

THRESHOLD DYNAMICS OF THE STOCHASTIC EPIDEMIC MODEL WITH JUMP-DIFFUSION INFECTION FORCE*

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Abstract This paper formulates a stochastic SIR epidemic model by supposing that the infection force is perturbed by Brown motion and Lévy jumps. The globally positive and bounded solution is proved firstly by constructing the suitable Lyapunov function. Then, a stochastic basic reproduction number R_0^L is derived, which is less than that for the deterministic model and the stochastic model driven by Brown motion. Analytical results show that the disease will die out if $R_0^L < 1$, and $R_0^L > 1$ is the necessary and sufficient condition for persistence of the disease. Theoretical results and numerical simulations indicate that the effects of Lévy jumps may lead to extinction of the disease while the deterministic model and the stochastic model driven by Brown motion both predict persistence. Additionally, the method developed in this paper can be used to investigate a class of related stochastic models driven by Lévy noise.

Keywords Stochastic epidemic model, jump-diffusion infection force, the threshold, extinction.

MSC(2010) 92D25, 34K20.

1. Introduction

To study dynamics of the epidemics, one of most popular epidemic models is the SIR model in [7] taking forms

$$\begin{cases} \dot{S} = \mu N - \mu S - \beta SI, \\ \dot{I} = \beta SI - (\mu + \delta + \gamma)I, \\ \dot{R} = \gamma I - \mu R, \end{cases} \quad (1.1)$$

where S, I and R denote the numbers of the susceptible, the infective and the recovered subpopulations. N is total population size, μ is the natural death rates, δ is the additional death rate induced by the infectious, γ is the recovery rate of infectious individuals.

In reality, the parameters in the population model are, more or less, perturbed by environmental noises [13]. Many scholars have tried to introduce noises into the underlying deterministic models and then studied effects of the noise on dynamics

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*The authors were supported by National Natural Science Foundation of China (11671260).

of the epidemics, see [3–5, 8, 9, 12, 14–17] for example. By supposing that the noise mainly has effects on the contact rate like

$$\beta \rightarrow \beta + \sigma \dot{B}(t),$$

Tornatore etc [15] formulated the stochastic SIR epidemic model as follows

$$\begin{cases} dS(t) = (\mu N - \mu S(t) - \beta S(t)I(t))dt - S(t)I(t)\sigma dB(t), \\ dI(t) = (\beta S(t) - (\mu + \delta + \gamma))I(t) dt + S(t)I(t)\sigma dB(t), \\ dR(t) = (\gamma I(t) - \mu R(t))dt. \end{cases} \quad (1.2)$$

The authors studied the existence of the positive solution and showed the threshold value by simulation. Sequently, Ji etc [5] proved the threshold of this stochastic SIR model when the noise is small. In [4], Gray et al. established and studied the dynamics of a stochastic SIS model. Zhao etc [16] show threshold for stochastic SIRS epidemic model with saturated incidence. Zhou etc study the global stability of a discrete multigroup SIR model with nonlinear incidence rate and a reaction-diffusion waterborne pathogen model respectively in [18, 19]. In [17], Zheng etc proposed a new way of investigating the asymptotic behaviour of a stochastic SIS system with multiplicative noise based on the solution of Langevin equation and Ornstein-Uhlenbeck process. More stochastic epidemic models can be found in [3, 8, 9, 14] and cited therein.

Jumps is frequently observed in the epidemic systems since the population may suffer sudden environmental shocks. Bhadra etc [2] applied a stochastic differential model driven by Lévy noise to study Malaria in Northwest India. For more about theoretical applications of the jump process to population models, one can refer to [1, 10, 11] and related references. To make system more realistic, we extend the system from additive noise on the Brown motion to the Brown motion with Lévy jumps $J(t) = \int_0^t \int_Z h(u)\tilde{N}(ds, du)$, and mainly consider its effects on the infection force parameter β such that

$$\beta \rightarrow \beta + \sigma \dot{B}(t) + \dot{J}(t).$$

Based on model (1.1), the studied model in this paper is formulated as that

$$\begin{cases} dS(t) = (\mu N - \mu S(t) - \beta S(t)I(t))dt - S(t^-)I(t^-)[\sigma dB(t) + \int_Z h(u)\tilde{N}(dt, du)], \\ dI(t) = (\beta S(t) - (\mu + \delta + \gamma))I(t) dt + S(t^-)I(t^-)[\sigma dB(t) + \int_Z h(u)\tilde{N}(dt, du)], \\ dR(t) = (\gamma I(t) - \mu R(t))dt. \end{cases} \quad (1.3)$$

