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ON POSITIVE SOLUTIONS OF EIGENVALUE PROBLEMS FOR A CLASS OF *P*-LAPLACIAN FRACTIONAL DIFFERENTIAL EQUATIONS*

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Abstract In this paper, we are concerned with the eigenvalue problem of a class of p-Laplacian fractional differential equations involving integral boundary conditions. New criteria are established for the existence of positive solutions of the problem under some superlinear and suberlinear conditions. The results of the existence of at least one, two and the nonexistence of positive solutions are also obtained by using the fixed point theory. Finally, several examples are provided to illustrate the obtained results.

Keywords Fractional differential equation, p-Laplacian operator, integral boundary condition, fixed point, eigenvalue.

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1. Introduction

Fractional differential equations appear in various fields such as physics, chemistry, and engineering. The theory of fractional differential equations has become an important aspect of differential equations, see [11, 14, 15]. Boundary value problem (BVP) of fractional differential equations has been investigated in the past years by many authors, see [7–10, 12, 13, 21] for example. In particular, in [7–10, 13], Jia, Liu and Jin considered the existence of solutions for fractional differential equations with integral boundary conditions, where the existence results were established by means of fixed point theory and the method of upper and lower solutions. On the other hand, some developments on the topic involving the p-Laplacian operators and complex boundary value conditions have also been reported in recent years, see [2, 16, 17, 20, 22]. As for the eigenvalue problems of fractional differential equations, there are also a few results, see [1, 4–6, 18, 19, 22, 23] among others. In [22], the existence of positive solutions for the eigenvalue problem

$$\begin{cases} -D_t^{\beta}\big(\varphi_p(D_t^{\alpha}x)\big)(t) = \lambda f(t, x(t)),\ t \in (0, 1), \\ x(0) = 0,\ D_t^{\alpha}x(t) = 0,\ x(1) = \int_0^1 x(s)\mathrm{d}A(s), \end{cases}$$

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was investigated by using the method of upper and lower solutions and Schauder fixed point theorem. Here D_t^β and D_t^α are standard Riemann-Liouville derivatives with $1 < \alpha \le 2$, $0 < \beta \le 1$. A is a function of bounded variation and $\int_0^1 x(s)\mathrm{d}A(s)$ denotes the Riemann-Stieltjes integral of x with respect to A, the p-Laplacian operator φ_p is defined as $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $f(t,x) : (0,1) \times (0,+\infty) \to [0,+\infty)$ is continuous and may be singular at $t = 0$, $t = 1$ and $x = 0$.

In [6], the authors studied the existence of positive solutions for eigenvalue problem of nonlinear fractional differential equation with generalized p-Laplacian operator

$$\begin{cases} D_{0^+}^\beta \Big(\phi\big(D_{0^+}^\alpha u(t)\big)\Big) = \lambda f\big(u(t)\big),\ t \in (0,1), \\ u(0) = u'(0) = u'(1) = 0,\ \phi(D_{0^+}^\alpha u(0)) = \Big(\phi\big(D_{0^+}^\alpha u(1)\big)\Big)' = 0, \end{cases}$$

where $2 < \alpha \le 3$, $1 < \beta \le 2$ are real numbers, ϕ is a generalized p-Laplacian operator, $\lambda > 0$ is a parameter, and $f : (0,+\infty) \to (0,+\infty)$ is continuous. By using the properties of Green function and Guo-Krasnosel'skii fixed-point theorem on cones, several existence results of at least one or two positive solutions in terms of different eigenvalue interval were obtained. Moreover, the nonexistence of positive solutions in term of the parameter λ was also considered there.

While in [19], the author considered the existence, multiplicity, and nonexistence of positive solutions for the system

$$(\mathbf{\Phi}(\mathbf{u}'))' + \lambda \mathbf{h}(t)\mathbf{f}(\mathbf{u}) = 0,\ 0 < t < 1,$$

with one of the following three sets of the boundary conditions:

$$\begin{cases} \mathbf{u}(0) = \mathbf{u}(1) = 0, \\ \mathbf{u}'(0) = \mathbf{u}(1) = 0, \\ \mathbf{u}(0) = \mathbf{u}'(1) = 0, \end{cases}$$

where $\mathbf{u} = (u_1, \cdots, u_n)$, $\mathbf{\Phi}(\mathbf{u}') = (\varphi(u_1'), \cdots, \varphi(u_n'))$, $\mathbf{h}(t) = \mathrm{diag}[h_1(t), \cdots, h_n(t)]$, and $\mathbf{f}(\mathbf{u}) = (f^1(u_1, \cdots, u_n), \cdots, f^n(u_1, \cdots, u_n))$. For this n-dimensional system, the author introduced the superlinearity and sublinearity of $\mathbf{f}(\mathbf{u})$ with respect to φ and got the results by using Krasnosel'skii's fixed point theorem on a cone.

However, as far as we know, there are few papers studying the eigenvalue problem for the p-Laplacian fractional differential equations involving the integral boundary condition. Inspired by the above work, in this paper, we will explore the eigenvalue problem for the following p-Laplacian fractional differential equation involving integral boundary conditions

$$\begin{cases} -{}^C D_{0^+}^\beta \big(\varphi_p({}^C D_{0^+}^\alpha u)\big)(t) = \lambda f(t, u(t)),\ t \in (0,1), \\ u(0) = \displaystyle\int_0^1 g_1(s)u(s)\mathrm{d}s,\ u(1) = \int_0^1 g_2(s)u(s)\mathrm{d}s,\ u''(0) = \int_0^1 g_3(s)u(s)\mathrm{d}s, \\ {}^C D_{0^+}^\alpha u(t)\ |_{t=0} = 0, \end{cases} \tag{1.1}$$

where ${}^C D_{0^+}^\beta$ and ${}^C D_{0^+}^\alpha$ are the standard Caputo derivatives with $2 < \alpha \le 3$, $0 < \beta \le 1$. The p-Laplacian operator φ_p is defined as $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $f(t,u) : [0,1] \times [0,+\infty) \to (0,+\infty)$ is continuous, $g_i(s) \in C[0,1] (i = 1,2,3)$, $J = [0,1]$, $\lambda > 0$ is a parameter.

By applying the properties of Green function and Guo-Krasnosel'skii fixed-point theorem on cones, we shall establish several new existence and nonexistence results for positive solutions in terms of different value of the parameter λ. Moreover, the existence of two positive solutions on the BVP (1.1) will be considered here as well.

Note that the order of the equation (1.1) is higher than that in [22], and also, compared to [5, 6, 19], our equation is inhomogeneous and there are some integral boundary value conditions. Hence the BVP (1.1) is more general than those considered in papers [5, 6, 19, 22]. As a result, the problem studied here is more complex and the computation becomes more difficult. Clearly, the obtained results in this paper extends directly the existing results appearing in Ref. [6, 19, 22].

This paper is organized as follows. In Section 2, we shall introduce some definitions of fractional integral and differential operators, we will also establish in this section some basic lemmas for the later discussion. Then, in Section 3, we investigate the existence of positive solutions for BVP (1.1). In Section 4, we explore the nonexistence of positive solutions for BVP (1.1) and obtain two nonexistence theorems. Finally, as applications, some examples are presented in Section 5 to illustrate the main results.

2. Preliminaries

In this section, we first collect some preliminaries on fractional order integral, derivative, the Green functions, and completely continuous operators. We then prove several lemmas here which will be utilized in our later discussion.

We start by the definitions of fractional order integrals.

Definition 2.1 (See [15]). Let $\alpha > 0$. For a function $u : (0, +\infty) \to \mathbb{R}$, the Riemann-Liouville fractional integral operator of order α of u is defined by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) \mathrm{d}s,$$

provided the integral exists.

