Journal of Applied Analysis and Computation
Volume 8, Number 1, February 2018, 123–151
Website:http://jaac-online.com/
DOI:10.11948/2018.123

GLOBAL BIFURCATIONS NEAR A DEGENERATE HETERODIMENSIONAL CYCLE*

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Abstract This article is devoted to investigating the bifurcations of a heterodimensional cycle with orbit flip and inclination flip, which is a highly degenerate singular cycle. We show the persistence of the heterodimensional cycle and the existence of bifurcation surfaces for the homoclinic orbits or periodic orbits. It is worthy to mention that some new features produced by the degeneracies that the coexistence of heterodimensional cycles and multiple periodic orbits are presented as well, which is different from some known results in the literature. Moreover, an example is given to illustrate our results and clear up some doubts about the existence of the system which has a heterodimensional cycle with both orbit flip and inclination flip. Our strategy is based on moving frame, the fundamental solution matrix of linear variational system is chose to be an active local coordinate system along original heterodimensional cycle, which can clearly display the non-generic properties-"orbit flip" and "inclination flip" for some sufficiently large time.

Keywords Heterodimensional cycle, inclination flip, orbit flip, moving frame.

MSC(2010) 34C23, 34C37, 37C29.

1. Introduction

It is well known that the homoclinic or heteroclinic orbits play an important role in the analysis of the mechanism for the existence of chaos and traveling wave problems associated with partial differential equations. And the analysis of bifurcations of homoclinic or heteroclinic orbits is crucial step towards the understanding of the global dynamics, which has attracted so much attentions that there are plenty of interesting results achieved in the literature [1, 2, 6, 13, 25, 29, 30]. As a special case of heteroclinic loops, the heterodimensional cycles are arousing more authors' interests since the initial investigation by Newhouse and Palis [21]. A heteroclinic cycle is said to be equi-dimensional if all the equilibria in the cycle have the same

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*the corresponding author is supported by National NSF (No. 11371140; 11671114), Shanghai Key Laboratory of PMMP; the first author is supported by the Fundamental Research Funds for the Central Universities (No. 2652017151).

index (dimension of the stable manifold). Otherwise, it is called heterodimensional cycle. Heterodimensional cycles can be arose in many practical model such as Chua's circuit [10], the modified Vander Pol-duffing electronic oscillator [1] and etc. Moreover, the heterodimensional cycles with saddle-foci always lead to extremely complex dynamics behaviors. In 2004, Wen [27] proved that diffeomorphisms exhibiting either a homoclinic tangency or a heterodimensional cycle are C^1 dense in the complement of the C^1 closure of hyperbolic systems. However, as we know, the codimension for the two orbits of the heterodimensional cycle are different, for example, may be one of the orbits is of co-dimensional 0 while the other orbit is codimensional-2, which makes the investigation much tougher. Consequently, it is interesting and challenging to study the heterodimensional cycles although it is difficult.

Fernández-Sánchez, *et al.* [11] investigated the T-point-Hopf bifurcation, in which the T-point heteroclinic cycles is actually a kind of heterodimensional cycles. In 2005, Lamb [14] declared that the reversible vector fields with heterodimensional cycles are dense near Hopf-zero bifurcation. In the same year, Rademacher [22] analyzed homoclinic orbits near heterodimensional cycles connecting an equilibrium point and a periodic orbit. Geng *et al* [12] devoted to the bifurcations of heterodimensional cycles under some generic conditions, for more researches on heterodimensional cycles, one may see [3–5, 8, 17–19] and the references cited there.

As is well known, there are many heteroclinic cycles with degeneracies such as resonance eigenvalues, orbit flip or inclination flip maybe appear in the practical system, and so are the heterodimensional cycles. So recently, some authors focus their attentions on the researches of the heterodimensional cycles with degeneracies. Lu [20] studied the heterodimensional cycle bifurcation with orbit-flip, they proved that the persistent heterodimensional cycles and periodic orbits can not coexist. Liu [17] investigated the heterodimensional cycle bifurcation with inclination flip, they revealed new features produced by the inclination flip that heterodimensional cycles and homoclinic orbits coexist. Some more studies on the degenerate heterodimensional cycle are recommended to see [18, 19, 28]. A natural question would then be asking what different bifurcation features can occur from the heterodimensional cycle with both orbit flip and inclination orbit. To answer this question, we devote to investigating the global bifurcations near a heterodimensional cycle with orbit flip and inclination flip.

As we all know that a common way to discuss the homoclinic or heteroclinic bifurcations is defining a suitable codim-1 transversal section to the unperturbed orbits and a Poincaré-map which is composed by two mappings. By virtue of the construction of the return map we may derive some information about the bifurcated periodic orbits, homoclinic orbits and heteroclinic orbits, the details one may see [23]. Of course, Lin's method is another effective way to discuss the homoclinic or heteroclinic bifurcations [15]. However it is tough to deal with the different degeneracy (including the inclination flip and the orbit flip). Our strategy is based on the moving coordinates, which was initiated by Zhu and Xia [30] and then improved in [18, 20, 28] and *et al*. A suitable fundamental solution matrix of linear variational system has been chosen to be an active local coordinate system along original heterodenmensional cycle, which can clearly display the degenerate properties-"orbit flip" and "inclination flip" when the time is large enough. The bifurcation equations which include important information can also be easily obtained by our method. By constructing the moving coordinates and Poincaré maps in a sufficiently small neighborhood of the original heterodimensional cycle, we achieve the surfaces for the perturbed parameter, on which the persistence of heterodimensional cycle, the existence of homoclinic orbits and periodic orbits are established. It is worthy to mention that some new features produced by the degeneracies that the coexistence of the persistent heterodimensional cycle and multiple periodic orbits are presented, which are different from the results obtained by Lu [20] and Liu [17]. Obviously, the bifurcations of heterodimensional cycles with both orbit flip and inclination flip have essential difference to that of heterodimensional cycle with only one orbit flip or inclination flip. Moreover, to illustrate our results and eliminate doubts about the existence of system which has a heterodimensional cycle with both orbit flip and inclination flip, we present an example at the end of the paper. Further more, we can point out that our results accomplished here can be extended to any higher dimensional systems.

2. Problem and Assumptions

Consider the following C^r system

$$\dot{z} = f(z) + g(z, \mu), \tag{2.1}$$

and its unperturbed system

$$\dot{z} = f(z), \tag{2.2}$$

where $r \geq 4,\ z \in \mathbb{R}^4,\ \mu \in \mathbb{R}^l,\ l > 2,\ 0 \leq |\mu| \ll 1,\ g(z,0) = 0,\ f(z)$ is C^r with respect to the phase variable z, $g(z,\mu)$ is C^r with respect to the phase variable z and the parameter μ. In this paper, we need the following assumptions.

(H_1) There are two hyperbolic equilibria $p_i,\ i = 1,2$ for system (2.2). And the linearization matrix $Df(p_1)$ has four simple real eigenvalues: $-\rho_1^1, \lambda_1^1, \lambda_1^2, \lambda_1^3$ fulfilling $-\rho_1^1 < 0 < \lambda_1^1 < \lambda_1^2 < \lambda_1^3;\ Df(p_2)$ has four simple real eigenvalues: $-\rho_2^1, -\rho_2^2, \lambda_2^1, \lambda_2^2$ satisfying $-\rho_2^2 < -\rho_2^1 < 0 < \lambda_2^1 < \lambda_2^2,\ \rho_2^2 \geq 3\rho_2^1,\ \lambda_2^2 \geq 3\lambda_2^1$.

(H_2) System (2.2) has a heteroclinic cycle $\Gamma = \Gamma_1 \cup \Gamma_2$ joining p_1 and p_2, where $\Gamma_i = \{z = r_i(t) : t \in \mathbb{R}\},\ r_1(+\infty) = r_2(-\infty) = p_2,\ r_1(-\infty) = r_2(+\infty) = p_1$, and

$$dim(T_{r_1(t)} W_{p_1}^u \cap T_{r_1(t)} W_{p_2}^s) = 1.$$

Here $r_i(t)$ denotes the flow of system (2.2), $t \in \mathbb{R},\ W_{p_i}^s$ and $W_{p_i}^u$ are the C^r stable and unstable manifolds of p_i. And $T_p M$ denotes the tangent space of the manifold M at p.

(H_3) Let $e_i^{\pm} = \lim\limits_{t \to \mp\infty} \frac{\dot{r}_i(t)}{|\dot{r}_i(t)|}$, then $e_1^+ \in T_{p_1} W_{p_1}^{uu},\ e_2^+ \in T_{p_2} W_{p_2}^u,\ e_1^- \in T_{p_2} W_{p_2}^s$, $e_2^- \in T_{p_1} W_{p_1}^s$ be unit eigenvectors corresponding to $\lambda_1^2, \lambda_2^1, -\rho_2^1, -\rho_1^1$, respectively, where $W_{p_1}^{uu}$ is the strong unstable manifold of p_1.

(H_4) $\lim\limits_{t \to -\infty} T_{r_1(t)} W_{p_2}^s = span\{e_1^+, e_2^-\},\ \ \lim\limits_{t \to +\infty} T_{r_1(t)} W_{p_1}^u = span\{e_1^-, e_2^+, e^{u+}\}$,

$$\lim_{t \to -\infty} T_{r_2(t)} W_{p_1}^s = span\{e_2^+\}, \qquad \lim_{t \to +\infty} T_{r_2(t)} W_{p_2}^u = span\{e_2^-, e^+\},$$

where e^+, e^{u+} is the unit eigenvector corresponding to λ_1^1, λ_2^2, respectively.

Remark 2.1. It is easy to see from (H_1) that Γ is a heterodimensional cycle. And the condition (H_2) means that Γ_1 is a transverse orbit, so it can be preserved under a small perturbation. That is, Γ_1 is of codimension 0 and Γ_2 is of codimension 2. (H_3) means that Γ_1 is in orbit-flip as $t \to -\infty$, namely, the heteroclinic orbit Γ_1

tends to p_1 along the strong unstable direction when $t \to -\infty$. While the fourth equation in (H_4) indicates that $W^u_{p_2}$ is in inclination flip as $t \to +\infty$. One may see Figure 1, where we draw the manifold $W^u_{p_2}$ only.

