Journal of Applied Analysis and Computation
Volume 8, Number 3, June 2018, 982–997
Website:http://jaac-online.com/
DOI:10.11948/2018.982

HOPF BIFURCATION ANALYSIS FOR A DELAYED PREDATOR-PREY SYSTEM WITH A PREY REFUGE AND SELECTIVE HARVESTING*

Miao Peng, Zhengdi Zhang†, Xuedi Wang and Xiuyu Liu

Abstract In this paper, a delayed predator-prey system with Holling type III functional response incorporating a prey refuge and selective harvesting is considered. By analyzing the corresponding characteristic equations, the conditions for the local stability and existence of Hopf bifurcation for the system are obtained, respectively. By utilizing normal form method and center manifold theorem, the explicit formulas which determine the direction of Hopf bifurcation and the stability of bifurcating period solutions are derived. Finally, numerical simulations supporting the theoretical analysis are given.

Keywords Predator-prey system, prey refuge, selective harvesting, local stability, Hopf bifurcation.

MSC(2010) 34C23, 34D20, 34K18, 34K20.

1. Introduction

In recent years, predator-prey system as the fundamental structure in population dynamics that has attracted much attention by researchers [1,3,10,12–14,19,20]. In these works, there are many factors which affect dynamical properties of predator-prey system such as time delay, functional response, stage structure, harvesting and prey refuge, etc., especially the joint effect of above factors. In [10], Li et al. considered a delayed predator-prey model with Holling type III functional response and stage structure for the prey:

$$\begin{aligned} \dot{x}_1(t) &= ax_2(t) - r_1x_1(t) - bx_1(t), \\ \dot{x}_2(t) &= bx_1(t) - r_2x_2(t) - b_1x_2^2(t) - \frac{a_1x_2^2(t)y(t)}{1+mx_2^2(t)}, \\ \dot{y}(t) &= \frac{a_2x_2^2(t-\tau)y(t-\tau)}{1+mx_2^2(t-\tau)} - ry(t), \end{aligned} \tag{1.1}$$

where $x_1(t)$, $x_2(t)$ represent the densities of the immature prey and the mature prey at time t, respectively; $y(t)$ represents the density of the predator at time t. All the parameters in system (1.1) are assumed positive. a is the birth rate of the

†the corresponding author. Email address: dyzhang@ujs.edu.cn(Z. Zhang)
Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu, 212013, China
*The authors were supported by National Natural Science Foundation of China (No. 11472116, No. 11472115), Postgraduate Research & Practice Innovation Program of Jiangsu Province (KYCX17_1784) and Qinglan project of Jiangsu.

immature prey, b denotes the rate of immature prey becoming mature prey, b_1 is the intraspecific competition coefficient of the mature prey, r_1 and r_2 are the death rates of the immature and the mature prey, respectively. r is the death rate of the predator, a_1 is the capturing rate, $\frac{a_2}{a_1}$ is the conversion rate of the predator, m is the half capturing saturation constant, and τ denotes a time delay due to the gestation of the predator. In [10], Li et al. was concerned with the combined effect of stage structure for the prey, Holling type III functional response and time delay in a predator-prey model.

Recently, the qualitative analysis of predator-prey systems incorporating a prey refuge has been developed by many scholars [5, 8, 11, 16, 18, 21]. It is well known that the refuge applied by prey has a stabilizing effect on the considered interactions and prey extinction can be prevented by the addition refuges. In [16], Sharma and Samanta studied a predator-prey Leslie-Gower model with disease in prey incorporating a prey refuge, they analyzed the existence of various equilibrium points and stability of the system at those equilibrium points.

In addition, the harvesting has an important impact on the dynamics of the systems [2, 4, 9, 22, 23]. In a harvesting system, the aim is to determine how much we can harvest without altering dangerously the harvested population. Gupta et al. [4] and Yuan et al. [22] described predator-prey models with harvesting for prey and discussed the related systems' dynamical behaviors.

In this paper, based on the above discussions and motivated by the work of Li et al. [10], we consider the following predator-prey system with intraspecific competition of the immature prey, a prey refuge, selective harvesting and two delays:

$$\begin{aligned}
\dot{x}_1(t) &= ax_2(t) - r_1x_1(t) - bx_1(t) - cx_1^2(t),\\
\dot{x}_2(t) &= bx_1(t) - r_2x_2(t) - b_1x_2(t)x_2(t-\tau_1) - \frac{a_1(1-m)^2x_2^2(t)y(t)}{1+r(1-m)^2x_2^2(t)},\\
\dot{y}(t) &= \frac{a_2(1-m)^2x_2^2(t-\tau_2)y(t-\tau_2)}{1+r(1-m)^2x_2^2(t-\tau_2)} - r_3y(t) - qEy(t-\tau_2),
\end{aligned}\tag{1.2}$$

where $x_1(t)$, $x_2(t)$ represent the densities of the immature prey and the mature prey at time t, $y(t)$ represents the density of the predator at time t, respectively. The parameters a, b, a_1, a_2, b_1, r_1 and r_2 are defined as in system (1.1). c is the intraspecific competition of the immature prey, r is the half capturing saturation constant, m is a constant number of prey using refuges, which protects m of prey from predation. r_3 is the death rate of the predator, q is the catch-ability coefficient of the predator species, E is the harvesting effort, τ_1 is the feedback delay of the mature prey, and τ_2 is a constant representing the assumption that the harvesting begins to occur after a certain age or size.

The initial conditions for system (1.2) take the form of

$$x_1(0) > 0, x_2(0) > 0, y(0) > 0. \tag{1.3}$$

According to the fundamental theory of functional differential equations [6], the system (1.2) has a unique solution $(x_1(t), x_2(t), y(t))$ satisfying initial conditions (1.3). It is easy to show that all solutions of system (1.2) with initial conditions (1.3) are defined on $[0, +\infty)$ and remain positive for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$.

The organization of this paper is as follows. In section 2, the local stability of the interior equilibrium and the existence of Hopf bifurcation for system (1.2) are discussed. The formulas for determining the direction of Hopf bifurcation and the stability of bifurcating period solutions are derived in section 3. In section 4, numerical simulations are carried out to illustrate the validity of the main results. Finally, a brief conclusion is given.

2. Local stability and Hopf bifurcation

It is obvious that system (1.2) has a unique interior equilibrium $E^*(x_1^*, x_2^*, y^*)$, where

$$\begin{aligned} x_1^* &= \frac{-(r_1+b)+\sqrt{(r_1+b)^2+4acx_2^*}}{2c}, \\ x_2^* &= \sqrt{\frac{r_3+qE}{(1-m)^2(a_2-rr_3-rqE)}}, \\ y^* &= \frac{(bx_1^*-r_2x_2^*-b_1(x_2^*)^2)(1+r(1-m)^2(x_2^*)^2)}{a_1(1-m)^2(x_2^*)^2}, \end{aligned}$$

if the following conditions satisfied:

$$(H1): a_2-rr_3-rqE>0, bx_1^*>r_2x_2^*+b_1(x_2^*)^2.$$

In this section, we only investigate the local stability of linearized system at the interior equilibrium and the existence of Hopf bifurcations for the system (1.2). Since the biological meaning of the interior equilibrium imply that immature prey, mature prey and predator all exist.