σ is a constant. $h(u)$ is continuous. $B(t)$ denotes the Brownian motion. $\tilde{N}(dt, du) = N(dt, du) - \pi(du)dt$ is compensating martingale where $N(t)$ is Poisson random measure, independent of $B(t)$, with the characteristic measure $\pi(du)$ such that $\pi(Z) < \infty$ for the measurable subset Z . Suppose that $(\Omega, \mathcal{F}, \mathcal{P})$ is a suitable filtered probability space. Note that the dynamic of R has no effect on the transmission dynamics, we have only to study the following model

$$\begin{cases} dS(t) = (\mu N - \mu S(t) - \beta S(t)I(t))dt - S(t^-)I(t^-)[\sigma dB(t) + \int_Z h(u)\tilde{N}(dt, du)], \\ dI(t) = (\beta S(t) - (\mu + \delta + \gamma))I(t) dt + S(t^-)I(t^-)[\sigma dB(t) + \int_Z h(u)\tilde{N}(dt, du)]. \end{cases} \quad (1.4)$$

The main aim of this paper is to investigate the effect of Lévy noise on the dynamics of this system, and further to derive the threshold value which can easily determine the extinction and persistence of the disease.

Remark 1.1. As far as Lévy noise is concerned, many results can be found in [1,2,6,10,11] and cited therein. In these literatures, some studied models are formulated by supposing that the death rate is perturbed by the Lévy noise with constant intensity; the other models, established by different method, have similar forms with perturbations: $S(t^-)[\sigma dB(t) + \int_Z h(u)\tilde{N}(dt, du)]$ and $I(t^-)[\sigma dB(t) + \int_Z h(u)\tilde{N}(dt, du)]$. These kinds of models can be studied by using the famous Ito formula to separate the state variable from the noise term. However, when the contact rate is perturbed by the Lévy noise, we have model (1.3), where the noise term has the form: $S(t^-)I(t^-)[\sigma dB(t) + \int_Z h(u)\tilde{N}(dt, du)]$. This means the method established previously can not be used directly to study model (1.3). That is also the main difficulty to be conquered in this paper. To the best of our knowledge, there are few literatures concerning the dynamics of the epidemic models when the infection force is perturbed by the jump process. The model is new, and the given method in this paper can be used to study a class of related models.

Before statement of the main results, we firstly make an assumption and prove one useful lemma.

Assumption A. $1 + h(u)N > 0$, $h(u) \leq K$ and $\int_Z [\log(1 + h(u)N)]^2 \pi(du) \leq K$ for some constant K .

Lemma 1.1. Let $g(\lambda x) = \log(1 + \lambda x) - \lambda x$ where λ, C are constants such that $0 < x \leq C$ and $\lambda C > -1$. Then

- (i) $0 \geq g(\lambda x) \geq g(\lambda C)$;
- (ii) $|g(\lambda x) - g(\lambda C)| \leq \frac{\lambda^2 C(C-x)}{1+\lambda C}$.

Proof. Compute the derivative $\frac{dg(\lambda x)}{dt} = \frac{\lambda}{1+\lambda x} - \lambda = \frac{-\lambda^2 x}{1+\lambda x}$, then

$$\frac{dg(\lambda x)}{dt} \leq 0 \text{ if } x \geq 0,$$

and

$$\frac{dg(\lambda x)}{dt} \geq 0 \text{ if } x < 0.$$

Clearly, $0 \geq g(\lambda x) \geq g(\lambda C)$ hold. By using Lagrange's Mean Value theorem, there is a $\xi \in [x, C]$ such that

$$|g(\lambda x) - g(\lambda C)| = \left| \frac{\lambda^2 \xi}{1 + \lambda \xi} \right| (C - x).$$