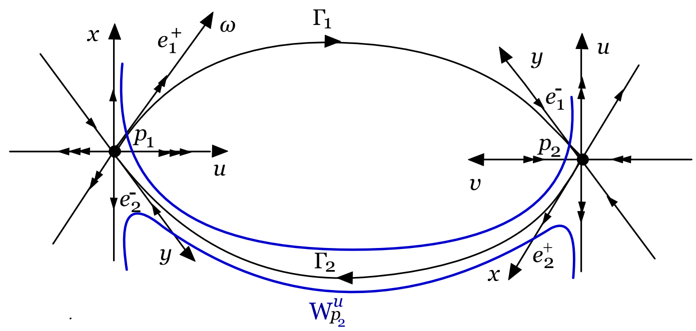

Figure 1. Heterodimensional cycle $\Gamma = \Gamma_1 \cup \Gamma_2$ with Γ_1 orbit flip and $W^u_{p_2}$ inclination flip.

Remark 2.2. In fact, the restriction on the dimension is not essential, we may extend our results to any higher dimensional system. For example, we can consider a 5-dimensional system with the numbers of unstable and stable eigenvalues are (2,3) and (3,2) respectively.

3. Local Coordinates and Poincaré map

In this section we shall achieve the normal form of system (2.1) and establish the Poincaré map near the original heterodimensional cycle Γ. And then the bifurcation equations will be obtained by virtue of the successor functions we define.

Firstly, we shall establish the normal form of system (2.1). Suppose that U_i is the sufficiently small neighborhood of p_i, then by using a translation and a linear transformation, system (2.1) turns to be the following form in U_1:

$$\begin{cases} \dot{x} = \lambda_1^1(\mu)x + O(2), \\ \dot{y} = -\rho_1^1(\mu)y + O(2), \\ \dot{u} = \lambda_1^3(\mu)u + O(2), \\ \dot{\omega} = \lambda_1^2(\mu)\omega + O(2), \end{cases}$$

and in the neighborhood U_2, system (2.1) becomes

$$\begin{cases} \dot{x} = \lambda_2^1(\mu)x + O(2), \\ \dot{y} = -\rho_2^1(\mu)y + O(2), \\ \dot{u} = \lambda_2^2(\mu)u + O(2), \\ \dot{v} = -\rho_2^2(\mu)v + O(2), \end{cases}$$

where $\lambda_1^i(0) = \lambda_1^i$, $\rho_1^1(0) = \rho_1^1$, $i = 1, 2, 3$. $\rho_2^j(0) = \rho_2^j$, $\lambda_2^j(0) = \lambda_2^j$, $j = 1, 2$. For notational convenience we use $\lambda_1^i(\mu)$, $-\rho_1^1(\mu)$, $i = 1, 2, 3$, and $\rho_2^j(\mu)$, $\lambda_2^j(\mu)$, $j = 1, 2$

as the corresponding eigenvalues of the linearization matrix of perturbed system (2.1), which depends on the small parameter μ obviously.

Next, according to the stable (unstable) and strong stable manifold theorem manifold theorems, we may choose two successive C^r and C^{r-1} transformations such that the local stable manifold, unstable manifold, strong unstable manifold can be straightened in the region of U_i, and they are rendered as

$$\begin{aligned}
&W^u_{p_1}=\{(x,y,u,\omega):y=0\}, && W^s_{p_1}=\{(x,y,u,\omega):x=u=\omega=0\},\\
&W^{uu}_{p_1}=\{(x,y,u,\omega):x=y=0\}, && W^s_{p_2}=\{(x,y,u,v):x=u=0\},\\
&W^u_{p_2}=\{(x,y,u,v):y=v=0\}, && W^{uu}_{p_2}=\{(x,y,u,v):x=y=v=0\}.
\end{aligned}$$

Also, we can straighten the orbit segments $\Gamma_i\cap U_1$, $\Gamma_i\cap U_2$, $i=1,2$.

Then due to the invariance of these manifolds, the system (2.1) has the following C^k normal form in U_1 of p_1:

$$\begin{cases}
\dot{x}=(\lambda_1^1(\mu)+o(1))x+O(y)[O(u)+O(\omega)],\\
\dot{y}=(-\rho_1^1(\mu)+o(1))y,\\
\dot{u}=(\lambda_1^3(\mu)+o(1))u+[O(x)+O(y)][O(x)+O(\omega)],\\
\dot{\omega}=(\lambda_1^2(\mu)+o(1))\omega+[O(x)+O(u)][O(x)+O(y)],
\end{cases}\tag{3.1}$$

and has C^k normal form in U_2 of p_2 as:

$$\begin{cases}
\dot{x}=(\lambda_2^1(\mu)+o(1))x+O(u)[O(y)+O(v)],\\
\dot{y}=(-\rho_2^1(\mu)+o(1))y+O(v)[O(x)+O(u)],\\
\dot{u}=(\lambda_2^2(\mu)+o(1))u+O(x)[O(y)+O(v)],\\
\dot{v}=(-\rho_2^2(\mu)+o(1))v+O(y)[O(x)+O(u)],
\end{cases}\tag{3.2}$$

where $k=\min\{r-2,\frac{\lambda_2^2}{\lambda_2^1}-1,\frac{\rho_2^2}{\rho_2^1}-1\}\geq 2$, which is owing to that the weak unstable manifold of p_1, and the weak stable manifold of P_2 are approximately $C^{\frac{\lambda_1^2}{\lambda_1^1}}$, $C^{\frac{\rho_2^2}{\rho_2^1}}$, respectively (see [24]). Of course, the same kind of change of variable can be achieved by using the theory of exponential dichotomies and weighted exponential dichotomies to get the normal form. But by [24], we know that the extra conditions $\lambda_1^2\geq 3\lambda_1^1$ and $\rho_2^2\geq 3\rho_2^1$ are necessary to ensure such change of coordinates are possible, so that the system (3.1),(3.2) are smooth enough.

Denote the orbits $r_i(t)$ by $r_i(t)=(r_i^x(t),\ r_i^y(t),\ r_i^u(t),r_i^\omega(t))^*$ in U_1, and $r_i(t)=(r_i^x(t),\ r_i^y(t),r_i^u(t),\ r_i^v(t))^*$ in U_2, $i=1,2$. Let T_i be large enough such that $r_1(-T_1)=(0,0,0,\delta)^*,r_1(T_1)=(0,\delta,0,0)^*,r_2(-T_2)=(\delta,0,0,0)^*,r_2(T_2)=(0,\delta,0,0)^*$, where "$*$" denotes the transposition, and $\delta>0$ is small enough such that $\{(x,y,u,\omega)^*:|x|,|y|,|u|,|\omega|<2\delta\}\subset U_1,\{(x,y,u,v)^*:|x|,|y|,|u|,|v|<2\delta\}\subset U_2$.

Take into account the linear variational system of (2.2)

$$\dot{Z}=Df(r_i(t))Z,\tag{3.3}$$

and its adjoint system

$$\dot{\Phi}=-(Df(r_i(t)))^*\Phi.\tag{3.4}$$

Let $Z(t)$ and $\Phi(t)$ be the fundamental solution matrixes of (3.3) and (3.4) respectively, the known results tell us that they have the relation as $(Z^{-1}(t))^* = \Phi(t)$.

Note that the assumption (H_1) means that the two equilibria are hyperbolic, which implies system (3.3) has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$, so the following properties can be guaranteed.

Lemma 3.1. *Assume (H_1)-(H_4) hold, then (1) there exists a fundamental solution matrix $Z_1(t) = (Z_1^1(t), Z_1^2(t), Z_1^3(t), Z_1^4(t))$ for system (3.3) satisfying*

$$
\begin{aligned}
&Z_1^1(t) = \frac{\dot{r}_1(t)}{|\dot{r}_1(-T_1)|} \in T_{r_1(t)}W^u_{p_1} \cap T_{r_1(t)}W^s_{p_2},\\
&Z_1^2(t), Z_1^3(t) \in T_{r_1(t)}W^u_{p_1} \cap (T_{r_1(t)}W^s_{p_2})^c,\\
&Z_1^4(t) \in (T_{r_1(t)}W^u_{p_1})^c \cap T_{r_1(t)}W^s_{p_2},
\end{aligned}
$$

such that

$$
Z_1(-T_1) = \begin{pmatrix} 0 & 1 & 0 & \omega_1^{41} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \omega_1^{43} \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; Z_1(T_1) = \begin{pmatrix} 0 & \omega_1^{21} & \omega_1^{31} & 0 \\ \omega_1^{12} & \omega_1^{22} & \omega_1^{32} & \omega_1^{42} \\ 0 & \omega_1^{23} & \omega_1^{33} & 0 \\ 0 & \omega_1^{24} & \omega_1^{34} & \omega_1^{44} \end{pmatrix},
$$

where $\omega_1^{12} < 0, \omega_1^{44} \neq 0, d_1 = \begin{vmatrix} \omega_1^{21} & \omega_1^{31} \\ \omega_1^{23} & \omega_1^{33} \end{vmatrix} \neq 0$. *The notation $(M)^c$ means subspace complementary to M.*

(2) there exists a fundamental solution matrix $Z_2(t) = (Z_2^1(t), Z_2^2(t), Z_2^3(t), Z_2^4(t))$ for system (3.3) satisfying

$$
\begin{aligned}
&Z_2^1(t), Z_2^2(t) \in (T_{r_2(t)}W^u_{p_2})^c,\\
&Z_2^3(t) = \frac{\dot{r}_2(t)}{|\dot{r}_2(-T_2)|} \in T_{r_2(t)}W^u_{p_2} \cap T_{r_2(t)}W^s_{p_1},\\
&Z_2^4(t) \in T_{r_2(t)}W^u_{p_2} \cap (T_{r_2(t)}W^s_{p_1})^c,
\end{aligned}
$$

$$
Z_2(-T_2) = \begin{pmatrix} \omega_2^{11} & \omega_2^{21} & 1 & 0 \\ \omega_2^{12} & \omega_2^{22} & 0 & 0 \\ \omega_2^{13} & \omega_2^{23} & 0 & 1 \\ \omega_2^{14} & \omega_2^{24} & 0 & 0 \end{pmatrix}, \; Z_2(T_2) = \begin{pmatrix} 0 & 0 & 0 & \omega_2^{41} \\ 0 & 0 & \omega_2^{32} & \omega_2^{42} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},
$$

where $\omega_2^{32} < 0, \omega_2^{41} \neq 0, d_2 = \begin{vmatrix} \omega_2^{12} & \omega_2^{22} \\ \omega_2^{14} & \omega_2^{24} \end{vmatrix} \neq 0.$

