules:

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \ \ \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \ \ \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}, \ \ \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{ijk} = -1.$$

For every quaternion $q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3$, the scalar and vector parts of q, are defined as $\Re(q) = q_0$ and $\Im(q) = q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$, respectively. If $q = \Im(q)$, then q is called pure imaginary quaternion. The quaternion conjugate is defined by $\bar{q} = q_0 - \boldsymbol{i}q_1 - \boldsymbol{j}q_2 - \boldsymbol{k}q_3$, and the norm $|q|$ of q is defined as $|q|^2 = q\bar{q} = \bar{q}q = \sum_{m=0}^{m=3} q_m^2$. Using the conjugate and norm of q, one can define the inverse of $q \in \mathbb{H}\backslash\{0\}$ by $q^{-1} = \bar{q}/|q|^2$. For each fixed unit pure imaginary quaternion $\boldsymbol{\varsigma}$, the quaternion has subset $\mathbb{C}_{\boldsymbol{\varsigma}} := \{a + b\boldsymbol{\varsigma} : a, b \in \mathbb{R}\}$ and $\mathbb{C}_{\boldsymbol{\varsigma}}$ is isomorphic to the complex numbers.

The quaternion exponential function $\exp(q)$ is defined by means of an infinite series as

$$\exp(q) := \sum_{n=0}^{\infty} \frac{q^n}{n!}.$$

Analogous to the complex case one may derive a closed-form representation:

$$\exp(q) = \exp(\Re(q))\left(\cos|\Im(q)| + \frac{\Im(q)}{|\Im(q)|}\sin|\Im(q)|\right).$$

For simplicity of notations, we sometimes use e^q to represent $\exp(q)$. For every $q \in \mathbb{H} \backslash \{0\}$, its principal argument is defined by

$$\mathrm{Arg}(q) := \arccos \frac{\Re(q)}{|q|} \in [0, \pi].$$

Then all possible values of the argument can be expressed as $\arg(q) = \mathrm{Arg}(q) + 2k\pi$, $k \in \mathbb{Z}$. It follows that the polar form of a non-real quaternion can be written as:

$$q = |q| \left(\frac{\Re(q)}{|q|} + \frac{\Im(q)}{|\Im(q)|} \cdot \frac{|\Im(q)|}{|q|} \right) = |q| \exp(\boldsymbol{\varsigma}\theta),$$

where $\boldsymbol{\varsigma} = \frac{\Im(q)}{|\Im(q)|}$ and $\theta = \arg(q)$. Accordingly, the principal logarithm function is defined by

$$\mathrm{Ln}(q) := \begin{cases} \ln(|q|) + \frac{\Im(q)}{|\Im(q)|}\mathrm{Arg}(q), & q \in \mathbb{H} \setminus \mathbb{R}, \\ \ln(|q|) + \boldsymbol{i}\pi\frac{1-\mathrm{sgn}(q)}{2}, & q \in \mathbb{R}. \end{cases}$$

Let $h > 0$, Georgiev and Morais [18] introduced the Hilger quaternion numbers

$$\mathbb{H}_h := \{p \in \mathbb{H} : p \neq -\frac{1}{h}\}.$$

They defined the addition $\oplus$ on $\mathbb{H}_h$ by $p \oplus q := p + q + pqh$ and proved that $(\mathbb{H}_h, \oplus)$ is a group. The generalized quaternion cylinder transformation was also given in [18]:

$$\xi_h(q) := \begin{cases} \frac{1}{h}\mathrm{Ln}(1 + qh), & h > 0; \\ q, & h = 0. \end{cases} \tag{2.1}$$

Next we recall an important transformation between quaternion and complex matrices which were studied in [6, 42]. Every quaternion matrix $A \in \mathbb{H}^{m \times n}$ can be expressed uniquely in the form of

$$A = \mathfrak{C}_1(A) + \mathfrak{C}_2(A)\boldsymbol{j}, \quad \text{where} \quad \mathfrak{C}_1(A), \mathfrak{C}_2(A) \in \mathbb{C}^{m \times n}.$$

So we can define $\mathfrak{G} : \mathbb{H}^{m \times n} \to \mathbb{C}^{2m \times 2n}$ by

$$\mathfrak{G}(A) := \begin{pmatrix} \mathfrak{C}_1(A) & \mathfrak{C}_2(A) \\ -\overline{\mathfrak{C}_2(A)} & \overline{\mathfrak{C}_1(A)} \end{pmatrix},$$

where $\mathfrak{G}(A)$ is called the *complex adjoint matrix* of the quaternion matrix A. For simplicity, $\mathfrak{G}(A)$ will be denoted by χ_A in the following.

From [28], we know that $\mathbb{H}^n$ over the division ring $\mathbb{H}$ is a right $\mathbb{H}$-module and $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \ldots, \boldsymbol{\eta}_k \in \mathbb{H}^n$ are right linearly independent if

$$\boldsymbol{\eta}_1\alpha_1 + \boldsymbol{\eta}_2\alpha_2 + \cdots + \boldsymbol{\eta}_k\alpha_k = 0, \alpha_i \in \mathbb{H} \quad \text{implies that} \quad \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0.$$

Let $A \in \mathbb{H}^{n \times n}$, a nonzero $\boldsymbol{\eta} \in \mathbb{H}^{n \times 1}$ is said to be a *right eigenvector* of A corresponding to the *right eigenvalue* $\lambda \in \mathbb{H}$ provided that

$$A\boldsymbol{\eta} = \boldsymbol{\eta}\lambda$$

holds. A matrix A_1 is said to be similar to a matrix A_2 if $A_2 = S^{-1}AS$ for some nonsingular matrix S. In particular, we say that two quaternions p, q are similar if $p = \alpha^{-1}q\alpha$ for some nonzero quaternion α. We recall some basic results about quaternion matrices which can be found, for instance, in [8, 33, 42].

Theorem 2.1. *Let $A \in \mathbb{H}^{n \times n}$, then the following statements hold.*

(i) *A has exactly n right eigenvalues (including multiplicity) which are complex numbers with nonnegative imaginary parts. These eigenvalues are called standard eigenvalues.*

(ii) *If λ is a eigenvalue of A, then there exists a standard eigenvalue λ' of A such that λ and λ' are similar.*

(iii) *A is invertible if and only if χ_A is invertible.*

(iv) *$\det \chi_A \geq 0$, and the characteristic polynomial of χ_A has real coefficients.*

(v) *Let $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \ldots, \boldsymbol{\eta}_k$ be eigenvectors of A that correspond to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, respectively. If these eigenvalues are pairwise non-similar. Then $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \ldots, \boldsymbol{\eta}_k$ are right linearly independent.*

(vi) *If A is (upper or lower) triangular, then the only eigenvalues are the diagonal elements (and the quaternions similar to them).*

2.2. Calculus on time scales

The theory of time scales has gained much popularity in recent years. Bohner and Peterson together with their research collaborators, such as Agarwal and Ahlbrandt, have greatly developed the theory of time scales (see e.g. [3,4,9,31]). An systematic introduction to dynamic equations on time scales was given by Bohner and Peterson [10]. We adopt the standard notations in [2, 10, 11]. A time scale is a nonempty closed subset of $\mathbb{R}$. There are some typical examples of time scales.

(i) $\mathbb{R}$ consists of all real numbers.

(ii) $h\mathbb{Z} := \{hk : k \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of integers.

(iii) $2^{N_0} := \{2^k : k \in N_0\}$, where N_0 is the set of nonnegative integers.

(iv) $\mathbb{P}_{a,b} := \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a]$, where a, b are positive real constants.

Throughout the paper, let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, the forward jump operator σ and the backward jump operator ρ are respectively defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t, \sigma(t) = t, \rho(t) < t, \rho(t) = t$, then t is said to be right-scattered, right-dense, left-scattered, left-dense, respectively. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ and the set $\mathbb{T}^{\kappa}$ are respectively defined by

$$\mu(t) := \sigma(t) - t \tag{2.2}$$

and

$$\mathbb{T}^{\kappa} := \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if} \quad \sup \mathbb{T} < \infty; \\ \mathbb{T}, & \text{if} \quad \sup \mathbb{T} = \infty. \end{cases}$$

The classical time scales calculus is only concerned with the real-valued functions. With minor adjustments, some basic concepts of time scales calculus can also be carried to quaternion-valued functions. We denote the set of all quaternion-valued functions which are defined on time scales $\mathbb{T}$ by $\mathbb{H} \otimes \mathbb{T}$.

Definition 2.1. Assume that $f \in \mathbb{H} \otimes \mathbb{T}$ and let $t \in \mathbb{T}^\kappa$. The delta derivative $f^\Delta(t)$ is defined to be the number (provided it exists) with property that given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \le \varepsilon \left| \sigma(t) - s \right|$$

holds for all $s \in U_\delta := (t - \delta, t + \delta) \cap \mathbb{T}$.

By writing $f \in \mathbb{H} \otimes \mathbb{T}$ in the form of $f(t) = f_0(t) + f_1(t)\boldsymbol{i} + f_2(t)\boldsymbol{j} + f_3(t)\boldsymbol{k}$ with $f_i \in \mathbb{R} \otimes \mathbb{T}$, it is easy to verify that f is delta differentiable if and only if f_0, f_1, f_2, f_3 are delta differentiable. Moreover, if f is delta differentiable, then

$$f^\Delta(t) = f_0^\Delta(t) + f_1^\Delta(t)\boldsymbol{i} + f_2^\Delta(t)\boldsymbol{j} + f_3^\Delta(t)\boldsymbol{k}.$$

It follows that some useful results concerning the delta derivative for real-valued functions in [10] can be carried to quaternion-valued functions.

Theorem 2.2. *Assume that $f, g \in \mathbb{H} \otimes \mathbb{T}$ are delta differentiable at $t \in \mathbb{T}^\kappa$, then the following statements hold.*

(i) $f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t)$.

(ii) $f + g$ *is delta differentiable at* t *and* $(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t)$.

(iii) *For any* $\alpha, \beta \in \mathbb{H}$, $\alpha f \beta$ *is delta differentiable at* t *and* $(\alpha f \beta)^\Delta(t) = \alpha f^\Delta(t)\beta$.

(iv) *The product* fg *is delta differentiable at* t *and*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).$$

(v) *If* $f(t)f(\sigma(t)) \ne 0$ *then* $\widetilde{f}(t) = (f(t))^{-1}$ *is delta differentiable at* t *and*

$$\widetilde{f}^\Delta(t) = -(f(\sigma(t)))^{-1} f^\Delta(t) (f(t))^{-1} = -(f(t))^{-1} f^\Delta(t) (f(\sigma(t)))^{-1}.$$

Remark 2.1. The statements 1, 2, 3 are easy to be understood. The equality $f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t))$ can be seen from statement 1. Since $\mathbb{H}$ is noncommutative, the equality $-(f(\sigma(t)))^{-1} f^\Delta(t) (f(t))^{-1} = -(f(t))^{-1} f^\Delta(t) (f(\sigma(t)))^{-1}$ is not obvious. But this equality is true by invoking statement 4 (let $g = \widetilde{f}$). We use an example to illustrate this result.

Example 2.1. Let $\mathbb{T} = \mathbb{Z}$, $f(t) = 1 + \boldsymbol{i} + t\boldsymbol{j}$. Then we have

$$\widetilde{f}(t) = (1 - \boldsymbol{i} - t\boldsymbol{j})(2 + t^2)^{-1},$$
$$f(\sigma(t)) = f(t + 1) = 1 + \boldsymbol{i} + (t + 1)\boldsymbol{j},$$
$$f^\Delta(t) = f(t + 1) - f(t) = \boldsymbol{j}.$$

By direct computation,

$$-(f(\sigma(t)))^{-1} f^\Delta(t) (f(t))^{-1} = -(f(t))^{-1} f^\Delta(t) (f(\sigma(t)))^{-1} = \widetilde{f}^\Delta(t) = \widetilde{f}(t+1) - \widetilde{f}(t).$$

They are equal to

$$(-(2t + 1) + (2t + 1)\boldsymbol{i} + (2 - (t^2 + t))\boldsymbol{j})(2 + t^2)^{-1}(2 + (t + 1)^2)^{-1}.$$

To describe integrable quaternion-valued functions on time scales, we need to introduce the concept of rd-continuous. The rd-continuity of real-valued functions was defined by Bohner *et al.* [10].

Definition 2.2. A real-valued function is called rd-continuous if it is continuous at right-dense points and its left-sided limits exist (finite) at left-dense points.

Bohner *et al.* [10] proved that every rd-continuous function has an antiderivative. Next we introduce the rd-continuity and integrability of quaternion-valued functions.