Definition 2.2 (See [15]). The Caputo derivative of order α for a function $u : (0, +\infty) \to \mathbb{R}$ is given by

$${}^{C}D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(s)}{(t-s)^{\alpha+1-n}} \mathrm{d}s,$$

provided the right side is pointwise defined on $(0, +\infty)$, where $n = [\alpha] + 1$ and $n - 1 < \alpha < n$.

It was shown in [15] that

Lemma 2.1. *For $\alpha > 0$ and $u \in AC^n(J)$. Then*

$$I_{0+}^{\alpha} {}^{C}D_{0+}^{\alpha} u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}, c_k \in \mathbb{R}, k = 0, 1, 2, \cdots, n-1,$$

where n is the smallest integer greater than or equal to α.

In order to get the expressions of solutions of the BVP (1.1), we need to prove the following lemma on the Green function of the linear BVP associated to (1.1).

Lemma 2.2. *For any $y \in C[0,1]$, the boundary value problem*

$$\begin{cases} {}^CD_{0+}^{\alpha} u(t) + y(t) = 0, \ t \in (0,1), \\ u(0) = \int_0^1 g_1(s)u(s)\mathrm{d}s, \ u(1) = \int_0^1 g_2(s)u(s)\mathrm{d}s, \ u''(0) = \int_0^1 g_3(s)u(s)\mathrm{d}s \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)\mathrm{d}s + \int_0^1 \Phi(t,s)u(s)\mathrm{d}s,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t(1-s)^{\alpha-1}, & 0 \le t \le s \le 1, \end{cases} \tag{2.2}$$

and

$$\Phi(t,s) = (1-t)g_1(s) + tg_2(s) + \frac{1}{2}(t^2 - t)g_3(s). \tag{2.3}$$

Proof. From Lemma 2.1, we have

$$u(t) = -I_{0+}^{\alpha} y(t) + c_0 + c_1 t + c_2 t^2,$$

that is,

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\mathrm{d}s + c_0 + c_1 t + c_2 t^2.$$

Using the boundary value conditions in (2.1), we can determine that

$$\begin{aligned} c_0 =& \int_0^1 g_1(s)u(s)\mathrm{d}s, \\ c_1 =& \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\mathrm{d}s + u(1) - u(0) - c_2 \\ =& \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\mathrm{d}s + \int_0^1 g_2(s)u(s)\mathrm{d}s \\ & - \int_0^1 g_1(s)u(s)\mathrm{d}s - \frac{1}{2}\int_0^1 g_3(s)u(s)\mathrm{d}s, \end{aligned} \tag{2.4}$$

and

$$c_2 = \frac{1}{2}\int_0^1 g_3(s)u(s)\mathrm{d}s.$$

Hence,

$$\begin{aligned} u(t) =& -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^1 t(1-s)^{\alpha-1} y(s)\mathrm{d}s \\ & + \int_0^1 (1-t)g_1(s)u(s)\mathrm{d}s + \int_0^1 tg_2(s)u(s)\mathrm{d}s + \frac{1}{2}\int_0^1 (t^2-t)g_3(s)u(s)\mathrm{d}s \\ =& \frac{1}{\Gamma(\alpha)} \int_0^t \big(t(1-s)^{\alpha-1} - (t-s)^{\alpha-1}\big) y(s)\mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_t^1 t(1-s)^{\alpha-1} y(s)\mathrm{d}s \\ & + \int_0^1 (1-t)g_1(s)u(s)\mathrm{d}s + \int_0^1 tg_2(s)u(s)\mathrm{d}s + \frac{1}{2}\int_0^1 (t^2-t)g_3(s)u(s)\mathrm{d}s \end{aligned}$$

$$= \int_0^1 G(t,s)y(s)\mathrm{d}s + \int_0^1 \Phi(t,s)u(s)\mathrm{d}s. \tag{2.5}$$

$\square$

Throughout this paper, we always assume the following hypothesis holds:

(H_0) $0 < m = \min\{\Phi(t,s) : (t,s) \in J \times J\} \leq \Phi(t,s) \leq M = \max\{\Phi(t,s) : (t,s) \in J \times J\} < 1$.

Lemma 2.3. *Suppose (H_0) holds. Let G and Φ be defined by (2.2) and (2.3). Then,*

(*i*) *for any $t,\ s \in (0,1)$, $G(t,s) > 0$ is continuous;*

(*ii*) *for any $s \in [0,1]$, $\max\limits_{t\in J} G(t,s) \leq \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}$, and for any $s \in [0,1]$, there exists $\epsilon \in (0, \frac{1}{2})$ such that*

$$\min_{t\in[\epsilon,1-\epsilon]} G(t,s) \geq \frac{\rho}{\Gamma(\alpha)}(1-s)^{\alpha-1},$$

where $\rho = \min\limits_{t\in[\epsilon,1-\epsilon]} q(t) = \min\{\epsilon-\epsilon^{\alpha-1}, (1-\epsilon)-(1-\epsilon)^{\alpha-1}\}$ with $q(t) = t-t^{\alpha-1}$;

(*iii*) *$\Phi(t,s) \in C([0,1]\times[0,1], [0,+\infty))$ is continuous.*

Proof. Assertion (*i*) follows readily from the expression of $G(t,s)$.

(*ii*) By the expression of $G(t,s)$, we can easily get

$$\max_{t\in J} G(t,s) \leq \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}.$$

We then show that $\min\limits_{t\in[\epsilon,1-\epsilon]} G(t,s) \geq \frac{\rho}{\Gamma(\alpha)}(1-s)^{\alpha-1}$, for any $s \in [0,1]$. In fact, when $t < s$, for $t \in [\epsilon, 1-\epsilon]$, we see

$$G(t,s) = \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} \geq \frac{\epsilon(1-s)^{\alpha-1}}{\Gamma(\alpha)} > \frac{\rho(1-s)^{\alpha-1}}{\Gamma(\alpha)}.$$

If $0 \leq s \leq t \leq 1$ and $t \in [\epsilon, 1-\epsilon]$, then $s \in [0, 1-\epsilon]$, and we have

$$\begin{aligned} G(t,s) &= \frac{t(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &= \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\big(t - (1 - \frac{1-t}{1-s})^{\alpha-1}\big) \\ &\geq \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}(t - t^{\alpha-1}). \end{aligned}$$

Denoted $q(t) = t - t^{\alpha-1}$, then q is a concave function. Let

$$\rho = \min_{t\in[\epsilon,1-\epsilon]} q(t) = \min_{\epsilon\in(0,\frac{1}{2})}\{\epsilon - \epsilon^{\alpha-1}, (1-\epsilon) - (1-\epsilon)^{\alpha-1}\},$$

then we have that

$$G(t,s) \geq \frac{\rho}{\Gamma(\alpha)}(1-s)^{\alpha-1}.$$

Therefore, $\min\limits_{t\in[\epsilon,1-\epsilon]} G(t,s) \geq \frac{\rho}{\Gamma(\alpha)}(1-s)^{\alpha-1}$ for any $s \in [0,1]$.