Proof. (1) Note that the heteroclinic orbit $r_1(t)$ tends to p_1 along the strong unstable manifold $W^{uu}_{p_1}$ when $t \to -\infty$, and tends to p_2 along the weak stable manifold $W^s_{p_2}$ as $t \to +\infty$, then based on the fact $Z_1^1(t) = \frac{\dot{r}_1(t)}{|\dot{r}_1(-T_1)|}$ and the orbit segments have been straightened, it is easy to have the expressions of $Z_1^1(-T_1), Z_1^1(T_1)$ and the fact $\omega_1^{12} < 0$. Choose $Z_1^2(t), Z_1^3(t) \in T_{r_1(t)}W^u_{p_1}$, then the strong inclination property guarantees that $d_1 \neq 0$. Let $\bar{Z}_1^4(t) \in (T_{r_1(t)}W_{p_1}^u)^c$ with $\bar{Z}_1^4(-T_1) = (0,1,0,0)^*$, we have $\bar{Z}_1^4(T_1) = (\bar{\omega}_1^{41}, \bar{\omega}_1^{42}, \bar{\omega}_1^{43}, \bar{\omega}_1^{44})^*$. Then $Z_1^4(t) = \bar{Z}_1^4(t) - d_1^{-1}(\bar{\omega}_1^{41}\omega_1^{33} - \bar{\omega}_1^{43}\omega_1^{31})Z_1^2(t) - d_1^{-1}(\bar{\omega}_1^{43}\omega_1^{21} - \bar{\omega}_1^{41}\omega_1^{23})Z_1^3(t)$, is also one solution in $(T_{r_1(t)}W_{p_1}^u)^c$ based on the property of the solution to the linear system Consequently, we achieve that $Z_1^4(-T_1) = (\omega_1^{41}, 1, \omega_1^{43}, 0)^*$, and $Z_1^4(T_1) = (0, \omega_1^{42}, 0, \omega_1^{44})^*$. Since $Z_1(t)$ is a fundamental solution matrix, we know that $\det Z_1(-T_1) \neq 0$, together with the Liouville formula, we have $\omega_1^{44} \neq 0$.

The proof of result (2) can be finished with similar argument of proof for result (1). □

Remark 3.1. The first columns of matrixes $Z_1(-T_1)$, $Z_1(T_1)$ clearly display the degenerate condition of "orbit flip", and the fourth columns of $Z_2(-T_2)$, $Z_2(T_2)$ clearly exhibit the degenerate condition of "inclination flip."

Take $(Z_i^1(t), Z_i^2(t), Z_i^3(t), Z_i^4(t)), i = 1, 2$ as a new local coordinate system along the original heterodimensional cycle Γ. Denote $\Phi_i(t) = (\phi_i^1, \phi_i^2, \phi_i^3, \phi_i^4)$, $\Phi_i(t)$ is defined as before. Take a coordinate transformation near the orbits Γ_i as

$$z(t) = S_i(t) \overset{def}{=} r_i(t) + Z_i(t)N_i(t),$$

where $N_1(t) = (0, n_1^2, n_1^3, n_1^4)^*$, $N_2(t) = (n_2^1, n_2^2, 0, n_2^4)^*$, and the components n_1^2, n_1^3, n_1^4 (resp. n_2^1, n_2^2, n_2^4) are the coordinate decomposition of system (2.1) in the new local coordinate system corresponding to $Z_1^2(t), Z_1^3(t), Z_1^4(t)$ (resp. $Z_2^1(t), Z_2^2(t), Z_2^4(t)$). Define the cross-sections as

$$\begin{aligned} S_1^0 &= \{z = S_1(-T_1) : |x|, |y|, |u|, |\omega| < 2\delta\}, \\ S_1^1 &= \{z = S_1(T_1) : |x|, |y|, |u|, |v| < 2\delta\}, \\ S_2^0 &= \{z = S_2(-T_2) : |x|, |y|, |u|, |v| < 2\delta\}, \\ S_2^1 &= \{z = S_2(T_2) : |x|, |y|, |u|, |\omega| < 2\delta\}, \end{aligned}$$

which intersect Γ_i transversally. (see Figure 2)

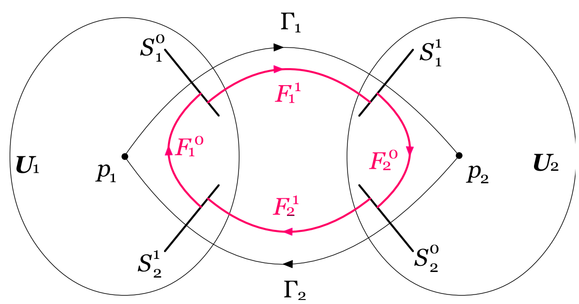

Figure 2. The cross sections and Poincaré map.

Next, we construct Poincaré map by two steps, which has four components $F_1^0 : S_2^1 \to S_1^0, F_1^1 : S_1^0 \to S_1^1, F_2^0 : S_1^1 \to S_2^0, F_2^1 : S_2^0 \to S_2^1$.

Step 1. Put $z(t) = S_i(t) \overset{def}{=} r_i(t) + Z_i(t)N_i(t)$ into equation (2.1), notice that $\dot{r}_i(t) = f(r_i(t))$, $\dot{Z}_i(t) = Df(r_i(t))Z_i(t)$, we have:

$$\dot{N}_i(t) = \Phi_i^*(t)g_\mu(r_i(t), 0)\mu + h.o.t.,$$

where g_μ is the partial derivation of $g(z,\mu)$ with respect to μ. Integrating both sides of the above equation from $-T_i$ to T_i, we obtain

$$N_i(T_i) = N_i(-T_i) + \int_{-T_i}^{T_i} \Phi_i^*(t) g_\mu(r_i(t),0)\mu \, \mathrm{d}t + h.o.t.,$$

which then defines the global map $F_1^1 : S_1^0 \longrightarrow S_1^1$ and $F_2^1 : S_2^0 \longrightarrow S_2^1$, as follows

$$F_1^1(0, n_1^{0,2}, n_1^{0,3}, n_1^{0,4})^* = (0, \bar{n}_1^{1,2}, \bar{n}_1^{1,3}, \bar{n}_1^{1,4})^*,$$
$$F_2^1(n_2^{0,1}, n_2^{0,2}, 0, n_2^{0,4})^* = (\bar{n}_2^{1,1}, \bar{n}_2^{1,2}, 0, \bar{n}_2^{1,4})^*,$$

where

$$\bar{n}_1^{1,j} = n_1^{0,j} + M_1^j \mu + h.o.t., \qquad \bar{n}_2^{1,k} = n_2^{0,k} + M_2^k \mu + h.o.t., \tag{3.5}$$

and $M_1^j = \int_{-T_1}^{T_1} \phi_1^{j*}(t) g_\mu(r_1(t),0)\,\mathrm{d}t$, $j = 2,3,4$; $M_2^k = \int_{-T_2}^{T_2} \phi_2^{k*}(t) g_\mu(r_2(t),0)\,\mathrm{d}t$, $k = 1,2,4$.

For the sake of simplicity for computation, we need the following result.

Lemma 3.2.

$$M_1^j = \int_{-T_1}^{T_1} \phi_1^{j*}(t) g_\mu(r_1(t),0)\,\mathrm{d}t = \int_{-\infty}^{+\infty} \phi_1^{j*}(t) g_\mu(r_1(t),0)\,\mathrm{d}t,\ \ j = 2,3,4;$$
$$M_2^k = \int_{-T_2}^{T_2} \phi_2^{k*}(t) g_\mu(r_2(t),0)\,\mathrm{d}t = \int_{-\infty}^{+\infty} \phi_2^{k*}(t) g_\mu(r_2(t),0)\,\mathrm{d}t,\ \ k = 1,2,4.$$

Proof. To avoid the redundant illustration, we only show that the equality

$$M_1^2 = \int_{-\infty}^{+\infty} \phi_1^{2*}(t) g_\mu(r_1(t),0)\,\mathrm{d}t \tag{3.6}$$

is true, the others can be obtained with similar arguments. To obtain (3.6) what we need to do is proving $\phi_1^{2*}(t) g_\mu(r_1(t),0) = 0$ when $|t| \geq T_1$. Set

$$\phi_1^{2*}(t) = (\phi_1^{21}, \phi_1^{22}, \phi_1^{23}, \phi_1^{24}),$$

Note that $\Phi_1^*(t) Z_1(t) = I$, it then follows that $\phi_1^{2*}(t) Z_1^1(t) = 0$. Together with $Z_1^1(T_1) = (0, \omega_1^{12}, 0, 0)^*$, $Z_1^1(-T_1) = (0,0,0,1)^*$, we have $\phi_1^{22}(T_1) = \phi_1^{24}(-T_1) = 0$.