We say that $f = f_0 + f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k} \in \mathbb{H} \otimes \mathbb{T}$ is rd-continuous provided that its every real components f_0, f_1, f_2, f_3 are rd-continuous. For every rd-continuous function f, we define the integral by

$$\int_r^s f(t)\Delta t := \int_r^s f_0(t)\Delta t + \boldsymbol{i} \int_r^s f_1(t)\Delta t + \boldsymbol{j} \int_r^s f_2(t)\Delta t + \boldsymbol{k} \int_r^s f_3(t)\Delta t$$
$$= F_0(s) - F_0(r) + (F_1(s) - F_1(r))\boldsymbol{i} + (F_2(s) - F_2(r))\boldsymbol{j} + (F_3(s) - F_3(r))\boldsymbol{k}$$
$$= F(s) - F(r),$$

where $F_i^\Delta(t) = f_i(t), (0 \le i \le 3)$ and $F^\Delta(t) = f(t)$ for $t \in \mathbb{T}^\kappa$.

From above discussion, we have the following two theorems.

Theorem 2.3. *If $f \in \mathbb{H} \otimes \mathbb{T}$ is rd-continuous and $t \in \mathbb{T}^\kappa$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

Theorem 2.4. *Let $a, b \in \mathbb{T}$ and suppose that $f \in \mathbb{H} \otimes \mathbb{T}$ is rd-continuous.*

1. If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt.$$

Namely, it is the classical integral from calculus.

2. If $[a, b] \cap \mathbb{T}$ contains only isolated points, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t \in [a,b) \cap \mathbb{T}} \mu(t)f(t), & if \quad a < b; \\ 0, & if \quad a = b; \\ -\sum_{t \in [b,a) \cap \mathbb{T}} \mu(t)f(t), & if \quad a > b. \end{cases}$$

3. First order linear QDETS

In this section, we will study the first order linear QDETS and its corresponding initial value problems. Firstly, we need to introduce some auxiliary concepts.

Definition 3.1. A function $p \in \mathbb{H} \otimes \mathbb{T}$ is said to be regressive if

$$1 + \mu(t)p(t) \ne 0, \quad \text{for all} \quad t \in \mathbb{T}^\kappa. \tag{3.1}$$

The set of all regressive and rd-continuous quaternion-valued functions is denoted by $\mathcal{R}(\mathbb{T}, \mathbb{H})$. By similar arguments to $(\mathbb{H}_h, \oplus)$, we know that $\mathcal{R}(\mathbb{T}, \mathbb{H})$ is a group under addition $\oplus$ which is defined by

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t), \quad \text{for all} \quad t \in \mathbb{T}^\kappa,$$

where $p, q \in \mathcal{R}(\mathbb{T}, \mathbb{H})$. Based on the definition of quaternion cylinder transformation (2.1), the generalized quaternion exponential function for $p \in \mathcal{R}(\mathbb{T}, \mathbb{H})$ is defined by

$$E_p(t, s) := \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right).$$

Clearly, the generalized quaternion exponential function never be zero for any $p \in \mathcal{R}(\mathbb{T}, \mathbb{H})$. We proceed by presenting some important properties of $E_p(t, s)$.

Lemma 3.1. *If $p(t) = \alpha \in \mathbb{H} \setminus \mathbb{R}$ is a quaternion constant, then $p \in \mathcal{R}(\mathbb{T}, \mathbb{H})$ and $E_p(t, r)E_p(r, s) = E_p(t, s)$ for all $r, s, t \in \mathbb{T}$.*

Proof. Obviously, $p \in \mathcal{R}(\mathbb{T}, \mathbb{H})$. Let $r, s, t \in \mathbb{T}$, then both $\int_r^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau$ and $\int_s^r \xi_{\mu(\tau)}(p(\tau))\Delta\tau$ are $\mathbb{C}_\varsigma$-valued, where $\varsigma = \frac{\mathfrak{J}(\alpha)}{|\mathfrak{J}(\alpha)|}$. Therefore

$$\begin{aligned}
E_p(t, r)E_p(r, s) &= \exp\left(\int_r^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)\exp\left(\int_s^r \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \\
&= \exp\left(\int_r^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau + \int_s^r \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \\
&= \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \\
&= E_p(t, s)
\end{aligned}$$

which completes the proof. $\qquad\square$

Lemma 3.2. *Suppose that $\alpha \in \mathbb{H} \setminus \mathbb{R}$ and $\mu(t) > 0$, then $E_\alpha(\sigma(t), t) - 1 = \alpha\mu(t)$.*

Proof. Since $\alpha \in \mathbb{H} \setminus \mathbb{R}$ and $\mu(t) > 0$, then by Theorem 2.3

$$\begin{aligned}
E_\alpha(\sigma(t), t) &= \exp\left(\int_t^{\sigma(t)} \xi_{\mu(\tau)}(\alpha)\Delta\tau\right) \\
&= \exp\left(\mu(t)\xi_{\mu(t)}(\alpha)\right) \\
&= \exp\left(\mathrm{Ln}(1 + \alpha\mu(t))\right).
\end{aligned}$$

Observe that $1 + \alpha\mu(t)$ is certainly not real-valued. It follows that

$$\exp\left(\mathrm{Ln}(1 + \alpha\mu(t))\right) = 1 + \alpha\mu(t),$$

which completes the proof. $\qquad\square$

Now we turn to study the following first order linear QDETS.

Definition 3.2. If $p \in \mathcal{R}(\mathbb{T}, \mathbb{H})$, then the first order linear quaternion dynamic equation

$$y^\Delta(t) = p(t)y(t) \tag{3.2}$$

is called regressive. For any fixed $t_0 \in \mathbb{T}$ and $c_0 \in \mathbb{H}$, the corresponding initial value problem is

$$y^\Delta(t) = p(t)y(t), \quad y(t_0) = c_0. \tag{3.3}$$

By similar arguments to Theorem 5.8 in [10], the initial value problem (3.3) has exactly a unique solution.

Remark 3.1. Let $\psi_p(t, t_0)$ denotes the unique solution of (3.3) with $c_0 = 1$. In the classical case, we know that $\psi_p(t, s) = E_p(t, s)$. This assertion, however, is no longer true in the quaternion case. Namely, $\psi_p(t, s)$ is not equal to $E_p(t, s)$ in general.

Example 3.1. Consider the time scale $\mathbb{T} = \mathbb{Z}$. Let $p(t) = 1 + \boldsymbol{i}t + \boldsymbol{j}$, then

$$
\begin{aligned}
\xi_{\mu(\tau)}(p(\tau)) &= \frac{1}{\mu(\tau)} \operatorname{Ln}(1 + p(\tau)) \\
&= \frac{1}{2} \ln(5 + \tau^2) + \frac{\boldsymbol{i}\tau + \boldsymbol{j}}{\sqrt{1 + \tau^2}} \arccos \frac{2}{\sqrt{5 + \tau^2}}.
\end{aligned}
$$

By direct computation, we have

$$
\int_0^1 \xi_{\mu(\tau)}(p(\tau)) \Delta\tau = \frac{1}{2} \ln 5 + \boldsymbol{j} \arccos \frac{2}{\sqrt{5}}.
$$

Thus

$$
E_p(1, 0) = \exp(\frac{1}{2} \ln 5 + \boldsymbol{j} \arccos \frac{2}{\sqrt{5}}) = 2 + \boldsymbol{j}.
$$

Let $y(t) = E_p(t, 0)$, we have

$$
y^\Delta(0) = y(1) - y(0) = 2 + \boldsymbol{j} \neq 1 + \boldsymbol{j} = p(0)y(0),
$$

which implies $\psi_p(t, 0) \neq E_p(t, 0)$.

Fortunately, under some suitable conditions, $\psi_p(t, s)$ is still equal to $E_p(t, s)$. To prove this result, we give an useful lemma first.

Lemma 3.3. *If* $\sigma(t) = t$ *and* $p(t) = \alpha \in \mathbb{H} \setminus \mathbb{R}$ *is a quaternion constant function. Then*

$$
\lim_{\tau \to t} \xi_{\mu(\tau)}(p(\tau)) = \alpha.
$$

Proof. If there exists a number $\delta_0 > 0$ such that $\mu(\tau) = 0$ for all $\tau \in U_{\delta_0}$. Then

$$
\lim_{\tau \to t} \xi_{\mu(\tau)}(p(\tau)) = \lim_{\tau \to t} \xi_0(\alpha) = \alpha.
$$

Otherwise, for any $\delta > 0$, there exists a number $\tau \in U_\delta$ such that $\mu(\tau) > 0$. Let $\delta_1 = 1$, define

$$
D_1 := \{\tau \in U_{\delta_1} : \mu(\tau) > 0\} \quad \text{and} \quad D_2 := \{\tau \in U_{\delta_1} : \mu(\tau) = 0\},
$$

then D_1 contains infinite elements. Now we claim that $\lim_{\tau \to t} \mu(\tau) = 0 = \mu(t)$. Since $\sigma(t) = t$, then there exists a strictly decreasing sequence $\{\tau_n\} \subset \mathbb{T}$ such that $\lim_{n \to \infty} \tau_n = t$. For any τ satisfying $0 < \tau - t < \tau_1$, there exists positive integer n_1 such that $\tau_{n_1+1} > \tau > \tau_{n_1}$. Thus $\sigma(\tau) \leq \tau_{n_1}$. It follows that $\mu(\tau) = \sigma(\tau) - \tau \leq \tau_{n_1} - \tau \leq \tau_{n_1+1} - \tau_{n_1}$. Therefore

$$
\lim_{\tau \to t^+} \mu(\tau) = \lim_{n_1 \to \infty} (\tau_{n_1+1} - \tau_{n_1}) = 0.
$$

For any $\tau < t$, we have $\tau \leq \sigma(\tau) \leq t$. Then $\mu(\tau) = \sigma(\tau) - \tau \leq t - \tau$. Thus

$$
0 \leq \lim_{\tau \to t^-} \mu(\tau) \leq \lim_{\tau \to t^-} (t - \tau) = 0.
$$

Then we have

$$
\lim_{\substack{\tau \to t \\ \tau \in D_1}} \xi_{\mu(\tau)}(\alpha) = \lim_{\substack{\tau \to t \\ \tau \in D_1}} \frac{1}{\mu(\tau)} \left(\ln|1 + \alpha\mu(\tau)| + \frac{\Im(\alpha)}{|\Im(\alpha)|} \arccos \frac{1 + \mu(\tau)\Re(\alpha)}{|1 + \alpha\mu(\tau)|} \right)
$$

$$
= \Re(\alpha) + \frac{\Im(\alpha)}{|\Im(\alpha)|} |\Im(\alpha)| = \alpha
$$

by invoking the following two limits

$$
\lim_{h \to 0} \frac{\ln\left(|1 + h\Re(\alpha)|^2 + |\Im(\alpha)h|^2\right)}{2h} = \Re(\alpha),
$$

$$
\lim_{h \to 0} \arccos \frac{1 + h\Re(\alpha)}{h\sqrt{|1 + h\Re(\alpha)|^2 + |\Im(\alpha)h|^2}} = |\Im(\alpha)|.
$$

In other words, for any $\varepsilon > 0$, there exists positive number $\delta_2 < \delta_1$ such that $\left|\xi_{\mu(\tau)}(\alpha) - \alpha\right| < \varepsilon$ for all $\tau \in U_{\delta_2} \cap D_1$. Note that for all $\tau \in U_{\delta_2} \cap D_2$, $\left|\xi_{\mu(\tau)}(\alpha) - \alpha\right| = \left|\xi_0(\alpha) - \alpha\right| = 0 < \varepsilon$. Hence $\left|\xi_{\mu(\tau)}(\alpha) - \alpha\right| < \varepsilon$ holds for all $\tau \in U_{\delta_2}$ which completes the proof. □

Theorem 3.1. *If $p \in \mathcal{R}(\mathbb{T}, \mathbb{H})$ is real-valued or $p(t) = \alpha \in \mathbb{H} \setminus \mathbb{R}$ is a quaternion constant function. Then $\psi_p(t, s) = E_p(t, s)$ for all $s, t \in \mathbb{T}$.*