Let $\bar{x}_1(t) = x_1(t) - x_1^*$, $\bar{x}_2(t) = x_2(t) - x_2^*$, $\bar{y}(t) = y(t) - y^*$and still denote $\bar{x}_1(t)$, $\bar{x}_2(t)$, $\bar{y}(t)$, respectively. Using Taylor expansion to expand the system (1.2) at the interior equilibrium $E^*(x_1^*, x_2^*, y^*)$, we have

$$\begin{aligned} \dot{x}_1(t) =& a_{11}x_1(t)+a_{12}x_2(t)+\sum_{i+j\geq 2} f_1^{(ij)}x_1^i(t)x_2^j(t), \\ \dot{x}_2(t) =& a_{21}x_1(t)+a_{22}x_2(t)+a_{23}y(t)+b_{22}x_2(t-\tau_1) \\ &+\sum_{i+j+k+l\geq 2} f_2^{(ijkl)}x_1^i(t)x_2^j(t)y^k(t)x_2^l(t-\tau_1), \\ \dot{y}(t) =& a_{33}y(t)+b_{32}x_2(t-\tau_2)+b_{33}y(t-\tau_2) \\ &+\sum_{i+j+k\geq 2} f_3^{(ijk)}y^i(t)x_2^j(t-\tau_2)y^k(t-\tau_2), \end{aligned} \tag{2.1}$$

where

$$a_{11}=-r_1-b-2cx_1^*, a_{12}=a, a_{21}=b, a_{22}=-r_2-b_1x_2^*-\frac{2a_1(1-m)^2x_2^*y^*}{(1+r(1-m)^2(x_2^*)^2)^2},$$
$$a_{23}=-\frac{a_1(1-m)^2(x_2^*)^2}{1+r(1-m)^2(x_2^*)^2}, a_{33}=-r_3, b_{22}=-b_1x_2^*, b_{32}=\frac{2a_2(1-m)^2x_2^*y^*}{(1+r(1-m)^2(x_2^*)^2)^2},$$
$$b_{33}=r_3,$$
$$f_1^{(ij)}=\frac{1}{i!j!}\frac{\partial^{i+j}f_1}{\partial x_1^i(t)\partial x_2^j(t)}\,|(x_1^*,x_2^*,y^*)\ ,$$

$$f_2^{(ijkl)} = \frac{1}{i!j!k!l!} \frac{\partial^{i+j+k+l} f_2}{\partial x_1^i(t) \partial x_2^j(t) \partial y^k(t) x_2^l(t-\tau_1)} |(x_1^*, x_2^*, y^*) ,$$
$$f_3^{(ijk)} = \frac{1}{i!j!k!} \frac{\partial^{i+j+k} f_3}{\partial y^i(t) \partial x_2^j(t-\tau_2) y^k(t-\tau_2)} |(x_1^*, x_2^*, y^*) ,$$
$$f_1 = ax_2(t) - r_1x_1(t) - bx_1(t) - cx_1^2(t),$$
$$f_2 = bx_1(t) - r_2x_2(t) - b_1x_2(t)x_2(t-\tau_1) - \frac{a_1(1-m)^2x_2^2(t)y(t)}{1+r(1-m)^2x_2^2(t)},$$
$$f_3 = \frac{a_2(1-m)^2x_2^2(t-\tau_2)y(t-\tau_2)}{1+r(1-m)^2x_2^2(t-\tau_2)} - r_3y(t) - qEy(t-\tau_2).$$

Then we can get the linearized system of system (2.1) as following:

$$\begin{aligned} \dot{x}_1(t) &= a_{11}x_1(t) + a_{12}x_2(t), \\ \dot{x}_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + a_{23}y(t) + b_{22}x_2(t-\tau_1), \\ \dot{y}(t) &= a_{33}y(t) + b_{32}x_2(t-\tau_2) + b_{33}y(t-\tau_2). \end{aligned} \tag{2.2}$$

Therefore, the corresponding characteristic equation of system (2.2) is given by

$$\begin{aligned} &\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + (n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau_1} \\ &+ (p_2\lambda^2 + p_1\lambda + p_0)e^{-\lambda\tau_2} + (q_1\lambda + q_0)e^{-\lambda(\tau_1+\tau_2)} = 0, \end{aligned} \tag{2.3}$$

where

$$m_0 = a_{33}(a_{12}a_{21} - a_{11}a_{22}), m_1 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21},$$
$$m_2 = -(a_{11} + a_{22} + a_{33}), n_0 = -a_{11}a_{33}b_{22}, n_1 = (a_{11} + a_{33})b_{22}, n_2 = -b_{22},$$
$$p_0 = -a_{11}a_{22}b_{33} + a_{11}a_{23}b_{32} + a_{12}a_{21}b_{33}, p_1 = a_{22}b_{33} + a_{11}b_{33} - a_{23}b_{32}, p_2 = -b_{33},$$
$$q_0 = -a_{11}b_{22}b_{33}, q_1 = b_{22}b_{33}.$$

In order to investigate the distribution of roots of the transcendental equation (2.3), we use the corollary 2.4 of Ruan and Wei [15]. Due to system (1.2) has two time delays, that is, τ_1 and τ_2, so we consider the following different cases.

Case 1: $\tau_1 = \tau_2 = 0$, the characteristic equation (2.3) reduces to

$$\lambda^3 + m_{12}\lambda^2 + m_{11}\lambda + m_{10} = 0, \tag{2.4}$$

where $m_{10} = m_0 + n_0 + p_0 + q_0, m_{11} = m_1 + n_1 + p_1 + q_1, m_{12} = m_2 + n_2 + p_2$.

It is not difficult to verify that $m_{10} > 0, m_{12} > 0$, thus, all the roots of Eq.(2.4) have negative real parts if the following condition holds:

$$(H11) : m_{11}m_{12} > m_{10}.$$

Namely, the equilibrium $E^*(x_1^*, x_2^*, y^*)$ is locally asymptotically stable when the condition (H11) satisfies.

Case 2: $\tau_1 > 0$, $\tau_2 = 0$. On substituting $\tau_2 = 0$, Eq. (2.3) reduces to

$$\lambda^3 + m_{22}\lambda^2 + m_{21}\lambda + m_{20} + (n_{22}\lambda^2 + n_{21}\lambda + n_{20})e^{-\lambda\tau_1} = 0, \tag{2.5}$$

where $m_{20} = m_0 + p_0, m_{21} = m_1 + p_1, m_{22} = m_2 + p_2, n_{20} = n_0 + q_0, n_{21} = n_1, n_{22} = n_2$.

For $\omega_1 > 0$, suppose $i\omega_1$ being a root of Eq.(2.5), and separate real and imaginary parts, we get

$$n_{21}\omega_1 \sin\omega_1\tau_1 + (n_{20} - n_{22}\omega_1^2)\cos\omega_1\tau_1 = m_{22}\omega_1^2 - m_{20},$$
$$n_{21}\omega_1 \cos\omega_1\tau_1 - (n_{20} - n_{22}\omega_1^2)\sin\omega_1\tau_1 = \omega_1^3 - m_{21}\omega_1. \tag{2.6}$$

Which leads to

$$\omega_1^6 + e_{22}\omega_1^4 + e_{21}\omega_1^2 + e_{20} = 0, \tag{2.7}$$

where $e_{20} = m_{20}^2 - n_{20}^2, e_{21} = m_{21}^2 - n_{21}^2 - 2m_{20}m_{22} + 2n_{20}n_{22}, e_{22} = m_{22}^2 - n_{22}^2 - 2m_{21}$.

Let $\omega_1^2 = v_1$, then Eq.(2.7) becomes

$$v_1^3 + e_{22}v_1^2 + e_{21}v_1 + e_{20} = 0. \tag{2.8}$$

Denote

$$f_1(v_1) = v_1^3 + e_{22}v_1^2 + e_{21}v_1 + e_{20}. \tag{2.9}$$

Since $f_1(0) = e_{20}$, $\lim\limits_{v_1\to+\infty} f_1(v_1) = +\infty$, and from Eq.(2.9), we have

$$f_1'(v_1) = 3v_1^2 + 2e_{22}v_1 + e_{21}. \tag{2.10}$$

After discussion about the roots of Eq.(2.10) is similar to that in [17], we have the following lemma.

