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EXACT TRAVELLING WAVE SOLUTIONS TO THE SPACE-TIME FRACTIONAL CALOGERO-DEGASPERIS EQUATION USING DIFFERENT METHODS

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Abstract In this paper, we employed the ansatz method, the exp-function method and the $\left(\frac{G'}{G}\right)$-expansion method for the first time to obtain the exact and traveling wave solutions of the space time fractional Calogero Degasperis equation. As a result, we obtained some soliton and traveling wave solutions for this equation by means of proposed three analytical methods and the aid of commercial software Maple. The results show that these methods are effective and powerful mathematical tool for solving nonlinear FDEs arising in mathematical physics.

Keywords Exact solution, exp-function method, $\left(\frac{G'}{G}\right)$-expansion method, ansatz method, space time fractional Calogero-Degasperis equation.

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1. Introduction

Fractional differential equations (FDEs) are generalizations of classical differential equations of integer order. In recent years, this equations have gained considerable interest. Many significant phenomena in natural science and engineering such as fluid dynamics, biology, polymeric materials, damping law, diffusion processes, quantum, system identification, electromagnetic, mechanics, fluid flow, finance, viscoelasticity, quantum, chemistry, signal processing, control theory and so on can be modeled by this equations [39, 43, 46, 47].

Beacaue of its potential applications, scientists have devoted remarkable attempt to study the numerical and analytic solutions of nonlinear FDEs. Nevertheless, not all nonlinear FDEs are solvable. During recent years, mathematicians and physicists have developed various techniques to find solutions of nonlinear FDEs such as the variational iteration method, adomian decomposition method, the homotopy perturbation method, the fractional sub-equation method, the first integral method, the $(G\prime/G)$-expansion method, the functional variable method, the exp-function method, the modified Kudryashov method, the fractional MSE method, the ansatz method and the modified trial equation method and so on [3–10, 15, 17–31, 35, 36, 40–42, 44, 48–50, 52, 54, 55].

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There are different kinds of fractional derivative operators. The most famous one is the Caputo definition that the function should be differentiable [16]. In [37, 38], modified Riemann-Liouville derivative is proposed by Jumarie. With this kind of fractional derivative and fractional complex transform, we can convert fractional differential equations into integer-order differential equations. The order α of Jumarie's derivative is defined by

$$D_t^\alpha u = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(u(\xi)-u(0))d\xi \quad , \ 0<\alpha<1 \\ (f^{(n)}(t))^{(\alpha-n)}, \ n\le\alpha<n+1, \ n\ge 1, \end{cases} \tag{1.1}$$

where $f:R\to R,\ t\to f(t)$ denotes a continuous (but not necessarily first-order-differentiable) function. Some useful formulas:

Property 1.

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}t^{r-\alpha}, \tag{1.2}$$

Property 2.

$$D_t^\alpha\{C_1 f(t)+C_2 g(t)\} = C_1 D_t^\alpha f(t) + C_2 D_t^\alpha g(t), \tag{1.3}$$

Property 3.

$$D_t^\alpha C = 0, \tag{1.4}$$

where $C,\ C_1$ and C_2 are constants.

In this line of thought, the structure of the article is as follows. In the next section, we briefly give the steps of the methods. In Section 3 we apply the methods to solve for the space–time fractional Calogero-Degasperis (CD) equation. Finally, conclusions are presented in last section.

2. Methods and theirs algorithms

We consider the following general (2+1) dimensional nonlinear space-time FDEs of the type

$$Q(u, D_t^\alpha u, D_x^\alpha u, D_y^\alpha u, D_t^{2\alpha} u, D_t^\alpha D_x^\alpha u, D_x^{2\alpha} u, D_x^\alpha D_y^\alpha u, D_y^{2\alpha} u, ...) = 0, \tag{2.1}$$

where $0<\alpha\le 1$ and u is an unknown function, and Q is a polynomial of u and its partial fractional derivatives.

The proper traveling wave variable is [32]

$$u(x,y,t) = f(\theta), \tag{2.2}$$

$$\theta = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{my^\alpha}{\Gamma(1+\alpha)} - \frac{nt^\alpha}{\Gamma(1+\alpha)}, \tag{2.3}$$

where $k,\ m$ and n are nonzero constants.

By using the chain rule

$$\begin{aligned} D_t^\alpha u &= \sigma_t \tfrac{df}{d\theta} D_t^\alpha\theta, \\ D_x^\alpha u &= \sigma_x \tfrac{df}{d\theta} D_x^\alpha\theta, \\ D_y^\alpha u &= \sigma_y \tfrac{df}{d\theta} D_y^\alpha\theta, \end{aligned} \tag{2.4}$$

where σ_t, σ_x and σ_y are called the sigma indexes [1], and it can be take $\sigma_t = \sigma_x = \sigma_y = L$, where L is a constant.