Notice that if $\lambda < 0$

$$\frac{\lambda^2 \xi}{1 + \lambda \xi} \leq \frac{\lambda^2 C}{1 + \lambda \xi} \leq \frac{\lambda^2 C}{1 + \lambda C}$$

and if $\lambda \geq 0$

$$\frac{\lambda^2 \xi}{1 + \lambda \xi} = \lambda \left(1 - \frac{1}{1 + \lambda \xi} \right) \leq \frac{\lambda^2 C}{1 + \lambda C},$$

then the desired result is proved. The proof is complete. \square

The left of this paper is organized as follows. Section 2 focuses on existence of the global positive and bounded solution. In Section 3, we present sufficient conditions for the extinction and persistence of the disease, where a stochastic threshold is identified. Section 4 shows existence of the sufficient and necessary condition for persistence in mean of the disease. Finally, Section 5 illustrates the obtained results with computer simulations, and then gives the conclusions in Section 6.

2. Globally positive and bounded solution

Since $S(t)$ and $I(t)$ are the numbers of the susceptible and the infective subpopulations, from the view of biological significance, it is of interest to study the existence of the positive solutions. The following theorem shows that the solution of (1.4) will not explode at a finite time and lies in a positive invariant set.

Theorem 2.1. *Assume that Assumption A holds. Then for any initial value $(S(0), I(0)) \in (0, N] \times (0, N]$, model (1.4) has a globally unique positive solution $(S(t), I(t)) \in (0, N] \times (0, N]$ on $[0, \infty)$ with probability 1.*

Proof. Since the coefficients are local Lipschitz continuous, then there exists a unique continuous positive solution of (1.4) for $t \in [0, \tau_e)$, where τ_e be the explosion time (see e.g. [12]). To show this solution is global, it need to prove $\tau_e = \infty$ a.s. Let $\bar{m} \in \mathbf{N}$ be sufficiently large such that $\frac{1}{\bar{m}} \leq S(0) \wedge I(0)$ and $S(0) + I(0) \leq N - \frac{1}{\bar{m}}$. Here and in the sequel $a \wedge b = \min\{a, b\}$. For any integer $m > \bar{m}$, define the stopping time $\tau_m = \inf\{t \in [0, \tau_e) : S(t) \wedge I(t) \leq \frac{1}{m} \text{ or } S(t) + I(t) \geq N - \frac{1}{m}\}$ where throughout this paper we set $\inf \emptyset = \infty$ (as usual \emptyset denotes the empty set). Clearly, τ_m is increasing as $m \rightarrow \infty$. Set $\tau_\infty = \lim_{m \rightarrow \infty} \tau_m$, then $\tau_\infty \leq \tau_e$ a.s. If $\tau_\infty = \infty$, then $\tau_e = \infty$ a.s. Suppose that $\tau_\infty = \infty$ a.s. does not hold, then there are constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $P(\tau_m \leq T) \geq \varepsilon$. For all $0 \leq t \leq \tau_m \wedge T$, we define a C^2 -function $V : R_+^2 \rightarrow R^+$ by

$$V(S, I) = S - 1 - \ln S + I - 1 - \ln I + \frac{1}{N - (S + I)}.$$

By using the generalized Itô formula to (1.4), we get that

$$\begin{aligned} & d(e^{-\mu t} V(S(t), I(t))) \\ &= [e^{-\mu t} \mathcal{L}(V(S(t), I(t))) - \mu e^{-\mu t} V(S(t), I(t))] dt - e^{-\mu t} \sigma(I(t) + S(t)) dB(t) \\ & \quad - e^{-\mu t} \int_{\mathcal{Z}} \log\{(1 + h(u)I(t^-))(1 + h(u)S(t^-))\} \tilde{N}(dt, du) \end{aligned}$$

where by (i) of Lemma 1.1 and $x - 1 - \ln x \geq 0$ for $x > 0$

$$\begin{aligned} & \mathcal{L}V(S(t), I(t)) - \mu V(S(t), I(t)) \\ &= (\Lambda - \mu S(t) - (\mu + \delta + \gamma)I(t)) - \left(\frac{\Lambda}{S(t)} - \mu - \beta I(t) - \frac{\sigma^2}{2}(I(t))^2\right) \\ & \quad - \int_{\mathcal{Z}} g(h(u)I(t)) \pi(du) - \left(\beta S(t) - (\mu + \delta + \gamma) - \frac{\sigma^2}{2}(S(t))^2\right) \\ & \quad - \int_{\mathcal{Z}} g(h(u)S(t)) \pi(du) + \frac{\mu(N - (S(t) + I(t))) - (\gamma + \delta)I(t)}{(N - (S(t) + I(t)))^2} \end{aligned}$$

$$\begin{aligned}
& -\mu V(S(t), I(t)) \\
& \leq \Lambda + \mu + \beta N + \sigma^2 N^2 + (\mu + \delta + \gamma) - 2 \int_Z g(h(u)N) \pi(du) := H.
\end{aligned}$$