(iii) According to the definition of $\Phi(t,s)$ and (H_0), we see $\Phi(t,s) \in C([0,1]\times[0,1],[0,+\infty))$ is continuous as desired. $\square$

Let $X = C(J)$ be the Banach space of all continuous functions from J to $\mathbb{R}$ with the norm $\|u\| = \max\limits_{t\in J}|u(t)|$, $P = \{u \in X : u(t) \geq 0, t \in J\}$, then P is a cone in X. We now define an operator $A : C(J) \to C(J)$ as

$$Au(t) = \int_0^1 \Phi(t,s)u(s)\mathrm{d}s. \tag{2.6}$$

Then we have that

Lemma 2.4. *Let (H_0) hold, the operator A has the following properties:*

- (i) *A is a bounded linear operator;*
- (ii) *$A(P) \subset P$;*
- (iii) *the operator A is reversible;*
- (iv) *$\|(I-A)^{-1}\| \leq \frac{1}{1-M}$.*

Proof. By (H_0), it is obvious that (i) and (ii) hold. Since $M < 1$, $\|Au\| \leq M\|u\| < \|u\|$, we get $\|A\| \leq M < 1$ and (iii) follows. To prove (iv), let $v(t) = u(t) - Au(t)$, then $v \in C[0,1]$, and

$$u(t) = (I-A)^{-1}v(t), \text{ for } t \in [0,1].$$

From the definition of the operator A, we have that

$$u(t) = v(t) + \int_0^1 \Phi(t,s)u(s)\mathrm{d}s.$$

Put $u_0(t) = u(t)$, $u_m(t) = v(t) + \int_0^1 \Phi(t,s)u_{m-1}(s)\mathrm{d}s$, $m = 1,2,\cdots$, and denote $\Phi_1(t,s) = \Phi(t,s)$. Then we can apply the method of iteration to get that

$$u(t) = v(t) + \int_0^1 R(t,s)v(s)\mathrm{d}s,$$

with

$$R(t,s) = \sum_{j=1}^{\infty}\Phi_j(t,s), \Phi_j(t,s) = \int_0^1 \Phi(t,\tau)\Phi_{j-1}(\tau,s)\mathrm{d}\tau, j = 2,3,\cdots.$$

Because $0 \leq \Phi(t,s) < M < 1$, we deduce that

$$0 \leq R(t,s) = \sum_{j=1}^{\infty}\Phi_j(t,s) < M + M^2 + \cdots + M^n + \cdots = \frac{M}{1-M}. \tag{2.7}$$

Since $(I-A)^{-1}v(t) = u(t)$, it yields that

$$|(I-A)^{-1}v(t)| \leq |v(t)| + \frac{M}{1-M}\left|\int_0^1 v(s)\mathrm{d}s\right| \leq \|v\| + \frac{M}{1-M}\|v\| = \frac{\|v\|}{1-M},$$

or

$$\|(I-A)^{-1}v(t)\| = \max_{t\in[0,1]}|(I-A)^{-1}v(t)| \leq \frac{\|v\|}{1-M},$$

which implies

$$\|(I-A)^{-1}\| \leq \frac{1}{1-M}.$$

Assertion (iv) is proved. □

With Lemma 2.2 and Lemma 2.4, we can verify the following lemma readily.

Lemma 2.5. *If* (H_0) *holds,* $u \in C(J)$, ${}^CD^{\alpha}_{0+}u \in C(J)$, *then the boundary value problem*

$$\begin{cases} -({}^CD^{\alpha}_{0+}u)(t) = f(t,u(t)),\ t \in (0,1), \\ u(0) = \int_0^1 g_1(s)u(s)\mathrm{d}s,\ u(1) = \int_0^1 g_2(s)u(s)\mathrm{d}s,\ u''(0) = \int_0^1 g_3(s)u(s)\mathrm{d}s \end{cases} \tag{2.8}$$

is equivalent to the integral equation

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s + \int_0^1 \Phi(t,s)u(s)\mathrm{d}s \\ &= \int_0^1 H(t,s)f(s,u(s))\mathrm{d}s, \end{aligned}$$

where

$$H(t,s) = G(t,s) + \int_0^1 R(t,\tau)G(\tau,s)\mathrm{d}\tau. \tag{2.9}$$

Proof. Define an nonlinear operator $K : C(J) \to C(J)$ as

$$Ku(t) = \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s. \tag{2.10}$$

Then from (2.6) and (2.10), the BVP (2.8) is equivalent to

$$u(t) = Ku(t) + Au(t).$$

By Lemma 2.4, we can get

$$(I-A)^{-1}Ku(t) = u(t),$$

that is

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s + \int_0^1 R(t,s)\int_0^1 G(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\mathrm{d}s \\ &= \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s + \int_0^1 \Big(\int_0^1 R(t,\tau)G(\tau,s)\mathrm{d}\tau\Big) f(s,u(s))\mathrm{d}s \\ &= \int_0^1 H(t,s)f(s,u(s))\mathrm{d}s. \end{aligned}$$

□

Lemma 2.6. *Suppose* (H_0) *holds, and* R *and* H *are defined by* (2.7) *and* (2.9). *Then the functions* R *and* H *have the following properties:*

(i) *for any* $t,s \in [0,1]$, $H(t,s) \geq 0$ *is continuous;*

(*ii*) *for any* $t, s \in [0,1]$, $\frac{m}{1-m} \le R(t,s) \le \frac{M}{1-M}$;

(*iii*) *for any* $t, s \in [0,1]$, $\frac{\rho m(1-2\epsilon)}{\Gamma(\alpha)(1-m)}(1-s)^{\alpha-1} \le H(t,s) \le \frac{1}{\Gamma(\alpha)(1-M)}(1-s)^{\alpha-1}$.

Proof. (*i*) follows from the expression of H, and (*ii*) from (2.7). Using (H_0), (2.7), (2.9) and Lemma 2.3, we have

$$\begin{aligned} H(t,s) &= G(t,s) + \int_0^1 R(t,\tau)G(\tau,s)\mathrm{d}\tau \\ &\ge \int_\epsilon^{1-\epsilon} R(t,\tau)G(\tau,s)\mathrm{d}\tau \\ &\ge \frac{\rho m}{\Gamma(\alpha)(1-m)} \int_\epsilon^{1-\epsilon} (1-s)^{\alpha-1}\mathrm{d}\tau \\ &= \frac{\rho m}{\Gamma(\alpha)(1-m)}(1-s)^{\alpha-1}(1-2\epsilon), \end{aligned}$$

and

$$\begin{aligned} H(t,s) &= G(t,s) + \int_0^1 R(t,\tau)G(\tau,s)\mathrm{d}\tau \\ &\le \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{M}{1-M}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}\tau \\ &= \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-M)}, \end{aligned}$$

thus,

$$\frac{\rho m}{\Gamma(\alpha)(1-m)}(1-2\epsilon)(1-s)^{\alpha-1} \le H(t,s) \le \frac{1}{\Gamma(\alpha)(1-M)}(1-s)^{\alpha-1}.$$

So we get (*iii*). □

Let $q > 1$ satisfy the relation $\frac{1}{q} + \frac{1}{p} = 1$, then $\varphi_p^{-1} = \varphi_q$, where p, φ_p are from (1.1). Next we consider the associated linear p-Laplacian fractional differential equations involving integral boundary conditions

$$\begin{cases} {}^CD_{0^+}^{\beta}\big(\varphi_p({}^CD_{0^+}^{\alpha}u)\big)(t) + y(t) = 0,\ t \in (0,1), \\ u(0) = \displaystyle\int_0^1 g_1(s)u(s)\mathrm{d}s,\ u(1) = \int_0^1 g_2(s)u(s)\mathrm{d}s,\ u''(0) = \int_0^1 g_3(s)u(s)\mathrm{d}s, \\ {}^CD_{0^+}^{\alpha}u(t)\mid_{t=0} = 0, \end{cases} \tag{2.11}$$

where $y \in C[0,1]$ and $y \ge 0$. For convenience, let $b = (\Gamma(\beta))^{-1}$, then we have the following lemma.