Substituting (2.2) with (1.2) and (2.4) into (2.1), we can rewrite Eq.(2.1) in the following nonlinear ODE;

$$H(f, f^{'}, f'', f''', ...) = 0, \tag{2.5}$$

where $\frac{df}{d\theta}$. Now we concider three different methods.

2.1. Basic idea of Ansatz method

For dark soliton solution, the starting hypothesis is in the form [11, 12, 51]

$$u(x, y, t) = A \tanh^p \theta \tag{2.6}$$

where $\theta = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{my^\alpha}{\Gamma(1+\alpha)} - \frac{nt^\alpha}{\Gamma(1+\alpha)}$ and A, k, c are nonzero constants. From the ansatz given above with expression (2.6), it is possible to obtain necessary derivatives. Then, the obtained derivatives are substituted in the Eq.(2.5) and we collect all terms with the same order of necessary terms. Then by equating each coefficient of the resulting polynomial to zero, we obtain a set of algebraic equations for; A, k and c. Finally solving the system of equations we can get exact solution of Eq.(2.1).

2.2. Basic idea of Exp-function method

This method is based on the assumption that traveling wave solutions can be expressed in the following form which was developed by He and Wu [33]

$$f(\theta) = \frac{\sum_{n=-c}^{d} a_n \exp[n\theta]}{\sum_{m=-p}^{q} b_m \exp[m\theta]} \tag{2.7}$$

where p, q, c and d are positive integers, a_n and b_m are unknown constants. Also this expression can write in the following equivalent form.

$$f(\theta) = \frac{a_{-c} \exp[-c\theta] + ... + a_d \exp[d\theta]}{b_{-p} \exp[-p\theta] + ... + b_q \exp[q\theta]}. \tag{2.8}$$

To determine the value of c and p, we balance the linear term of lowest order of equation Eq.(2.5) with the lowest order nonlinear term. Similarly, to determine the value of d and q, we balance the linear term of highest order of Eq.(2.5) with highest order nonlinear term [2, 13, 34, 56].

2.3. Basic idea of $\left(\frac{G'}{G}\right)$-expansion method

Step 1: According to the this method which was developed by Wang [53], we look for its solution $f(\theta)$ in the polynomial form

$$f(\theta) = \sum_{i=0}^{z} a_i \left(\frac{G'}{G}\right)^i, \quad a_z \neq 0, \tag{2.9}$$

where a_i are constants, while $G(\theta)$ is the solution of the auxiliary linear second order ODE

$$G''(\theta) + \lambda G'(\theta) + \mu G(\theta) = 0, \tag{2.10}$$

with λ and μ are being constants.

Step 2: z is a positive integer which is determined by the homogeneous balancing method in Eq.(2.5).

Step 3: By substituting Eqs.(2.9) and (2.10) into Eq.(2.5) with the value of z obtained in Step 2, and collecting all terms with the same order of $\left(\frac{G'}{G}\right)$ together. Then setting each coefficient to zero, we obtained a set of algebraic equations for a_i, k, m and n.

Step 4: Solve the system of algebraic equations obtained in step 3 for a_i $(i = 0, 1, 2, ..., z)$, k, m and n by use of Maple. Then we substitute a_i $(i = 0, 1, 2, ..., z)$, k, m, n and the solutions of Eq.(2.10) into Eq.(2.9), we can obtain a series of fundamental solutions of Eq.(2.1) [14].

3. New solutions of the space–time fractional Calogero-Degasperis (CD) equation

We consider the space–time fractional Calogero-Degasperis (CD) equation [45]

$$D_t^{\alpha} D_x^{\alpha} u - 4D_x^{\alpha} u D_x^{2\alpha} u - 2D_y^{\alpha} u D_x^{2\alpha} u + D_y^{\alpha} D_x^{3\alpha} u = 0, \tag{3.1}$$

where $0 < \alpha \leq 1$.

By using the transformations (2.2) and (2.3) with (2.4), then by once integrating, Eq.(3.1) reduced into following ODE

$$-nkf' - (2k^3L + mk^2L)(f')^2 + mk^3L^2 f''' = 0, \tag{3.2}$$

where "f'" $= \frac{df}{d\theta}$ and constant of integration is taken to be zero.