Proof. For real-valued $p \in \mathcal{R}(\mathbb{T}, \mathbb{H})$, please refer to Theorem 2.33 in [10]. For $p(t) = \alpha \in \mathbb{H} \setminus \mathbb{R}$, observe first that $E_\alpha(s, s) = 1$. We only need to show that $E_\alpha(t, s)$ satisfies (3.2). Let $t \in \mathbb{T}^\kappa$ be right-scattered, that is, $\mu(t) = \sigma(t) - t > 0$. By applying Lemma 3.1 and 3.2,

$$
E_\alpha^\Delta(t, s) = \frac{E_\alpha(\sigma(t), s) - E_\alpha(t, s)}{\mu(t)}
$$

$$
= \frac{E_\alpha(\sigma(t), t) - 1}{\mu(t)} E_\alpha(t, s)
$$

$$
= \alpha E_\alpha(t, s).
$$

Let $t \in \mathbb{T}^\kappa$ be right-dense. For any given positive number $\varepsilon < |E_\alpha(t, s)|$. Lemma 3.3 implies that there exists $\delta_1 > 0$ such that

$$
\left|\xi_{\mu(\tau)}(\alpha) - \alpha\right| < \frac{\varepsilon}{2 |E_\alpha(t, s)|} < 1
$$

for all $\tau \in U_{\delta_1}$. Then $\left|\int_{t'}^t \xi_{\mu(\tau)}(\alpha)\Delta\tau\right| \le (1 + |\alpha|) |t - t'|$ and

$$
|E_\alpha(t, s)| \cdot \left|\int_{t'}^t \left(\xi_{\mu(\tau)}(\alpha) - \alpha\right) \Delta\tau\right| \le \frac{\varepsilon}{2} |t - t'|
$$

for all $t' \in U_{\delta_1}$. Observe that $\lim_{\substack{q \to 0 \\ q \in \mathbb{C}_\zeta}} (1 - q - \exp(-q)) q^{-1} = 0$ where $\zeta = \frac{\Im(\alpha)}{|\Im(\alpha)|}$. Then there exists $\delta_1 > \delta_2 > 0$ such that

$$
\left|1 - \int_{t'}^t \xi_{\mu(\tau)}(\alpha)\Delta\tau - E_\alpha(t', t)\right| \le \frac{\varepsilon}{2(1 + |\alpha|) |E_\alpha(t, s)|} \cdot \left|\int_{t'}^t \xi_{\mu(\tau)}(\alpha)\Delta\tau\right|
$$

for all $t' \in U_{\delta_2}$. Therefore, by Lemma 3.1

$$
\begin{aligned}
&|E_\alpha(t,s) - E_\alpha(t',s) - \alpha E_\alpha(t,s)(t-t')| \\
&= |1 - E_\alpha(t',t) - \alpha(t-t')| \cdot |E_\alpha(t,s)| \\
&\leq |E_\alpha(t,s)| \cdot \left| \int_{t'}^{t} \left(\xi_{\mu(\tau)}(\alpha) - \alpha \right) \Delta\tau \right| \\
&\quad + |E_\alpha(t,s)| \cdot \left| 1 - \int_{t'}^{t} \xi_{\mu(\tau)}(\alpha) \Delta\tau - E_\alpha(t',t) \right| \\
&\leq \frac{\varepsilon}{2} |t-t'| + \frac{\varepsilon}{2} |t-t'| \\
&= \varepsilon |t-t'|
\end{aligned}
$$

for all $t' \in U_{\delta_2}$. This implies that $E_\alpha^\Delta(t,s) = \alpha E_\alpha(t,s)$. The proof is complete. $\square$

Proposition 3.1. *Let $p \in \mathcal{R}(\mathbb{T}, \mathbb{H})$, then the following assertions hold.*

(i) *If $\phi(t)$ satisfies (3.2) and $\phi(t_0) = 0$ for some $t_0 \in \mathbb{T}$, then $\phi(t) = 0$ for all $t \in \mathbb{T}$. In other words, if $\phi(t_0) \neq 0$ for some $t_0 \in \mathbb{T}$, then $\phi(t) \neq 0$ for all $t \in \mathbb{T}$.*

(ii) *If $\phi(t)$ is a nonzero solution of (3.2), then for any $t, r, s \in \mathbb{T}$, $\psi_p(t,s) = \phi(t)\phi^{-1}(s) = \psi_p(t,r)\psi_p(r,s) \neq 0$ and $\psi_p^{-1}(t,s) = \psi_p(s,t)$.*

(iii) *For any given IVP $y(t_0) = c_0$, the solution is $\psi_p(t,t_0)c_0$.*

(iv) *$\psi_0(t,s) = 1$ and $\psi_p(t,t) = 1$.*

Proof. In the classical case, for any $y = \phi(t)$ satisfying (3.2) never vanishes, because it is just a composition of exponential and $\xi_{\mu(\cdot)}(p(\cdot))$. But Example 3.1 indicates that $\psi_p(t,s)$ is not necessarily to be $E_p(t,s)$ in the quaternion case. So assertion 1 is not a trivial result. Let $w(t) = |\phi(t)|^2 = \overline{\mathfrak{C}_1(\phi(t))}\mathfrak{C}_1(\phi(t)) + \overline{\mathfrak{C}_2(\phi(t))}\mathfrak{C}_2(\phi(t))$. Then by Theorem 2.2,

$$
\begin{aligned}
w^\Delta &= \overline{\mathfrak{C}_1(\phi)}\mathfrak{C}_1(\phi)^\Delta + \overline{\mathfrak{C}_1(\phi)}^\Delta \mathfrak{C}_1(\phi)^\sigma + \overline{\mathfrak{C}_2(\phi)}\mathfrak{C}_2(\phi)^\Delta + \overline{\mathfrak{C}_2(\phi)}^\Delta \mathfrak{C}_2(\phi)^\sigma \\
&= \overline{\mathfrak{C}_1(\phi)}\left(\mathfrak{C}_1(p)\mathfrak{C}_1(\phi) - \mathfrak{C}_2(p)\overline{\mathfrak{C}_2(\phi)} \right) + \left(\overline{\mathfrak{C}_1(p)\mathfrak{C}_1(\phi)} - \overline{\mathfrak{C}_2(p)}\mathfrak{C}_2(\phi) \right) \left(\mathfrak{C}_1(\phi) + \mu \mathfrak{C}_1(\phi)^\Delta \right) \\
&\quad + \overline{\mathfrak{C}_2(\phi)}\left(\mathfrak{C}_1(p)\mathfrak{C}_2(\phi) + \mathfrak{C}_2(p)\overline{\mathfrak{C}_1(\phi)} \right) + \left(\overline{\mathfrak{C}_1(p)\mathfrak{C}_2(\phi)} + \overline{\mathfrak{C}_2(p)}\mathfrak{C}_1(\phi) \right) \left(\mathfrak{C}_2(\phi) + \mu \mathfrak{C}_2(\phi)^\Delta \right) \\
&= 2\Re(p) \left(|\mathfrak{C}_1(\phi)|^2 + |\mathfrak{C}_2(\phi)|^2 \right) \\
&\quad + \mu \left(\overline{\mathfrak{C}_1(p)\mathfrak{C}_1(\phi)} - \overline{\mathfrak{C}_2(p)}\mathfrak{C}_2(\phi) \right) \left(\mathfrak{C}_1(p)\mathfrak{C}_1(\phi) - \mathfrak{C}_2(p)\overline{\mathfrak{C}_2(\phi)} \right) \\
&\quad + \left(\overline{\mathfrak{C}_1(p)\mathfrak{C}_2(\phi)} + \overline{\mathfrak{C}_2(p)}\mathfrak{C}_1(\phi) \right) \left(\mathfrak{C}_1(p)\mathfrak{C}_2(\phi) + \mathfrak{C}_2(p)\overline{\mathfrak{C}_1(\phi)} \right) \\
&= 2 \left(\Re(p) + \mu |p|^2 \right) w.
\end{aligned}
$$

Therefore $w(t_0) = 0$ implies that $w(t) = 0$ for all $t \in \mathbb{T}$ by Theorem 3.1. This completes the proof of assertion 1. The rest of assertions follows from first assertion. $\square$

Example 3.2. Let $\mathbb{T} = 2^{\mathbb{N}_0}$ and $\alpha \in \mathbb{H} \setminus \mathbb{R}$ be a quaternion constant. Then $\mu(2^m) = 2^{m+1} - 2^m = 2^m$ and therefore

$$
\xi_{\mu(2^m)}(\alpha) = 2^{-m} \mathrm{Ln}(1 + 2^m \alpha).
$$

It follows that

$$\int_1^{2^k} \xi_{\mu(\tau)}(\alpha)\Delta\tau = \sum_{m=0}^{k-1} \mu(2^m)\xi_{\mu(2^m)}(\alpha) = \sum_{m=0}^{k-1} \text{Ln}(1 + 2^m\alpha).$$

Thus

$$E_\alpha(2^k, 1) = \prod_{m=0}^{k-1} (1 + 2^m\alpha),$$

Example 3.3. Let $\mathbb{T} = h\mathbb{Z}$ with $h > 0$ and $\alpha \in \mathbb{H} \setminus \mathbb{R}$ be a quaternion constant. Then

$$E_\alpha(t, 0) = |1 + \alpha h|^{\frac{t}{h}} \left(\cos\left(\frac{t}{h} \arccos \frac{1 + \alpha_0 h}{|1 + \alpha h|} \right) + \frac{\mathfrak{I}(\alpha)}{|\mathfrak{I}(\alpha)|} \sin\left(\frac{t}{h} \arccos \frac{1 + \alpha_0 h}{|1 + \alpha h|} \right) \right)$$

$$= (1 + \alpha h)^{\frac{t}{h}}.$$

4. Linear systems of QDETS

Let $A(t) = (a_{ij}(t))_{m \times n}$ be an $m \times n$ quaternion-matrix-valued function with $a_{ij}(t) \in \mathbb{H} \otimes \mathbb{T}$. We denote the set of such quaternion-matrix-valued functions by $\mathbb{H}^{m \times n} \otimes \mathbb{T}$. We say $A(t)$ is rd-continuous (delta differentiable) on $\mathbb{T}$ if all of $a_{ij}(t)$ $(1 \leq i \leq m, 1 \leq j \leq n)$ is rd-continuous (delta differentiable) on $\mathbb{T}$. If $A(t)$ is delta differentiable on $\mathbb{T}$, we put $A^\Delta(t) := (a_{ij}^\Delta(t))_{n \times n}$.

In order to state our results, we introduce some notations which are analogous to those used in [15, 24]. Let $\Omega_k^{(m)}$ be the set consisting of all possible k-combinations of the set $\{1, 2, \ldots, m\}$. The number of elements of $\Omega_k^{(m)}$ is C_m^k. Let $\Lambda_k = \{i_1, i_2, \ldots, i_k\} \in \Omega_k^{(m)}$ be an index set. For any $B \in \mathbb{C}^{m \times m} \otimes \mathbb{T}$, $B(\Lambda_k)$ denotes the principal sub-matrix that lies in the rows and columns of B indexed by Λ_k and

$$\mathcal{V}_k(B) := \sum_{\Lambda_k \in \Omega_k^{(m)}} \det B(\Lambda_k) \tag{4.1}$$

denotes the sum of determinants of all $B(\Lambda_k), \Lambda_k \in \Omega_k^{(m)}$. Similarly, let $B(\Lambda_k, \Delta)$ be the $m \times m$ matrix generated from B by replacing original entries with delta derivatives on the rows indexed by Λ_k.

We consider the linear nonhomogeneous quaternion dynamic equations

$$\phi^\Delta(t) = A(t)\phi(t) + \boldsymbol{f}(t) \tag{4.2}$$

and the linear homogeneous quaternion dynamic equations

$$\phi^\Delta(t) = A(t)\phi(t), \tag{4.3}$$

where $\phi(t), \boldsymbol{f}(t) \in \mathbb{H}^{n \times 1} \otimes \mathbb{T}$ and $A(t) \in \mathbb{H}^{n \times n} \otimes \mathbb{T}$. Let $t_0 \in \mathbb{T}$ and $\boldsymbol{\eta} \in \mathbb{H}^n$, then the corresponding initial value problem is

$$\phi(t_0) = \boldsymbol{\eta}. \tag{4.4}$$

We are in a position to introduce the concept of regressivity of $A(t)$.

Definition 4.1. We say $A(t) \in \mathbb{H}^{n \times n} \otimes \mathbb{T}$ is regressive provided that

$$I_n + \mu(t)A(t) \quad \text{is invertible for all} \quad t \in \mathbb{T}^\kappa. \tag{4.5}$$

The totality of all such regressive and rd-continuous quaternion-valued functions is denoted by $\mathcal{R}(\mathbb{T}, \mathbb{H}^{n \times n})$. If $A \in \mathcal{R}(\mathbb{T}, \mathbb{H}^{n \times n})$, we say system (4.2) is regressive.

Lemma 4.1. *Let $A(t) \in \mathbb{H}^{n \times n} \otimes \mathbb{T}$. Then for any fixed $t \in \mathbb{T}$, $I_n + \mu(t)A(t)$ is invertible if and only if $I_{2n} + \mu(t)\chi_A(t)$ is invertible.*

Proof. Let $\widetilde{A}(t) = I_n + \mu(t)A(t)$, it is easy to see that $\chi_{\widetilde{A}}(t) = I_{2n} + \mu(t)\chi_A(t)$. Then by statement 3 of Theorem 2.1, we complete the proof. $\qquad\square$

We need a lemma about the equivalent conditions of regressivity in the classical case. This result can be found in [10, 15].

Lemma 4.2. *Let $B(t) \in \mathbb{C}^{m \times m} \otimes \mathbb{T}$, then for any fixed $t \in \mathbb{T}$, the following statements are equivalent.*

- (i) $I_m + \mu(t)B(t)$ *is invertible.*
- (ii) *The eigenvalues $\lambda_i(t)$ of $B(t)$ is regressive for all $1 \le i \le m$. Namely, $1 + \mu(t)\lambda_i(t) \ne 0$ for all $1 \le i \le m$.*
- (iii) *$u(t)$ is regressive, namely, $1 + \mu(t)u(t) \ne 0$, where $u(t) = \sum_{k=1}^{2n} \mu^{k-1}(t)\mathcal{V}_k(B(t))$ and $\mathcal{V}_k$ is defined by (4.1).*

An immediate consequence of Lemma 4.1 and 4.2 is the equivalent conditions of regressivity for $A(t) \in \mathbb{H}^{n \times n} \otimes \mathbb{T}$.