Lemma 2.1. *For the polynomial equation (2.8), we have the following results:*

(i) If (H21) $e_{20} \geq 0, \Delta = e_{22}^2 - 3e_{21} \leq 0$ holds, then Eq.(2.8) has no positive root;

(ii) If (H22) $e_{20} \geq 0, \Delta = e_{22}^2 - 3e_{21} > 0, v_1^* = \frac{-e_{21}+\sqrt{\Delta}}{3} > 0$ and $f_1(v_1^*) \leq 0$ or (H23) $e_{20} < 0$ holds, then Eq.(2.8) has positive root.

Suppose that Eq.(2.8) has positive roots. Without loss of generality, we assume that it has three positive roots, which are denoted by v_{11}, $v_{12}$and v_{13}. Then Eq.(2.7) has three positive roots $\omega_{1k} = \sqrt{v_{1k}}, k = 1, 2, 3$. The corresponding critical value of time delay $\tau_{1k}^{(j)}$ is

$$\tau_{1k}^{(j)} = \frac{1}{\omega_{1k}} \arccos\left\{\frac{A_{24}\omega_{1k}^4 + A_{22}\omega_{1k}^2 + A_{20}}{B_{24}\omega_{1k}^4 + B_{22}\omega_{1k}^2 + B_{20}}\right\} + \frac{2\pi j}{\omega_{1k}}, \tag{2.11}$$

where $A_{20} = -m_{20}n_{20}, A_{22} = n_{20}m_{22} + n_{20}m_{20} - n_{21}m_{21}, A_{24} = n_{21} - m_{22}n_{22}, B_{20} = n_{20}^2$, $B_{22} = n_{21}^2 - 2n_{20}n_{22}, B_{24} = n_{22}^2$.

Thus $\pm\omega_{1k}$ is a pair of purely imaginary roots of Eq.(2.5) with $\tau_1 = \tau_{1k}^{(j)}$, and let $\tau_{10} = \min\{\tau_{1k}^{(0)}\}, (k = 1, 2, 3\cdots)$, $\omega_{10} = \omega_{1k_0}$.

According to the Hopf Bifurcation Theorem [7,24], we need to verify the transversality condition. Differentiating Eq.(2.5) with respect to τ_1, and noticing that λ is a function of τ_1, we obtain

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1} = -\frac{3\lambda^2 + 2m_{22}\lambda + m_{21}}{\lambda(\lambda^3 + m_{22}\lambda^2 + m_{21}\lambda + m_{20})} + \frac{2n_{22}\lambda + n_{21}}{\lambda(n_{22}\lambda^2 + n_{21}\lambda + n_{20})} - \frac{\tau_1}{\lambda}. \tag{2.12}$$

Which leads to

$$
\begin{aligned}
Re(\frac{d\lambda}{d\tau_1})^{-1} =&Re(-\frac{3\lambda^2+2m_{22}\lambda+m_{21}}{\lambda(\lambda^3+m_{22}\lambda^2+m_{21}\lambda+m_{20})})_{\lambda=i\omega_{10}}+\\
&Re(\frac{2n_{22}\lambda+n_{21}}{\lambda(n_{22}\lambda^2+n_{21}\lambda+n_{20})})_{\lambda=i\omega_{10}}\\
=&\frac{3\omega_{10}^4+2(m_{22}^2-n_{22}^2-2m_{21})\omega_{10}^2+m_{21}^2-2m_{20}m_{22}}{(\omega_{10}^3-m_{21}\omega_{10})^2+(m_{20}-m_{22}\omega_{10}^2)^2}-\\
&\frac{2n_{22}^2\omega_{10}^2+n_{21}^2-2n_{20}n_{22}}{(n_{22}\omega_{10}^2-n_{20})^2+n_{21}^2\omega_{10}^2}.
\end{aligned}
$$

From Eq.(2.6), we have

$$
(\omega_{10}^3-m_{21}\omega_{10})^2+(m_{20}-m_{22}\omega_{10}^2)^2=(n_{22}\omega_{10}^2-n_{20})^2+n_{21}^2\omega_{10}^2. \tag{2.13}
$$

Noting that $\left\{\frac{d(Re\lambda)}{d\tau_1}\right\}_{\lambda=i\omega_{10}}$ and $\left\{Re(\frac{d\lambda}{d\tau_1})^{-1}\right\}_{\lambda=i\omega_{10}}$ have the same sign, then

$$
\begin{aligned}
sign\left\{\frac{d(Re\lambda)}{d\tau_1}\right\}_{\lambda=i\omega_{10}}=sign\left\{Re(\frac{d\lambda}{d\tau_1})^{-1}\right\}_{\lambda=i\omega_{10}}=&\frac{3(\omega_{10}^2)^2+2e_{22}\omega_{10}^2+e_{21}}{n_{21}^2\omega_{10}^2+(n_{22}\omega_{10}^2-n_{20})^2}\\
=&\frac{f_1'(\omega_{10}^2)}{n_{21}^2\omega_{10}^2+(n_{22}\omega_{10}^2-n_{20})^2}.
\end{aligned} \tag{2.14}
$$

Therefore, $\left\{\frac{d(Re\lambda)}{d\tau_1}\right\}_{\lambda=i\omega_{10}}\neq 0$ if the following condition holds:

$$
(H24): f_1'(\omega_{10}^2)\neq 0.
$$

According to the above analysis, we have the following results.

Theorem 2.1. *For system (1.2), τ_2=0.*
(i) If (H21) holds, then the interior equilibrium $E^*(x_1^*,x_2^*,y^*)$ is asymptotically stable for all $\tau_1\geq 0$.
(ii) If (H22) or (H23) and (H24) holds, then the interior equilibrium $E^*(x_1^*,x_2^*,y^*)$ is asymptotically stable for all $\tau_1\in[0,\tau_{10})$ and unstable for $\tau_1>\tau_{10}$. Furthermore, the system (1.2) undergoes a Hopf bifurcation at the interior equilibrium $E^*(x_1^*,x_2^*,y^*)$ when $\tau_1=\tau_{10}$.

Case 3: $\tau_1=0$, $\tau_2>0$. The calculation is very similar to case 2, we obtain the following results.

Theorem 2.2. *For system (1.2), τ_1=0. The interior equilibrium $E^*(x_1^*,x_2^*,y^*)$ is asymptotically stable for all $\tau_2\in[0,\tau_{20})$ and unstable for $\tau_2>\tau_{20}$. Furthermore, the system (1.2) undergoes a Hopf bifurcation at the interior equilibrium $E^*(x_1^*,x_2^*,y^*)$ when $\tau_2=\tau_{20}$, where τ_{20} represents the minimum critical value of time delay τ_2 for the occurrence of Hopf bifurcation when $\tau_1=0$.*

Case 4: $\tau_1=\tau_2=\tau\neq 0$.

Theorem 2.3. *For system (1.2), $\tau_1 = \tau_2 = \tau \neq 0$. The interior equilibrium $E^*(x_1^*, x_2^*, y^*)$ is asymptotically stable for all $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$. Furthermore, the system (1.2) undergoes a Hopf bifurcation at the positive equilibrium $E^*(x_1^*, x_2^*, y^*)$ when $\tau = \tau_0$, where τ_0 represents the minimum critical value of time delay τ for the occurrence of Hopf bifurcation.*

Case 5: $\tau_1 > 0, \tau_2 \in [0, \tau_{20})$ and $\tau_1 \neq \tau_2$.