3.1. Exact solution by ansatz method

From the ansatz given above Eq.(2.6), we get necessary derivatives and these are substituted in the Eq.(3.1) we get algebraic equation. By use this equation, in order to reduce the number of coefficients of the powers of $\tanh\theta$, we determine the balance value of p. Equating $2(p+1) = p+3$ yields $p = 1$. Using this value of $p = 1$, this algebraic equation reduces to

$$\begin{aligned}(8A^2n^2k - 4A^2mkn + 24Ak^3m)\tanh^5 \\ +(8A^2mkn - 40Ak^3m - 16A^2n^2k - 2Akn)\tanh^3 \\ +(16Ak^3m + 2Akn - 4A^2mkn + 8A^2n^2k)\tanh = 0.\end{aligned} \tag{3.3}$$

So, we obtain a system of algebraic equations

$$\begin{aligned}8A^2n^2k - 4A^2mkn + 24AL^2k^3m = 0, \\ 8A^2mkn - 40AL^2k^3m - 16A^2n^2k - 2Akn = 0, \\ 16AL^2k^3m + 2Akn - 4A^2mkn + 8A^2n^2k = 0.\end{aligned} \tag{3.4}$$

Solving this system for A and n gives

$$A = \frac{6L^2k^2m}{n(m-2n)}, \tag{3.5}$$

$$n = 4L^2k^2m. \tag{3.6}$$

Equation (3.5) prompts the constraint $m \neq 2n$. Thus finally, the dark soliton solution for Eq.(3.1) is given by

$$u(x,y,t) = A\tanh\left(\frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{my^\alpha}{\Gamma(1+\alpha)} - \frac{nt^\alpha}{\Gamma(1+\alpha)}\right), \tag{3.7}$$

where A is given in Eq.(3.5) and n is given in Eq.(3.6).

3.2. Exact solutions by Exp-function method

By balancing the order of f''' and $(f')^2$ in Eq.(3.2), we obtain

$$f''' = \frac{c_1\exp[-(c+7p)\theta] + ...}{c_2\exp[-8p\theta] + ...}, \tag{3.8}$$

and

$$(f')^2 = \frac{c_3\exp[-(2c+2p)\theta] + ...}{c_4\exp[-4p\theta] + ...}, \tag{3.9}$$

where c_i are determined coefficients only for simplicity. Balancing lowest order of exp-function in Eqs.(3.8) and (3.9) we get

$$-(c+7p) = -(2c+6p), \tag{3.10}$$

and

$$p = c. \tag{3.11}$$

Similar process we balance the linear term of highest order in Eq.(3.2), we obtain

$$q = d. \tag{3.12}$$

The simplest choice for $c,\ p,\ q$ and d is $p=c=1$ and $q=d=1$. According to $p=c=1$ and $q=d=1$, Eq.(2.8) becomes

$$f(\theta) = \frac{a_1\exp(\theta) + a_0 + a_{-1}\exp(-\theta)}{b_1\exp(\theta) + b_0 + b_{-1}\exp(-\theta)}. \tag{3.13}$$

Substituting Eq.(3.13) into Eq.(3.2), and solving this system of algebraic equations by the help of Maple, we get the following results

Case 1:

$$\begin{aligned} &a_1 = 0, \quad a_0 = a_0, \ a_{-1} = a_{-1}, \\ &b_1 = 0, \quad b_0 = 0, \quad b_{-1} = b_{-1}, \\ &k = -\tfrac{m}{2}, \ m = m, \ n = \tfrac{m^3L^2}{4}. \end{aligned} \tag{3.14}$$

Substituting these results into (3.13), we get the following exact solution

$$u_1(x,y,t)=\frac{a_0+a_{-1}\exp\left(-\left(\frac{-mx^\alpha}{2\Gamma(1+\alpha)}+\frac{my^\alpha}{\Gamma(1+\alpha)}-\frac{m^3L^2t^\alpha}{4\Gamma(1+\alpha)}\right)\right)}{b_{-1}\exp\left(-\left(\frac{-mx^\alpha}{2\Gamma(1+\alpha)}+\frac{my^\alpha}{\Gamma(1+\alpha)}-\frac{m^3L^2t^\alpha}{4\Gamma(1+\alpha)}\right)\right)}. \tag{3.15}$$

Case 2:

$$\begin{aligned} &a_1=a_1, \quad a_0=a_0, \; a_{-1}=a_{-1},\\ &b_1=0, \quad\;\; b_0=b_0, \;\; b_{-1}=0,\\ &k=-\tfrac{m}{2}, \;\; m=m, \;\; n=\tfrac{m^3L^2}{4}. \end{aligned} \tag{3.16}$$