Theorem 4.1. *Let $A(t) \in \mathbb{H}^{n \times n} \otimes \mathbb{T}$. Then for any fixed $t \in \mathbb{T}$, $A(t)$ is regressive if and only if all of its eigenvalues are regressive.*

Proof. By Lemma 4.1 and 4.2, we know that $A(t) \in \mathbb{H}^{n \times n} \otimes \mathbb{T}$ is regressive if and only if all the standard eigenvalues of $A(t)$ are regressive. Note that if $\lambda \in \mathbb{H}$ is regressive, then $\alpha^{-1}\lambda\alpha$ is also regressive for any nonzero $\alpha \in \mathbb{H}$. Then by statement 2 of Theorem 2.1, we complete the proof. $\qquad\square$

By similar arguments to Theorem 5.8 in [10], we have the following existence and uniqueness theorem.

Theorem 4.2. *If $A \in \mathcal{R}(\mathbb{T}, \mathbb{H}^{n \times n})$ and $\boldsymbol{f}$ is rd-continuous. Then the initial value problem (4.2), (4.4) has exactly a unique solution.*

To study the properties of solutions of (4.3), we should define the concept of *Wronskian* for quaternion dynamic equations. Due to the noncommutative property of quaternons, there is no unified definition of determinant of quaternion matrix. Many researchers have proposed different definitions. But, as mentioned in [28], some definitions of determinant may be not suitable to define Wronskian. Kou *et al.* [27] adopted the definition of determinant based on permutation proposed by Chen [13]. The proof of Liouville's formula in [27] is complicated. In this paper, we adopt an alternative definition of determinant called *q-determinant* [42] for $A \in \mathbb{H}^{n \times n}$. It is defined by

$$\widehat{\det}A := \det \chi_A. \tag{4.6}$$

We know that the product rule for delta derivative is more tedious than traditional one. But the Wronskian defined in [27] contains many product operations.

So we use q-determinant to define the Wronskian for quaternion dynamic equations on time scales.

Definition 4.2. Let $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_n(t)$, $\boldsymbol{x}_i(t) \in \mathbb{H}^n \otimes \mathbb{T}$ be n quaternion-vector-valued functions. Denote

$$M(t) := (\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_n(t)) = \begin{pmatrix} x_{11}(t) \ x_{12}(t) \ \cdots \ x_{1n}(t) \\ x_{21}(t) \ x_{22}(t) \ \cdots \ x_{2n}(t) \\ \cdots \\ x_{n1}(t) \ x_{n2}(t) \ \cdots \ x_{nn}(t) \end{pmatrix}.$$

The Wronskian of $M(t)$ is defined by

$$W(t) := \widehat{\det} M(t) = \det \chi_M(t).$$

If $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_n(t)$ are solutions of (4.3), then we call $M(t)$ a solution matrix of (4.3).

Theorem 4.3. *If $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t) \in \mathbb{H}^n \otimes \mathbb{T}$ are solutions of (4.3), then the right linear combination $\boldsymbol{x}_1(t)\alpha_1 + \boldsymbol{x}_2(t)\alpha_2$ for any $\alpha_1, \alpha_2 \in \mathbb{H}$ is also a solution of (4.3).*

Theorem 4.4. *If $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_n(t)$, $\boldsymbol{x}_i(t) \in \mathbb{H}^n \otimes \mathbb{T}$ are right linearly dependent on $\mathbb{T}$, then $W(t) = 0$ for all $t \in \mathbb{T}$. In other words, if $W(t) \neq 0$ for some $t_0 \in \mathbb{T}$, then $x_1(t), x_2(t), \cdots, x_n(t)$ are right linearly independent on $\mathbb{T}$.*

Proof. If $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_n(t)$ are right linearly dependent on $\mathbb{T}$, then there exists a nonzero vector $\boldsymbol{q} = (q_1, q_2, \cdots, q_n)^\top \in \mathbb{H}^n$ such that

$$M(t)\boldsymbol{q} = 0, \quad \text{for all} \quad t \in \mathbb{T}.$$

If there exists $t_0 \in \mathbb{T}$ such that $W(t_0) \neq 0$, by definition of Wronskian, we have $\chi_M(t_0)$ is invertible. From Theorem 2.1, it follows that $M(t_0)$ is invertible. Thus $M(t_0)\boldsymbol{\eta} = 0$ has a unique solution $\boldsymbol{\eta} = 0$ by Theorem 4.3 of [42]. This contradicts the fact that $\boldsymbol{\eta} = \boldsymbol{q}$ is a nonzero solution of $M(t_0)\boldsymbol{\eta} = 0$. $\square$

Theorem 4.5. *If $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_n(t)$ are n right linearly independent solutions of (4.3), then $W(t) \neq 0$ for all $t \in \mathbb{T}$.*

Proof. Assume that there exists $t_0 \in \mathbb{T}$ such that $W(t_0) = 0$. By similar arguments to Theorem 4.4, it is easy to see that there is a nonzero vector $\boldsymbol{q} = (q_1, q_2, \cdots, q_n)^\top \in \mathbb{H}^n$ such that $M(t_0)\boldsymbol{q} = 0$. Define $\boldsymbol{x}(t) := M(t)\boldsymbol{q}$ for all $t \in \mathbb{T}$. By Theorem 4.3, $\boldsymbol{x}(t)$ is a solution of (4.3) with $\boldsymbol{x}(t_0) = 0$. Note that $\boldsymbol{y}(t) \equiv 0$ is the unique solution of (4.3) and initial condition (4.4) with $\boldsymbol{\eta} = 0$. Therefore $M(t)\boldsymbol{q} = \boldsymbol{x}(t) = \boldsymbol{y}(t) \equiv 0$. This implies that $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_n(t)$ are right linearly dependent, which is contradiction to the hypotheses of the theorem. $\square$

To deduce the Liouville's formula, we need an important Lemma.

Lemma 4.3. *Let $M(t)$ be a solution matrix of (4.3) and $W(t)$ be the corresponding Wronskian. Then for any index set $\Lambda_k \in \Omega_k^{(m)}$ with $m = 2n$, we have*

$$\det \chi_M(\Lambda_k, \Delta) = \det \chi_A(\Lambda_k) \cdot W.$$

Proof. Without loss of generality, we may assume $\Lambda_k = \{i_1, i_2, \cdots, i_k\}$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq m$. We only consider the case of $i_k \leq n$, the other cases can be similarly proved.

Observe that $M^\Delta(t) = A(t)M(t)$, then (i_r, j)-component of $\chi_M(\Lambda_k, \Delta)$ is

$$
\begin{cases}
\displaystyle\sum_{s=1}^{n} \mathfrak{C}_1(a_{i_r s})\mathfrak{C}_1(x_{sj}) - \mathfrak{C}_2(a_{i_r s})\overline{\mathfrak{C}_2(x_{sj})}, & j \leq n; \\
\displaystyle\sum_{s=1}^{n} \mathfrak{C}_1(a_{i_r s})\mathfrak{C}_2(x_{sj}) + \mathfrak{C}_2(a_{i_r s})\overline{\mathfrak{C}_1(x_{sj})}, & j > n.
\end{cases}
$$

If $i \leq n$ and $i \notin \Lambda_k$, the (i, j)-component of $\chi_M(\Lambda_k, \Delta)$ is

$$
\begin{cases}
\mathfrak{C}_1(x_{ij}), & j \leq n; \\
\mathfrak{C}_2(x_{ij}), & j > n.
\end{cases}
$$

If $i > n$, the (i, j)-component of $\chi_M(\Lambda_k, \Delta)$ is

$$
\begin{cases}
-\overline{\mathfrak{C}_2(x_{i-n,j})}, & j \leq n; \\
\overline{\mathfrak{C}_1(x_{i-n,j})}, & j > n.
\end{cases}
$$

Then for $s \leq n$ and $s \notin \Lambda_k$, we do row operation of $-\mathfrak{C}_1(a_{i_r s})R_s + R_{i_r}$ for $\chi_M(\Lambda_k, \Delta)$. And for $s > n$, we do row operation of $-\mathfrak{C}_2(a_{i_r, s-n})R_s + R_{i_r}$ for $\chi_M(\Lambda_k, \Delta)$. After doing this procedure for $1 \leq r \leq k$, we obtain a new matrix $\chi_M(\Lambda_k, \Delta, new)$. The (i_r, j)-component of $\chi_M(\Lambda_k, \Delta, new)$ is

$$
\begin{cases}
\displaystyle\sum_{s=1}^{k} \mathfrak{C}_1(a_{i_r i_s})\mathfrak{C}_1(x_{i_s j}), & j \leq n; \\
\displaystyle\sum_{s=1}^{k} \mathfrak{C}_1(a_{i_r i_s})\mathfrak{C}_2(x_{i_s j}), & j > n.
\end{cases}
$$

Construct a block diagonal matrix

$$
S := \begin{pmatrix} I_{i_1-1} & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & I_{m-i_k} \end{pmatrix},
$$

where I_s is the s-order identity matrix and

$$
H := \begin{pmatrix}
\mathfrak{C}_1(a_{i_1 i_1}) & 0 & \mathfrak{C}_1(a_{i_1 i_2}) & 0 & \mathfrak{C}_1(a_{i_1 i_3}) & \ldots & \mathfrak{C}_1(a_{i_1 i_k}) \\
0 & I_{i_2-i_1-1} & 0 & 0 & 0 & \ldots & 0 \\
\mathfrak{C}_1(a_{i_2 i_1}) & 0 & \mathfrak{C}_1(a_{i_2 i_2}) & 0 & \mathfrak{C}_1(a_{i_2 i_3}) & \ldots & \mathfrak{C}_1(a_{i_2 i_k}) \\
0 & 0 & 0 & I_{i_3-i_2-1} & 0 & \ldots & 0 \\
\mathfrak{C}_1(a_{i_3 i_1}) & 0 & \mathfrak{C}_1(a_{i_3 i_2}) & 0 & \mathfrak{C}_1(a_{i_3 i_3}) & \ldots & \mathfrak{C}_1(a_{i_3 i_k}) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathfrak{C}_1(a_{i_k i_1}) & 0 & \mathfrak{C}_1(a_{i_k i_2}) & 0 & \mathfrak{C}_1(a_{i_k i_3}) & \ldots & \mathfrak{C}_1(a_{i_k i_k})
\end{pmatrix}.
$$

Then $\chi_M(\Lambda_k, \Delta, new) = S\chi_M$. Therefore

$$\det \chi_M(\Lambda_k, \Delta) = \det \chi_M(\Lambda_k, \Delta, new) = \det S \cdot \det \chi_M = (\det H) \cdot W.$$

Observe that $\det H = \det \chi_A(\Lambda_k)$, the proof is complete.   □

We need a lemma about delta derivative of determinant from [10, Theorem 5.105].

Lemma 4.4 ([10]). *Let $C = (c_{ij})_{1 \leq i,j \leq m} \in \mathbb{R}^n \otimes \mathbb{T}$ be delta differentiable. For $1 \leq k \leq m$, define $B^{(k)} = (b_{ij}^{(k)})_{1 \leq i,j \leq m}$ by*

$$b_{ij}^{(k)} := \begin{cases} c_{ij}^{\sigma} & if \quad i < k, \\ c_{ij}^{\Delta} & if \quad i = k, \\ c_{ij} & if \quad i > k. \end{cases}$$

Then $\det C \in \mathbb{R} \otimes \mathbb{T}$ is delta differentiable, and

$$(\det C)^{\Delta} = \sum_{k=1}^{m} \det B^{(k)}.$$

Remark 4.1. Although Lemma 4.4 is established for real-valued functions, it is easy to verify that Lemma 4.4 is also valid for complex-valued functions.

Now we present the Liouville's formula.

Theorem 4.6. *The Wronskian of solution matrix $M(t)$ of (4.3) satisfies the following Liouville's formula.*

$$W(t) = E_u(t, t_0)W(t_0), \tag{4.7}$$

where $u = \sum_{k=1}^{2n} \mu^{k-1} \mathcal{V}_k(\chi_A)$ and $\mathcal{V}_k$ is defined by (4.1).