Lemma 2.7. *The associated linear p-Laplacian fractional differential equations* (2.11) *has the unique solution*

$$u(t) = \int_0^1 H(t,s)\varphi_q\Big(\int_0^s b(s-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau\Big)\mathrm{d}s,\ t \in [0,1]. \tag{2.12}$$

Proof. In fact, let $w = {}^C D^\alpha_{0+} u$, $v = \varphi_p(w)$. Then the solution of the initial value problem

$$\begin{cases} {}^C D^\beta_{0+} v(t) + y(t) = 0, \ t \in (0,1), \\ v(0) = 0, \end{cases} \tag{2.13}$$

is given by $v(t) = -I^\beta_{0+} y(t) + c_0$, $t \in [0,1]$. Since $v(0) = 0$, $0 < \beta \leq 1$, we get $c_0 = 0$, and consequently,

$$v(t) = -I^\beta_{0+} y(t), t \in [0,1]. \tag{2.14}$$

Noting that ${}^C D^\alpha_{0+} u = w$, $w = \varphi_q(v)$, we derive from (2.13) that the solution of the BVP (2.11) satisfies

$$\begin{cases} {}^C D^\alpha_{0+} u(t) = \varphi_q\big(- I^\beta_{0+} y(t)\big), \ t \in (0,1), \\ u(0) = \int_0^1 g_1(s) u(s) \mathrm{d}s, \ u(1) = \int_0^1 g_2(s) u(s) \mathrm{d}s, \ u''(0) = \int_0^1 g_3(s) u(s) \mathrm{d}s. \end{cases} \tag{2.15}$$

Since $y(t) \geq 0$, we have $\varphi_q\big(- I^\beta_{0+} y(t)\big) = -\varphi_q\big(I^\beta_{0+} y(t)\big)$. Thus by Lemma 2.5, the solution of the BVP (2.15) can be written as

$$u(t) = -\int_0^1 H(t,s)\Big(- \varphi_q\big(I^\beta_{0+} y(s)\big)\Big) \mathrm{d}s,$$

which implies (2.12). □

By Lemma 2.7, we have immediately that

Lemma 2.8. *If (H_0) holds and $u \in C(J)$, ${}^C D^\alpha_{0+} u \in C(J)$, then the boundary value problem*

$$\begin{cases} -{}^C D^\beta_{0+}\big(\varphi_p({}^C D^\alpha_{0+} u)\big)(t) = \lambda f(t, u(t)), \ t \in (0,1), \\ u(0) = \int_0^1 g_1(s) u(s) \mathrm{d}s, \ u(1) = \int_0^1 g_2(s) u(s) \mathrm{d}s, \ u''(0) = \int_0^1 g_3(s) u(s) \mathrm{d}s, \\ {}^C D^\alpha_{0+} u(t) \mid_{t=0} = 0, \end{cases} \tag{2.16}$$

is equivalent to the integral equation

$$u(t) = \int_0^1 H(t,s) \varphi_q\big(\lambda \int_0^s b(s-\tau)^{\beta-1} f(\tau, u(\tau)) \mathrm{d}\tau\big) \mathrm{d}s, \ t \in [0,1].$$

In the light of Lemma 2.6 and Lemma 2.3, we can now introduce a cone P_0 in X. Let

$$P_0 = \{u \in P : u(t) \geq \sigma \|u\|, t \in J\},$$

then $P_0 \subset X$ is a reproducing cone, with $\sigma = \frac{\rho m(1-M)}{1-m}(1-2\epsilon)$. Clearly, $0 < \sigma < 1$. Define now an operator $T_\lambda : X \to X$ as

$$T_\lambda u(t) = \int_0^1 H(t,s) \varphi_q\big(\lambda \int_0^s b(s-\tau)^{\beta-1} f(\tau, u(\tau)) \mathrm{d}\tau\big) \mathrm{d}s.$$

Then one has that

Lemma 2.9. *Assume (H_0) holds, then $T_\lambda : P_0 \to P_0$ is completely continuous.*

Proof. We prove this assertion by the well-known Arzela-Ascoli theorem. First, from Lemma 2.6, we have

$$\begin{aligned} T_\lambda u(t) &= \int_0^1 H(t,s)\varphi_q\big(\lambda\int_0^s b(s-\tau)^{\beta-1}f(\tau,u(\tau))\mathrm{d}\tau\big)\mathrm{d}s \\ &\geq \frac{\rho m(1-2\epsilon)}{\Gamma(\alpha)(1-m)}\int_0^1 (1-s)^{\alpha-1}\varphi_q\big(\lambda\int_0^s b(s-\tau)^{\beta-1}f(\tau,u(\tau))\mathrm{d}\tau\big)\mathrm{d}s, \end{aligned}$$

and

$$T_\lambda u(t) \leq \frac{1}{\Gamma(\alpha)(1-M)}\int_0^1 (1-s)^{\alpha-1}\varphi_q\big(\lambda\int_0^s b(s-\tau)^{\beta-1}f(\tau,u(\tau))\mathrm{d}\tau\big)\mathrm{d}s.$$

Then,

$$T_\lambda u(t) \geq \frac{\rho m(1-M)}{1-m}(1-2\epsilon)\|T_\lambda u\| = \sigma\|T_\lambda u\|,$$

this implies

$$T_\lambda(P_0) \subset P_0.$$

In view of non-negativeness and continuity of H and f, $T_\lambda : P_0 \to P_0$ is clearly continuous. Next, let $\Omega \subset P_0$ be bounded, i.e., there exists a constant $r > 0$ such that $\|u\| \leq r$, for $u \in \Omega$. Set $L = \max_{0\leq u\leq r, 0\leq t\leq 1} |f(t,u(t))| + 1$. Then for $u \in \Omega$ and $t \in J$,

$$\begin{aligned} |T_\lambda u(t)| &= \big|\int_0^1 H(t,s)\varphi_q\big(\lambda\int_0^s b(s-\tau)^{\beta-1}f(\tau,u(\tau))\mathrm{d}\tau\big)\mathrm{d}s\big| \\ &\leq \frac{L^{q-1}}{\Gamma(\alpha)(1-M)}\int_0^1 (1-s)^{\alpha-1}\varphi_q\big(\lambda\int_0^s b(s-\tau)^{\beta-1}\mathrm{d}\tau\big)\mathrm{d}s \\ &= \frac{B(\alpha,\beta(q-1)+1)}{\Gamma(\alpha)(1-M)}\cdot(\frac{\lambda Lb}{\beta})^{q-1} \\ &< +\infty, \end{aligned}$$

hence, $T_\lambda(\Omega)$ is uniformly bounded.

It remains to prove the equi-continuity of T_λ. Since $H(t,s)$ is continuous and hence uniformly continuous on $[0,1]\times[0,1]$, for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for $t_1,\ t_2 \in J$ with $|t_1 - t_2| < \delta$ imply

$$|H(t_1,s) - H(t_2,s)| < \varepsilon.$$

Then, for all $u \in P_0$,

$$\begin{aligned} |T_\lambda u(t_2) - T_\lambda u(t_1)| &\leq \int_0^1 |H(t_2,s)-H(t_1,s)|\varphi_q\big(\lambda\int_0^s b(s-\tau)^{\beta-1}f(\tau,u(\tau))\mathrm{d}\tau\big)\mathrm{d}s \\ &\leq \varepsilon\int_0^1 \varphi_q(\lambda L\int_0^s b(s-\tau)^{\beta-1}\mathrm{d}\tau)\mathrm{d}s \\ &= \frac{\varepsilon}{\beta(q-1)+1}\cdot(\frac{\lambda Lb}{\beta})^{q-1}, \end{aligned}$$

which shows T_λ is equicontinuous on P_0.

Therefore, by Arzela-Ascoli theorem, we infer that $T_\lambda : P_0 \to P_0$ is completely continuous. $\square$

Finally, for convenience, we introduce here the following notations to be used in the subsequent sections.

$$f^0 = \limsup_{u\to 0^+} \sup_{t\in J} \frac{f(t,u)}{\varphi_p(u)}, \qquad f_\infty = \liminf_{u\to +\infty} \inf_{t\in J} \frac{f(t,u)}{\varphi_p(u)},$$

$$f_0 = \liminf_{u\to 0^+} \inf_{t\in J} \frac{f(t,u)}{\varphi_p(u)}, \qquad f^\infty = \limsup_{u\to +\infty} \sup_{t\in J} \frac{f(t,u)}{\varphi_p(u)},$$

$$A_1 = \frac{1}{\Gamma(\alpha)(1-M)} B(\alpha, \beta(q-1)+1)(\Gamma(\beta+1))^{1-q},$$

$$A_2 = \frac{\rho m(1-2\epsilon)\sigma}{\Gamma(\alpha)(1-m)} B(\alpha, \beta(q-1)+1)(\Gamma(\beta+1))^{1-q},$$

$$A_3 = \frac{\rho m(1-2\epsilon)}{\Gamma(\alpha)(1-m)} B(\alpha, \beta(q-1)+1)(\Gamma(\beta+1))^{1-q}.$$

It is easy to see that, from (H_0) and the definition of ρ in Lemma 2.3 there holds for $\epsilon \in (0, \frac{1}{2})$,

$$\frac{A_1}{A_3} = \frac{1-m}{\rho m(1-2\epsilon)(1-M)} > 1. \tag{2.17}$$

3. Existence of positive solutions

In this section, we study the existence of positive solutions for BVP (1.1), by applying fixed point theorem on a cone. We will obtain sufficient conditions for the problem having at least one positive solution and two positive solutions, respectively.