Substituting these results into (3.13), we obtain

$$u_2(x,y,t)=\frac{a_1\exp\left(\frac{-mx^\alpha}{2\Gamma(1+\alpha)}+\frac{my^\alpha}{\Gamma(1+\alpha)}-\frac{m^3L^2t^\alpha}{4\Gamma(1+\alpha)}\right)+a_0+a_{-1}\exp\left(-\left(\frac{-mx^\alpha}{2\Gamma(1+\alpha)}+\frac{my^\alpha}{\Gamma(1+\alpha)}-\frac{m^3L^2t^\alpha}{4\Gamma(1+\alpha)}\right)\right)}{b_0}. \tag{3.17}$$

Case 3:

$$\begin{aligned} &a_1=a_1, \; a_0=0, \quad\; a_{-1}=a_{-1},\\ &b_1=0, \quad b_0=0, \quad\; b_{-1}=b_{-1},\\ &k=k, \quad\; m=-2k, \; n=-8k^3L^2. \end{aligned} \tag{3.18}$$

Substituting these results into (3.13), we get the following exact solution

$$u_3(x,y,t)=\frac{a_1\exp\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{2ky^\alpha}{\Gamma(1+\alpha)}+\frac{8k^3L^2t^\alpha}{\Gamma(1+\alpha)}\right)+a_{-1}\exp\left(-\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{2ky^\alpha}{\Gamma(1+\alpha)}+\frac{8k^3L^2t^\alpha}{\Gamma(1+\alpha)}\right)\right)}{b_{-1}\exp\left(-\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{2ky^\alpha}{\Gamma(1+\alpha)}+\frac{8k^3L^2t^\alpha}{\Gamma(1+\alpha)}\right)\right)}. \tag{3.19}$$

Case 4:

$$\begin{aligned} &a_1=0, \; a_0=-\tfrac{b_0(6kLmb_{-1}-(2k+m)a_{-1})}{b_{-1}(2k+m)}, \; a_{-1}=a_{-1},\\ &b_1=0, \; b_0=b_0, \qquad\qquad\qquad\qquad\; b_{-1}=b_{-1},\\ &k=k \quad m=m, \qquad\qquad\qquad\qquad\;\; n=mk^2L^2, \end{aligned} \tag{3.20}$$

where a_{-1}, b_0 and b_{-1} are free parameters. When we substitute these results into (3.13), we get the following exact solution

$$u_4(x,y,t)=\frac{-\frac{b_0(6kLmb_{-1}-(2k+m)a_{-1})}{b_{-1}(2k+m)}+a_{-1}\exp\left(-\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{my^\alpha}{\Gamma(1+\alpha)}-\frac{mk^2L^2t^\alpha}{\Gamma(1+\alpha)}\right)\right)}{b_0+b_{-1}\exp\left(-\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{my^\alpha}{\Gamma(1+\alpha)}-\frac{mk^2L^2t^\alpha}{\Gamma(1+\alpha)}\right)\right)}. \tag{3.21}$$

Case 5:

$$\begin{aligned} &a_1=a_1, \; a_0=\tfrac{a_1b_{-1}^2+a_{-1}b_0^2}{b_{-1}b_0}, \; a_{-1}=a_{-1},\\ &b_1=0, \quad b_0=b_0, \qquad\qquad b_{-1}=b_{-1},\\ &k=k \quad\;\; m=-2k, \qquad\quad\; n=-2k^3L^2, \end{aligned} \tag{3.22}$$

where $a_{-1},\ b_0$ and b_{-1} are free parameters. When we substitute these results into (3.13), we have

$$u_5(x,y,t)=\frac{\begin{array}{c}a_1\exp\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{2ky^\alpha}{\Gamma(1+\alpha)}+\frac{2k^3L^2t^\alpha}{\Gamma(1+\alpha)}\right)+\frac{a_1b_{-1}^2+a_{-1}b_0^2}{b_{-1}b_0}\\+a_{-1}\exp\left(-\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{2ky^\alpha}{\Gamma(1+\alpha)}+\frac{2k^3L^2t^\alpha}{\Gamma(1+\alpha)}\right)\right)\end{array}}{b_0+b_{-1}\exp\left(-\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{2ky^\alpha}{\Gamma(1+\alpha)}+\frac{2k^3L^2t^\alpha}{\Gamma(1+\alpha)}\right)\right)}. \tag{3.23}$$