By (i) of Lemma 1.1 and **Assumption A**,

$$\begin{aligned}
0 & \leq -2 \int_Z g(h(u)N) \pi(du) \\
& \leq \int_Z \left[(\log(h(u)N))^2 + (h(u)N)^2 \right] \pi(du) + 2\pi(Z) < \infty,
\end{aligned}$$

then $H < \infty$ and hence

$$\begin{aligned}
EV(S(\tau_m \wedge T), I(\tau_m \wedge T)) & \leq V(S(0), I(0)) e^{\mu T} + E \int_0^{\tau_m \wedge T} e^{\mu(T-s)} H ds \\
& \leq (V(S(0), I(0)) + TH) e^{\mu T}.
\end{aligned} \tag{2.1}$$

For all $\omega \in \{\tau_m \leq T\}$, $\min\{S(\tau_m), I(\tau_m)\} = \frac{1}{m}$ or $S(\tau_m) + I(\tau_m) = N - \frac{1}{m}$ holds, which leads to

$$\begin{aligned}
EV(S(\tau_m \wedge T), I(\tau_m \wedge T)) & \geq E [I_{\{\tau_m \leq T\}} V(S(\tau_m), I(\tau_m))] \\
& \geq \left[\left(\frac{1}{m} - 1 + \ln m \right) \wedge m \right] \varepsilon,
\end{aligned} \tag{2.2}$$

where $I_{\{\tau_m \leq T\}}$ is an indicator function. By letting $m \rightarrow \infty$, from (2.1) and (2.2) we get the contradiction

$$\infty > V(S(0), I(0)) e^{\mu T} + TH e^{\mu T} = \infty.$$

Therefore $\tau_e = \infty$ *a.s.* The proof is complete. \square

Remark 2.1. Obviously, $(N, 0)$ is a trivial solution of (1.4). In the following, we always set $S(0) > 0$, $I(0) > 0$ and $S(0) + I(0) < N$. From Theorem 2.1, a positive invariant set of (1.4) can be defined as

$$\Gamma = \{(S, I) : S > 0, I > 0, S + I < N \text{ a.s.}\}.$$

Then

$$\lim_{t \rightarrow \infty} \frac{1}{t} [S(t) + I(t) - S(0) - I(0)] = 0 \text{ a.s.}$$

Integrating (1.4) from 0 to t gives that

$$\mu \int_0^t S(s) ds + (\mu + \delta + \gamma) \int_0^t I(s) ds = \mu N t - [S(t) + I(t) - S(0) - I(0)]. \tag{2.3}$$

3. The threshold for extinction and persistence

When dynamics of the deterministic model (1.1) are studied, the basic reproduction number $R_0 = \frac{\beta N}{\mu + \delta + \gamma}$ is the key parameter. A stochastic basic reproduction number $R_0^S = R_0 - \frac{1}{\mu + \delta + \gamma} \frac{\sigma^2}{2} N^2$ has been obtained in [3–5, 8, 9, 14–16], which has similar properties as R_0 in case that the noise is small. To study extinction and persistence of (1.4), let us firstly define a new parameter

$$R_0^L = R_0 - \frac{1}{\mu + \delta + \gamma} \left[\frac{\sigma^2}{2} N^2 - \int_Z (\log(1 + h(u)N) - h(u)N) \pi(du) \right].$$

Lemma 3.1. (See e.g. [Lemma 3.1, [1]]). Let $M(t)$, $t \geq 0$ be a local martingale vanishing at time 0 and define

$$\rho_M(t) := \int_0^t \frac{d\langle M, M \rangle(s)}{(1+s)^2}, \quad t \geq 0$$

where $\langle M, M \rangle(t)$ is Meyers angle bracket process. Then

$$\lim_{t \rightarrow \infty} \frac{M(t)}{t} = 0 \text{ a.s. provided that } \lim_{t \rightarrow \infty} \rho_M(t) < \infty \text{ a.s.}$$

Theorem 3.1. Let $(S(t), I(t))$ be solution of (1.4) with $(S(0), I(0)) \in \Gamma$. Assume that **Assumption A** holds.