We consider Eq.(2.3) with τ_2 in its stable interval, and τ_1 is regarded as the parameter. Let $i\omega_{1_*} (\omega_{1_*} > 0)$ be the root of Eq.(2.3), then we obtain

$$\begin{aligned} &E_{51} \sin \omega_{1_*} \tau_1 + E_{52} \cos \omega_{1_*} \tau_1 = E_{53}, \\ &E_{51} \cos \omega_{1_*} \tau_1 - E_{52} \sin \omega_{1_*} \tau_1 = E_{54}, \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} E_{51} &= n_1 \omega_{1_*} - q_0 \sin \omega_{1_*} \tau_2 + q_1 \omega_{1_*} \cos \omega_{1_*} \tau_2, \\ E_{52} &= n_0 - n_2 \omega_{1_*}^2 + q_0 \cos \omega_{1_*} \tau_2 + q_1 \omega_{1_*} \sin \omega_{1_*} \tau_2, \\ E_{53} &= m_2 \omega_{1_*}^2 - m_0 - p_1 \omega_{1_*} \sin \omega_{1_*} \tau_2 + (p_2 \omega_{1_*}^2 - p_0) \cos \omega_{1_*} \tau_2, \\ E_{54} &= \omega_{1_*}^3 - m_1 \omega_{1_*} - p_1 \omega_{1_*} \cos \omega_{1_*} \tau_2 - (p_2 \omega_{1_*}^2 - p_0) \sin \omega_{1_*} \tau_2. \end{aligned}$$

From Eq.(2.15), we have

$$\begin{aligned} &\omega_{1_*}^6 + e_{52} \omega_{1_*}^4 + e_{51} \omega_{1_*}^2 + e_{50} + (c_{54} \omega_{1_*}^4 + c_{52} \omega_{1_*}^2 + c_{50}) \cos \omega_{1_*} \tau_2 \\ &+ (c_{55} \omega_{1*}^5 + c_{53} \omega_{1_*}^3 + c_{51} \omega_{1_*}) \sin \omega_{1_*} \tau_2 = 0, \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} &e_{50} = m_0^2 + p_0^2 - n_0^2 - q_0^2, e_{51} = m_1^2 + p_1^2 - n_1^2 - q_1^2 + 2n_0 n_2 - 2m_0 m_2 - 2p_0 p_2, \\ &e_{52} = m_2^2 + p_2^2 - n_2^2 - 2m_1, c_{50} = 2m_0 p_0 - 2n_0 q_0, \\ &c_{51} = 2p_1 m_0 - 2p_0 m_1 + 2n_1 q_0 - 2n_0 q_1, \\ &c_{52} = 2p_1 m_1 - 2p_0 m_2 + 2n_2 q_0 - 2m_0 p_2 - 2n_1 q_1, \\ &c_{53} = 2m_1 p_2 + 2n_2 q_1 + 2p_0 - 2p_1 m_2, c_{54} = 2m_2 p_2 - 2p_1, c_{55} = -2p_2. \end{aligned}$$

In order to give the main results, we give the following assumption.

(H51): Equation (2.16) has at least finite positive root.

Suppose that (H51) holds, we denote the positive roots of Eq.(2.16) as $\omega_{1_*}^{(1)}$, $\omega_{1_*}^{(2)}$, $\omega_{1_*}^{(3)}$, $\omega_{1_*}^{(4)}$, $\omega_{1_*}^{(5)}$ and $\omega_{1_*}^{(6)}$. For every $\omega_{1_*}^{(i)} (i = 1, 2, 3, 4, 5, 6)$, the corresponding critical value of time delay $\tau_{1i}^{(j)}$, $j = 1, 2, 3 \dots$ is

$$\tau_{1i}^{(j)} = \frac{1}{\omega_{1_*}} \arccos \left\{ \frac{E_{51} E_{54} + E_{52} E_{53}}{E_{51}^2 + E_{52}^2} + 2\pi j \right\}_{\omega_{1_*} = \omega_{1_*}^i}, i = 1, 2, 3, 4, 5, 6; j = 0.1, 2 \dots \tag{2.17}$$

Let $\tau'_{10} = \min \left\{ \tau_{1i}^{(0)} \, | i = 1, 2, \dots 6; j = 0, 1, 2 \dots \right\}$, ω'_{10} is the corresponding root of Eq.(2.16) with τ'_{10}.

In the following, we differentiate the two sides of Eq.(2.3) with respect to τ_1 to verify the transversality condition.

Taking the derivative of λ with respect to τ_1 in Eq.(2.3) and substituting $\lambda = i\omega'_{10}$, we get

$$Re(\frac{d\lambda}{d\tau_1})^{-1}_{\lambda=i\omega'_{10}} = Re(\frac{A'+B'i}{C'+D'i}) = \frac{A'C'+B'D'}{C'^2+D'^2}, \tag{2.18}$$

where

$$\begin{aligned}
A' =& m_1 - 3(\omega'_{10})^2 + 2n_2\omega'_{10}\sin\omega'_{10}\tau'_{10} + n_1\cos\omega'_{10}\tau'_{10} \\
&+ \sin\omega'_{10}\tau_2(2p_2\omega'_{10} - p_1\omega'_{10}\tau_2 - q_1\sin\omega'_{10}\tau'_{10}) \\
&+ \cos\omega'_{10}\tau_2[p_2\tau_2(\omega'_{10})^2 + p_1 - p_0\tau_2 + q_1\cos\omega'_{10}\tau'_{10}], \\
B' =& 2m_2\omega'_{10} - n_1\sin\omega'_{10}\tau'_{10} + 2n_2\omega'_{10}\cos\omega'_{10}\tau'_{10} \\
&+ \sin\omega'_{10}\tau_2[-p_1 + p_0\tau_2 - p_2\tau_2(\omega'_{10})^2 - q_1\cos\omega'_{10}\tau'_{10}] \\
&+ \cos\omega'_{10}\tau_2(-p_1\omega'_{10}\tau_2 + 2p_2\omega'_{10} - q_1\sin\omega'_{10}\tau'_{10}), \\
C' =& [n_0\omega'_{10} - n_2(\omega'_{10})^3]\sin\omega'_{10}\tau'_{10} - n_1(\omega'_{10})^2\cos\omega'_{10}\tau'_{10} + \sin\omega'_{10}\tau_2 \\
&\times [q_0\omega'_{10}\cos\omega'_{10}\tau'_{10} + q_1(\omega'_{10})^2\sin\omega'_{10}\tau'_{10}] + \cos\omega'_{10}\tau_2 \\
&\times [q_0\omega'_{10}\sin\omega'_{10}\tau'_{10} - q_1(\omega'_{10})^2\cos\omega'_{10}\tau'_{10}], \\
D' =& [n_0\omega'_{10} - n_2(\omega'_{10})^3]\cos\omega'_{10}\tau'_{10} + n_1(\omega'_{10})^2\sin\omega'_{10}\tau'_{10} + \cos\omega'_{10}\tau_2 \\
&\times [q_1(\omega'_{10})^2\sin\omega'_{10}\tau'_{10} + q_0\omega'_{10}\cos\omega'_{10}\tau'_{10}] + \sin\omega'_{10}\tau_2 \\
&\times [-q_0\omega'_{10}\sin\omega'_{10}\tau'_{10} + q_1(\omega'_{10})^2\cos\omega'_{10}\tau'_{10}].
\end{aligned}$$

Obviously, if the following condition holds: $(H52) A'C' + B'D' \neq 0$.

Then, we have $\left\{\frac{d(Re\lambda)}{d\tau_1}\right\}_{\lambda=i\omega'_{10}} \neq 0$. By the above analysis, we have the following theorem.

Theorem 2.4. *For system (1.2), $\tau_1 > 0, \tau_2 \in [0, \tau_{20})$ and $\tau_1 \neq \tau_2$. Suppose that the conditions (H51) and (H52) hold, then the interior equilibrium $E^*(x_1^*, x_2^*, y^*)$ is asymptotically stable for all $\tau_1 \in [0, \tau'_{10})$ and unstable for $\tau_1 > \tau'_{10}$. Furthermore, the system (1.2) undergoes a Hopf bifurcation at the interior equilibrium $E^*(x_1^*, x_2^*, y^*)$ when $\tau_1 = \tau'_{10}$.*

3. Direction and stability of Hopf bifurcation

In the previous section, we have shown that the system (1.2) undergoes Hopf bifurcation for different combinations of τ_1 and τ_2. In this section, we shall study the direction of Hopf bifurcation and the stability of bifurcating periodic solutions of system (1.2) with respect to τ_1 and $\tau_2 \in [0, \tau_{20})$. The theoretical approach we will apply is based on the normal form theory and center manifold theorem [7]. It is considered that system (1.2) undergoes Hopf bifurcation at $\tau_1 = \tau'_{10}, \tau_2 \in [0, \tau_{20})$. Without loss of generality, we assume that $\tau'_{10} > \tau'_2$.