If we take $a_0=1,\ a_{-1}=1$ and $b_{-1}=1$, solution (3.15) becomes

$$u_1(x,y,t)=\frac{1+\cosh\left(\frac{-mx^\alpha}{2\Gamma(1+\alpha)}+\frac{my^\alpha}{\Gamma(1+\alpha)}-\frac{m^3L^2t^\alpha}{4\Gamma(1+\alpha)}\right)-\sinh\left(\frac{-mx^\alpha}{2\Gamma(1+\alpha)}+\frac{my^\alpha}{\Gamma(1+\alpha)}-\frac{m^3L^2t^\alpha}{4\Gamma(1+\alpha)}\right)}{\cosh\left(\frac{-mx^\alpha}{2\Gamma(1+\alpha)}+\frac{my^\alpha}{\Gamma(1+\alpha)}-\frac{m^3L^2t^\alpha}{4\Gamma(1+\alpha)}\right)-\sinh\left(\frac{-mx^\alpha}{2\Gamma(1+\alpha)}+\frac{my^\alpha}{\Gamma(1+\alpha)}-\frac{m^3L^2t^\alpha}{4\Gamma(1+\alpha)}\right)}. \tag{3.24}$$

Similarly; if we take $a_0=2,\ a_1=1,\ a_{-1}=1$ and $b_0=2$, solution (3.17) becomes

$$u_2(x,y,t)=2\cosh^2\left(-\frac{mx^\alpha}{4\Gamma(1+\alpha)}+\frac{my^\alpha}{2\Gamma(1+\alpha)}-\frac{m^3L^2t^\alpha}{8\Gamma(1+\alpha)}\right). \tag{3.25}$$

When we take $a_1=1,\ a_{-1}=1$ and $b_{-1}=1$, solution (3.19) becomes

$$\begin{aligned}u_3(x,y,t)=&1+\cosh\left(\frac{2kx^\alpha}{\Gamma(1+\alpha)}-\frac{4ky^\alpha}{\Gamma(1+\alpha)}+\frac{16k^3L^2t^\alpha}{\Gamma(1+\alpha)}\right)\\&+\sinh\left(\frac{2kx^\alpha}{\Gamma(1+\alpha)}-\frac{4ky^\alpha}{\Gamma(1+\alpha)}+\frac{16k^3L^2t^\alpha}{\Gamma(1+\alpha)}\right).\end{aligned} \tag{3.26}$$

Also if we take $a_{-1}=1$, $b_{-1}=1$ and $6kLm=2k+m$, solution (3.21) becomes

$$u_4(x,y,t)=\frac{\cosh\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{my^\alpha}{\Gamma(1+\alpha)}-\frac{mk^2L^2t^\alpha}{\Gamma(1+\alpha)}\right)-\sinh\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{my^\alpha}{\Gamma(1+\alpha)}-\frac{mk^2L^2t^\alpha}{\Gamma(1+\alpha)}\right)}{1+\cosh\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{my^\alpha}{\Gamma(1+\alpha)}-\frac{mk^2L^2t^\alpha}{\Gamma(1+\alpha)}\right)-\sinh\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{my^\alpha}{\Gamma(1+\alpha)}-\frac{mk^2L^2t^\alpha}{\Gamma(1+\alpha)}\right)}. \tag{3.27}$$

Finally, when we take $a_1=1,\ a_{-1}=-1,\ b_0=1$ and $b_{-1}=1$, solution (3.23) becomes

$$\begin{aligned}u_5(x,y,t)=&1+\cosh\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{2ky^\alpha}{\Gamma(1+\alpha)}+\frac{2k^3L^2t^\alpha}{\Gamma(1+\alpha)}\right)\\&-\sinh\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{2ky^\alpha}{\Gamma(1+\alpha)}+\frac{2k^3L^2t^\alpha}{\Gamma(1+\alpha)}\right).\end{aligned} \tag{3.28}$$

3.3. Exact solution by $\left(\frac{G'}{G}\right)$-expansion method

According to homogeneous balancing method, with balancing the f''' and $(f')^2$, we get $2z+2=3+z$, hence $z=1$. Then we suppose that Eq.(3.2) has the following formal solution

$$f(\theta)=a_0+a_1\left(\frac{G'}{G}\right),\ a_1\neq 0. \tag{3.29}$$

By using Eq.(3.29) and Eq.(2.10) we have necessary derivaties and substituting them into Eq.(3.2), collecting the coefficients of $\left(\frac{G'}{G}\right)^i$ $(i=0,...,4)$ and setting it to zero we derive a set of algebraic equations and solving this system by Maple we get

$$\begin{aligned}a_0&=a_0, & a_1&=-\frac{6mkL}{m+2k},\\ k&=k, & m&=m,\\ n&=\left(\lambda^2-4\mu\right)mk^2L^2,\end{aligned} \tag{3.30}$$

λ and μ are arbitrary constants. By using (3.30) with expression (3.2) can be written as

$$f(\theta) = a_0 - \tfrac{6mkL}{m+2k}\left(\tfrac{G'}{G}\right). \tag{3.31}$$

When $\lambda^2 - 4\mu > 0$, substituting the general solution of (2.10) into (3.31), we obtain the following traveling wave solution of space–time fractional CD equation

$$f_1(\theta) = \quad a_0 + \frac{3mkL\lambda}{m+2k} - \frac{3mkL\sqrt{\lambda^2-4\mu}}{m+2k}\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\theta + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\theta}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\theta + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\theta}\right), \tag{3.32}$$

where $\theta = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{my^\alpha}{\Gamma(1+\alpha)} - \frac{(\lambda^2-4\mu)mk^2L^2t^\alpha}{\Gamma(1+\alpha)}$.