Proof.   From Lemma 4.4, we see that

$$\begin{aligned}
W^{\Delta} &= (\det \chi_M)^{\Delta} \\
&= \det B^{(1)} + \det B^{(2)} + \cdots + \det B^{(2n)} \\
&= \det \begin{pmatrix} \chi_M^{\Delta}(1,:) \\ \chi_M(2,:) \\ \vdots \\ \chi_M(2n,:) \end{pmatrix} + \det \begin{pmatrix} \chi_M^{\sigma}(1,:) \\ \chi_M^{\Delta}(2,:) \\ \vdots \\ \chi_M(2n,:) \end{pmatrix} + \cdots + \det \begin{pmatrix} \chi_M^{\sigma}(1,:) \\ \chi_M^{\sigma}(2,:) \\ \vdots \\ \chi_M^{\Delta}(2n,:) \end{pmatrix},
\end{aligned}$$

where

$$B^{(s)} = \begin{pmatrix} \chi_M^{\sigma}(1,:) \\ \chi_M^{\sigma}(2,:) \\ \vdots \\ \chi_M^{\Delta}(s,:) \\ \vdots \\ \chi_M(2n,:) \end{pmatrix}$$

and $\chi_M(j,:)$, $(1 \le j \le 2n)$ is the j-th row of χ_M. Observe that

$$\chi_M^\sigma(k,:) = \chi_M(k,:) + \mu\chi_M^\Delta(k,:) \text{ for } 1 \le k < s.$$

Thus

$$\det B^{(s)} = \sum_{k=1}^{s} \sum_{\substack{\Lambda_k \in \Omega_k^{(2n)} \\ \max \Lambda_k = s}} \mu^{k-1} \det \chi_M(\Lambda_k, \Delta).$$

By Lemma 4.3, we obtain

$$W^\Delta = \sum_{s=1}^{2n} \det B^{(s)} = \sum_{s=1}^{2n}\sum_{k=1}^{s} \sum_{\substack{\Lambda_k \in \Omega_k^{(2n)} \\ \max \Lambda_k = s}} \mu^{k-1} \det \chi_A(\Lambda_k) \cdot W$$

$$= \sum_{k=1}^{2n}\sum_{s=k}^{2n} \sum_{\substack{\Lambda_k \in \Omega_k^{(2n)} \\ \max \Lambda_k = s}} \mu^{k-1} \det \chi_A(\Lambda_k) \cdot W$$

$$= \sum_{k=1}^{2n} \sum_{\Lambda_k \in \Omega_k^{(2n)}} \mu^{k-1} \det \chi_A(\Lambda_k) \cdot W$$

$$= uW.$$

In fact, $\mathcal{V}_1(\chi_A), \mathcal{V}_2(\chi_A), \cdots, \mathcal{V}_{2n}(\chi_A)$ are coefficients of the characteristic polynomial of χ_A. Then, by Theorem 2.1 and the definition of u, we conclude that u is real-valued. From Theorem 3.1 follows the Liouville's formula (4.7). $\qquad\square$

Remark 4.2. If $\mathbb{T} = \mathbb{R}$, then the graininess function μ vanishes identically. Thus

$$u(t) = \mathcal{V}_1(\chi_A(t)) = tr\chi_A(t) = 2\Re\left(trA(t)\right).$$

Therefore the Liouville's formula becomes

$$W(t) = \exp\left(2\int_{t_0}^{t} \Re(trA(\tau))d\tau\right) W(t_0).$$

If $\mathbb{T} = \mathbb{Z}$, then the graininess function μ is identically equal to 1. Therefore $u(t)$ equals to the sum of all coefficients of the characteristic polynomial of $\chi_A(t)$. In this case, the Liouville's formula becomes

$$W(t) = \exp\left(\sum_{\tau=t_0}^{t-1} \xi_1(u(\tau))\right) W(t_0).$$

Corollary 4.1. *Let $W(t)$ be the Wronskian of solution matrix $M(t)$ of (4.3). Then $W(t) = 0$ at same $t_0 \in \mathbb{T}$ if and only if $W(t) = 0$ for all $t \in \mathbb{T}$.*

As an immediate consequence of Theorem 4.4, 4.5 and Corollary 4.1, we have

Theorem 4.7. *Let $A \in \mathcal{R}(\mathbb{T}, \mathbb{H}^{n \times n})$ and $W(t)$ be the Wronskian of solution matrix $M(t)$ of (4.3). Then $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_n(t)$ are right linearly dependent on $\mathbb{T}$ if and only if $W(t) = 0$ at some $t_0 \in T$. And $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_n(t)$ are right linearly independent on $\mathbb{T}$ if and only if $W(t) \neq 0$ at some $t_0 \in T$.*

Let $A \in \mathcal{R}(\mathbb{T}, \mathbb{H}^{n \times n})$ and $\boldsymbol{e}_k$ be the k-th column of I_n. Then there exists a unique solution $\boldsymbol{x}_k(t)$ of (4.3) satisfying $\boldsymbol{x}_k(t_0) = \boldsymbol{e}_k$. Thus $M(t) = (\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_n(t))$ is a solution matrix of (4.3). Let $W(t)$ be the Wronskian of $M(t)$. It is easy to see that $W(t_0) = 1 \neq 0$. Then we immediately have

Theorem 4.8. *If $A \in \mathcal{R}(\mathbb{T}, \mathbb{H}^{n \times n})$, then (4.3) has exactly n right linearly independent solutions $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_n(t)$. The corresponding solution matrix $M(t)$ is called fundamental solution matrix of (4.3).*

Theorem 4.8 illustrates the existence of fundamental solution matrix. In fact, fundamental solution matrix is not unique. Suppose that $M(t)$ is a fundamental solution matrix of (4.3). Multiplying $M(t)$ with an arbitrary non-singular quaternion matrix on the right-side, it remains to be a fundamental solution matrix. Using Theorem 4.7, we can easily determine whether $M(t)$ is a fundamental solution matrix or not.

Corollary 4.2. *A solution matrix $M(t)$ of (4.3) is a fundamental solution matrix if and only if its corresponding Wronskian $W(t) \neq 0$ for some $t_0 \in \mathbb{T}$. Moreover, if $M(t)$ is a fundamental solution matrix, then $M(t)$ is invertible for all $t \in \mathbb{T}$.*

Next result describes the structure of the general solution of (4.3). It is a right $\mathbb{H}$-module. That implies that if we know n right linearly independent solutions of (4.3), then we actually know all possible solutions of (4.3), since any other solution is just a right linear combination of known solutions.

Theorem 4.9. *Let $M(t) = (\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \cdots, \boldsymbol{x}_n(t))$ be a fundamental solution matrix of (4.3). Then any solution of (4.3) can be written as*

$$\boldsymbol{x}(t) = M(t)\boldsymbol{q}, \tag{4.8}$$

where $\boldsymbol{q} = (q_1, q_2, \cdots, q_n)^\top \in \mathbb{H}^n$ is undetermined quaternion vector. The totality of the solutions form a right $\mathbb{H}$-module.

Proof. For any $t_0 \in \mathbb{T}$, by Corollary 4.2, $M(t_0)$ is invertible. Thus $M(t_0)\boldsymbol{\eta} = \boldsymbol{x}(t_0)$ has a unique solution $\boldsymbol{\eta} = \boldsymbol{q}$. Note that $M(t)\boldsymbol{q}$ is also a solution of 4.3 with initial condition $\boldsymbol{\phi}(t_0) = \boldsymbol{x}(t_0)$. By the uniqueness theorem, the equality (4.8) holds. $\qquad \square$

Let $A \in \mathcal{R}(\mathbb{T}, \mathbb{H}^{n \times n})$ and $M(t)$ be a fundamental solution matrix of (4.3). Define the state-transition matrix

$$\Psi_A(t, s) := M(t)M^{-1}(s).$$

It is not difficult to verify that $\Psi_A(t, s)$ is well-defined. Suppose that $M_1(t)$ be a fundamental solution matrix which is different from $M(t)$. It is easy to see that $M_1(t)M_1^{-1}(s) = M(t)M^{-1}(s)$ by uniqueness theorem. By the similar arguments to Proposition 3.1, we have

Proposition 4.1. *Let $A \in \mathcal{R}(\mathbb{T}, \mathbb{H}^{n \times n})$ and $t, r, s \in \mathbb{T}$, then the following assertions hold.*

  (i) *$\Psi_A(t, s)$ is invertible.*

  (ii) *$\Psi_0(t, s) \equiv I_n$ and $\Psi_A(t, t) = I_n$.*

  (iii) *$\Psi_A(t, s)\Psi_A(s, r) = \Psi_A(t, r)$. In particular, $\Psi_A^{-1}(t, s) = \Psi_A(s, t)$.*

  (iv) *Any solution $\boldsymbol{x}(t)$ of (4.3) with $\boldsymbol{x}(t_0) = \boldsymbol{\eta}_0$ can be expressed by $\boldsymbol{x}(t) = \Psi_A(t, t_0)\boldsymbol{\eta}_0$.*

After studying the homogeneous equations (4.3), we also consider the nonhomogeneous equations (4.2). It is easy to verify the following two results.

Lemma 4.5. *If $\phi^{NH}(t)$ and $\phi^H(t)$ are solutions of (4.2) and (4.3) respectively, then $\phi^{NH}(t) + \phi^H(t)$ is a solution of (4.2).*

Lemma 4.6. *If $\phi_1^{NH}(t)$ and $\phi_2^{NH}(t)$ are two solutions of (4.2), then $\phi_1^{NH}(t) - \phi_2^{NH}(t)$ is a solution of (4.3).*

Next result describes the structure of the general solution of (4.2).

Theorem 4.10. *Suppose that $M(t)$ be a fundamental solution matrix of (4.3) and $\phi_0^{NH}(t)$ be a solution of (4.2). Then any solution of (4.2) can be expressed by*

$$\phi^{NH}(t) = M(t)\boldsymbol{q} + \phi_0^{NH}(t), \tag{4.9}$$

where $\boldsymbol{q} = (q_1, q_2, \cdots, q_n)^\top$ is a constant quaternion vector.

Proof. For any solution $\phi^{NH}(t)$ of (4.2), we know that $\phi^{NH}(t) - \phi_0^{NH}(t)$ is a solution of (4.3) by Lemma 4.6. From Theorem 4.9, it follows that there exists $\boldsymbol{q} \in \mathbb{H}^n$ such that $\phi^{NH}(t) - \phi_0^{NH}(t) = M(t)\boldsymbol{q}$. Then we have equality (4.9). $\qquad\square$

Theorem 4.10 indicates that if we know a fundamental solution matrix of (4.3) and a particular solution of (4.2), then we actually know all possible solutions of (4.2). The following result further points out that if a fundamental solution matrix of (4.3) $M(t)$ is known, then the general solution of (4.2) can be specifically described by method of variation of constants.

Theorem 4.11. *Let $A \in \mathcal{R}(\mathbb{T}, \mathbb{H}^{n \times n})$ and $M(t)$ be a fundamental solution matrix of (4.3). Suppose that $\boldsymbol{f}$ is rd-continuous. Then the general solution of (4.2) is given by*

$$\phi^{NH}(t) = M(t)\boldsymbol{q} + M(t) \int_{t_0}^t M^{-1}(\sigma(\tau))\boldsymbol{f}(\tau)\Delta\tau, \tag{4.10}$$

where $\boldsymbol{q} = (q_1, q_2, \cdots, q_n)^\top$ is a constant quaternion vector.

Proof. Let us look for a solution of (4.2) in a form similar to (4.8). Suppose that

$$\phi^{NH}(t) = M(t)\boldsymbol{q}(t). \tag{4.11}$$

By differentiating both sides of (4.11), we obtain

$$A(t)M(t)\boldsymbol{q}(t) + \boldsymbol{f}(t) = M^\Delta(t)\boldsymbol{q}(t) + M(\sigma(t))\boldsymbol{q}^\Delta(t).$$

Observe that $M^\Delta(t) = A(t)M(t)$. Thus

$$\boldsymbol{q}^\Delta(t) = M^{-1}(\sigma(t))\boldsymbol{f}(t).$$

Therefore

$$\boldsymbol{q}(t) = \int_{t_0}^t M^{-1}(\sigma(\tau))\boldsymbol{f}(\tau)\Delta\tau + \boldsymbol{q},$$

where $\boldsymbol{q}$ is a constant quaternion vector. Then we obtain an expression of $\phi^{NH}(t)$ as follows:

$$\phi^{NH}(t) = M(t)\boldsymbol{q} + M(t) \int_{t_0}^t M^{-1}(\sigma(\tau))\boldsymbol{f}(\tau)\Delta\tau.$$

Now it remains to show that (4.10) is exactly a solution of (4.2). We use product rule to differentiate $\phi^{NH}(t)$:

$$\left(\phi^{NH}\right)^{\Delta}(t) = M^{\Delta}(t)\boldsymbol{q} + M^{\Delta}(t)\int_{t_0}^t M^{-1}(\sigma(\tau))\boldsymbol{f}(\tau)\Delta\tau + M(\sigma(t))M^{-1}(\sigma(t))\boldsymbol{f}(t)$$

$$= A(t)\left(M(t)\boldsymbol{q} + M(t)\int_{t_0}^t M^{-1}(\sigma(\tau))\boldsymbol{f}(\tau)\Delta\tau\right) + \boldsymbol{f}(t)$$

$$= A(t)\phi^{NH}(t) + \boldsymbol{f}(t).$$

The proof is complete. □

Corollary 4.3. *The solution of initial value problem (4.2) and (4.4) is given by*

$$\phi^{NH}(t) = \Psi_A(t, t_0)\eta + \int_{t_0}^t \Psi_A(t, \sigma(\tau))\boldsymbol{f}(\tau)\Delta\tau. \tag{4.12}$$