Since $r_1(t) = (0, r_1^y(t), 0, 0)^*$ as $t \geq T_1$, where $|r_1^y(t)| = O(\delta)$. Note (3.2), we obtain

$$Df(r_1(t)) = \begin{pmatrix} \lambda_2^1 + O(\delta) & 0 & O(\delta) & 0 \\ O(\delta) & -\rho_2^1 + O(\delta) & O(\delta) & O(\delta) \\ O(\delta) & 0 & \lambda_2^2 + O(\delta) & 0 \\ O(\delta) & 0 & O(\delta) & -\rho_2^2 + O(\delta) \end{pmatrix}.$$

As $\phi_1^2(t)$ is a solution of $\dot{\Phi} = -(Df(r_1(t)))^*\Phi$, then $\dot{\phi}_1^{22}(t) = -[-\rho_2^1 + O(\delta)]\phi_1^{22}(t)$. According to $\phi_1^{22}(T_1) = 0$, it follows $\phi_1^{22}(t) = 0$ for $t \geq T_1$. Similarly, as $r_1(t) = (0,0,0,r_1^\omega(t))^*$ for $t \leq -T_1$, we have $\phi_1^{24}(t) = 0$ as $t \leq -T_1$.

Based on the normal forms (3.1) and (3.2), we get

$$g_\mu(r_1(t),0) = (0, O(\delta), 0, 0)^*,\ \text{ for } t \geq T_1,$$

$$g_\mu(r_1(t),0) = (0,0,0,O(\delta))^*, \text{ for } t \le -T_1.$$

It then yields $\phi_1^{2*}(t)g_\mu(r_1(t),0) = 0$, $|t| \ge T_1$. The conclusion is verified. □

Step 2. Next we shall establish the local maps $F_1^0 : q_2^1 \in S_2^1 \longrightarrow q_1^0 \in S_1^0$ and $F_2^0 : q_1^1 \in S_1^1 \longrightarrow q_2^0 \in S_2^0$ induced by flows in the neighborhood U_i.

Let τ_i $(i = 1,2)$ be the time spent from q_{i-1}^1 to q_i^0, $q_0^1 = q_2^1$. Suppose $\rho_1^1 > \lambda_1^1$, $\lambda_2^1 > \rho_2^1$, then we select $s_1 = e^{-\lambda_1^1(\mu)\tau_1}$, $s_2 = e^{-\rho_2^1(\mu)\tau_2}$ (if $\rho_1^1 < \lambda_1^1$, $\lambda_2^1 < \rho_2^1$, then it turns to $s_1 = e^{-\rho_1^1(\mu)\tau_1}$, $s_2 = e^{-\lambda_2^1(\mu)\tau_2}$). According to the normal forms (3.1), (3.2), the local map $F_1^0 : q_2^1(x_2^1, y_2^1, u_2^1, \omega_2^1) \in S_2^1 \to q_1^0(x_1^0, y_1^0, u_1^0, \omega_1^0) \in S_1^0$ can be expressed as

$$\begin{aligned} &x_2^1 = x(T_2) \approx s_1 x_1^0,\ y_1^0 = y(T_2+\tau_1) \approx \delta s_1^{\beta_1(\mu)}, \\ &u_2^1 = u(T_2) \approx s_1^{\frac{\lambda_1^3(\mu)}{\lambda_1^1(\mu)}} u_1^0,\ \omega_2^1 = \omega(T_2) \approx \delta s_1^{\frac{\lambda_1^2(\mu)}{\lambda_1^1(\mu)}}, \end{aligned} \tag{3.7}$$

and the local map $F_2^0 : q_1^1(x_1^1, y_1^1, u_1^1, v_1^1) \in S_1^1 \to q_2^0(x_2^0, y_2^0, u_2^0, v_2^0) \in S_2^0$ can be expressed as

$$\begin{aligned} &x_1^1 = x(T_1) \approx \delta s_2^{\frac{1}{\beta_2(\mu)}},\ y_2^0 = y(T_1+\tau_2) \approx \delta s_2, \\ &u_1^1 = u(T_1) \approx s_2^{\frac{\lambda_2^2(\mu)}{\rho_2^1(\mu)}} u_2^0,\ v_2^0 = v(T_1+\tau_2) \approx s_2^{\frac{\rho_2^2(\mu)}{\rho_2^1(\mu)}} v_1^1, \end{aligned} \tag{3.8}$$

where $\beta_1(\mu) = \dfrac{\rho_1^1(\mu)}{\lambda_1^1(\mu)}, \dfrac{1}{\beta_2(\mu)} = \dfrac{\lambda_2^1(\mu)}{\rho_2^1(\mu)}$, we call $(s_1, s_2, x_1^0, u_1^0, u_2^0, v_1^1)$ Shilnikov variables.

To get the Poincaré map, we still need to establish the relationship between the old coordinates

$$q_1^0(x_1^0, y_1^0, u_1^0, \omega_1^0)^*,\ q_1^1(x_1^1, y_1^1, u_1^1, v_1^1)^*,\ q_2^0(x_2^0, y_2^0, u_2^0, v_2^0)^*,\ q_2^1(x_2^1, y_2^1, u_2^1, \omega_2^1)^*$$

and their new coordinates

$$q_1^0(0, n_1^{0,2}, n_1^{0,3}, n_1^{0,4}),\ q_1^1(0, n_1^{1,2}, n_1^{1,3}, n_1^{1,4}),\ q_2^0(n_2^{0,1}, n_2^{0,2}, 0, n_2^{0,4}),\ q_2^1(n_2^{1,1}, n_2^{1,2}, 0, n_2^{1,4}).$$

Based on $S_i(t) = r_i(t) + Z_i(t)N_i(t)$, and the expressions of $Z_i(-T_i)$, $Z_i(T_i)$, $(i = 1,2)$, we obtain

$$\begin{cases} n_1^{0,2} = x_1^0 - \omega_1^{41} y_1^0, \\ n_1^{0,3} = u_1^0 - \omega_1^{43} y_1^0, \\ n_1^{0,4} = y_1^0, \\ n_2^{0,1} = d_2^{-1}(\omega_2^{24} y_2^0 - \omega_2^{22} v_2^0), \\ n_2^{0,2} = d_2^{-1}(\omega_2^{12} v_2^0 - \omega_2^{14} y_2^0), \\ n_2^{0,4} = u_2^0 + d_2^{-1}[(\omega_2^{23}\omega_2^{14} - \omega_2^{13}\omega_2^{24}) y_2^0 + (\omega_2^{13}\omega_2^{22} - \omega_2^{23}\omega_2^{12}) v_2^0], \end{cases} \tag{3.9}$$

and

$$\begin{cases} n_1^{1,2} = d_1^{-1}(\omega_1^{33}x_1^1 - \omega_1^{31}u_1^1), \\ n_1^{1,3} = d_1^{-1}(\omega_1^{21}u_1^1 - \omega_1^{23}x_1^1), \\ n_1^{1,4} = (\omega_1^{44})^{-1}v_1^1 + (\omega_1^{44})^{-1}d_1^{-1}[(\omega_1^{23}\omega_1^{34} - \omega_1^{24}\omega_1^{33})x_1^1 + (\omega_1^{24}\omega_1^{31} - \omega_1^{21}\omega_1^{34})u_1^1], \\ n_2^{1,1} = u_2^1, \\ n_2^{1,2} = \omega_2^1, \\ n_2^{1,4} = (\omega_2^{41})^{-1}x_2^1. \end{cases} \tag{3.10}$$

Together with equations (3.5), (3.7), (3.9), we have the Poincaré map $F_1 = F_1^1 \circ F_1^0$: $S_2^1 \to S_1^1$ as follows

$$\begin{cases} \bar{n}_1^{1,2} = x_1^0 - \delta\omega_1^{41}s_1^{\beta_1(\mu)} + M_1^2\mu + h.o.t., \\ \bar{n}_1^{1,3} = u_1^0 - \delta\omega_1^{43}s_1^{\beta_1(\mu)} + M_1^3\mu + h.o.t., \\ \bar{n}_1^{1,4} = \delta s_1^{\beta_1(\mu)} + M_1^4\mu + h.o.t.. \end{cases} \tag{3.11}$$

and by (3.5), (3.8), (3.9), we obtain the Poincaré map $F_2 = F_2^1 \circ F_2^0$: $S_1^1 \to S_2^1$ as follows

$$\begin{cases} \bar{n}_2^{1,1} = d_2^{-1}(\delta\omega_2^{24}s_2 - \omega_2^{22}s_2^{\frac{\rho_2^2(\mu)}{\rho_2^1(\mu)}}v_1^1) + M_2^1\mu + h.o.t., \\ \bar{n}_2^{1,2} = d_2^{-1}(\omega_2^{12}s_2^{\frac{\rho_2^2(\mu)}{\rho_2^1(\mu)}}v_1^1 - \delta\omega_2^{14}s_2) + M_2^2\mu + h.o.t., \\ \bar{n}_2^{1,4} = u_2^0 + d_2^{-1}[\delta(\omega_2^{23}\omega_2^{14} - \omega_2^{13}\omega_2^{24})s_2 + (\omega_2^{13}\omega_2^{22} - \omega_2^{23}\omega_2^{12})s_2^{\frac{\rho_2^2(\mu)}{\rho_2^1(\mu)}}v_1^1] \\ \qquad + M_2^4\mu + h.o.t.. \end{cases} \tag{3.12}$$