Similarly when $\lambda^2 - 4\mu < 0$, we get

$$f_2(\theta) = a_0 + \frac{3mkL\lambda}{m+2k} - \frac{3mkL\sqrt{4\mu-\lambda^2}}{m+2k}\left(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\theta + C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\theta}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\theta + C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\theta}\right), \tag{3.33}$$

where $\theta = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{my^\alpha}{\Gamma(1+\alpha)} - \frac{(\lambda^2-4\mu)mk^2L^2t^\alpha}{\Gamma(1+\alpha)}$.

In particular, if $C_1 \neq 0, C_2 = 0,\ \lambda > 0,\ \mu = 0,\ a_0 = 0$ then f_1 and f_2 become

$$u(x,y,t) = \frac{3mkL\lambda}{m+2k}\left\{1 - \tanh\left(\frac{\lambda kx^\alpha}{2\Gamma(1+\alpha)} + \frac{\lambda my^\alpha}{2\Gamma(1+\alpha)} - \frac{\lambda^3mk^2L^2t^\alpha}{2\Gamma(1+\alpha)}\right)\right\}. \tag{3.34}$$

4. Conclusion

In this paper, based on the the ansatz method, the exp-function method and the $(G\prime/G)-$expansion method, the space time fractional CD equation is solved exactly. The obtained exact solutions are either hyperbolic function solutions or turned into hyperbolic function solutions when suitable parameters are chosen. Comparing our results to the Mohyud-Din's results [45] it can be seen that these results are new. Moreover, when the established solutions are compared with each other, it can be seen that solutions (3.7), (3.24), (3.25), (3.26), (3.27), (3.28) and (3.34) are different and never been obtained. When we choose $\lambda = 2,\ a_0 = -\frac{3mkL\lambda}{m+2k}$ and $L = -\frac{m+2k}{4mk(m-2n)}$, solution (3.34) can be convert solution (3.7). Also, we can compare methods. While the ansatz method and the $(G\prime/G)-$expansion method only give one solution, the exp-function method gives different and variety of travelling wave solutions. The performance of these methods are found to be reliable, effective,.very powerful and convenient for solving nonlinear FDEs.

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References

[1] E. Aksoy, M. Kaplan and A. Bekir, *Exponential rational functionmethod for space–time fractional differential equations*, Waves in Random and Complex Media, 2016, 26(2), 142–151.

[2] I. Aslan, *On the application of the Exp-function method to the KP equation for N-soliton solutions*, Applied Mathematics and Computation, 2012, 219, 2825–2828.

[3] A. Bekir, E. Aksoy and A. C. Cevikel, *Exact solutions of nonlinear time fractional partial differential equations by sub-equation method*, Mathematical Methods in the Applied Sciences, 2015, 38(13), 2779–2784.

[4] A. Bekir, O. Guner and O. Unsal, *The First Integral Method for Exact Solutions of Nonlinear Fractional Differential Equations*, Journal of Computational and Nonlinear Dynamics, 2015, 10(021020), 463–470.

[5] A. Bekir and O. Guner, *Exact solutions of nonlinear fractional differential equations by (G'/G)-expansion method*, Chin. Phys. B, 2013, 22(11), 110202.

[6] A. Bekir, O. Guner, O. Unsal and M. Mirzazadeh, *Applications of fractional complex transform and (G'/G)-expansion method for time-fractional differential equations*, Journal of Applied Analysis and Computation, 2016, 6(1), 131–144.

[7] A. Bekir, O. Guner and E. Aksoy, *Periodic and hyperbolic solutions of nonlinear fractional differential equations*, Appl. Comput. Math., 2016, 15(1), 88–95.

[8] A. Bekir, O. Guner, A. H. Bhrawy and A. Biswas, *Solving nonlinear fractional differential equations using exp-function and (G'/G)-expansion methods*, Rom. Journ. Phys., 2015, 60(3–4), 360–378.

[9] A. Bekir and O. Guner, *Bright and dark soliton solutions for some nonlinear fractional differential equations*, Chin. Phys. B, 2016, 25(3), 030203.

[10] H. Bulut, H. M. Baskonus and Y. Pandir, *The Modified Trial Equation Method for Fractional Wave Equation and Time Fractional Generalized Burgers Equation*, Abstract and Applied Analysis, 2013, 2013, 636802.