Let $\tau_1 = \tau'_{10} + \mu$, $\mu \in R$, $t = s\tau_1, x_1(s\tau_1) = \hat{x}_1(s)$, $x_2(s\tau_1) = \hat{x}_2(s)$, $y(s\tau_1) = \hat{y}(s)$. Denotes $x_1 = \hat{x}_1, x_2 = \hat{x}_2, y = \hat{y}$ and $t = s$, then system (1.2) can be written as a functional differential equation (FDE) in $C = C([-1, 0], R^3)$:

$$u'(t) = L_\mu(u_t) + F(\mu, u_t), \tag{3.1}$$

where $u(t) = (x_1(t), x_2(t), y(t))^T \in C$, and $u_t(\theta) = u(t+\theta) = (x_1(t+\theta), x_2(t+\theta), y(t+\theta))^T \in C$, and $L_\mu : C \to R^3, F : R \times C \to R^3$ are gived by

$$L_\mu(\varphi) = (\tau'_{10}+\mu)\tilde{A}\varphi(0) + (\tau'_{10}+\mu)\tilde{B}\varphi(-\frac{\tau'_2}{\tau'_{10}}) + (\tau'_{10}+\mu)\tilde{C}\varphi(-1), \tag{3.2}$$

and

$$F(\mu,\varphi) = (\tau'_{10}+\mu)(F_1, F_2, F_3)^T, \tag{3.3}$$

where

$$\varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta), \varphi_3(\theta))^T \in C,$$

$$\tilde{A} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}, \tilde{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_{32} & b_{33} \end{pmatrix}, \tilde{C} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned} F_1 =& a'_{11}\varphi_1^2(0) + \cdots, \\ F_2 =& a'_{21}\varphi_2^2(0) + a'_{22}\varphi_2(0)\varphi_2(-1) + a'_{23}\varphi_2(0)\varphi_3(0) + a'_{24}\varphi_2^3(0) + a'_{25}\varphi_2^2(0)\varphi_3(0) + \cdots, \\ F_3 =& a'_{31}\varphi_2^2(-\frac{\tau'_2}{\tau'_{10}}) + a'_{32}\varphi_2(-\frac{\tau'_2}{\tau'_{10}})\varphi_3(-\frac{\tau'_2}{\tau'_{10}}) + a'_{33}\varphi_2^3(-\frac{\tau'_2}{\tau'_{10}}) \\ &+ a'_{34}\varphi_2^2(-\frac{\tau'_2}{\tau'_{10}})\varphi_3(-\frac{\tau'_2}{\tau'_{10}}) + \cdots, \end{aligned}$$

and

$$\begin{aligned} a'_{11} &= -c, a'_{21} = \frac{3a_1 r(1-m)^4 (x_2^*)^2 y^* - a_1(1-m)^2 y^*}{(1+r(1-m)^2(x_2^*)^2)^3}, a'_{22} = -b_1, \\ a'_{23} &= -\frac{2a_1(1-m)^2 x_2^*}{(1+r(1-m)^2(x_2^*)^2)^2}, a'_{24} = \frac{4a_1 r(1-m)^4 x_2^* y^*(1-r(1-m)^2(x_2^*)^2)}{(1+r(1-m)^2(x_2^*)^2)^4}, \\ a'_{25} &= \frac{3a_1 r(1-m)^4(x_2^*)^2 - a_1(1-m)^2}{(1+r(1-m)^2(x_2^*)^2)^3}, \\ a'_{31} &= \frac{-3a_2 r(1-m)^4(x_2^*)^2 y^* + a_2(1-m)^2 y^*}{(1+r(1-m)^2(x_2^*)^2)^3}, \\ a'_{32} &= \frac{2a_2(1-m)^2 x_2^*}{(1+r(1-m)^2(x_2^*)^2)^2}, a'_{33} = \frac{4a_2 r(1-m)^4 x_2^* y^*(r(1-m)^2(x_2^*)^2-1)}{(1+r(1-m)^2(x_2^*)^2)^4}, \\ a'_{34} &= \frac{-3a_2 r(1-m)^4(x_2^*)^2 + a_2(1-m)^2}{(1+r(1-m)^2(x_2^*)^2)^3}. \end{aligned}$$

Hence, by the Riesz representation theorem, there exists a 3×3 matrix function $\eta(\theta,\mu)$ of bounded variation for $\theta \in [-1, 0]$, such that

$$L_\mu\varphi = \int_{-1}^0 d\eta(\theta,\mu)\varphi(\theta), \varphi \in C. \tag{3.4}$$

In fact, we can choose

$$\eta(\theta,\mu)=\begin{cases}(\tau_{10}'+\mu)(\tilde{A}+\tilde{B}+\tilde{C}), \theta=0,\\ (\tau_{10}'+\mu)(\tilde{B}+\tilde{C}), & \theta\in[-\frac{\tau_2'}{\tau_{10}},0),\\ (\tau_{10}+\mu)\tilde{C}, & \theta\in(-1,-\frac{\tau_2'}{\tau_{10}}),\\ 0, & \theta=-1.\end{cases} \tag{3.5}$$

For $\varphi\in C([-1,0],R^3)$, define

$$A(\mu)\varphi=\begin{cases}\frac{d\varphi(\theta)}{d\theta}, & -1\le\theta<0,\\ \int_{-1}^{0}d\eta(\theta,\mu)\varphi(\theta), \theta=0,\end{cases} \tag{3.6}$$

and

$$R_\mu(\varphi)=\begin{cases}0, & -1\le\theta<0,\\ F(\mu,\varphi), \theta=0.\end{cases} \tag{3.7}$$

Then Eq.(3.1) can be transformed into the following operator equation

$$u_t'=A(\mu)u_t+R(\mu)u_t. \tag{3.8}$$

For $\phi\in C([-1,0],(R^3)^*)$, where $(R^3)^*$ is the 3-dimensional space of row vectors, we further define the adjoint operator A^* of $A(0)$:

$$A^*\phi(s)=\begin{cases}-\frac{d\phi(s)}{ds}, & s\in(0,1],\\ \int_{-1}^{0}d\eta^T(t,0)\phi(-t), s=0.\end{cases} \tag{3.9}$$

For $\varphi\in C([-1,0],R^3)$ and $\phi\in C([-1,0],(R^3)^*)$, define the bilinear form

$$\langle\phi(s),\varphi(s)\rangle=\bar{\phi}(0)\varphi(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\bar{\phi}(\xi-\theta)d\eta(\theta)\varphi(\xi)d\xi, \tag{3.10}$$

where $\eta(\theta)=\eta(\theta,0)$, $A=A(0)$ and A^* are adjoint operators.

From the discussions in Section 2, we know that $\pm i\omega_{10}'\tau_{10}'$ are eigenvalues of $A(0)$. Thus they are also the eigenvalues of A^*.