In particular, let $\alpha \in \mathbb{H}$ be a quaternion constant, then the solution of $y^{\Delta}(t) = \alpha y(t) + f(t)$, $y(t_0) = 0$ is

$$y(t) = \int_{t_0}^t E_\alpha(t, \sigma(\tau))f(\tau)\Delta\tau$$

$$= E_\alpha(t, 0)\int_{t_0}^t E_\alpha(0, \sigma(\tau))f(\tau)\Delta\tau$$

$$= E_\alpha(t, 0)\int_{t_0}^t E_\alpha^{-1}(\sigma(\tau), 0)f(\tau)\Delta\tau.$$

5. Linear QDETS with constant coefficients

Let $A \in \mathcal{R}(\mathbb{T}, \mathbb{H}^{n \times n})$ be a constant quaternion matrix and suppose that $\boldsymbol{f} \in \mathbb{H}^{n \times 1} \otimes \mathbb{T}$ is rd-continuous. In this section, we consider the following quaternion-valued linear equations

$$\phi^{\Delta}(t) = A\phi(t) + \boldsymbol{f}(t) \tag{5.1}$$

and its corresponding homogeneous equations

$$\phi^{\Delta}(t) = A\phi(t). \tag{5.2}$$

Theorem 5.1. *If λ is a right eigenvalue of A and $\boldsymbol{\eta}$ is an eigenvector corresponding to λ. Then $\phi(t) = \boldsymbol{\eta}E_\lambda(t, 0)$ is a solution of (5.2).*

Proof. Suppose that A is regressive, then $\lambda \in \mathcal{R}(\mathbb{T}, \mathbb{H})$ by Theorem 4.1. Thus $E_\lambda(t, 0)$ is well-defined and therefore

$$\phi^{\Delta}(t) = \boldsymbol{\eta}\lambda E_\lambda(t, 0) = A\boldsymbol{\eta}E_\lambda(t, 0) = A\phi(t)$$

for $t \in \mathbb{T}^\kappa$. The proof is complete. □

Theorem 5.2. *If A has n right linearly independent eigenvectors $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \cdots, \boldsymbol{\eta}_n$ corresponding to right eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ (no matter whether they are similar). Then*

$$M(t) = (\boldsymbol{\eta}_1 E_{\lambda_1}(t, 0), \boldsymbol{\eta}_2 E_{\lambda_2}(t, 0), \cdots, \boldsymbol{\eta}_n E_{\lambda_n}(t, 0))$$

is a fundamental solution matrix of (5.2). In particular, if A has n distinct standard eigenvalues, then $\lambda_1, \lambda_2, \cdots, \lambda_n$ can be chosen to be the standard eigenvalues of A.

Proof.   By Theorem 5.1, we see that $\boldsymbol{\eta}_1 E_{\lambda_1}(t,0), \boldsymbol{\eta}_2 E_{\lambda_2}(t,0), \cdots, \boldsymbol{\eta}_n E_{\lambda_n}(t,0)$ are solutions of (5.2). It remains to show that they are right linearly independent. Let $W(t)$ be the Wronskian of $M(t)$. Since $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \cdots, \boldsymbol{\eta}_n$ are right linearly independent, then

$$W(0) = \widehat{\det} M(0) = \widehat{\det}(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \cdots, \boldsymbol{\eta}_n) \neq 0.$$

Thus $M(t)$ is a fundamental solution matrix of (5.2) by Theorem 4.2. If A has n distinct standard eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, then they are pairwise non-similar. By Theorem 2.1, we know that their corresponding eigenvectors are right linearly independent. This completes the proof. $\qquad\square$

Example 5.1. Find a fundamental solution matrix of

$$\boldsymbol{\phi}^{\Delta}(t) = A\boldsymbol{\phi}(t) = \begin{pmatrix} \boldsymbol{i} & 1 \\ 0 & 1+\boldsymbol{i} \end{pmatrix} \boldsymbol{\phi}(t) \tag{5.3}$$

for the special time scales $\mathbb{T} = \mathbb{Z}$.

By Theorem 2.1, we know that A has two distinct standard eigenvalues $\lambda_1 = \boldsymbol{i}, \lambda_2 = 1 + \boldsymbol{i}$. Their corresponding eigenvectors are $\boldsymbol{\eta}_1 = (1,0)^{\top}, \boldsymbol{\eta}_2 = (1,1)^{\top}$ respectively. By Example 3.3, we have $E_{\lambda_1}(t,0) = (1+\boldsymbol{i})^t$ and $E_{\lambda_2}(t,0) = (2+\boldsymbol{i})^t$. Therefore

$$\begin{pmatrix} (1+\boldsymbol{i})^t & (2+\boldsymbol{i})^t \\ 0 & (2+\boldsymbol{i})^t \end{pmatrix}$$

is a fundamental solution matrix of (5.3).

Example 5.2. Find a fundamental solution matrix of

$$\boldsymbol{\phi}^{\Delta}(t) = A\boldsymbol{\phi}(t) = \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} \\ 0 & \boldsymbol{i} \end{pmatrix} \boldsymbol{\phi}(t) \tag{5.4}$$

for the special time scales $\mathbb{T} = \mathbb{Z}$.

By Theorem 2.1, we know that A has a repeated standard eigenvalue $\lambda = \boldsymbol{i}$. Although A only has one standard eigenvalue, there are two right linearly independent eigenvectors $\boldsymbol{\eta}_1 = (1,0)^{\top}, \boldsymbol{\eta}_2 = (\frac{\boldsymbol{k}}{2}, 1)^{\top}$ corresponding to $\lambda = \boldsymbol{i}$. Therefore

$$\begin{pmatrix} (1+\boldsymbol{i})^t & \frac{\boldsymbol{k}(1+\boldsymbol{i})^t}{2} \\ 0 & (1+\boldsymbol{i})^t \end{pmatrix}$$

is a fundamental solution matrix of (5.4).

From Example 7.4 in [42], we know that not every $n \times n$ constant quaternion matrix has n right linearly independent eigenvectors. In this case, Theorem 5.2 is of no use any more. The Putzer's algorithm for the classical case in [10] is generalized to quaternion dynamic equations on time scales. Since Putzer's algorithm avoids the computing of eigenvectors. So it is particularly useful for quaternion matrices that do not have n right linearly independent eigenvectors.

Theorem 5.3. *Let $A \in \mathcal{R}(\mathbb{T}, \mathbb{H}^{n \times n})$ be a constant quaternion matrix. If there exists quaternion constants $\alpha_1, \alpha_2, \cdots, \alpha_m \in \mathcal{R}(\mathbb{T}, \mathbb{H})$ such that $P_m = 0$, where $P_0, P_1, P_2, \cdots, P_m$ are recursively defined by $P_0 = I_n$ and*

$$
\begin{aligned}
P_k &= AP_{k-1} - P_{k-1}\alpha_k \\
&= A^k + \sum_{j=1}^{k}(-1)^j A^{k-j} \sum_{1 \leq i_1 < i_2 \cdots < i_j \leq k} \alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_j}
\end{aligned}
$$

for $1 \leq k \leq m$. Then

$$
\Psi_A(t, t_0) = \sum_{k=0}^{m-1} P_k \varphi_{k+1}(t), \tag{5.5}
$$

where $\boldsymbol{\varphi}(t) := (\varphi_1(t), \varphi_1(t), \cdots, \varphi_m(t))^\top$ is the solution of the following initial value problem

$$
\boldsymbol{\varphi}^\Delta(t) = \begin{pmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ 1 & \alpha_2 & 0 & \ddots & \vdots \\ 0 & 1 & \alpha_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \alpha_m \end{pmatrix} \boldsymbol{\varphi}(t), \quad \boldsymbol{\varphi}(t_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{5.6}
$$

Proof. From statement 6 of Theorem 2.1 and Lemma 4.2 we see that the coefficient matrix of (5.6) is regressive and therefore (5.6) has a unique solution $\boldsymbol{\varphi}(t) := (\varphi_1(t), \varphi_1(t), \cdots, \varphi_m(t))^\top$. Let $\boldsymbol{\phi}(t)$ be the right-hand side of (5.5). Then

$$
\begin{aligned}
\boldsymbol{\phi}^\Delta(t) - A\boldsymbol{\phi}(t) &= \sum_{k=0}^{m-1} P_k \varphi_{k+1}^\Delta(t) - A \sum_{k=0}^{m-1} P_k \varphi_{k+1}(t) \\
&= P_0 \alpha_1 \varphi_1(t) + \sum_{k=1}^{m-1} P_k \left(\varphi_k(t) + \alpha_{k+1}\varphi_{k+1}(t)\right) - A \sum_{k=0}^{m-1} P_k \varphi_{k+1}(t) \\
&= \sum_{k=1}^{m-1} P_k \varphi_k(t) - \sum_{k=0}^{m-1} \left(AP_k - P_k \alpha_{k+1}\right) \varphi_{k+1}(t) \\
&= \sum_{k=1}^{m-1} P_k \varphi_k(t) - \sum_{k=0}^{m-1} P_{k+1} \varphi_{k+1}(t) \\
&= -P_m \varphi_m(t) = 0.
\end{aligned}
$$

The last equality is a consequence of $P_m = 0$. Since $\boldsymbol{\varphi}(t_0) = \varphi_1(t_0)P_0 = I$. Thus $\boldsymbol{\varphi}(t)$ is a fundamental solution matrix of (4.3). Therefore

$$
\Psi_A(t, t_0) = \boldsymbol{\varphi}(t)\boldsymbol{\varphi}(t_0) = \sum_{k=0}^{m-1} P_k \varphi_{k+1}(t),
$$

which completes the proof. $\square$

Example 5.3. Find the state-transition matrix of the quaternion dynamic equations

$$\phi^\Delta(t) = A\phi(t) = \begin{pmatrix} i & 1 & 0 \\ 0 & j & 0 \\ 0 & 1 & k \end{pmatrix} \phi(t)$$

for the special time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

Since χ_A is not diagonalizable, then A does not have n right linearly independent eigenvectors belonging to right eigenvalues. So Theorem 5.2 does not apply to this problem.

Let $\alpha_1 = i, \alpha_2 = -i, \alpha_3 = j$ and $P_0 = I_3$. Then we can easily get that

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & j-i & 0 \\ 0 & 1 & k-i \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & j+i & 0 \\ 0 & 0 & 0 \\ 0 & j+k & 0 \end{pmatrix}$$

and $P_3 = 0$. Now we need to solve IVP

$$\varphi^\Delta(t) = \begin{pmatrix} i & 0 & 0 \\ 1 & -i & 0 \\ 0 & 1 & j \end{pmatrix} \varphi(t), \quad \varphi(t_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

If $\mathbb{T} = \mathbb{R}$ and $t_0 = 0$, then $\varphi_1(t) = e^{it}$ and

$$\varphi_2^\Delta(t) = -i\varphi_2(t) + e^{it}, \quad \varphi_2(0) = 0.$$

Then by Corollary 4.3, we have $\varphi_2(t) = \sin t$ and therefore

$$\varphi_3^\Delta(t) = j\varphi_3(t) + \sin t, \quad \varphi_3(0) = 0.$$

Thus

$$\varphi_3(t) = \frac{1}{4}e^{-jt}\left(-2je^{2jt}t + e^{2jt} - 1\right).$$

Then we obtain

$$\Psi_A(t,0) = \sum_{k=0}^{2} P_k \varphi_{k+1}(t) = \begin{pmatrix} e^{it} & \frac{t}{2}\left(e^{it} - ke^{-it}\right) + \frac{1+k}{2}\sin t & 0 \\ 0 & e^{jt} & 0 \\ 0 & \frac{t}{2}\left(e^{jt} + ie^{jt}\right) + \frac{1-i}{2}\sin t & e^{kt} \end{pmatrix}. \qquad (5.7)$$

By direct computation, we see that both $\Psi_A^\Delta(t,0)$ and $A\Psi_A(t,0)$ are equal to

$$\begin{pmatrix} ie^{it} & \frac{t}{2}\left(ie^{it} + je^{-it}\right) + \frac{1}{2}\left(e^{it} + e^{jt}\right) & 0 \\ 0 & je^{jt} & 0 \\ 0 & \frac{t}{2}\left(je^{jt} + ke^{jt}\right) + \frac{1}{2}\left(e^{jt} + e^{kt}\right) & ke^{kt} \end{pmatrix}.$$

That means that (5.7) is exactly the state-transition matrix.