(i) If $R_0^L < 1$ and $\sigma^2 N + \int_Z \frac{h^2(u)N}{1+h(u)N} \pi(du) \leq \beta$ hold, then the disease I dies out exponentially with an exact decay rate, i.e.,

$$\limsup_{t \rightarrow \infty} \log \frac{I(t)}{I(0)} = (\mu + \gamma + \delta) (R_0^L - 1) < 0 \text{ a.s.}$$

(ii) If $R_0^L > 1$, then the disease will be persistent in mean, i.e.,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t I(s) ds \geq \frac{\mu}{\beta} (R_0^L - 1) \text{ a.s.}$$

Proof. By applying the generalized Itô formula to (1.4), we get

$$\log I(t) = \log I(0) + \int_0^t \mathcal{L} \log I(s) ds + M_1(t) + M_2(t), \quad (3.1)$$

where

$$\mathcal{L} \log I(t) = \beta S(t) - (\mu + \delta + \gamma) - \frac{\sigma^2}{2} (S(t))^2 + \int_Z g(h(u) S(t)) \pi(du), \quad (3.2)$$

$$M_1(t) = \int_0^t \sigma S(s) dB(s) \text{ and } M_2(t) = \int_0^t \int_Z \log(1 + h(u) S(s^-)) \tilde{N}(ds, du).$$

Since $M_1(t)$ and $M_2(t)$ are two martingales with the quadratic forms

$$\langle M_1, M_1 \rangle(t) = \int_0^t \sigma^2 S^2(s) ds \leq \sigma^2 N^2 t \text{ a.s.},$$

and

$$\langle M_2, M_2 \rangle(t) = \int_0^t \int_Z [\log(1 + h(u) S(s))]^2 \pi(du) ds \leq Kt \text{ a.s.}$$

Due to Lemma 3.1, we have

$$\lim_{t \rightarrow \infty} \frac{M_1(t)}{t} = 0 \text{ and } \lim_{t \rightarrow \infty} \frac{M_2(t)}{t} = 0 \text{ a.s.} \quad (3.3)$$

By the fact that $(S(t), I(t)) \in \Gamma$, applying (ii) of Lemma 1.1 to (3.2) yields

$$\begin{aligned} \mathcal{L} \log I(t) &\leq \beta N - (\mu + \delta + \gamma) - \frac{\sigma^2}{2} N^2 + \int_Z g(h(u) N) \pi(du) - \beta(N - S(t)) \\ &\quad + \frac{\sigma^2}{2} (N + S(t))(N - S(t)) + \int_Z \frac{h^2(u) N}{1 + h(u) N} \pi(du) (N - S(t)) \\ &\leq (\mu + \delta + \gamma) (R_0^L - 1) \\ &\quad + \left(\sigma^2 N + \int_Z \frac{h^2(u) N}{1 + h(u) N} \pi(du) - \beta \right) (N - S(t)). \end{aligned} \tag{3.4}$$

In view of the condition $\sigma^2 N + \int_Z \frac{h^2(u) N}{1 + h(u) N} \pi(du) \leq \beta$, it follows that

$$\mathcal{L} \log I(t) \leq (\mu + \delta + \gamma) (R_0^L - 1) \text{ a.s.}$$

This, together with (3.1) and (3.3), leads to

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \frac{I(t)}{I(0)} \leq (\mu + \delta + \gamma) (R_0^L - 1) < 0 \text{ a.s.} \tag{3.5}$$

On the other hand, from (3.2) we have

$$\mathcal{L} \log I(t) = (\mu + \delta + \gamma) (R_0^L - 1) + H(t) \tag{3.6}$$

where

$$H(t) = \left(\frac{\sigma^2}{2} (N + S(t)) - \beta \right) (N - S(t)) + \int_Z (g(h(u) S(t)) - g(h(u) N)) \pi(du)$$

satisfying

$$\begin{aligned} |H(t)| &\leq (\sigma^2 N + \beta) (N - S(t)) + \int_Z |g(h(u) S(t)) - g(h(u) N)| \pi(du) \\ &\leq \left(\sigma^2 N + \beta + \int_Z \frac{h^2(u) N}{1 + h(u) N} \pi(du) \right) (N - S(t)). \end{aligned}$$