We will employ the following well known fixed point principle founded in [3] to prove the existence results of positive solutions.

Lemma 3.1. *Let X be a Banach space and let $P_0 \subset X$ be a cone in X. Assume Ω_1, Ω_2 are bounded open subsets of X with $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : P_0 \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P_0$ be a completely continuous operator such that either*

(i) $\|Tu\| \le \|u\|$, for any $u \in P_0 \cap \partial\Omega_1$, and $\|Tu\| \ge \|u\|$, for any $u \in P_0 \cap \partial\Omega_2$, or

(ii) $\|Tu\| \ge \|u\|$, for any $u \in P_0 \cap \partial\Omega_1$, and $\|Tu\| \le \|u\|$, for any $u \in P_0 \cap \partial\Omega_2$, then T has a fixed point in $P_0 \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

We first present the result of the existence of at least one positive solution in the following theorem under the conditions on f^0 and f_∞.

Theorem 3.1. *Suppose (H_0) holds. If there exist ξ, $N > 0$ such that $f^0 < \xi$, $f_\infty > N$. Then for each λ satisfying*

$$\frac{1}{A_2^{p-1} N} \le \lambda \le \frac{1}{A_1^{p-1} \xi}, \tag{3.1}$$

the BVP (1.1) has at least one positive solution.

Proof. Let λ satisfy (3.1). By the definition of $f^0 < \xi$, there exists $r_1 > 0$ such that

$$f(t,u) \leq \xi\varphi_p(u), \text{ for } t \in [0,1] \text{ and } u \in (0, r_1]. \tag{3.2}$$

Put

$$\Omega_1 = \{u \in X : \|u\| \leq r_1\}.$$

If $u \in P_0$ with $\|u\| = r_1$, from (3.1) and (3.2), it follows that

$$\begin{aligned}
\|T_\lambda u\| &\leq \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-M)}\varphi_q\big(\lambda\int_0^s b(s-\tau)^{\beta-1}f(\tau,u(\tau))\mathrm{d}\tau\big)\mathrm{d}s \\
&\leq \frac{1}{\Gamma(\alpha)(1-M)}\int_0^1 (1-s)^{\alpha-1}\varphi_q\big(\lambda\int_0^s b(s-\tau)^{\beta-1}\xi\varphi_p(r_1)\mathrm{d}\tau\big)\mathrm{d}s \\
&\leq \lambda^{q-1}\xi^{q-1}\frac{B(\alpha,\beta(q-1)+1)}{\Gamma(\alpha)(1-M)}(\frac{b}{\beta})^{q-1}\cdot r_1 \\
&\leq \lambda^{q-1}\xi^{q-1}A_1 r_1 \\
&\leq r_1 = \|u\|.
\end{aligned}$$

Hence,

$$\|T_\lambda u\| \leq \|u\|, \text{ for } u \in P_0 \cap \partial\Omega_1. \tag{3.3}$$

By the definition of f_∞, there exists $\bar{r} > 0$ such that

$$f(t,u) \geq N\varphi_p(u), \text{ for } t \in [0,1] \text{ and } u \in [\bar{r}, +\infty). \tag{3.4}$$

Let $r_2 > \frac{1}{\sigma}\bar{r}$ and $r_2 = \max\{2r_1, \frac{1}{\sigma}\bar{r}\}$ and take

$$\Omega_2 = \{u \in X : \|u\| \leq r_2\}.$$

Thus, for any $u \in P_0 \cap \partial\Omega_2$, we have $u \in P_0$ and $\|u\| = r_2$, so

$$u(t) \geq \sigma\|u\| = \sigma r_2 > \bar{r}.$$

Then, by (3.1) and (3.4), we compute that

$$\begin{aligned}
\|T_\lambda u\| \geq |T_\lambda u| &\geq \int_0^1 H(t,s)\varphi_q\big(\lambda\int_0^s b(s-\tau)^{\beta-1}N\varphi_p(u(\tau))\mathrm{d}\tau\big)\mathrm{d}s \\
&\geq \frac{\rho m(1-2\epsilon)}{\Gamma(\alpha)(1-m)}\int_0^1 (1-s)^{\alpha-1}\varphi_q\big(\lambda\int_0^s b(s-\tau)^{\beta-1}N\varphi_p(\sigma\|u\|)\mathrm{d}\tau\big)\mathrm{d}s \\
&\geq \lambda^{q-1}N^{q-1}\frac{\rho m(1-2\epsilon)\sigma}{\Gamma(\alpha)(1-m)}(\frac{b}{\beta})^{q-1}B(\alpha,\beta(q-1)+1)\cdot r_2 \\
&\geq \lambda^{q-1}N^{q-1}\cdot A_2\cdot r_2 \\
&\geq r_2 = \|u\|.
\end{aligned}$$

Therefore,

$$\|T_\lambda u\| \geq \|u\|, \text{ for } u \in P_0 \cap \partial\Omega_2. \tag{3.5}$$

By (3.3), (3.5) and Lemma 3.1, we conclude T_λ has a fixed point $u \in P_0 \cap (\Omega_2 \setminus \overline{\Omega}_1)$ with $r_1 \leq \|u\| \leq r_2$, and clearly u is a positive solution for the BVP (1.1). □

Similarly, employing conditions for f_0 and f^∞, we can obtain that

Theorem 3.2. *Suppose* (H_0) *holds. If there exist* $\xi,\ N > 0$ *such that* $f_0 > N$, $f^\infty < \xi$ *hold. Then for each*

$$\frac{1}{A_2^{p-1}N} \le \lambda \le \frac{1}{A_1^{p-1}\xi}, \tag{3.6}$$

the BVP (1.1) *has at least one positive solution.*

Proof. Let λ satisfy (3.6). By the definition of f_0, we see that there exists $r_1 > 0$ such that

$$f(t,u) \ge N\varphi_p(u),\ \text{for } t \in [0,1] \text{ and } u \in (0, r_1].$$

So, if $u \in P_0$ with $\|u\| = r_1$, then as in the proof of Theorem 3.1, if we take

$$\Omega_1 = \{u \in X : \|u\| \le r_1\},$$

then,

$$\|T_\lambda u\| \ge \|u\|,\ \text{for } u \in P_0 \cap \partial\Omega_1. \tag{3.7}$$

By the definition of f^∞, there exists $r_2 > 0$ such that

$$f(t,u) \le \xi\varphi_p(u),\ \text{for } t \in [0,1] \text{ and } u \in [r_2, +\infty).$$

So, if we choose

$$\Omega_2 = \{u \in X : \|u\| \le r_2\},$$

then similar to that in the proofs of Theorem 3.1 we can show that, for $u \in P_0$ with $\|u\| = r_2$,

$$\|T_\lambda u\| \le \|u\|,\ \text{for } u \in P_0 \cap \partial\Omega_2. \tag{3.8}$$

By (3.7), (3.8) and Lemma 3.1 again, we deduce that T_λ has a fixed point $u \in P_0 \cap (\Omega_2 \setminus \overline{\Omega}_1)$ with $r_1 \le \|u\| \le r_2$, and clearly u is a positive solution for the BVP (1.1). □

The following theorem plays an important role in proving of the subsequent two theorems in this part.