If $\mathbb{T} = \mathbb{Z}$ and $t_0 = 0$, then $\varphi_1(t) = (1 + \boldsymbol{i})^t$ and

$$\varphi_2^\Delta(t) = -\boldsymbol{i}\varphi_2(t) + (1 + \boldsymbol{i})^t, \quad \varphi_2(0) = 0.$$

Then we have $\varphi_2(t) = \frac{\boldsymbol{i}}{2}\left((1 - \boldsymbol{i})^t - (1 + \boldsymbol{i})^t\right)$ and therefore

$$\varphi_3^\Delta(t) = \boldsymbol{j}\varphi_3(t) + \frac{1}{2}\boldsymbol{i}\left((1 - \boldsymbol{i})^t - (1 + \boldsymbol{i})^t\right), \quad \varphi_3(0) = 0.$$

Thus

$$\varphi_3(t) = \frac{1}{4}\left(1 - 2^{-t}(1 - \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})^t\right)(1 + \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$$
$$+ \frac{1}{4}\left(1 - 2^{-t}(1 + \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})^t\right)(-1 + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}).$$

Then we obtain

$$\Psi_A(t, 0) = \begin{pmatrix} (1 + \boldsymbol{i})^t \; \frac{\boldsymbol{i}}{2}\gamma_1 + 2^{-t}(1 + \boldsymbol{i})\gamma_2 - \frac{1 - \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}}{2} & 0 \\ 0 & (1 + \boldsymbol{i})^t + \frac{1 - \boldsymbol{k}}{2}\gamma_1 & 0 \\ 0 & \frac{\boldsymbol{i}}{2}\gamma_1 + 2^{-t}(\boldsymbol{i} + \boldsymbol{k})\gamma_2 - \frac{1 - \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}}{2} & (1 + \boldsymbol{i})^t + \frac{1 + \boldsymbol{j}}{2}\gamma_1 \end{pmatrix}, \quad (5.8)$$

where $\gamma_1 = (1 - \boldsymbol{i})^t - (1 + \boldsymbol{i})^t$, $\gamma_2 = (1 + \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})^{t-1} - (1 - \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})^{t-1}$. By direct computation, we see that both $\Psi_A^\Delta(t, 0)$ and $A\Psi_A(t, 0)$ are equal to

$$\begin{pmatrix} \boldsymbol{i}(1 + \boldsymbol{i})^t & (1 + \boldsymbol{i})^t - \frac{\boldsymbol{k}}{2}\gamma_1 + 2^{-t}(\boldsymbol{i} - 1)\gamma_2 - \frac{1 + \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}}{2} & 0 \\ 0 & \boldsymbol{j}(1 + \boldsymbol{i})^t + \frac{\boldsymbol{j} - \boldsymbol{i}}{2}\gamma_1 & 0 \\ 0 & (1 + \boldsymbol{i})^t + \frac{1 + \boldsymbol{j} - \boldsymbol{k}}{2}\gamma_1 + 2^{-t}(\boldsymbol{j} - 1)\gamma_2 + \frac{1 - \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}}{2} & \boldsymbol{k}(1 + \boldsymbol{i})^t + \frac{\boldsymbol{k} - \boldsymbol{i}}{2}\gamma_1 \end{pmatrix}.$$

That means that (5.8) is exactly the state-transition matrix.

Example 5.4. Find the state-transition matrix of the quaternion dynamic equations

$$\phi^\Delta(t) = A\phi(t) = \begin{pmatrix} \boldsymbol{i} \; \boldsymbol{j} \; \boldsymbol{j} \\ \boldsymbol{k} \; 1 \; \boldsymbol{k} \\ 0 \; 0 \; 1 \end{pmatrix}\phi(t)$$

for the special time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

Let $\alpha_1 = 1, \alpha_2 = 0, \alpha_3 = 1 + \boldsymbol{i}, \alpha_4 = 1 - \boldsymbol{i}$ and $P_0 = I_3$. Then we can easily get that

$$P_1 = \begin{pmatrix} \boldsymbol{i} - 1 \; \boldsymbol{j} \; \boldsymbol{j} \\ \boldsymbol{k} \; 0 \; \boldsymbol{k} \\ 0 \; 0 \; 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} -1 \; \boldsymbol{k} \; \boldsymbol{i} + \boldsymbol{k} \\ \boldsymbol{j} \; -\boldsymbol{i} \; \boldsymbol{k} - \boldsymbol{i} \\ 0 \; 0 \; 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 \; -2\boldsymbol{j} \; -2\boldsymbol{j} \\ 0 \; -2 \; -2 \\ 0 \; 0 \; 0 \end{pmatrix}$$

and $P_4 = 0$. Now we need to solve IVP

$$\varphi^\Delta(t) = \begin{pmatrix} 1 \; 0 \; 0 \; 0 \\ 1 \; 0 \; 0 \; 0 \\ 0 \; 1 \; 1 + \boldsymbol{i} \; 0 \\ 0 \; 0 \; 1 \; 1 - \boldsymbol{i} \end{pmatrix}\varphi(t), \quad \varphi(t_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

If $\mathbb{T} = \mathbb{R}$ and $t_0 = 0$, then $\varphi_1(t) = e^t$, $\varphi_2(t) = e^t - 1$ and

$$\varphi_3^{\Delta}(t) = (1 + \boldsymbol{i})\varphi_3(t) + e^t - 1, \quad \varphi_3(0) = 0.$$

Then we have $\varphi_3(t) = \left(-\frac{1}{2} - \frac{\boldsymbol{i}}{2}\right)\left(-(1 + \boldsymbol{i})e^t + e^{(1+\boldsymbol{i})t} + \boldsymbol{i}\right)$ and therefore

$$\varphi_4^{\Delta}(t) = (1 - \boldsymbol{i})\varphi_4(t) + \left(-\frac{1}{2} - \frac{\boldsymbol{i}}{2}\right)\left(-(1 + \boldsymbol{i})e^t + e^{(1+\boldsymbol{i})t} + \boldsymbol{i}\right), \quad \varphi_4(0) = 0.$$

Thus

$$\varphi_4(t) = -\frac{1 + \boldsymbol{i}}{4}e^{(1-\boldsymbol{i})t} + e^t - \frac{1 - \boldsymbol{i}}{4}e^{(1+\boldsymbol{i})t} - \frac{1}{2}.$$

Using the Putzer's algorithm (5.5), we can obtain

$$\Psi_A(t, 0) = \begin{pmatrix} \frac{1-\boldsymbol{i}}{2} + \frac{1+\boldsymbol{i}}{2}\gamma_1 & \frac{\boldsymbol{k}-\boldsymbol{j}}{2} + \gamma_2 & \boldsymbol{j}\gamma_3 + \gamma_4 - e^t \\ \frac{\boldsymbol{j}-\boldsymbol{k}}{2} + \frac{\boldsymbol{k}-\boldsymbol{j}}{2}\gamma_1 & \frac{1-\boldsymbol{i}}{2} - \boldsymbol{j}\gamma_2 & \boldsymbol{i}\gamma_3 - \boldsymbol{j}\gamma_4 - (1 - \boldsymbol{j} - \boldsymbol{k})e^t \\ 0 & 0 & e^t \end{pmatrix}, \qquad (5.9)$$

where $\gamma_1 = e^{(1+\boldsymbol{i})t}$, $\gamma_2 = \frac{\boldsymbol{j}-\boldsymbol{k}}{2}e^{(1-\boldsymbol{i})t}$, $\gamma_3 = \frac{\boldsymbol{k}-1-\boldsymbol{i}-\boldsymbol{j}}{2}$, $\gamma_4 = e^{(1+\boldsymbol{i})t}\frac{1-\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}}{2}$. The result is consistent with the result of Example 6.3 in [27].

If $\mathbb{T} = \mathbb{Z}$ and $t_0 = 0$, then $\varphi_1(t) = 2^t$, $\varphi_2(t) = 2^t - 1$ and

$$\varphi_3^{\Delta}(t) = (1 + \boldsymbol{i})\varphi_3(t) + 2^t - 1, \quad \varphi_3(0) = 0.$$

Then we have

$$\begin{aligned}
\varphi_3(t) &= (2 + \boldsymbol{i})^t \int_0^t (2 + \boldsymbol{i})^{-(\tau+1)}(2^\tau - 1)\Delta\tau \\
&= (2 + \boldsymbol{i})^t \sum_{\tau=0}^{t-1} (2 + \boldsymbol{i})^{-(\tau+1)}(2^\tau - 1) \\
&= -\frac{1 + \boldsymbol{i}}{2}\left(\boldsymbol{i} - (1 + \boldsymbol{i})2^t + (2 + \boldsymbol{i})^t\right),
\end{aligned}$$

and therefore

$$\varphi_4^{\Delta}(t) = (1 - \boldsymbol{i})\varphi_4(t) - \frac{1 + \boldsymbol{i}}{2}\left(\boldsymbol{i} - (1 + \boldsymbol{i})2^t + (2 + \boldsymbol{i})^t\right), \quad \varphi_4(0) = 0.$$

Thus

$$\begin{aligned}
\varphi_4(t) &= (2 - \boldsymbol{i})^t \int_0^t (2 - \boldsymbol{i})^{-\tau-1}\varphi_3(\tau)\Delta\tau \\
&= \frac{1}{4}\left(2^{2+t} - (1 + \boldsymbol{i})(2 - \boldsymbol{i})^t + (\boldsymbol{i} - 1)(2 + \boldsymbol{i})^t - 2\right).
\end{aligned}$$

Then we obtain

$$\Psi_A(t, 0) = \begin{pmatrix} \frac{1-\boldsymbol{i}}{2} + \frac{1+\boldsymbol{i}}{2}\gamma_1 & \frac{\boldsymbol{k}-\boldsymbol{j}}{2} + \gamma_2 & \boldsymbol{j}\gamma_3 + \gamma_4 - 2^t \\ \frac{\boldsymbol{j}-\boldsymbol{k}}{2} + \frac{\boldsymbol{k}-\boldsymbol{j}}{2}\gamma_1 & \frac{1-\boldsymbol{i}}{2} - \boldsymbol{j}\gamma_2 & \boldsymbol{i}\gamma_3 - \boldsymbol{j}\gamma_4 - (1 - \boldsymbol{j} - \boldsymbol{k})2^t \\ 0 & 0 & 2^t \end{pmatrix}, \qquad (5.10)$$

where $\gamma_1 = (2 + i)^t$, $\gamma_2 = \frac{j-k}{2}(2 - i)^t$, $\gamma_3 = \frac{k-1-i-j}{2}$, $\gamma_4 = (2 + i)^t \frac{1-i+j-k}{2}$. By direct computation, we see that both $\Psi_A^\Delta(t, 0)$ and $A\Psi_A(t, 0)$ are equal to

$$
\begin{pmatrix}
i(2+i)^t & j(2-i)^t & (1+i)\gamma_4 - 2^t \\
k(2+i)^t & (2-i)^t & (k-j)\gamma_4 - (1-j-k)2^t \\
0 & 0 & 2^t
\end{pmatrix}.
$$

That means that (5.10) is exactly the state-transition matrix.

Theorem 5.3 is a generalization of Theorem 5.35 in [10]. In the classical case, by Cayley-Hamilton theorem,

$$
\prod_{j=1}^{n}(A - \lambda_j I) = 0,
$$

where $A \in \mathbb{C}^{n \times n}$ and $\lambda_j, (1 \leq j \leq n)$ are eigenvalues of A. So $\alpha_1, \alpha_2, \cdots, \alpha_m$ can be chosen as the eigenvalues of $A \in \mathbb{C}^{n \times n}$. In the quaternion case, however, the selection of $\alpha_k, (1 \leq k \leq m)$ is more difficult. What should be clear to us is that the less m the less calculation.

We say that $h(z) = z^m + z^{m-1}\beta_1 + \cdots + \beta_m$ is an annihilating polynomial of quaternion matrix $A \in \mathbb{H}^{n \times n}$ if

$$
h(A) = A^m + A^{m-1}\beta_1 + \cdots + I_n\beta_m = 0,
$$

where $\beta_1, \beta_2, \cdots, \beta_m \in \mathbb{H}$. To authors' best knowledge, there are (at least) two annihilating polynomials for every $A \in \mathbb{H}^{n \times n}$. The first one which was presented by Zhang [42] is $\mathrm{ch}_{\chi_A}(z)$, the characteristic polynomial of χ_A. In this case, $m = 2n$ and $\alpha_1, \alpha_2, \cdots, \alpha_m$ are exactly the standard eigenvalues of A. The other one is called minimal polynomial which was given by Rodman [33]. The coefficients of minimal polynomial in [33] are confined to be real. Thus, there may be some other annihilating polynomials (with quaternion coefficients), which possess less degree than minimal polynomial. Since m-degree minimal polynomial possesses real coefficients, then it has m complex roots and the Vieta's formula holds. Therefore, $\alpha_1, \alpha_2, \cdots, \alpha_m$ can be chosen as the complex roots of minimal polynomial. Unfortunately, there is no explicit expression for minimal polynomial of quaternion matrices until now. On the other hand, even when we know an annihilating polynomial $h(z) = z^m + z^{m-1}\beta_1 + \cdots + \beta_m$ (with quaternion coefficients) of A, we still can not find $\alpha_1, \alpha_2, \cdots, \alpha_m$. In fact, we need to find $\alpha_1, \alpha_2, \cdots, \alpha_m$ such that

$$
(-1)^k \sum_{1 \leq i_1 < i_2 \cdots < i_k \leq m} \alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_k} = \beta_k \tag{5.11}
$$

for $1 \leq k \leq m$. Thus, $\alpha_1, \alpha_2, \cdots, \alpha_m$ may not exist. Even if they exist, we can not conclude that they are roots of $h(z)$ (see Example 5.5). Even if they are roots of $h(z)$, we still can not find them. This is because that the number of zeros of quaternion polynomials is indeterminate and the computing of zeros of quaternion polynomials is complicated. For details of zeros of quaternion polynomials, please refer to [25, 32, 34, 35].