Suppose that $q(\theta)=(1,q_2,q_3)^Te^{i\omega_{10}'\tau_{10}'\theta}$ is the eigenvector of $A(0)$ corresponding to $i\omega_{10}'\tau_{10}'$ and $q^*(s)=1/\rho(1,q_2^*,q_3^*)e^{i\omega_{10}'\tau_{10}'s}$ is the eigenvector of A^* corresponding to $-i\omega_{10}'\tau_{10}'$. By the direct calculation, we obtain

$$\begin{aligned}q_2&=\frac{i\omega_{10}'-a_{11}}{a_{12}},q_3=\frac{b_{32}(i\omega_{10}'-a_{11})e^{-i\omega_{10}'\tau_2'}}{a_{12}(i\omega_{10}'-a_{33}-b_{33}e^{-i\omega_{10}'\tau_2'})},\\ q_2^*&=-\frac{a_{11}+i\omega_{10}'}{a_{21}},q_3^*=\frac{a_{23}(a_{11}-i\omega_{10}')}{a_{12}(a_{33}+b_{33}e^{-i\omega_{10}'\tau_2'}+i\omega_{10}')}.\end{aligned} \tag{3.11}$$

Then, from Eq.(3.10), we have

$$\begin{aligned}\langle q^*(s), q(\theta)\rangle =&\bar{q}^*(0)q(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta}\bar{q}^*(\xi-\theta)d\eta(\theta)q(\xi)d\xi\\ =&\frac{1}{\bar{\rho}}[1 + q_2\bar{q}_2^* + q_3\bar{q}_3^* - \int_{-1}^{0}(1,\bar{q}_2^*,\bar{q}_3^*)\theta e^{i\omega_{10}'\tau_{10}'\theta}d\eta(\theta)(1,q_2,q_3)^T]\\ =&\frac{1}{\bar{\rho}}[1 + q_2\bar{q}_2^* + q_3\bar{q}_3^* + b_{22}\tau_{10}'q_2\bar{q}_2^*e^{-i\omega_{10}'\tau_{10}'}\\ &+\tau_2'e^{-i\omega_{10}'\tau_2'}\bar{q}_3^*(b_{32}q_2 + b_{33}q_3)].\end{aligned} \tag{3.12}$$

Therefore, we can choose

$$\bar{\rho} = 1 + q_2\bar{q}_2^* + q_3\bar{q}_3^* + b_{22}\tau_{10}'q_2\bar{q}_2^*e^{-i\omega_{10}'\tau_{10}'} + \tau_2'(b_{32}q_2 + b_{33}q_3)\bar{q}_3^*e^{-i\omega_{10}'\tau_{10}'}, \tag{3.13}$$

such that $\langle q^*(s), q(\theta)\rangle = 1$, $\langle q^*(s), \bar{q}(\theta)\rangle = 0$.

In the remained of this section, by using the algorithms in [7] and using a similar calculation process in [17], we can get the coefficients used in determining the direction of Hopf bifurcation and the stability of the bifurcation periodic solutions:

$$\begin{aligned}g_{20} =&\frac{2\tau_{10}'}{\bar{\rho}}[a_{11}' + \bar{q}_2^*(a_{21}'q_2^2 + a_{22}'q_2^2e^{-i\omega_{10}'\tau_{10}'} + a_{23}'q_2q_3)\\ &+\bar{q}_3^*(a_{31}'q_2^2e^{-2i\omega_{10}'\tau_2'} + a_{32}'q_2q_3e^{-2i\omega_{10}'\tau_2'})],\\ g_{11} =&\frac{\tau_{10}'}{\bar{\rho}}[2a_{11}' + \bar{q}_2^*(2a_{21}'q_2\bar{q}_2 + a_{22}'q_2\bar{q}_2(e^{-i\omega_{10}'\tau_{10}'} + e^{i\omega_{10}'\tau_{10}'}) + a_{23}'(q_2\bar{q}_3 + \bar{q}_2q_3))\\ &+\bar{q}_3^*(2a_{31}'q_2\bar{q}_2 + a_{32}'(q_2\bar{q}_3 + \bar{q}_2q_3))],\\ g_{02} =&\frac{2\tau_{10}'}{\bar{\rho}}[a_{11}' + \bar{q}_2^*(a_{21}'\bar{q}_2^2 + a_{22}'\bar{q}_2^2e^{i\omega_{10}'\tau_{10}'} + a_{23}'\bar{q}_2\bar{q}_3)\\ &+\bar{q}_3^*(a_{31}'\bar{q}_2^2e^{2i\omega_{10}'\tau_2'} + a_{32}'\bar{q}_2\bar{q}_3e^{2i\omega_{10}'\tau_2'})],\\ g_{21} =&\frac{2\tau_{10}'}{\bar{\rho}}[a_{11}'(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)) + \bar{q}_2^*(a_{21}'(2q_2W_{11}^{(2)}(0) + \bar{q}_2W_{20}^{(2)}(0))\\ &+a_{22}'(q_2W_{11}^{(2)}(-1) + \tfrac{1}{2}\bar{q}_2W_{20}^{(2)}(-1) + \tfrac{1}{2}\bar{q}_2W_{20}^{(2)}(0)e^{i\omega_{10}'\tau_{10}'}\\ &+q_2W_{11}^{(2)}(0)e^{-i\omega_{10}'\tau_{10}'}) + a_{23}'(q_2W_{11}^{(3)}(0) + \tfrac{1}{2}\bar{q}_2W_{20}^{(3)}(0) + q_3W_{11}^{(2)}(0)\\ &+\tfrac{1}{2}\bar{q}_3W_{20}^{(2)}(0)) + 3a_{24}'q_2^2\bar{q}_2 + a_{25}'(q_2^2\bar{q}_3 + 2q_2\bar{q}_2q_3)) + \bar{q}_3^*(a_{31}'(2q_2W_{11}^{(2)}(-\tfrac{\tau_2'}{\tau_{10}'})e^{-i\omega_{10}'\tau_2'}\\ &+\bar{q}_2W_{20}^{(2)}(-\frac{\tau_2'}{\tau_{10}'})e^{i\omega_{10}'\tau_2'}) + a_{32}'(q_2W_{11}^{(3)}(-\frac{\tau_2'}{\tau_{10}'})e^{-i\omega_{10}'\tau_2'}\\ &+\tfrac{1}{2}\bar{q}_2W_{20}^{(3)}(-\tfrac{\tau_2'}{\tau_{10}'})e^{i\omega_{10}'\tau_2'} + \tfrac{1}{2}\bar{q}_3W_{20}^{(2)}(-\tfrac{\tau_2'}{\tau_{10}'})e^{i\omega_{10}'\tau_2'} + q_3W_{11}^{(2)}(-\tfrac{\tau_2'}{\tau_{10}'})e^{-i\omega_{10}'\tau_2'})\\ &+3a_{33}'q_2^2\bar{q}_2e^{-i\omega_{10}'\tau_2'} + a_{34}'(q_2^2\bar{q}_3e^{-i\omega_{10}'\tau_2'} + 2q_2\bar{q}_2q_3e^{-i\omega_{10}'\tau_2'}))].\end{aligned} \tag{3.14}$$

However

$$\begin{aligned}W_{20}(\theta) &= \frac{ig_{20}}{\omega_{10}'\tau_{10}'}q(0)e^{i\omega_{10}'\tau_{10}'\theta} + \frac{i\bar{g}_{02}}{3\omega_{10}'\tau_{10}'}\bar{q}(0)e^{-i\omega_{10}'\tau_{10}'\theta} + E_1e^{2i\omega_{10}'\tau_{10}'\theta},\\ W_{11}(\theta) &= -\frac{ig_{11}}{\omega_{10}'\tau_{10}'}q(0)e^{i\omega_{10}'\tau_{10}'\theta} + \frac{i\bar{g}_{11}}{\omega_{10}'\tau_{10}'}\bar{q}(0)e^{-i\omega_{10}'\tau_{10}'\theta} + E_2.\end{aligned} \tag{3.15}$$