By taking integration, let $H_0 = \left(\sigma^2 N + \beta + \int_Z \frac{h^2(u) N}{1 + h(u) N} \pi(du) \right) \frac{\mu + \delta + \gamma}{\mu}$, from (2.3) and (3.5) we compute that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |H(s)| ds \leq H_0 \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t I(s) ds = 0. \tag{3.7}$$

The proof on extinction of the disease is completed by substituting (3.6) and (3.7) into (3.1).

Next, we prove the persistence in mean of the disease. By (3.6) and (i) of Lemma 1.1, we have

$$\begin{aligned} \mathcal{L} \log I(t) &= (\mu + \delta + \gamma) (R_0^L - 1) + H(t) \\ &\geq (\mu + \delta + \gamma) (R_0^L - 1) - \beta(N - S(t)). \end{aligned} \tag{3.8}$$

By inserting (3.8) into (3.1) and then using (2.3) and (3.3), it is derived that

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t I(s) ds \geq \frac{\mu}{\beta} (R_0^L - 1) \text{ a.s.}$$

The proof is complete. □

Remark 3.1. In Theorem 3.1, under $\sigma^2 N + \int_Z \frac{h^2(u)N}{1+h(u)N} \pi(du) \leq \beta$, the disease goes extinct if $R_0^L < 1$. When $R_0^L > 1$, the disease will persist in mean. In case that the noise is small, R_0^L has similar properties as R_0 to the corresponding deterministic SIR epidemic model, thus we consider R_0^L as the stochastic basic reproduction number for (1.4). Clearly, R_0^L is less than R_0 and R_0^S , which means that the jump process has negative effects on prevailing of the epidemics.

Remark 3.2. Let $h(u) \equiv 0$, model (1.4) reduces to the stochastic SIR model studied in [5, 15], Theorem 3.1 remains consistent with the previously results. However, different from results in [5, 15], the exact decay rate for extinction is given in this paper. In this sense, the previously known results are improved and generalized.

As a consequence of Theorem 3.1, we directly have the following corollary.

Corollary 3.1. *If $R_0^L > 1$, then the disease modeled by (1.4) with initial value $(S(0), I(0)) \in \Gamma$ is weakly permanent with probability one, i.e.,*

$$\limsup_{t \rightarrow \infty} I(t) > 0 \text{ a.s.}$$

4. Non-persistence in mean under $R_0^L = 1$

In this section, we discuss the condition for non-persistence in mean of the disease. To begin with, we firstly prove a lemma.

Lemma 4.1. *Let λ be a positive constant and $F(t)$ be a function such that $\lim_{t \rightarrow \infty} \frac{F(t)}{t} = 0$. If there is a positive function $f(t)$ satisfying*

$$\log f(t) \leq -\lambda \int_0^t f(s) ds + F(t),$$

then $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(s) ds = 0$.

Proof. The condition $\lim_{t \rightarrow \infty} \frac{F(t)}{t} = 0$ implies that for any $\varepsilon > 0$, there is a constant T , for all $t > T$

$$|F(t)| < \varepsilon t. \tag{4.1}$$

Denote $\eta(t) = \int_0^t f(s) ds$, then

$$\log \frac{d\eta(t)}{dt} \leq -\lambda \eta(t) + F(t).$$

By using (4.1), integrating both sides of the above equation yields that

$$e^{\lambda \eta(t)} \leq 1 + \lambda \int_0^t e^{F(s)} ds \leq 1 + \lambda \int_0^T e^{F(s)} ds + \lambda \int_T^t e^{\varepsilon s} ds,$$

and hence

$$\eta(t) \leq \frac{1}{\lambda} \log \left(1 + \lambda \int_0^T e^{F(s)} ds + \frac{\lambda}{\varepsilon} e^{\varepsilon(t-T)} \right).$$