Theorem 3.3. *Suppose* (H_0) *holds. If there exist* $r_2, r_1 > 0$ *with* $r_2 > \frac{A_1}{A_3}\cdot r_1$ *(*$> r_1$ *by* (2.17)*) such that*

$$\lambda \min_{\sigma r_1 \le u \le r_1,\, 0 \le t \le 1} f(t,u(t)) \ge \varphi_p\Big(\frac{r_1}{A_3}\Big),\ \textit{and}\ \lambda \max_{0 \le u \le r_2,\, 0 \le t \le 1} f(t,u(t)) \le \varphi_p\Big(\frac{r_2}{A_1}\Big),$$

then the BVP (1.1) *has at least one positive solution* $u \in P_0$ *with* $r_1 \le \|u\| \le r_2$.

Proof. Put $\Omega_1 = \{u \in X : \|u\| \le r_1\}$. Then for $u \in P_0 \cap \partial\Omega_1$, we have

$$\begin{aligned}
T_\lambda u(t) &= \int_0^1 H(t,s)\varphi_q\big(\lambda \int_0^s b(s-\tau)^{\beta-1} f(\tau,u(\tau))\mathrm{d}\tau\big)\mathrm{d}s \\
&\ge \frac{\rho m(1-2\epsilon)}{\Gamma(\alpha)(1-m)} \int_0^1 (1-s)^{\alpha-1} \\
&\quad \cdot \varphi_q\big(\lambda \int_0^s b(s-\tau)^{\beta-1} \min_{\sigma r_1 \le u \le r_1,\, 0 \le \tau \le 1} f(\tau,u(\tau))\mathrm{d}\tau\big)\mathrm{d}s \\
&\ge \varphi_q\big(\lambda \min_{\sigma r_1 \le u \le r_1,\, 0 \le \tau \le 1} f(\tau,u(\tau))\big) \frac{\rho m(1-2\epsilon)}{\Gamma(\alpha)(1-m)} B(\alpha, \beta(q-1)+1)\Big(\frac{b}{\beta}\Big)^{q-1}
\end{aligned}$$

$$= \varphi_q\big(\lambda \min_{\sigma r_1 \le u \le r_1,\, 0 \le \tau \le 1} f(\tau, u(\tau))\big) \cdot A_3$$
$$\ge r_1 = \|u\|,$$

that is,

$$\|T_\lambda u\| \ge \|u\|, \text{ for } u \in P_0 \cap \partial\Omega_1. \tag{3.9}$$

On the other hand, let $\Omega_2 = \{u \in X : \|u\| \le r_2\}$, then for $u \in P_0 \cap \partial\Omega_2$, we see that

$$\begin{aligned}
T_\lambda u(t) &= \int_0^1 H(t,s)\varphi_q\big(\lambda \int_0^s b(s-\tau)^{\beta-1} f(\tau, u(\tau))\mathrm{d}\tau\big)\mathrm{d}s \\
&\le \frac{1}{\Gamma(\alpha)(1-M)} \int_0^1 (1-s)^{\alpha-1} \\
&\quad \cdot \varphi_q\big(\lambda \int_0^s b(s-\tau)^{\beta-1} \max_{0 \le u \le r_2,\, 0 \le \tau \le 1} f(\tau, u(\tau))\mathrm{d}\tau\big)\mathrm{d}s \\
&\le \varphi_q\big(\lambda \max_{0 \le u \le r_2,\, 0 \le \tau \le 1} f(\tau, u(\tau))\big) \frac{B(\alpha, \beta(q-1)+1)}{\Gamma(\alpha)(1-M)} (\frac{b}{\beta})^{q-1} \\
&= \varphi_q\big(\lambda \max_{0 \le u \le r_2,\, 0 \le \tau \le 1} f(\tau, u(\tau))\big) \cdot A_1 \\
&\le r_2 = \|u\|.
\end{aligned}$$

So,

$$\|T_\lambda u\| \le \|u\|, \text{ for } u \in P_0 \cap \partial\Omega_2. \tag{3.10}$$

Thus, from (3.9), (3.10) and Lemma 3.1, it follows that BVP (1.1) has at least one positive solution $u \in P_0$ with $r_1 \le \|u\| \le r_2$. $\square$

With the help of Theorem 3.3, we will discuss in the sequel the existence of at least two positive solutions for the BVP (1.1).

Theorem 3.4. *Suppose* (H_0) *holds, let* $\lambda_1 = \sup_{r>0} \frac{\varphi_p(r)}{\varphi_p(A_1) \max_{0 \le u \le r,\, 0 \le t \le 1} f(t,u)}$. *If* $f_0 = +\infty$ *and* $f_\infty = +\infty$*, then the BVP* (1.1) *has at least two positive solutions for each* $\lambda \in (0, \lambda_1)$.

Proof. Define

$$x(r) = \frac{\varphi_p(r)}{\varphi_p(A_1) \max_{0 \le u \le r,\, 0 \le t \le 1} f(t,u)}.$$

In view of the continuity of f, $f_0 = +\infty$ and $f_\infty = +\infty$, we know that $x(r) : (0, +\infty) \to (0, +\infty)$ is continuous and

$$\lim_{r \to 0^+} x(r) = \lim_{r \to +\infty} x(r) = 0.$$

So there exists $r_0 \in (0, +\infty)$ such that

$$x(r_0) = \sup_{r>0} x(r) = \lambda_1.$$

Therefore, for $\lambda \in (0, \lambda_1)$, there exist constants a_1, a_2 $(0 < a_1 < r_0 < a_2 < +\infty)$ with

$$x(a_1) = x(a_2) = \lambda.$$

Hence,

$$\lambda f(t,u(t)) \leq \varphi_p(\frac{a_1}{A_1}),\ \text{for } t\in[0,1],\ u\in[0,a_1], \tag{3.11}$$

$$\lambda f(t,u(t)) \leq \varphi_p(\frac{a_2}{A_1}),\ \text{for } t\in[0,1],\ u\in[0,a_2]. \tag{3.12}$$

On the other hand, as $f_0 = +\infty$ and $f_\infty = +\infty$, there exists constants b_1, $b_2\,(0 < b_1 < a_1 < r_0 < a_2 < b_2 < +\infty)$ such that

$$\frac{f(t,u)}{\varphi_p(u)} \geq \frac{1}{\lambda\varphi_p(A_3)},\ \text{for } t\in[0,1],\ u\in(0,b_1]\cup[\sigma b_2,+\infty),$$

and so

$$\lambda \min_{\sigma b_1\leq u\leq b_1,\,0\leq t\leq 1} f(t,u(t)) \geq \varphi_p(\frac{b_1}{A_3}), \tag{3.13}$$

$$\lambda \min_{\sigma b_2\leq u\leq b_2,\,0\leq t\leq 1} f(t,u(t)) \geq \varphi_p(\frac{b_2}{A_3}). \tag{3.14}$$

By (3.11), (3.12), (3.13) and (3.14), and applying Theorem 3.3 and Lemma 3.1, we can deduce that the BVP (1.1) has at least two positive solutions for each $\lambda\in(0,\lambda_1)$, and $b_1 < u_1 < a_1$, $a_2 < u_2 < b_2$. □

Theorem 3.5. *Suppose (H_0) holds, let $\lambda_2 = \inf_{r>0} \frac{\varphi_p(r)}{\varphi_p(A_3)\min_{\sigma r\leq u\leq r,\,0\leq t\leq 1} f(t,u)}$. If $f_0 = 0$ and $f_\infty = 0$, then the BVP (1.1) has at least two positive solutions for each $\lambda\in(\lambda_2,+\infty)$.*