Where $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^T \in R^3$ and $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T \in R^3$ are also constant vectors and can be determined by the following equations, respectively

$$\begin{pmatrix} 2i\omega'_{10} - a_{11} & -a_{12} & 0 \\ -a_{21} & 2i\omega'_{10} - a_{22} - b_{22}e^{-2i\omega'_{10}\tau'_{10}} & -a_{23} \\ 0 & -b_{32}e^{-2i\omega'_{10}\tau'_2} & 2i\omega'_{10} - a_{33} - b_{33}e^{-2i\omega'_{10}\tau'_2} \end{pmatrix} E_1 = 2\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix},$$

$$\begin{pmatrix} -a_{11} & -a_{12} & 0 \\ -a_{21} & -a_{22} - b_{22} & -a_{23} \\ 0 & -b_{32} & -a_{33} - b_{33} \end{pmatrix} E_2 = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}, \tag{3.16}$$

with

$$\begin{aligned} &H_1 = a'_{11}, H_2 = a'_{21}q_2^2 + a'_{22}q_2^2e^{-i\omega'_{10}\tau'_{10}} + a'_{23}q_2q_3, \\ &H_3 = a'_{31}q_2^2e^{-2i\omega'_{10}\tau'_2} + a'_{32}q_2q_3e^{-2i\omega'_{10}\tau'_2}, \\ &P_1 = 2a'_{11}, P_2 = 2a'_{21}q_2\bar{q}_2 + a'_{22}q_2\bar{q}_2(e^{-i\omega'_{10}\tau'_{10}} + e^{i\omega'_{10}\tau'_{10}}) + a'_{23}(q_2\bar{q}_3 + \bar{q}_2q_3), \\ &P_3 = 2a'_{31}q_2\bar{q}_2 + a'_{32}(q_2\bar{q}_3 + \bar{q}_2q_3). \end{aligned}$$

Therefore, we can calculate g_{21} and compute the following values:

$$\begin{aligned} &c_1(0) = \frac{i}{2\omega'_{10}\tau'_{10}}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2}, \\ &\mu_2 = -\frac{Re\{c_1(0)\}}{Re\{\lambda'(\tau'_{10})\}}, \\ &\beta_2 = 2Re(c_1(0)), \\ &T_2 = -\frac{Im\{c_1(0)\} + \mu_2 Im\{\lambda'(\tau_{10})\}}{\omega'_{10}\tau'_{10}}. \end{aligned} \tag{3.17}$$

Which determine the properties of bifurcation period solutions at $\tau = \tau'_{10}$ on the center manifold. From the discussion above, we have the following results.

Theorem 3.1. *For system (1.2), the direction of Hopf bifurcation is determined by the sign of μ_2: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical). The stability of the bifurcating periodic solutions is determined by the sign of β_2: if $\beta_2 < 0$ ($\beta_2 > 0$), the bifurcating periodic solutions are stable (unstable). The period of the bifurcating periodic solutions is determined by the sign of T_2: if $T_2 > 0$ ($T_2 < 0$), the bifurcating periodic solutions increase (decrease).*

4. Numerical examples

In this section, we present some numerical simulations by using Matlab to illustrate the analytical results, and the corresponding waveform and the phase plots of system (1.2) are drawn.

Let $a = 8$, $a_1 = 5.25$, $a_2 = 4$, $r_1 = 1$, $r_2 = 2$, $r_3 = 1$, $r = 3$, $b = 5$, $b_1 = 1$, $c = 0.5$, $m = 0.1$, $qE = 0.1$. Then, we have the following particular example of system (1.2):

$$\begin{aligned}
\dot{x}_1(t) &= 8x_2(t) - x_1(t) - 5x_1(t) - 0.5x_1^2(t),\\
\dot{x}_2(t) &= 5x_1(t) - 2x_2(t) - x_2(t)x_2(t-\tau_1) - \frac{5.25(1-0.1)^2x_2^2(t)y(t)}{1+3(1-0.1)^2x_2^2(t)},\\
\dot{y}(t) &= \frac{4(1-0.1)^2x_2^2(t-\tau_2)y(t-\tau_2)}{1+3(1-0.1)^2x_2^2(t-\tau_2)} - y(t) - 0.1y(t-\tau_2).
\end{aligned} \tag{4.1}$$

It is not difficult to verify that the (H1) holds, we can get the interior equilibrium $E^*(1.6346, 1.3929, 2.3875)$.

For $\tau_1 > 0, \tau_2 = 0$, we obtain $\omega_{10} = 1.1497$, $\tau_{10} = 0.7842$. From Theorem 2.1, we know that the interior equilibrium E^* is asymptotically stable when $\tau_1 \in [0, \tau_{10})$, when the time delay τ_1 passes through the critical value τ_{10}, the interior equilibrium E^* will lose its stability and a Hopf bifurcation occurs, and a family of periodic solutions bifurcate from the interior equilibrium E^*. The corresponding waveform and the phase plots are depicted in Figure 1 and Figure 2. Similarly, we have $\omega_{20} = 0.3924$, $\tau_{20} = 2.2209$.

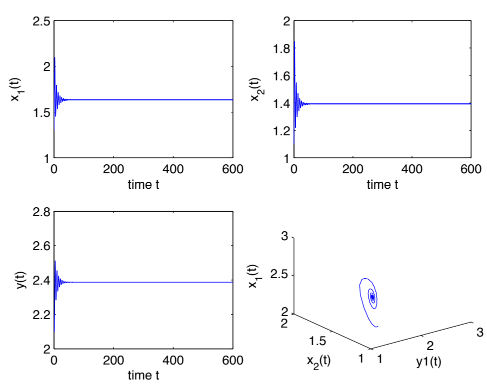

Figure 1. When $\tau_2 = 0$, E^* is asymptotically stable for $\tau_1 = 0.65 < \tau_{10} = 0.7842$.

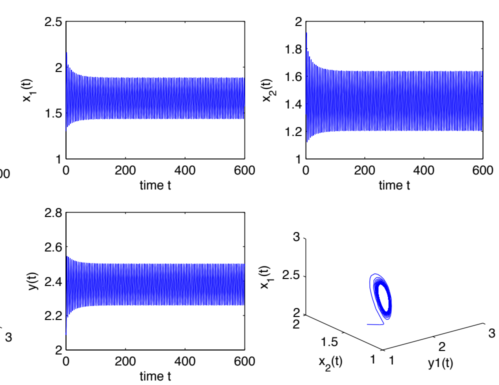

Figure 2. When $\tau_2 = 0$, E^* undergoes a Hopf bifurcation for $\tau_1 = 0.8 > \tau_{10} = 0.7842$.

For $\tau_1 = \tau_2 = \tau \neq 0$, we obtain $\omega_0 = 0.8051$, $\tau_0 = 0.6326$. From Theorem 2.3, we know that, when the time delay τ increases from zero to τ_0, the interior equilibrium E^* is asymptotically stable. Once the time delay τ passes through the critical value τ_0, the interior equilibrium E^* will lose its stability and a Hopf bifurcation occurs.

For $\tau_1 > 0$, $\tau_2' = 1.8 \in [0, \tau_{20}]$, we have $\omega_{10}' = 0.3780$, $\tau_{10}' = 0.3619$. According to Theorem 2.4, E^* is asymptotically stable when $\tau_1 \in [0, \tau_{10}')$ and unstable when $\tau_1 > \tau_{10}'$. After the computation of Eq.(3.17), we obtain $c_1(0) = -1.9358 - 8.0829i$, $\mu_2 = 54.6836$, $\beta_2 = -3.8716$. From theorem 3.1, the Hopf bifurcation is supercritical, the bifurcating periodic solutions are stable, which can be depicted in Figure 3 and Figure 4.

These numerical simulation results constitute excellent validations of our theoretical analysis. Due to the bifurcation periodic solutions are stable, the species in system (1.2) can coexist in an oscillatory mode from the viewpoint of biology.