[11] A. Biswas, H. Triki and M. Labidi, *Bright and Dark Solitons of the Rosenau-Kawahara Equation with Power Law Nonlinearity*, Physics of Wave Phenomena, 2011, 19(1), 24–29.

[12] A. Bekir and O. Guner, *Bright and dark soliton solutions of the (3 + 1)-dimensional generalized Kadomtsev–Petviashvili equation and generalized Benjamin equation*, Pramana-J. Phys., 2013, 81(2), 203–214.

[13] A. Bekir and A. Boz, *Application of He's exp-function method for nonlinear evolution equations*, Computers and Mathematics with Applications, 2009, 58, 2286–2293.

[14] A. Bekir, *Application of the (G'/G)-expansion method for nonlinear evolution equations*, Physics Letters A, 2008, 372, 3400–3406.

[15] J. Choi, H. Kim and R. Sakthivel, *Exact travelling wave solutions of reaction-diffusion models of fractional order*, Journal of Applied Analysis and Computation, 2017, 7(1), 236–248.

[16] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent II*, Geophys. J. Royal Astronom. Soc, 1967, 13, 529–539.

[17] V. Daftardar-Gejji and S. Bhalekar, *Solving multi-term linear and non-linear diffusion wave equations of fractional order by adomian decomposition method*, Appl. Math. Comput., 2008, 202, 113–120.

[18] A. A. Elbeleze, A. Kılıçman and B. M. Taib, *Fractional Variational Iteration Method and Its Application to Fractional Partial Differential Equation*, Mathematical Problems in Engineering, 2013, 2013, 543848.

[19] M. Eslami, B. F. Vajargah, M. Mirzazadeh and A. Biswas, *Application of first integral method to fractional partial differential equations*, Indian J Phys, 2014, 88(2), 177–184.

[20] S. M. Ege and E. Mısırlı, *The modified Kudryashov method for solving some fractional-order nonlinear equations*, Advances in Difference Equations, 2014, 2014, 135.

[21] K. A. Gepreel, T. A. Nofal and A. A. Al-Thobaiti, *Numerical solutions for the nonlinear partial fractional Zakharov-Kuznetsov equations with time and space fractional*, Scientific Research and Essays, 2014, 9, 471–482.

[22] K. A. Gepreel, T. A. Nofal and A. A. Alasmari, *Exact solutions for nonlinear integro-partial differential equations using the generalized Kudryashov method*, Journal of the Egyptian Mathematical Society, 2017, 25 (4), 438–444.

[23] K. A. Gepreel, T. A. Nofal and N. S. Al-Sayali, *Optical Soliton Solutions for Nonlinear Evolution Equations in Mathematical Physics by Using the Extended (G'/G) Expansion Function Method*, Journal of Computational and Theoretical Nanoscience, 2017, 14(2), 979–990.

[24] K. A. Gepreel, *The homotopy perturbation method to the nonlinear fractional Kolmogorov-Petrovskii-Piskunov equations*, Applied Math. Letters, 2011, 24, 1428–1434.

[25] K. A. Gepreel, *Exact solutions for nonlinear integral member of Kadomtsev-Petviashvili hierarchy differential equations using the modified (w/g)-expansion method*, Computers and Mathematics with Applications, 2016, 72(9), 2072–2083.

[26] K. A. Gepreel and A. A. Al-Thobaiti, *Exact solution of nonlinear partial fractional differential equations using the fractional sub-equation method*, Indian Journal of Phys., 2014, 88(3), 293–300.

[27] K. A. Gepreel and S. Omran, *Exact solutions for nonlinear partial fractional differential equations*, Chin. Phys. B, 2012, 21, 110204.

[28] O. Guner and D. Eser, *Exact Solutions of the Space Time Fractional Symmetric Regularized Long Wave Equation Using Different Methods*, Advances in Mathematical Physics, 2014, 2014, 456804.

[29] K. A. Gepreel, T. A. Nofal and N. S. Al-Sayali, *Exact solutions to the generalized Hirota-Satsuma KdV equations using the extended trial equation method*, Eng. Lett., 2016, 24(3) 274–283.

[30] K. A. Gepreel, *Extended trial equation method for nonlinear coupled Schrodinger Boussinesq partial differential equations*, Journal of the Egyptian Mathematical Society, 2016, 24(3), 381–391.

[31] O. Guner, *Singular and non-topological soliton solutions for nonlinear fractional differential equations*, Chin. Phys. B, 2015, 24(10), 100201.

[32] J. H. He, S. K. Elegan and Z. B. Li, *Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus*, Physics Letters A, 2012, 376, 257–259.