Then

$$\limsup_{t \rightarrow \infty} \frac{\eta(t)}{t} \leq \limsup_{t \rightarrow \infty} \frac{\log \left(\frac{2\lambda}{\varepsilon} e^{\varepsilon(t-T)} \right)}{\lambda t} = \varepsilon.$$

Note the fact that $f(t)$ is positive, let $\varepsilon \rightarrow 0$ we get

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(s) ds = 0.$$

The proof is complete. \square

Theorem 4.1. *Assume that Assumption A holds. If the conditions $R_0^L = 1$ and $\sigma^2 N + \int_Z \frac{h^2(u)N}{1+h(u)N} \pi(du) < \beta$ hold, then the disease I is non-persistent in mean, that is, $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t I(s) ds = 0$ a.s.*

Proof. Denote $\lambda_0 = \beta - \sigma^2 N - \int_Z \frac{h^2(u)N}{1+h(u)N} \pi(du)$, then $\lambda_0 > 0$ and

$$\mathcal{L} \log I(t) \leq -\lambda_0 (N - S(t)). \quad (4.2)$$

Inserting (4.2) into (3.1), by (2.3) we can show that

$$\log I(t) \leq -\frac{\lambda_0 (\mu + \delta + \gamma)}{\mu} \int_0^t I(s) ds + F(t),$$

where $F(t) = S(t) + I(t) - S(0) - I(0) + \log I(0) + M_1(t) + M_2(t)$ satisfying $\lim_{t \rightarrow \infty} \frac{F(t)}{t} = 0$ a.s. Then by applying Lemma 4.1, we complete the proof. \square

Remark 4.1. Let $\sigma^2 N + \int_Z \frac{h^2(u)N}{1+h(u)N} \pi(du) < \beta$ hold, by combining Theorem 3.1 and Theorem 4.1, $R_0^L > 1$ is a sufficient and necessary condition for persistence in mean of the disease.

5. Computer simulations

In this paper, we use a Lévy jump process on the infection force to model the abrupt environmental perturbations and consider a stochastic SIR model. In order to illustrate the theoretical results on the effects of Lévy jumps, we numerically simulate the solutions of stochastic model (1.4) by the famous Milstein method with two cases where the disease does not die out under Brown perturbations.

Example 5.1. In model (1.4), set $N = 1$, $\beta = 0.3$, $\mu = 0.05$, $\gamma = 0.1$, $\delta = 0.05$, and $\pi(Z) = 1$. The initial value is $(S(0), I(0)) = (0.5, 0.1)$. To show the effect of the noise on dynamic of the system, we consider the following two cases: (B1) $\sigma = 0.1$, $h(u) \equiv 0.2$ and (B2) $\sigma = 0.1$, $h(u) \equiv 0.6$.

Compute that $R_0 = 1.5 > 1$, then the disease modeled by the corresponding deterministic model will persist, and the system converges to its positive equilibrium. In case of (B1), we have $R_0^S = 1.475$ and $R_0^L = 1.386$, which means that

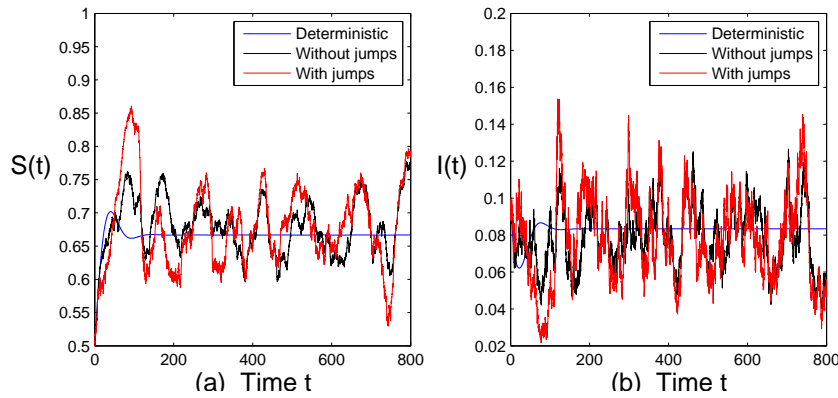


Figure 1. Paths of model (1.4) in case of (B1) with step size $\Delta t = 0.001$. The blue lines are the samples of the corresponding deterministic model, the black lines show the paths of model (1.4) without jumps and the red lines represent the paths of model (1.4) with jumps.