Consequently, the successor functions

$$\begin{aligned} (G_1, G_2) &\overset{def}{=} G(s_1, s_2, x_1^0, u_1^0, u_2^0, v_1^1) \\ &= (G_1^2, G_1^3, G_1^4, G_2^1, G_2^2, G_2^4) = (F_1(q_2^1) - q_1^1,\ F_2(q_1^1) - q_2^1) \end{aligned}$$

as follows

$$\begin{cases} G_1^2 = x_1^0 - \delta\omega_1^{41}s_1^{\beta_1(\mu)} - d_1^{-1}(\delta\omega_1^{33}s_2^{\frac{1}{\beta_2(\mu)}} - \omega_1^{31}s_2^{\frac{\lambda_2^2(\mu)}{\rho_2^1(\mu)}}u_2^0) + M_1^2\mu + h.o.t., \\ G_1^3 = u_1^0 - \delta\omega_1^{43}s_1^{\beta_1(\mu)} - d_1^{-1}(\omega_1^{21}s_2^{\frac{\lambda_2^2(\mu)}{\rho_2^1(\mu)}}u_2^0 - \delta\omega_1^{23}s_2^{\frac{1}{\beta_2(\mu)}}) + M_1^3\mu + h.o.t., \\ G_1^4 = \delta s_1^{\beta_1(\mu)} - (\omega_1^{44})^{-1}v_1^1 - (\omega_1^{44})^{-1}d_1^{-1}[\delta(\omega_1^{23}\omega_1^{34} - \omega_1^{24}\omega_1^{33})s_2^{\frac{1}{\beta_2(\mu)}} + (\omega_1^{24}\omega_1^{31} \\ \qquad - \omega_1^{21}\omega_1^{34})s_2^{\frac{\lambda_2^2(\mu)}{\rho_2^1(\mu)}}u_2^0] + M_1^4\mu + h.o.t., \\ G_2^1 = \delta\omega_2^{24}d_2^{-1}s_2 - d_2^{-1}\omega_2^{22}s_2^{\frac{\rho_2^2(\mu)}{\rho_2^1(\mu)}}v_1^1 - s_1^{\frac{\lambda_1^3(\mu)}{\lambda_1^1(\mu)}}u_1^0 + M_2^1\mu + h.o.t., \\ G_2^2 = d_2^{-1}\omega_2^{12}s_2^{\frac{\rho_2^2(\mu)}{\rho_2^1(\mu)}}v_1^1 - \delta\omega_2^{14}d_2^{-1}s_2 - \delta s_1^{\frac{\lambda_1^2(\mu)}{\lambda_1^1(\mu)}} + M_2^2\mu + h.o.t., \\ G_2^4 = u_2^0 - (\omega_2^{41})^{-1}s_1x_1^0 + d_2^{-1}[\delta(\omega_2^{23}\omega_2^{14} - \omega_2^{13}\omega_2^{24})s_2 + (\omega_2^{13}\omega_2^{22} - \omega_2^{23}\omega_2^{12})s_2^{\frac{\rho_2^2(\mu)}{\rho_2^1(\mu)}}v_1^1] \\ \qquad + M_2^4\mu + h.o.t., \end{cases}$$

can be achieved by using (3.10), (3.11), (3.12).

As we know, the non-generic conditions of heterodimensional cycle Γ can yield that

$$W=\frac{\partial G}{\partial Q}|_{Q=0,\mu=0}=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & -(\omega_1^{44})^{-1} & 0\\ 0 & \delta d_2^{-1}\omega_2^{24} & 0 & 0 & 0 & 0\\ 0 & -\delta d_2^{-1}\omega_2^{14} & 0 & 0 & 0 & 0\\ 0 & \delta d_2^{-1}(\omega_2^{23}\omega_2^{14}-\omega_2^{13}\omega_2^{24}) & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.13}$$

is degenerate at $Q=(s_1,s_2,x_1^0,u_1^0,v_1^1,u_2^0)=0$. Which implies the implicit function theorem is not work here. That is, the uniqueness of heteroclinic loop, homoclinic loop or periodic orbit cannot be guaranteed, in another words, their coexistence may be possible.

Next, we denote $\lambda_1^i=\lambda_1^i(\mu), i=1,2,3; \rho_1^1=\rho_1^1(\mu), \beta_1=\frac{\rho_1^1(\mu)}{\lambda_1^1(\mu)}, \beta_2=\frac{\rho_2^1(\mu)}{\lambda_2^1(\mu)}; \rho_2^j(\mu)=\rho_2^j, \lambda_2^j=\lambda_2^j(\mu), j=1,2.$

Notice that the four columns of (3.13), we know that $(x_1^0,u_1^0,v_1^1,u_2^0,)$ can be solved uniquely from $(G_1^2,\ G_1^3,\ G_1^4,\ G_2^4)=0$. And then put it into $(G_2^1,G_2^2)=0$, we obtain the bifurcation equations:

$$\begin{cases} \omega_2^{24}s_2=-\delta^{-1}d_2M_2^1\mu+\omega_1^{43}d_2s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\beta_1}+\omega_2^{22}s_2^{\frac{\rho_2^2}{\rho_2^1}}[\omega_1^{44}s_1^{\beta_1}-d_1^{-1}(\omega_1^{23}\omega_1^{34}-\omega_1^{24}\omega_1^{33})s_2^{\frac{1}{\beta_2}}\\ \qquad +\delta^{-1}\omega_1^{44}M_1^4\mu]-\omega_2^{23}d_1^{-1}d_2s_1^{\frac{\lambda_1^3}{\lambda_1^1}}s_2^{\frac{1}{\beta_2}}-\delta^{-1}d_2M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}}+h.o.t.,\\ \omega_2^{14}s_2=\delta^{-1}d_2M_2^2\mu-d_2s_1^{\frac{\lambda_1^2}{\lambda_1^1}}+\omega_2^{12}s_2^{\frac{\rho_2^2}{\rho_2^1}}[\omega_1^{44}s_1^{\beta_1}-d_1^{-1}(\omega_1^{23}\omega_1^{34}-\omega_1^{24}\omega_1^{33})s_2^{\frac{1}{\beta_2}}\\ \qquad +\delta^{-1}\omega_1^{44}M_1^4\mu]+h.o.t.. \end{cases} \tag{3.14}$$

Remark 3.2. From the expression of $Z_2(-T_2)$ in Lemma 1, we have $d_2=\begin{vmatrix}\omega_2^{12} & \omega_2^{22}\\ \omega_2^{14} & \omega_2^{24}\end{vmatrix}$ $\neq 0$, that is, $(\omega_2^{14})^2+(\omega_2^{24})^2\neq 0$. In other words, there are three possible situations: $\omega_2^{14}\omega_2^{24}\neq 0; \omega_2^{14}=0,\omega_2^{24}\neq 0; \omega_2^{14}\neq 0,\omega_2^{24}=0$.

4. Main Results

In this section, we can discuss the persistence of heterodimensional cycles, the existence of homoclinic orbits and periodic orbits by the existence of solution $s_1=s_2=0$, $s_1>0, s_2=0$ (or $s_1=0, s_2>0$) and $s_1>0, s_2>0$ for (3.14). Moreover, we will establish the coexistence of the persistent hyterodimensional cycle and periodic orbits or homoclinic orbits.

Firstly, we establish the persistence of the heterodimensional cycle under small perturbation.

If $s_1 = s_2 = 0$ is the solution of equation (3.14), we obtain $M_2^1\mu + h.o.t. = 0$, $M_2^2\mu + h.o.t. = 0$. Assume $rank(M_2^1, M_2^2) = 2$, then we have

$$L_{12} = \{\mu : \ M_2^1\mu + h.o.t. = M_2^2\mu + h.o.t. = 0\},$$

such that for $\mu \in L_{12}$ and $0 < |\mu| \ll 1$ system (2.1) has a unique heteroclinic loop $\Gamma_\mu = \Gamma_1^\mu \cup \Gamma_2^\mu$. L_{12} is a codimension 2 surface with normal plane spanned by M_2^1, M_2^2 at $\mu = 0$. By $G_1^2 = 0$, we know that $M_1^2\mu \neq 0$ corresponds to $x_1^0 \neq 0$, which means that the persistent heteroclinic orbit Γ_1^μ enters p_1 along the leading unstable manifold as $t \to -\infty$. Then we have the following results.

Theorem 4.1. *Suppose that hypotheses (H_1)-(H_4) are satisfied, and $Rank(M_2^1, M_2^2) = 2$, then there exists a $(l-2)$-dimensional surface*

$$L_{12} = \{\mu : \ M_2^1\mu + h.o.t. = M_2^2\mu + h.o.t. = 0\},$$

with a normal plane spanned by $\Sigma_{12} = span\{M_2^1, M_2^2\}$ at $\mu = 0$, such that system (2.1) has a unique heterodimensional cycles $\Gamma_\mu = \Gamma_1^\mu \cup \Gamma_2^\mu$ as $\mu \in L_{12}$ and $0 < |\mu| \ll 1$. Furthermore, the persistent heteroclinic orbit Γ_1^μ has no orbit-flip as $t \to -\infty$ if $M_1^2\mu \neq 0$.

A corresponding results about the existence of the homoclinic orbit connecting p_i is contained in the next two theorems.

Theorem 4.2. *Suppose the conditions (H_1)-(H_4) are satisfied, $Rank(M_2^1,\ M_2^2) \geq 1$, then for $0 < |\mu| \ll 1$, the following results hold.*

(1) If $\omega_2^{14}\omega_2^{24} \neq 0$, then there exists an $(l-1)$-dimensional surface

$$L_1^1 = \{\mu : \ W_1^1(\mu) \overset{def}{=} (\omega_2^{14}M_2^1 + \omega_2^{24}M_2^2)\mu + h.o.t. = 0, \omega_2^{14}d_2M_2^2\mu > 0\}$$

such that system (2.1) has a unique orbit Γ_1^1 homoclinic to p_1 as $\mu \in L_1^1$. Meanwhile, the surface L_1^1 is tangent to the surface L_{12} at $\mu = 0$.

(2) If $\omega_2^{14} = 0, \omega_2^{24} \neq 0$, then there exists an $(l-1)$-dimensional surface

$$\begin{aligned} L_1^2 = \{\mu : \ W_1^2(\mu) \overset{def}{=} & M_2^2\mu + \omega_1^{44}\omega_2^{12}d_2^{-1}s^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu \\ & + \delta\omega_2^{12}(\omega_1^{33}\omega_1^{24} - \omega_1^{23}\omega_1^{34})(d_1d_2)^{-1}s^{\frac{\rho_2^2}{\rho_2^1}+\frac{1}{\beta_2}} + h.o.t. = 0, \\ & s = -\delta^{-1}(\omega_2^{24})^{-1}d_2M_2^1\mu, \omega_2^{24}d_2M_2^1\mu < 0\} \end{aligned}$$

such that system (2.1) has a unique orbit Γ_1^2 homoclinic to p_1 as $\mu \in L_1^2$. Meanwhile, the surface L_1^2 is tangent to the surface L_{12} at $\mu = 0$.