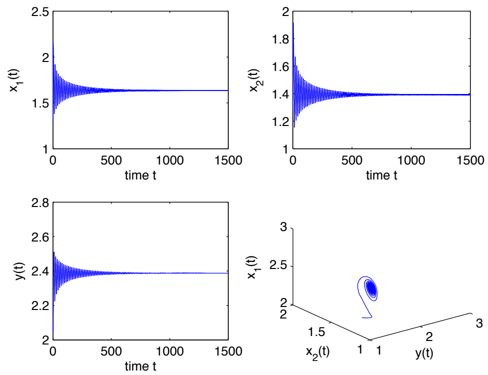

Figure 3. E^* is asymptotically stable for $\tau_1 = 0.21 < \tau'_{10} = 0.3619$ and $\tau'_2 = 1.8$.

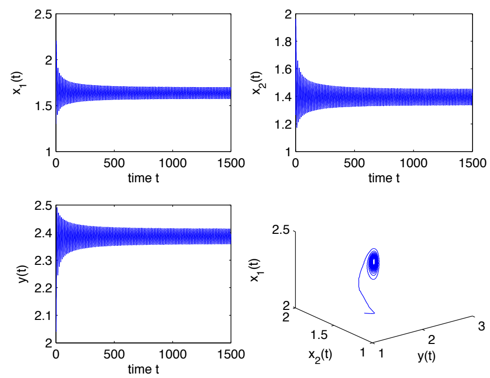

Figure 4. E^* undergoes a Hopf bifurcation for $\tau_1 = 0.4 > \tau'_{10} = 0.3619$ and $\tau'_2 = 1.8$.

5. Conclusions

In this paper, we have incorporated two time delays, a prey refuge and selective harvesting into a stage structured predator-prey system. By analyzing the associated characteristic equation, its local stability and the existence of Hopf bifurcation with respect to delay are established. By using the normal form theory and center manifold theorem, the explicit formulas which determine the direction of Hopf bifurcation and stability of the bifurcating periodic solution are derived. The numerical results which the Hopf bifurcation is supercritical and the bifurcation periodic solutions are stable are in accord with the theoretical analysis.

In addition, stage structure for the prey is considered in this paper due to the lack of hunting ability for the immature prey. If we investigate stage structure for the predator or stage structure for the prey and the predator together, what will the dynamical behavior of system is? This is very valuable from the perspective of biological diversity, and we leave it for the future work.

Acknowledgements

The authors would like to thank the referees for the careful reading of the manuscript and valuable suggestions. This work was supported by National Natural Science Foundation of China (No. 11472116, No. 11472115), Postgraduate Research & Practice Innovation Program of Jiangsu Province (KYCX17_1784) and Qinglan project of Jiangsu.

References

[1] C. Celik, *Hopf bifurcation of a ratio-dependent predator-prey system with time delay*, Chaos Solitons and Fractals, 2009, 42(3), 1474–1484.

[2] K. Chakraborty, S. Jana and T. K. Kar, *Global dynamics and bifurcation in a stage structured prey-predator fishery model with harvesting*, Applied Mathematics and Computation, 2012, 218(18), 9271–9290.

[3] L. W. Deng, X. D. Wang and M. Peng, *Hopf bifurcation analysis for a ratio-dependent predator-prey system with two delays and stage structure for the predator*, Applied mathematics and computation, 2014, 231, 214–230.

[4] R. P. Gupta and P. Chandra, *Bifurcation analysis of modified Leslie-Gower predator-prey model with Michaelis-Menten type prey harvesting*, Journal of Mathematical Analysis and Applications, 2013, 398(1), 278–295.

[5] M. Haquea, M. S. Rahman, E. Venturino and B. L. Li, *Effect of a functional response-dependent prey refuge in a predator-prey model*, Ecological Complexity, 2014, 20, 248–256.

[6] J. K. Hale, *Theory of Functional Differential Equations*, Springer, New York, 1977.

[7] B. D. Hassard and N. D. Kazarinoff and Y. H. Wan, *Theory and applications of Hopf bifurcation*, Cambridge University Press, Cambridge, UK, 1981.

[8] S. Jana, M. Chakraborty, K. Chakraborty and T. K. Kar, *Global stability and bifurcation of time delayed prey-predator system incorporating prey refuge*, Mathematics and Computers in Simulation, 2012, 85(3), 57–77.

[9] T. K. Kar and A. Ghorai, *Dynamic behaviour of a delayed predator-prey model with harvesting*, Applied Mathematics and Computation, 2011, 217(22), 9085–9104.

[10] F. Li and H. W. Li, *Hopf bifurcation of a predator-prey model with time delay and stage structure for the prey*, Mathematical and Computer Modelling, 2012, 55, 672–679.

[11] X. Liu and M. A. Han, *Chaos and Hopf bifurcation analysis for a two species predator-prey system with prey refuge and diffusion*, Nonlinear Analysis Real World Applications, 2011, 12(2), 1047–1061.

[12] X. Y. Meng, H. F. Huo and X. B. Zhang, *Stability and global Hopf bifurcation in a delayed food web consisting of a prey and two predator*, Communications in Nonlinear Science Numerical Simulation, 2011, 16(11), 4335–4348.

[13] X. Y. Meng, H. F. Huo and X. B. Zhang, *Stability and Hopf bifurcation in a three-species system with feedback delays*, Nonlinear Dynamics, 2011, 64(4), 349–364.

[14] M. Peng, Zh. D. Zhang and X. D. Wang, *Hybrid control of Hopf bifurcation in a Lotka-Volterra predator-prey model with two delays*, Advances in Difference Equations, 2017, 387. DOI: 10.1186/s13662-017-1434-5.

[15] S. Ruan and J. Wei, *On the zero of some transcendential functions with applications to stability of delay differential equations with two delays*, Dynamics of Continuous Discrete and Impulsive Systems Series A Mathematical Analysis, 2003, 10(6), 863–874.

[16] S. Sharma and G. P. Samanta, *A Leslie-Gower predator-prey model with disease in prey incorporating a prey refuge*, Chaos Solitons and Fractals, 2015, 70(1), 69–84.

[17] Y. Song and J. Wei, *Bifurcation analysis for Chen's System with delayed feedback and its application to Control of chaos*, Chaos, Solitons and Fractals, 2004, 22(1), 75–91.

[18] J. P. Tripathi, S. Abbas and M. Thakur, *A density dependent delayed predator-prey model with Beddington-DeAngelis type function response incorporating a prey refuge*, Communications in Nonlinear Science Numerical Simulation, 2015, 22, 427–450.

[19] P. J. Wangersky and W. J. Cunningham, *Time lag in prey-predator population models*, Ecology, 1957, 38(1), 136–139.

[20] X. D. Wang, M. Peng and X. Y. Liu, *Stability and Hopf bifurcation analysis of a ratio-dependent predator-prey model with two time delays and Holling typeIII functional response*, Applied mathematics and computation, 2015, 268, 496–508.

[21] Y. M. Wu, F. D. Chen, W. L. Chen and Y. H. Lin, *Dynamic Behaviors of a Nonautonomous DiscretePredator-Prey System Incorporating a Prey Refuge and Holling Type II Functional Response*, Discrete Dynamics in Nature and Society, 2012. DOI:10.1155/2012/508962.

[22] R. Yuan, W. H. Jiang and Y. Wang, *Saddle-node-Hopf bifurcation in a modified Leslie-Gower predator-prey model with time-delay and prey harvesting*, Journal of Mathematical Analysis and Applications, 2015, 422(2), 1072–1090.

[23] H. Y. Zhao, X. X. Huang and X. B. Zhang, *Hopf bifurcation and harvesting control of a bioeconomic plankton model with delay and diffusion terms*, Physica A, 2015, 421(52), 300–315.

[24] Zh. D. Zhang and Q. S. Bi, *Bifurcation in a piecewise linear circuit with switching boundaries*, International Journal of Bifurcation and Chaos, 2012, 22(2). DOI: 10.1142/S0218127412500344.