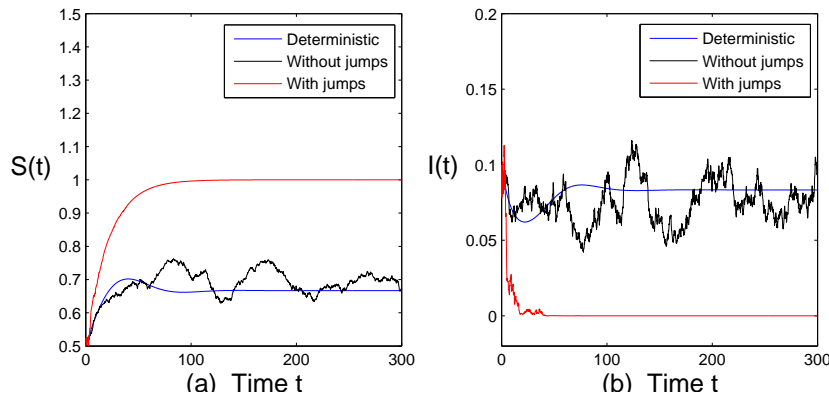


Figure 2. Paths of model (1.4) in case of (B2) with step size $\Delta t = 0.001$.

both noises are so small that the disease still persist due to (II) of Theorem 3.1. Fig 1 confirms this. When we let the jumps size to be large as in case (B2), $\sigma^2 N + \int_{\mathcal{Z}} \frac{h^2(u)N}{1+h(u)N} \pi(du) = 0.23 \leq 0.3 = \beta$ and $R_0^L = 0.825 < 1$ hold. Then by (I) of Theorem 3.1, the disease will die out. See Fig 2. By comparing the given two cases, it is easy to see that jumps have negative effects on prevailing of the epidemics.

6. Conclusions

This paper formulates a stochastic SIR epidemic model with the infection force driven by Lévy jumps. Existence of the globally positive and bounded solution is proved by constructing the suitable Liapovon function. Then under assumption that the noises are small, a modified basic reproduction number R_0^L for (1.4) is defined, which has similar properties as R_0 to the underlying deterministic SIR epidemic model. In details,

- $R_0^L < 1$ and $\sigma^2 N + \int_Z \frac{h^2(u)N}{1+h(u)N} \pi(du) \leq \beta$, then $\lim_{t \rightarrow \infty} I(t) = 0$ a.s.
- $R_0^L = 1$ and $\sigma^2 N + \int_Z \frac{h^2(u)N}{1+h(u)N} \pi(du) < \beta$, then $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t I(s) ds = 0$ a.s.
- $R_0^L > 1$, then $\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t I(s) ds \geq \frac{\mu}{\beta} (R_0^L - 1) > 0$ a.s.

Apart from introduction of Lévy jumps, an exact decay rate for extinction is derived which improves the known results. Let Brown noise be small such that $R_0 - \frac{1}{\mu+\delta+\gamma} \frac{\sigma^2 N^2}{2} > 1$, by [15], the disease will persist in mean. However, when we choose the jump size to satisfy $R_0^L < 1$ and $\sigma^2 N + \int_Z \frac{h^2(u)N}{1+h(u)N} \pi(du) \leq \beta$, the disease will die out exponentially. This means that jumps can change dynamics of the system significantly.

Remark 6.1. From the above discussions, an interesting and important open problem arises. That is, whether the disease will die out if one of the following conditions is satisfied

$$(A1) \quad R_0^L < 1 \text{ and } \sigma^2 N + \int_Z \frac{h^2(u)N}{1+h(u)N} \pi(du) \leq \beta;$$

$$(A2) \quad R_0^L = 1 \text{ and } \sigma^2 N + \int_Z \frac{h^2(u)N}{1+h(u)N} \pi(du) \geq \beta.$$

Finally, we mention that, basic on models in [3, 8, 9, 14, 16–19], more generalized epidemic models with jumps will be formulated by introducing jumps into the infection force. Then the analogue of the basic reproduction number for thus established model can be obtained respectively by using the method given in this paper.

Acknowledgements

The authors would like to the anonymous reviewers for their pertinent comments and suggestions, which greatly improves the earlier manuscript.

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