Proof. Define

$$x(r) = \frac{\varphi_p(r)}{\varphi_p(A_3)\min_{\sigma r\leq u\leq r,\,0\leq t\leq 1} f(t,u)},$$

then $x(r): (0,+\infty)\to(0,+\infty)$ is continuous and

$$\lim_{r\to 0^+} x(r) = \lim_{r\to+\infty} x(r) = +\infty.$$

So there exists $r_0\in(0,+\infty)$ such that

$$x(r_0) = \inf_{r>0} x(r) = \lambda_2.$$

Then, for $\lambda\in(\lambda_2,+\infty)$, one has d_1, $d_2\,(0 < d_1 < r_0 < d_2 < +\infty)$ with

$$x(d_1) = x(d_2) = \lambda,$$

thus,

$$\lambda f(t,u(t)) \geq \varphi_p(\frac{d_1}{A_3}),\ \text{for } t\in[0,1],\ u\in[\sigma d_1,d_1], \tag{3.15}$$

$$\lambda f(t,u(t)) \geq \varphi_p(\frac{d_2}{A_3}),\ \text{for } t\in[0,1],\ u\in[\sigma d_2,d_2]. \tag{3.16}$$

In addition, since $f_0 = 0$, there exists $c_1\,(0 < c_1 < d_1 < +\infty)$ such that

$$\frac{f(t,u)}{\varphi_p(u)} \leq \frac{1}{\lambda\varphi_p(A_1)},\ \text{for } t\in[0,1],\ u\in(0,c_1],$$

and so

$$\lambda \max_{0\le u\le c_1,\,0\le t\le 1} f(t,u(t)) \le \varphi_p(\frac{c_1}{A_1}). \tag{3.17}$$

Analogously, because $f_\infty = 0$, there exists a constant $c_2 \in (d_2,+\infty)$ such that

$$\frac{f(t,u)}{\varphi_p(u)} \le \frac{1}{\lambda\varphi_p(A_1)},\ \text{for } t\in[0,1],\ u\in(c_2,+\infty).$$

Let $M' = \max\limits_{0\le u\le c_2,\,0\le t\le 1} f(t,u)$ and $\varphi_p(c_2) \ge \lambda\varphi_p(A_1)$. Then

$$\lambda \max_{0\le u\le c_2,\,0\le t\le 1} f(t,u(t)) \le \varphi_p(\frac{c_2}{A_1}). \tag{3.18}$$

Therefore, from (3.15), (3.16), (3.17) and (3.18), together with Theorem 3.3 and Lemma 3.1, we infer that BVP (1.1) has at least two positive solutions for each $\lambda \in (\lambda_2,+\infty)$ and $c_1 < u_1 < d_1$, $c_2 < u_2 < d_2$. □

4. Nonexistence of positive solutions

In this section, we turn to consider the problem of nonexistence of positive solutions for the BVP (1.1). The first result is that

Theorem 4.1. *If $f^0 < +\infty$ and $f^\infty < +\infty$, then there exists $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$, the BVP (1.1) has no positive solution.*

Proof. Since $f^0 < +\infty$ and $f^\infty < +\infty$, there exist positive constants M_1, M_2, r_1 and r_2 such that $r_1 < r_2$ and

$$\begin{aligned} &f(t,u) \le M_1\varphi_p(u),\ \text{for } t\in[0,1],\ u\in[0,r_1],\\ &f(t,u) \le M_2\varphi_p(u),\ \text{for } t\in[0,1],\ u\in[r_2,+\infty). \end{aligned}$$

Denote $\overline{M} = \max\{M_1, M_2, \max\limits_{r_1\le u\le r_2,\,0\le t\le 1} \frac{f(t,u)}{\varphi_p(u)}\}$, then

$$f(t,u) \le \overline{M}\varphi_p(u),\ \text{for } t\in[0,1],\ u\in[0,+\infty).$$

Assume conversely that $v(t)$ is a positive solution of the BVP (1.1). We will show that this leads to a contradiction when $0 < \lambda < \lambda_0 := \overline{M}^{-1}(A_1^{-1})^{p-1}$. In fact, since $T_\lambda v(t) = v(t)$ for $t \in [0,1]$, it yields

$$\begin{aligned} \|v\| = \|T_\lambda v\| &\le \frac{1}{\Gamma(\alpha)(1-M)}\int_0^1 (1-s)^{\alpha-1}\varphi_q\big(\lambda\int_0^s b(s-\tau)^{\beta-1} f(\tau,v(\tau))\mathrm{d}\tau\big)\mathrm{d}s\\ &\le \frac{1}{\Gamma(\alpha)(1-M)}\int_0^1 (1-s)^{\alpha-1}\varphi_q\big(\lambda\int_0^s b(s-\tau)^{\beta-1}\overline{M}\varphi_p\big(v(\tau)\big)\mathrm{d}\tau\big)\mathrm{d}s\\ &\le (\lambda\overline{M})^{q-1}\|v\|\cdot A_1\\ &< \|v\|, \end{aligned}$$

this is a contradiction. Therefore, the BVP (1.1) has no positive solution in this situation. □

Theorem 4.2. *If $f_0 > 0$ and $f_\infty > 0$, then there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, the BVP (1.1) has no positive solution.*

Proof. Due to $f_0 > 0$ and $f_\infty > 0$, there exist positive constants m_1, m_2, r_3 and r_4 such that $r_3 < r_4$ and

$$f(t,u) \geq m_1\varphi_p(u), \text{ for } t \in [0,1], \; u \in [0, r_3].$$
$$f(t,u) \geq m_2\varphi_p(u), \text{ for } t \in [0,1], \; u \in [r_4, +\infty).$$

Let $\overline{m} = \min\{m_1, m_2, \min\limits_{r_3 \leq u \leq r_4,\, 0 \leq t \leq 1} \frac{f(t,u)}{\varphi_p(u)}\}$, we then obtain that

$$f(t,u) \geq \overline{m}\varphi_p(u), \text{ for } t \in [0,1], \; u \in [0, +\infty).$$

We prove this theorem by contradiction when $\lambda > \lambda_0 := \overline{m}^{-1}(A_2^{-1})^{p-1}$ again. If $v(t)$ is a positive solution of the BVP (1.1), then

$$\begin{aligned}
\|v\| = \|T_\lambda v\| &\geq \frac{\rho m(1-2\epsilon)}{\Gamma(\alpha)(1-m)} \int_0^1 (1-s)^{\alpha-1}\varphi_q\big(\lambda \int_0^s b(s-\tau)^{\beta-1} f(\tau, v(\tau))\mathrm{d}\tau\big)\mathrm{d}s \\
&\geq \frac{\rho m(1-2\epsilon)}{\Gamma(\alpha)(1-m)} \int_0^1 (1-s)^{\alpha-1}\varphi_q\big(\lambda \int_0^s b(s-\tau)^{\beta-1} \overline{m}\varphi_p(v(\tau))\mathrm{d}\tau\big)\mathrm{d}s \\
&\geq (\lambda\overline{m})^{q-1}\|v\| \cdot A_2 \\
&> \|v\|,
\end{aligned}$$

which is impossible. Hence, the BVP (1.1) has no positive solution. $\square$

5. Examples

In order to illustrate the applications of the main results obtained in Section 3 and 4, we present some examples in this part.

Example 5.1. Consider the boundary value problem of fractional differential equation

$$\begin{cases}
-{}^CD_{0^+}^{\frac{1}{2}}\big(\varphi_p({}^CD_{0^+}^{\frac{5}{2}}u)\big)(t) = \lambda(t^2+1)(65u - \dfrac{8449}{130}\sin u), \; t \in (0,1), \\
u(0) = \displaystyle\int_0^1 \frac{200}{201}u(s)\mathrm{d}s, \; u(1) = \int_0^1 \frac{200}{201}u(s)\mathrm{d}s, \; u''(0) = \int_0^1 \frac{4}{201}u(s)\mathrm{d}s, \\
{}^CD_{0^+}^{\alpha}u(t)\mid_{t=0} = 0,
\end{cases} \tag{5.1}$$

where $g_1(s) = g_2(s) = \frac{200}{201}$, $g_3(s) = \frac{4}{201}$, $\alpha = \frac{5}{2}$, $\beta = \frac{1}{2}$, $p = 2$, $\epsilon = \frac{1}{4}$, $f(t, u(t)) = (t^2+1)(65u - \frac{8449}{130}\sin u)$.