Example 5.5. Let $h(z) = z^2 + z\beta_1 + \beta_2$ and $\alpha_1 = i, \alpha_2 = -j$. Then $\alpha_1, \alpha_2, \beta_1, \beta_2$ satisfy (5.11), but

$$
h(\alpha_1) = i^2 + i(j - i) + k = 2k \neq 0.
$$

In practice, $\alpha_1, \alpha_2, \cdots, \alpha_m$ are usually chosen to be the eigenvalues of $A \in \mathbb{H}^{n \times n}$ (see Example 5.3 and 5.4). The value of m does not need to be as large as $2n$. By iterative computing, we can always get more and more succinct P_k as k increases. Although there are some theoretical challenges on selections of $\alpha_1, \alpha_2, \cdots, \alpha_m$, the Putzer's algorithm is still feasible. The theoretical challenges give us something to focus on and work toward.

The method of variation of constants for one-dimensional case has been used many times in Example 5.3 and 5.4. We now present an example to illustrate the feasibility of the method of variation of constants for higher dimensional case.

Example 5.6. Find the solution of the initial value problem

$$\phi^\Delta(t) = A\phi(t) + \boldsymbol{f}(t) = \begin{pmatrix} \boldsymbol{j} & 0 \\ 0 & \boldsymbol{k} \end{pmatrix} \phi(t) + \begin{pmatrix} \boldsymbol{i} \\ t\boldsymbol{j} \end{pmatrix}, \quad \phi(0) = \begin{pmatrix} \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix} \tag{5.12}$$

for the special time scale $\mathbb{T} = \mathbb{Z}$.

It is easy to get that

$$\Psi_A(t, 0) = \begin{pmatrix} (1+\boldsymbol{j})^t & 0 \\ 0 & (1+\boldsymbol{k})^t \end{pmatrix}.$$

By Corollary 4.3,

$$\phi(t) = \Psi_A(t,0)\phi(0) + \Psi_A(t,0) \int_0^t \Psi_A^{-1}(\sigma(\tau),0)\boldsymbol{f}(\tau)\Delta\tau$$

$$= \Psi_A(t,0)\phi(0) + \Psi_A(t,0) \int_0^t \begin{pmatrix} (1+\boldsymbol{j})^{-\tau-1} & 0 \\ 0 & (1+\boldsymbol{k})^{-\tau-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{i} \\ t\boldsymbol{j} \end{pmatrix} \Delta\tau$$

$$= \Psi_A(t,0)\phi(0) + \Psi_A(t,0) \begin{pmatrix} \sum_{\tau=0}^{t-1}(1+\boldsymbol{j})^{-\tau-1}\boldsymbol{i} \\ \sum_{\tau=0}^{t-1}\tau(1+\boldsymbol{k})^{-\tau-1}\boldsymbol{j} \end{pmatrix}$$

$$= \Psi_A(t,0)\phi(0) + \begin{pmatrix} (1+\boldsymbol{j})^t & 0 \\ 0 & (1+\boldsymbol{k})^t \end{pmatrix} \begin{pmatrix} (1-(1+\boldsymbol{j})^{-t})\boldsymbol{k} \\ (1+\boldsymbol{k})^{-t}(1-(1+\boldsymbol{k})^t+t\boldsymbol{k})\boldsymbol{j} \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{j}(1+\boldsymbol{j})^t \\ \boldsymbol{k}(1+\boldsymbol{k})^t \end{pmatrix} + \begin{pmatrix} ((1+\boldsymbol{j})^t-1)\boldsymbol{k} \\ (1-(1+\boldsymbol{k})^t+t\boldsymbol{k})\boldsymbol{j} \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{j}(1+\boldsymbol{j})^t + ((1+\boldsymbol{j})^t-1)\boldsymbol{k} \\ \boldsymbol{k}(1+\boldsymbol{k})^t + (1-(1+\boldsymbol{k})^t+t\boldsymbol{k})\boldsymbol{j} \end{pmatrix}.$$

Note that

$$\begin{aligned} \phi_1^\Delta(t) &= \phi_1(t+1) - \phi_1(t) \\ &= \boldsymbol{j}(1+\boldsymbol{j})^{t+1} + ((1+\boldsymbol{j})^{t+1}-1)\boldsymbol{k} - \boldsymbol{j}(1+\boldsymbol{j})^t + ((1+\boldsymbol{j})^t-1)\boldsymbol{k} \\ &= (1+\boldsymbol{j})^t(1-\boldsymbol{i}) \\ &= \boldsymbol{j}\phi_1(t) + \boldsymbol{i} \end{aligned}$$

and

$$\phi_2^\Delta(t) = \phi_2(t+1) - \phi_2(t)$$
$$= \boldsymbol{k}(1+\boldsymbol{k})^{t+1} + \big(1 - (1+\boldsymbol{k})^{t+1} + (t+1)\boldsymbol{k}\big)\boldsymbol{j} - \boldsymbol{k}(1+\boldsymbol{k})^t + \big(1 - (1+\boldsymbol{k})^t + t\boldsymbol{k}\big)\boldsymbol{j}$$
$$= (1+\boldsymbol{k})^t(\boldsymbol{i}-1) - \boldsymbol{i}$$
$$= \boldsymbol{k}\phi_2(t) + t\boldsymbol{j}.$$

Thus $\boldsymbol{\phi}(t)$ is exactly the solution of (5.12).

6. Conclusion

In this paper, we establish the basic theory of linear quaternion dynamic equations on time scales (QDETS). It not only generalizes the theory of quaternion differential equations (QDEs) but also extends the theory of dynamic equations on time scales (DETS). Employing the newly defined Wronskian determinant, the Liouville's formula for QDETS is derived, thereby giving the structure of general solutions of QDETS. We present the Putzer's algorithm to compute fundamental solution matrix for homogeneous QDETS. The Putzer's algorithm is applicable to all homogeneous QDETS with constant coefficient matrices. It is particularly useful for quaternion coefficient matrices which are not diagonalizable. Furthermore, the variation of constants formula of solving the nonhomogeneous QDETS is also derived. Importantly, examples are given in each sections to illustrate our results.

Acknowledgements

References

[1] S. L. Adler, *Quaternionic quantum mechanics and quantum fields*, Oxford Univ. Press, 1995.

[2] R. Agarwal, M. Bohner, D. O'Regan and A. Peterson, *Dynamic equations on time scales: a survey*, J. Comput. Appl. Math., 2002, 141(1), 1–26.

[3] R. P. Agarwal, M. Bohner and P. J. Wong, *Sturm-Liouville eigenvalue problems on time scales*, Appl. Math. Comput., 1999, 99(2), 153–166.

[4] C. D. Ahlbrandt, M. Bohner and J. Ridenhour, *Hamiltonian systems on time scales*, J. Math. Anal. Appl., 2000, 250(2), 561–578.

[5] S. L. Alder, *Quaternionic quantum field theory*, Commun. Math. Phys., 1986, 104(4), 611–656.

[6] H. Aslaksen, *Quaternionic determinants*, Math. Intell., 1996, 18(3), 57–65.

[7] F. M. Atici, D. C. Biles and A. Lebedinsky, *An application of time scales to economics*, Math. Comput. Model., 2006, 43(7), 718–726.

[8] A. Baker, *Right eigenvalues for quaternionic matrices: a topological approach*, Linear Alg. Appl., 1999, 286(1), 303–309.

[9] M. Bohner and G. S. Guseinov, *The laplace transform on isolated time scales*, Comput. Math. Appl., 2010, 60(6), 1536–1547.

[10] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction With Applications*, Springer Science & Business Media, 2001.

[11] M. Bohner and A. C. Peterson, *Advances in dynamic equations on time scales*, Springer Science & Business Media, 2002.

[12] J. Campos and J. Mawhin, *Periodic solutions of quaternionic-valued ordinary differential equations*, Ann. Mat. Pura Appl., 2006, 185, S109–S127.

[13] L. X. Chen, *Definition of determinant and cramer solutions over the quaternion field*, Acta Mathematica Sinica, 1991, 7(2), 171–180.

[14] J. C. Chou, *Quaternion kinematic and dynamic differential equations*, IEEE Trans. Robot. Autom., 1992, 8(1), 53–64.

[15] V. Cormani, *Liouville's formula on time scales*, Dynamic Systems and Applications, 2003, 12(1/2), 79–86.

[16] F. Farid, Q.-W. Wang and F. Zhang, *On the eigenvalues of quaternion matrices*, Lin. Multilin. Alg., 2011, 59(4), 451–473.

[17] A. Gasull, J. Llibre and X. Zhang, *One-dimensional quaternion homogeneous polynomial differential equations*, J. Math. Phys., 2009, 50(8), 082705.

[18] S. Georgiev and J. Morais, *An introduction to the Hilger quaternion numbers*, in *AIP Conference Proceedings*, 1558, 2013, 550–553.

[19] J. Gibbon, *A quaternionic structure in the three-dimensional euler and ideal magneto-hydrodynamics equations*, Physica D: Nonlinear Phenomena, 2002, 166(1), 17–28.

[20] J. D. Gibbon, D. D. Holm, R. M. Kerr and I. Roulstone, *Quaternions and particle dynamics in the euler fluid equations*, Nonlinearity, 2006, 19(8), 1969.

[21] S. Gupta, *Linear quaternion equations with application to spacecraft attitude propagation*, in *Aerospace Conference, 1998 IEEE*, 1, IEEE, 1998, 69–76.

[22] S. Hilger, *Ein Maßkettenkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, Ph.D. thesis, Universität Würzburg, 1988.

[23] S. Hilger, *Analysis on measure chains – a unified approach to continuous and discrete calculus*, Results Math., 1990, 18(1–2), 18–56.

[24] R. A. Horn and C. R. Johnson, *Matrix analysis*, Cambridge university press, 1985.

[25] D. Janovská and G. Opfer, *A note on the computation of all zeros of simple quaternionic polynomials*, SIAM J. Numer. Anal., 2010, 48(1), 244–256.

[26] I. Klapper and H. Qian, *Remarks on discrete and continuous large-scale models of DNA dynamics*, Biophysical journal, 1998, 74(5), 2504–2514.

[27] K. I. Kou, W. K. Liu and Y. H. Xia, *Linear quaternion differential equations: Basic theory and fundamental results II*, arXiv preprint arXiv:1602.01660, 2016.

[28] K. I. Kou and Y. H. Xia, *Linear quaternion differential equations: Basic theory and fundamental results I*, arXiv preprint arXiv:1510.02224, 2015.

[29] R. J. Marks, I. A. Gravagne and J. M. Davis, *A generalized fourier transform and convolution on time scales*, J. Math. Anal. Appl., 2008, 340(2), 901–919.

[30] D. Metaxas and G. Tsechpenakis, *Dynamic data driven coupling of continuous and discrete methods for 3D tracking*, in *Computational Science–ICCS 2005*, Springer, 2005, 712–720.

[31] A. Peterson and B. Thompson, *Henstock–Kurzweil delta and nabla integrals*, J. Math. Anal. Appl., 2006, 323(1), 162–178.

[32] A. Pogorui* and M. Shapiro, *On the structure of the set of zeros of quaternionic polynomials*, Complex Variables, Theory and Application: An International Journal, 2004, 49(6), 379–389.

[33] L. Rodman, *Topics in quaternion linear algebra*, Princeton University Press, 2014.

[34] R. Serôdio, E. Pereira and J. Vitória, *Computing the zeros of quaternion polynomials*, Comput. Math. Appl., 2001, 42(8), 1229–1237.

[35] R. Serôdio and L.-S. Siu, *Zeros of quaternion polynomials*, Appl. Math. Lett., 2001, 14(2), 237–239.

[36] J. Sola, *Quaternion kinematics for the error-state KF*, Laboratoire d'Analyse et d'Architecture des Systemes-Centre national de la recherche scientifique (LAAS-CNRS), Toulouse, France, Tech. Rep, 2012.

[37] A. Sudbery, *Quaternionic analysis*, in *Mathematical Proceedings of the Cambridge Philosophical Society*, 85, Cambridge Univ Press, 1979, 199–225.

[38] Q.-W. Wang, *The general solution to a system of real quaternion matrix equations*, Comput. Math. Appl., 2005, 49(5), 665–675.

[39] P. Wilczyński, *Quaternionic-valued ordinary differential equations. the riccati equation*, J. Differ. Equ., 2009, 247(7), 2163–2187.

[40] P. Wilczyński, *Quaternionic-valued ordinary differential equations ii. coinciding sectors*, J. Differ. Equ., 2012, 252(8), 4503–4528.

[41] Y. H. Xia, H. Huang and K. I. Kou, *An algorithm for solving linear non-homogeneous quaternion-valued differential equations*, arXiv preprint arXiv:1602.08713, 2016.

[42] F. Zhang, *Quaternions and matrices of quaternions*, Linear Algebra Appl., 1997, 251, 21–57.

[43] X. Zhang, *Global structure of quaternion polynomial differential equations*, Commun. Math. Phys., 2011, 303(2), 301–316.