(3) If $\omega_2^{14} \neq 0, \omega_2^{24} = 0$, then there exists an $(l-1)$-dimensional surface

$$\begin{aligned} L_1^3 = \{\mu : \ W_1^3(\mu) \overset{def}{=} & M_2^1\mu - \omega_1^{44}\omega_2^{22}d_2^{-1}s^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu \\ & - \delta\omega_2^{22}(\omega_1^{33}\omega_1^{24} - \omega_1^{23}\omega_1^{34})(d_1d_2)^{-1}s^{\frac{\rho_2^2}{\rho_2^1}+\frac{1}{\beta_2}} + h.o.t. = 0, \\ & s = \delta^{-1}(\omega_2^{14})^{-1}d_2M_2^2\mu, \omega_2^{14}d_2M_2^2\mu > 0\} \end{aligned}$$

such that system (2.1) has a unique orbit Γ_1^3 homoclinic to p_1 as $\mu \in L_1^3$. Meanwhile, the surface L_1^3 is tangent to the surface L_{12} at $\mu = 0$.

Proof. Assume has a solution satisfying $s_1 = 0, 0 < s_2 \ll 1$, the equation (3.14) then turns into

$$\begin{cases} \omega_2^{24} s_2 = -\delta^{-1} d_2 M_2^1 \mu + \omega_2^{22} s_2^{\frac{\rho_2^2}{\rho_2^1}} [d_1^{-1}(\omega_1^{33}\omega_1^{24} - \omega_1^{23}\omega_1^{34}) s_2^{\frac{1}{\beta_2}} + \delta^{-1}\omega_1^{44} M_1^4 \mu] + h.o.t., \\ \omega_2^{14} s_2 = \delta^{-1} d_2 M_2^2 \mu + \omega_2^{12} s_2^{\frac{\rho_2^2}{\rho_2^1}} [d_1^{-1}(\omega_1^{33}\omega_1^{24} - \omega_1^{23}\omega_1^{34}) s_2^{\frac{1}{\beta_2}} + \delta^{-1}\omega_1^{44} M_1^4 \mu] + h.o.t.. \end{cases} \tag{4.1}$$

(1) If $\omega_2^{14}\omega_2^{24} \neq 0$, then (4.1) can be reduced to

$$\begin{cases} \omega_2^{24} s_2 = -\delta^{-1} d_2 M_2^1 \mu + h.o.t., \\ \omega_2^{14} s_2 = \delta^{-1} d_2 M_2^2 \mu + h.o.t.. \end{cases} \tag{4.2}$$

The second equation of (4.2) then yields to

$$s_2 = \delta^{-1} d_2 (\omega_2^{14})^{-1} M_2^2 \mu + h.o.t.$$

Obviously, $0 < s_2 \ll 1$ when $\omega_2^{14} d_2 M_2^2 \mu > 0$ and $0 < |\mu| \ll 1$. Substituting s_2 into the first equation of (4.2), we obtain the bifurcation surface

$$L_1^1 = \{\mu : \ W_1^1(\mu) \stackrel{def}{=} (\omega_2^{14} M_2^1 + \omega_2^{24} M_2^2)\mu + h.o.t. = 0, \omega_2^{24} d_2 M_2^2 \mu > 0\}$$

with a common normal plane $\omega_2^{14} M_2^1 + \omega_2^{24} M_2^2 \in \Sigma_{12}$, which is tangent to L_{12} at $\mu = 0$.

(2) If $\omega_2^{14} = 0, \omega_2^{24} \neq 0$, equation (4.1) becomes

$$\begin{cases} \omega_2^{24} s_2 = -\delta^{-1} d_2 M_2^1 \mu + h.o.t., \\ \delta^{-1} d_2 M_2^2 \mu + \omega_2^{12} s_2^{\frac{\rho_2^2}{\rho_2^1}} [d_1^{-1}(\omega_1^{33}\omega_1^{24} - \omega_1^{23}\omega_1^{34}) s_2^{\frac{1}{\beta_2}} + \delta^{-1}\omega_1^{44} M_1^4 \mu] + h.o.t. = 0, \end{cases} \tag{4.3}$$

The first equation of (4.3) implies that there exists one sufficiently small positive solution

$$0 < s_2 = -\delta^{-1} (\omega_2^{24})^{-1} d_2 M_2^1 \mu + h.o.t. \ll 1$$

as $\omega_2^{24} d_2 M_2^1 \mu < 0$. And then put s_2 into the second equation, we obtain the bifurcation surface L_1^2 with normal vector $M_2^2 \in \Sigma_{12}$ at $\mu = 0$, such that there exists a unique loop Γ_1^2 homoclinic to p_1 for $\mu \in L_1^2$ and $0 < |\mu| \ll 1$.

(3) If $\omega_2^{14} \neq 0, \omega_2^{24} = 0$, then (4.1) turns to:

$$\begin{cases} \delta^{-1} d_2 M_2^1 \mu - \omega_2^{22} s_2^{\frac{\rho_2^2}{\rho_2^1}} [d_1^{-1}(\omega_1^{33}\omega_1^{24} - \omega_1^{23}\omega_1^{34}) s_2^{\frac{1}{\beta_2}} + \delta^{-1}\omega_1^{44} M_1^4 \mu] h.o.t. = 0, \\ \omega_2^{14} s_2 = \delta^{-1} d_2 M_2^2 \mu + h.o.t.. \end{cases} \tag{4.4}$$

Notice that the second equation of (4.4), we have $0 < s_2 = \delta^{-1}(\omega_2^{14})^{-1} d_2 M_2^2 \mu + h.o.t. \ll 1$ as $\omega_2^{14} d_2 M_2^2 \mu > 0$. Substituting s_2 into the first equation, the bifurcation surface L_1^3 is then obtained. It is easy to see that $\dfrac{\partial W_1^3(\mu)}{\partial \mu}|_{\mu=0} = M_2^1$, which means that L_1^3 is tangent to L_{12} at $\mu = 0$. □

Theorem 4.3. *Suppose the conditions* (H_1)*-*(H_4) *are satisfied,* Rank$(M_2^1, M_2^2) \geq 1$, *then for* $0 < |\mu| \ll 1$, *the following results hold.*

(1) If $\omega_1^{43} \neq 0$, then when μ satisfies $|M_2^1\mu| \ll |M_1^3\mu|^{\frac{\alpha}{\alpha-1}}$, there exists a bifurcation surface

$$L_2^1 = \{\mu :\ W_2^1(\mu) \overset{def}{=} M_2^2\mu - \delta\tilde{s}_1^{\frac{\lambda_1^2}{\lambda_1^1}} + h.o.t. = 0, \omega_1^{43}M_1^3\mu > 0\},$$

such that system (2.1) has a unique orbit Γ_2^1 homoclinic to p_2 in the small neighborhood of Γ for $\mu \in L_2^1$ and $0 < |\mu| \ll 1$, where $\alpha = \dfrac{\lambda_1^3 + \lambda_1^1\beta_1}{\lambda_1^3} > 1$, $\tilde{s}_1 = [\delta^{-1}(\omega_1^{43})^{-1}M_1^3\mu]^{\frac{1}{\beta_1}} + h.o.t.$.

(2) If $\omega_1^{43} \neq 0$, then when μ satisfies $|M_2^1\mu| \gg |M_1^3\mu|^{\frac{\alpha}{\alpha-1}}$, there exists an bifurcation surface

$$L_2^2 = \{\mu :\ W_2^2(\mu) \overset{def}{=} M_2^2\mu - \delta\hat{s}_1^{\frac{\lambda_1^2}{\lambda_1^1}} + h.o.t. = 0, \omega_1^{43}M_2^1\mu > 0\},$$

such that system (2.1) has a unique orbit homoclinic to p_2 in the small neighborhood of Γ for $\mu \in L_2^2$ and $0 < |\mu| \ll 1$, where $\hat{s}_1 = [\delta^{-1}(\omega_1^{43})^{-1}M_2^1\mu]^{\frac{\lambda_1^1}{\lambda_1^3+\lambda_1^1\beta_1}} + h.o.t.$.

(3) If $\omega_1^{43} = 0$, then when μ satisfies $|M_2^1\mu| \ll |M_1^3\mu|$, there exists a bifurcation surface

$$L_2^3 = \{\mu :\ W_2^3(\mu) \overset{def}{=} M_2^2\mu - \delta\bar{s}_1^{\frac{\lambda_1^2}{\lambda_1^1}} + h.o.t. = 0, M_2^1\mu M_1^3\mu < 0\},$$

such that system (2.1) has a unique orbit homoclinic to p_2 in the small neighborhood of Γ for $\mu \in L_2^3$ and $0 < |\mu| \ll 1$, where $\bar{s}_1 = (-\dfrac{M_2^1\mu}{M_1^3\mu})^{\frac{\lambda_1^1}{\lambda_1^3}}$.

(4) If $\omega_1^{43} = 0$, then when μ satisfies $|M_2^1\mu| \gg |M_1^3\mu|$, system (2.1) has no homoclinic loop associated to p_2 near Γ.

Proof. Let $s_2 = 0$ in system (3.14), then we have

$$\begin{cases} \omega_1^{43}s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\beta_1} - \delta^{-1}M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}} - \delta^{-1}M_2^1\mu + h.o.t. \quad = 0, \\ s_1^{\frac{\lambda_1^2}{\lambda_1^1}} - \delta^{-1}M_2^2\mu + h.o.t. = 0. \end{cases} \tag{4.5}$$

Take $t = s_1^{\frac{\lambda_1^3}{\lambda_1^1}}, \alpha = \dfrac{\lambda_1^3 + \lambda_1^1\beta_1}{\lambda_1^3}$ in the first equation of (4.5), we have

$$\omega_1^{43}t^{\alpha} = \delta^{-1}M_2^1\mu + \delta^{-1}M_1^3\mu t + h.o.t. \tag{4.6}$$

(1) If $\omega_1^{43} \neq 0$, then when $|M_2^1\mu| \ll |M_1^3\mu|^{\frac{\alpha}{\alpha-1}}$ and $\omega_1^{43}M_1^3\mu > 0$ are valid, we can conclude that system (4.6) has a unique sufficiently small positive solution

$$t = [\delta^{-1}(\omega_1^{43})^{-1}M_1^3\mu]^{\frac{1}{\alpha-1}} + h.o.t.$$

This follows the fact that $|M_2^1\mu| \ll |M_1^3\mu t|$. Then we have

$$\tilde{s}_1 = t^{\frac{\lambda_1^1}{\lambda_1^3}} = [\delta^{-1}(\omega_1^{43})^{-1}M_1^3\mu]^{\frac{1}{\beta_1}} + h.o.t..$$

Putting this solution $\tilde{s}_1$ into the second equation of (4.5), we obtain the bifurcation surface L_2^1, which is tangent to L_{12} at $\mu = 0$.