By a simple calculation, we obtain that

$$\begin{aligned}
&m = \frac{133}{134}, \quad M = \frac{200}{201}, \quad \rho = \frac{3(2-\sqrt{3})}{8}, \quad \sigma = \frac{399(2-\sqrt{3})}{3216}, \\
&A_1^{p-1} = \frac{67}{2}, \; A_2^{p-1} = \frac{1270.49474}{34304}, \quad f^0 = \frac{1}{65} < \xi = \frac{1}{64}, \quad f_\infty = 65 > N = 64.
\end{aligned}$$

Hence

$$\frac{1}{A_2^{p-1}N} < \frac{1}{A_1^{p-1}\xi}.$$

Thus, by Theorem 3.1, the BVP (5.1) has a positive solution for each $\lambda \in (0.42188, 1.91045)$.

Example 5.2. Consider the boundary value problem of fractional differential e-quation

$$\begin{cases} -{}^{C}D_{0^+}^{\frac{1}{2}}\big(\varphi_p({}^{C}D_{0^+}^{\frac{5}{2}}u)\big)(t) = \lambda \dfrac{e^t(u^2(t)+u(t))(51+\sin u)}{3120u(t)+1},\ t\in(0,1),\\ u(0)=\displaystyle\int_0^1 \frac{200}{201}u(s)\mathrm{d}s,\ u(1)=\int_0^1\frac{200}{201}u(s)\mathrm{d}s,\ u''(0)=\int_0^1\frac{4}{201}u(s)\mathrm{d}s,\\ {}^{C}D_{0^+}^{\alpha}u(t)\mid_{t=0}=0, \end{cases} \tag{5.2}$$

where $g_1(s)=g_2(s)=\frac{200}{201}$, $g_3(s)=\frac{4}{201}$, $\alpha=\frac{5}{2}$, $\beta=\frac{1}{2}$, $p=2$, $\epsilon=\frac{1}{4}$, $f(t,u(t))=\frac{e^t(u^2(t)+u(t))(51+\sin u)}{3120u(t)+1}$.

It is easy to compute that

$$\begin{aligned} &m=\frac{133}{134},\quad M=\frac{200}{201},\ \rho=\frac{3(2-\sqrt{3})}{8},\ \sigma=\frac{399(2-\sqrt{3})}{3216},\\ &A_1^{p-1}=\frac{67}{2},\ A_2^{p-1}=\frac{1270.49474}{34304},\ f_0=51>N=50,\ f^\infty=\frac{52e}{3120}<\xi=\frac{1}{20}. \end{aligned}$$

So we get

$$\frac{1}{A_2^{p-1}N}<\frac{1}{A_1^{p-1}\xi}.$$

By Theorem 3.1, the BVP (5.2) has a positive solution for each $\lambda\in(0.54001,0.59701)$.

Example 5.3. Consider the boundary value problem of fractional differential e-quation

$$\begin{cases} -{}^{C}D_{0^+}^{\frac{1}{2}}\big(\varphi_p({}^{C}D_{0^+}^{\frac{5}{2}}u)\big)(t) = \lambda \dfrac{e^t(100u^2(t)+u(t))(3+\sin u)}{\frac{3}{100}u(t)+1},\ t\in(0,1),\\ u(0)=\displaystyle\int_0^1 \frac{200}{201}u(s)\mathrm{d}s,\ u(1)=\int_0^1\frac{200}{201}u(s)\mathrm{d}s,\ u''(0)=\int_0^1\frac{4}{201}u(s)\mathrm{d}s,\\ {}^{C}D_{0^+}^{\alpha}u(t)\mid_{t=0}=0, \end{cases} \tag{5.3}$$

where $g_1(s)=g_2(s)=\frac{200}{201}$, $g_3(s)=\frac{4}{201}$, $\alpha=\frac{5}{2}$, $\beta=\frac{1}{2}$, $p=2$, $\epsilon=\frac{1}{4}$, $f(t,u(t))=\frac{e^t(100u^2(t)+u(t))(3+\sin u)}{\frac{3}{100}u(t)+1}$.

For this situation, we have

$$\begin{aligned} &m=\frac{133}{134},\ M=\frac{200}{201},\ \rho=\frac{3(2-\sqrt{3})}{8},\ \sigma=\frac{399(2-\sqrt{3})}{3216},\\ &A_1^{p-1}=\frac{67}{2},\ A_2^{p-1}=\frac{1270.49474}{34304},\ f^0=3e<\xi=10,\ f_\infty=\frac{40000}{3}>N=10000,\\ &f_0=3,\ f^\infty=\frac{40000e}{3}, \end{aligned}$$

and $3u<f(t,u)<40000u$. Hence we can conclude that

(*i*) By Theorem 3.1, the BVP (5.3) has a positive solution for each $\lambda\in(0.00270, 0.00299)$.

(*ii*) By Theorem 4.1, the BVP (5.3) has no positive solution for all $\lambda\in(0,\frac{3}{1340000e})$.

(*iii*) By Theorem 4.2, the BVP (5.3) has no positive solution for all $\lambda\in(9.00017,+\infty)$.

Example 5.4. Consider the boundary value problem of fractional differential equation

$$\begin{cases} -{}^C D_{0^+}^{\frac{1}{2}}\big(\varphi_p({}^C D_{0^+}^{\frac{5}{2}} u)\big)(t) = \lambda \dfrac{(t+1)(u^2(t)+u(t))(31+\sin u)}{2112u(t)+1}, \ t \in (0,1), \\ u(0) = \displaystyle\int_0^1 \frac{200}{201} u(s)\mathrm{d}s, \ u(1) = \int_0^1 \frac{200}{201} u(s)\mathrm{d}s, \ u''(0) = \int_0^1 \frac{4}{201} u(s)\mathrm{d}s, \\ {}^C D_{0^+}^{\alpha} u(t) \mid_{t=0} = 0, \end{cases} \tag{5.4}$$

where $g_1(s) = g_2(s) = \frac{200}{201}$, $g_3(s) = \frac{4}{201}$, $\alpha = \frac{5}{2}$, $\beta = \frac{1}{2}$, $p = 2$, $\epsilon = \frac{1}{4}$, $f(t, u(t)) = \frac{(t+1)(u^2(t)+u(t))(31+\sin u)}{2112u(t)+1}$.

As above we can compute for (5.4) that

$$\begin{aligned} & m = \frac{133}{134}, \ M = \frac{200}{201}, \ \rho = \frac{3(2-\sqrt{3})}{8}, \ \sigma = \frac{399(2-\sqrt{3})}{3216}, \\ & A_1^{p-1} = \frac{67}{2}, \ A_2^{p-1} = \frac{1270.49474}{34304}, \ f_0 = 31 > N = 30, \ f^\infty = \frac{1}{33} < \xi = \frac{1}{32}, \\ & f^0 = 62, \ f_\infty = \frac{1}{66}, \end{aligned}$$

and $\frac{1}{66}u < f(t,u) < 62u$. Therefore, we deduce that

(i) By Theorem 3.1, the BVP (5.4) has a positive solution for each $\lambda \in (0.90002, 0.95522)$.

(ii) By Theorem 4.1, the BVP (5.4) has no positive solution for all $\lambda \in (0, 0.00048)$.

(iii) By Theorem 4.2, the BVP (5.4) has no positive solution for all $\lambda \in (1782.03335, +\infty)$.

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