[33] J. H. He and M. A. Abdou, *New periodic solutions for nonlinear evolution equations using Exp-function method*, Chaos Solitons Fractals, 2007, 34(5), 1421–1429.

[34] J. H. He and X. H. Wu, *Exp-function method for nonlinear wave equations*, Chaos Solitons Fractals, 2006, 30, 700–708.

[35] M. Javidi and B. Ahmad, *Numerical solution of fourth-order time-fractional partial differential equations with variable coefficients*, Journal of Applied Analysis and Computation, 2015, 5(1), 52–63.

[36] H. Jafari, H. Tajadodi, D. Baleanu, A. A. Al-Zahrani, Y. A. Alhamed and A. H. Zahid, *Fractional sub-equation method for the fractional generalized reaction Duffing model and nonlinear fractional Sharma-Tasso-Olver equation*, Central European Journal of Physics, 2013, 11(10), 1119–1124.

[37] G. Jumarie, *Modified Riemann–Liouville derivative and fractional Taylor series of nondifferentiable functions further results*, Comput. Math. Appl., 2006, 51, 1367–1376.

[38] G. Jumarie, *Table of some basic fractional calculus formulae derived from a modified Riemann-Liouvillie derivative for nondifferentiable functions*, Appl. Maths. Lett., 2009, 22, 378–385.

[39] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[40] K. Khan and M. A. Akbar, *Exact and solitary wave solutions for the Tzitzeica–Dodd–Bullough and the modified KdV–Zakharov–Kuznetsov equations using the modified simple equation method*, Ain Shams Engineering Journal, 2013, 4(4), 903–909.

[41] K. Khan and M. A. Akbar, *Exact solutions of the (2+1)-dimensional cubic Klein–Gordon equation and the (3+1)-dimensional Zakharov–Kuznetsov equation using the modified simple equation method*, Journal of the Association of Arab Universities for Basic and Applied Sciences, 2014, 15, 74–81.

[42] W. Liu and K. Chen, *The functional variable method for finding exact solutions of some nonlinear time-fractional differential equations*, Pramana-J. Phys., 2013, 81(3), 377–384.

[43] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[44] M. S. Mohamed and K. A. Gepreel, *Reduced differential transform method for nonlinear integral member of Kadomtsev–Petviashvili hierarchy differential equations*, Journal of the Egyptian Mathematical Society, 2017, 25(1), 1–7.

[45] S. T. Mohyud-Din, T. Nawaz, E. Azhar and M. A. Akbar, *Fractional sub-equation method to space–timefractional Calogero-Degasperis and potential Kadomtsev-Petviashvili equations*, Journal of Taibah University for Science, 2017, 11(2), 258–263.

[46] I. Podlubny, *Fractional Differential Equations*, Academic Press, California, 1999.

[47] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Switzerland, 1993.

[48] S. Sahoo and S. S. Ray, *Improved fractional sub-equation method for (3+1)-dimensional generalized fractional KdV-Zakharov-Kuznetsov equations*, Computers and Mathematics with Applications, 2015, 70, 158–166.

[49] N. Shang and B. Zheng, *Exact Solutions for Three Fractional Partial Differential Equations by the (G'/G) Method*, International Journal of Applied Mathematics, 2013, 43, 3–10.

[50] N. Taghizadeh, M. Mirzazadeh, M. Rahimian and M. Akbari, *Application of the simplest equation method to some time-fractional partial differential equations*, Ain Shams Engineering Journal, 2013, 4(4), 897–902.

[51] H. Triki and A. M. Wazwaz, *Bright and dark soliton solutions for a K(m,n) equation with t-dependent coefficients*, Phys. Lett. A, 2009, 373, 2162–2165.

[52] G. Wu and E. W. M. Lee, *Fractional variational iteration method and its application*, Physics Letters A, 2010, 374, 2506–2509.

[53] M. Wang, X. Li and J. Zhang, The (G'/G)-expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics, Phys. Lett. A, 2008, 372, 417–423.

[54] L. G. Yuan and Z. Alam, *An optimal homotopy analysis method based on particle swarm optimization: application to fractional-order differential equation*, Journal of Applied Analysis and Computation, 2016, 6 (1), 103–118.

[55] X. J. Yang, D. Baleanu, Y. Khan and S. T. Mohyud-din, *Local fractional variational iteration method for Diffusion and wave equations on cantor sets*, Rom. J. Phys., 2014, 59, 36–48.

[56] S. Zhang and H. Q. Zhang, *An Exp-function method for new N-soliton solutions with arbitrary functions of a (2 +1)-dimensional vcBK system*, Computers and Mathematics with Applications, 2011, 61, 1923–1930.