(2) If $\omega_1^{43} \neq 0$, then when μ satisfies $|M_2^1\mu| \gg |M_1^3\mu|^{\frac{\alpha}{\alpha-1}}$ and $\omega_1^{43}M_2^1\mu > 0$, (4.6) becomes

$$\omega_1^{43}t^\alpha = \delta^{-1}M_2^1\mu + h.o.t.,$$

which has a unique sufficiently small positive solution

$$t = [\delta^{-1}(\omega_1^{43})^{-1}M_2^1\mu]^{\frac{1}{\alpha}} + h.o.t.$$

This follows the fact that $|M_2^1\mu| \gg |M_1^3\mu t|$. Now we have

$$s_1 = \hat{s}_1 = [\delta^{-1}(\omega_1^{43})^{-1}M_2^1\mu]^{\frac{\lambda_1^1}{\lambda_1^3+\lambda_1^1\beta_1}} + h.o.t..$$

Substituting $\hat{s}_1$ into the second equation of (4.5), we obtain the bifurcation surface L_2^2.

(3) If $\omega_1^{43} = 0$, then by the first equation of (4.5),we have

$$M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}} = -M_2^1\mu + h.o.t.,$$

which has a unique sufficiently small positive solution $s_1 = \bar{s}_1 = (-\dfrac{M_2^1\mu}{M_1^3\mu})^{\frac{\lambda_1^1}{\lambda_1^3}} + h.o.t.$ if $M_1^3\mu M_2^1\mu < 0$ and $|M_2^1\mu| \ll |M_1^3\mu|$. Substituting $\bar{s}_1$ into the second equation of (4.5), we obtain the bifurcation surface L_2^3.

(4) If $\omega_1^{43} = 0$, then when μ satisfies $|M_2^1\mu| \gg |M_1^3\mu|$, by the discussion of (3), we know that (4.5) has no sufficiently small positive solution s_1. This ends the proof of conclusion (4). □

Remark 4.1. If we discuss the bifurcation problem according to the second equation of (4.5), we obtain that

$$s_1 = [\delta^{-1}M_2^2\mu]^{\frac{\lambda_1^1}{\lambda_1^2}} + h.o.t.$$

Then when $M_2^2\mu > 0$ we have $0 < s_1 \ll 1$. Substituting s_1 into the first equation of (4.5), we can obtain the following bifurcation surface

$$\begin{aligned} L_2^4 = \{\mu :\ W_2^4(\mu) &\stackrel{def}{=} \omega_1^{43}[\delta^{-1}M_2^2\mu]^{\frac{\lambda_1^2+\lambda_1^1\beta_1}{\lambda_1^2}} - \delta^{-1}M_1^3\mu[\delta^{-1}M_2^2\mu]^{\frac{\lambda_1^3}{\lambda_1^2}} \\ &- \delta^{-1}M_2^1\mu + h.o.t. = 0, M_2^2\mu > 0\}, \end{aligned}$$

such that for $\mu \in L_2^4$ and $0 < |\mu| \ll 1$, system (2.1) has a unique homoclinic orbit connecting p_2 in a neighborhood of the heterodimensional cycle Γ.

Next, relying on the analysis for the bifurcation equations (3.14), we discuss the existence of the periodic orbit under small perturbation.

Theorem 4.4. *Suppose that hypotheses (H_1)-(H_4) are valid, Rank$(M_2^1,\ M_2^2) \geq 1$, and $\omega_2^{14}\omega_2^{24} \neq 0$, then for $0 < |\mu| \ll 1$, the following results hold.*

(1) If $M_2^2\mu < 0, d_2\omega_2^{14} < 0$, then when μ satisfies $\omega_2^{24}W_1^1(\mu) > 0$, system (2.1) has one unique periodic orbit near Γ; when μ satisfies $\omega_2^{24}W_1^1(\mu) < 0$, system (2.1) has no periodic orbits near Γ.

(2) If $M_2^2\mu > 0, d_2\omega_2^{14} < 0$, when μ satisfies $\omega_2^{24}d_2W_2^4(\mu) > 0$, there exists one periodic orbit near Γ; when μ satisfies $\omega_2^{24}d_2W_2^2(\mu) < 0$, system (2.1) *has no periodic orbit near Γ.*

(3) If $M_2^2\mu > 0, d_2\omega_2^{14} > 0$, when μ satisfies $\omega_2^{24}W_1^1(\mu) > 0, d_2\omega_2^{24}W_2^4(\mu) > 0$, system (2.1) *has one unique periodic orbit near Γ; otherwise, there exists no periodic orbit near Γ.*

(4) If $M_2^2\mu < 0, d_2\omega_2^{14} > 0$, then system (2.1) *has no periodic orbit near Γ.*

Proof. When $\omega_2^{14}\omega_2^{24} \neq 0$, bifurcation equations (3.14) are reduced to

$$\begin{cases} \omega_2^{24}s_2 = -\delta^{-1}d_2M_2^1\mu + \omega_1^{43}d_2s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\beta_1} - \delta^{-1}d_2M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}} + h.o.t., \\ \omega_2^{14}s_2 = \delta^{-1}d_2M_2^2\mu - d_2s_1^{\frac{\lambda_1^2}{\lambda_1^1}} + h.o.t.. \end{cases} \tag{4.7}$$

By the second equation of (4.7), we have

$$s_2 = (\omega_2^{14})^{-1}d_2(\delta^{-1}M_2^2\mu - s_1^{\frac{\lambda_1^2}{\lambda_1^1}}) + h.o.t.$$

(1) In case $M_2^2\mu < 0, \omega_2^{14}d_2 < 0$, and $0 < s_1 \ll 1$, we have $0 < s_2 \ll 1$. Substituting the expression of s_2 into the first equation of (4.7), we have

$$\begin{aligned} F(s_1,\mu) \overset{def}{=} &(\omega_2^{14})^{-1}(\delta^{-1}M_2^2\mu - s_1^{\frac{\lambda_1^2}{\lambda_1^1}}) + (\omega_2^{24})^{-1}\delta^{-1}M_2^1\mu - \omega_1^{43}(\omega_2^{24})^{-1}s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\beta_1} \\ &+ \delta^{-1}(\omega_2^{24})^{-1}M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}} + h.o.t. = 0. \end{aligned}$$

Setting $t_1 = s_1^{\frac{\lambda_1^2}{\lambda_1^1}}$, then

$$\begin{aligned} F(t_1,\mu) =&\delta^{-1}(\omega_2^{14}\omega_2^{24})^{-1}[\omega_2^{14}M_2^1\mu + \omega_2^{24}M_2^2\mu] - (\omega_2^{14})^{-1}t_1 \\ &- \omega_1^{43}(\omega_2^{24})^{-1}t_1^{\frac{\lambda_1^3+\lambda_1^1\beta_1}{\lambda_1^2}} + \delta^{-1}(\omega_2^{24})^{-1}M_1^3\mu t_1^{\frac{\lambda_1^3}{\lambda_1^2}} + h.o.t. = 0. \end{aligned}$$

By $F(0,\mu) = \delta^{-1}(\omega_2^{14}\omega_2^{24})^{-1}W_1^1(\mu), F'_{t_1}(t_1,\mu) = -(\omega_2^{14})^{-1} + h.o.t.$, then we know that $F(t_1,\mu) = 0$ has a unique sufficiently small positive solution $t_1 = t_1(\mu) > 0$ as $\omega_2^{24}W_1^1(\mu) > 0$, then system (2.1) has one unique periodic orbit. If $\omega_2^{24}W_1^1(\mu) < 0$, then (2.1) has no periodic orbits near Γ.

(2) If $M_2^2\mu > 0, d_2\omega_2^{14} < 0$, then by the second equation of (4.7), we obtain

$$s_1 = [\delta^{-1}M_2^2\mu - d_2^{-1}\omega_2^{14}s_2]^{\frac{\lambda_1^1}{\lambda_1^2}} + h.o.t. > 0.$$

Putting this simple expression of s_1 into the first equation, we have

$$\begin{aligned} G(s_2,\mu) \overset{def}{=} &\omega_2^{24}s_2 + \delta^{-1}d_2M_2^1\mu - \omega_1^{43}d_2[\delta^{-1}M_2^2\mu - d_2^{-1}\omega_2^{14}s_2]^{\frac{\lambda_1^3+\lambda_1^1\beta_1}{\lambda_1^2}} \\ &+ \delta^{-1}d_2M_1^3\mu[\delta^{-1}M_2^2\mu - d_2^{-1}\omega_2^{14}s_2]^{\frac{\lambda_1^3}{\lambda_1^2}} + h.o.t. = 0. \end{aligned}$$

Then by $G(0,\mu) = -d_2W_2^4(\mu), G'_{s_2}(s_2,\mu) = \omega_2^{24} + h.o.t.$, we know that $G(s_2,\mu) = 0$ has a unique sufficiently small positive solution $0 < s_2 = s_2(\mu) \ll 1$ as $\omega_2^{24}d_2W_2^4(\mu) > 0$.

Otherwise, $G(s_2,\mu)=0$ no sufficiently small positive solutions. We finish the proof of the conclusion of (2).

(3) If $M_2^2\mu>0, d_2\omega_2^{14}>0$, then it is easy to see that $s_1>0$ only if $0\le s_2<\tilde{s}_2=\delta^{-1}d_2(\omega_2^{14})^{-1}M_2^2\mu$. So to show the existence of the periodic orbit, it suffices to find a positive solution $s_2=s_2(\mu)$ of $G(s_2,\mu)=0$ such that $0\le s_2(\mu)<\tilde{s}_2$. By $G'_{s_2}(s_2,\mu)=\omega_2^{24}+h.o.t.\neq 0$, we know that $G(s_2,\mu)$ is monotone with respect to s_2. By some simple computation, we have $G(\tilde{s}_2,\mu)=\delta^{-1}d_2(\omega_2^{14})^{-1}W_1^1\mu$, then if $\omega_2^{24}W_1^1(\mu)>0, d_2\omega_2^{24}W_2^4(\mu)>0$, we can get $G'_{s_2}(s_2,\mu)G(0,\mu)<0, G(0,\mu)G(\tilde{s}_2,\mu)<0$, which means that $G(s_2,\mu)=0$ has a unique sufficiently small positive solution satisfying $0\le s_2(\mu)<\tilde{s}_2$. Otherwise, $G(s_2,\mu)=0$ has no sufficiently small positive solution satisfying $0\le s_2(\mu)<\tilde{s}_2$. we obtain the conclusion (3).

(4) If $M_2^2\mu<0, d_2\omega_2^{14}>0$, then we have $s_2=(\omega_2^{14})^{-1}d_2(\delta^{-1}M_2^2\mu-s_1^{\frac{\lambda_1^2}{\lambda_1^1}})+h.o.t.<0$. Then the bifurcation equation (3.14) has no sulutions satisfying $0<s_1\ll 1, 0<s_2\ll 1$, that is, system (2.1) has no periodic orbits near Γ. □

Theorem 4.5. *Suppose that hypotheses* (H_1)-(H_4) *are valid,* $Rank(M_2^1,\ M_2^2)\ge 1$, $\omega_2^{24}\neq 0$ *and* $\omega_2^{14}=\omega_1^{43}=0$. *Then for* $0<|\mu|\ll 1$*, the following results hold.*

(1) If $\omega_2^{24}d_2M_2^1\mu<0, \omega_2^{24}d_2M_1^3\mu<0$, *then system* (2.1) *has a unique periodic orbit near* Γ *as* μ *lies in the small one-sided neighborhood of* L_1^2 *satisfying* $W_1^2(\mu)>0$*; System* (2.1) *has no periodic orbits near* Γ *as* $W_1^2(\mu)<0$.

(2) If $\omega_2^{24}d_2M_2^1\mu<0, \omega_2^{24}d_2M_1^3\mu>0$, *then when* μ *lies in the region* $\{\mu: |M_2^1\mu|\ll|M_1^3\mu|,\ W_1^2(\mu)>0, W_2^3(\mu)<0\}$, *system* (2.1) *has a unique periodic orbit near* Γ*; Otherwise, System* (2.1) *has no periodic orbits near* Γ.

(3) If $\omega_2^{24}d_2M_2^1\mu>0$, *system* (2.1) *has no periodic orbits near* Γ.

Proof. For $0<|\mu|\ll 1$, when $\omega_2^{14}=\omega_1^{43}=0$ but $\omega_2^{24}\neq 0$, bifurcation equations (3.14) are changed into

$$\begin{cases} \omega_2^{24}s_2=-\delta^{-1}d_2M_2^1\mu-\delta^{-1}d_2M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}}+h.o.t., \\ \delta^{-1}d_2M_2^2\mu-d_2s_1^{\frac{\lambda_1^2}{\lambda_1^1}}+\omega_2^{12}s_2^{\frac{\rho_2^2}{\rho_2^1}}[\omega_1^{44}s_1^{\beta_1}-d_1^{-1}(\omega_1^{23}\omega_1^{34}-\omega_1^{24}\omega_1^{33})s_2^{\frac{1}{\beta_2}} \\ \quad +\delta^{-1}\omega_1^{44}M_1^4\mu]+h.o.t.=0. \end{cases} \tag{4.8}$$

By the first equation of (4.8), we have

$$s_2=-\delta^{-1}(\omega_2^{24})^{-1}d_2[M_2^1\mu+M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}}]+h.o.t..$$

(1) If $\omega_2^{24}d_2M_2^1\mu<0, \omega_2^{24}d_2M_1^3\mu<0$, then for $0<s_1\ll 1$, we have $0<s_2\ll 1$. Substituting the expression of s_2 into the second equation of (4.8), we have

$$\begin{aligned} F(s_1,\mu)\overset{def}{=}&\delta^{-1}M_2^2\mu-s_1^{\frac{\lambda_1^2}{\lambda_1^1}}+\omega_2^{12}d_2^{-1}[-\delta^{-1}(\omega_2^{24})^{-1}d_2]^{\frac{\rho_2^2}{\rho_2^1}}[M_2^1\mu+M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}}]^{\frac{\rho_2^2}{\rho_2^1}}[\omega_1^{44}s_1^{\beta_1} \\ &-d_1^{-1}(\omega_1^{23}\omega_1^{34}-\omega_1^{24}\omega_1^{33})[-\delta^{-1}(\omega_2^{24})^{-1}d_2]^{\frac{1}{\beta_2}}(M_2^1\mu+M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}})^{\frac{1}{\beta_2}} \\ &+\delta^{-1}\omega_1^{44}M_1^4\mu]+h.o.t.=0. \end{aligned}$$

Take $t_1 = s_1^{\frac{\lambda_1^2}{\lambda_1^1}}$, then we have

$$\begin{aligned}F(t_1,\mu) = {} & \delta^{-1}M_2^2\mu - t_1 + \omega_2^{12}d_2^{-1}[-\delta^{-1}(\omega_2^{24})^{-1}d_2]^{\frac{\rho_2^2}{\rho_2^1}}[M_2^1\mu + M_1^3\mu t_1^{\frac{\lambda_1^3}{\lambda_1^2}}]^{\frac{\rho_2^2}{\rho_2^1}}[\omega_1^{44}t_1^{\frac{\lambda_1^1\beta_1}{\lambda_1^2}} \\ & - d_1^{-1}(\omega_1^{23}\omega_1^{34} - \omega_1^{24}\omega_1^{33})[-\delta^{-1}(\omega_2^{24})^{-1}d_2]^{\frac{1}{\beta_2}}(M_2^1\mu + M_1^3\mu t_1^{\frac{\lambda_1^3}{\lambda_1^2}})^{\frac{1}{\beta_2}} \\ & + \delta^{-1}\omega_1^{44}M_1^4\mu] + h.o.t..\end{aligned}$$

Notice that $F(0,\mu) = \delta^{-1}W_1^2(\mu), F'_{t_1}(t_1,\mu) = -1 + h.o.t..$ In case $W_1^2(\mu) > 0$, then $F(t_1,\mu) = 0$ has a unique sufficiently small positive solution $0 < t_1 \ll 1$, thus system (2.1) has a unique periodic orbit near Γ; In case $W_1^2(\mu) < 0$, then $F(t_1,\mu) = 0$ has no sufficiently small positive solution, thus system (2.1) has no periodic orbits near Γ.

(2) If $\omega_2^{24}d_2M_2^1\mu < 0, \omega_2^{24}d_2M_1^3\mu > 0$, then we have $M_2^1\mu M_1^3\mu < 0$. It is easy to see that $0 < s_2 \ll 1$ only if $0 < s_1 < \bar{s}_1 = (-\dfrac{M_2^1\mu}{M_1^3\mu})^{\frac{\lambda_1^1}{\lambda_1^3}} \ll 1$. To assure the existence of small enough positive solutions, $|M_1^3\mu| \gg |M_2^1\mu|, M_2^1\mu M_1^3\mu < 0$ must be valid. Next we shall look for the small positive solution $s_1 = s_1(\mu)$ of $F(s_1,\mu)$ such that $0 < s_1 < \bar{s}_1$. That is, we need to find small positive solution $t_1 = t_1(\mu)$ of $F(t_1,\mu)$ satisfying $0 < t_1 < \bar{t}_1 = (-\dfrac{M_2^1\mu}{M_1^3\mu})^{\frac{\lambda_1^2}{\lambda_1^3}}$. In case of $W_1^2(\mu) > 0, W_2^3(\mu) < 0$, then by $F'_{t_1}(t_1,\mu) = -1 + h.o.t.$, we obtain that $F(0,\mu)F((\bar{t}_1,\mu) < 0, F(0,\mu)F'_{t_1}(t_1,\mu) < 0$, where $F(\bar{t}_1,\mu) = \delta^{-1}W_2^3(\mu)$. Therefore, $F(t_1,\mu)$ has a small positive solution. We obtain the conclusion (2).

(3) If $\omega_2^{24}d_2M_2^1\mu > 0$, then when $\omega_2^{24}d_2M_1^3\mu < 0$, we have

$$s_2 = -\delta^{-1}(\omega_2^{24})^{-1}d_2M_2^1\mu - \delta^{-1}(\omega_2^{24})^{-1}d_2M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}} + h.o.t. < 0$$

for $0 < s_1 < \bar{s}_1 = (-\dfrac{M_2^1\mu}{M_1^3\mu})^{\frac{\lambda_1^1}{\lambda_1^3}}$. On the other hand, notice that $F(s_1,\mu)$ has no positive solution $s_1 = s_1(\mu)$ for $s_1 > \bar{s}_1$, then equation (4.8) has no positive solutions; when $\omega_2^{24}d_2M_1^3\mu > 0$, by the expression of s_2, we know that $s_2 < 0$ for $s_1 \geq 0$,then equation (4.8) also has no positive solutions. The proof is complete. □

Next, relying on the analysis for the bifurcation equations (3.14), we discuss the coexistence of the heterodimensional cycle, homoclinic orbit and periodic orbit under small perturbation

Theorem 4.6. *Suppose that hypotheses (H_1)-(H_4) are valid, Rank$(M_2^1,\ M_2^2) = 2$, and $\omega_2^{24} \neq 0$, then for $\mu \in L_{12}$ and $0 < |\mu| \ll 1$, the heterodimensional cycle can not coexistence with homoclinic orbit and periodic orbit.*

Proof. (1) When $\omega_2^{14}\omega_2^{24} \neq 0$, bifurcation equations (3.14) are reduced to

$$\begin{cases} \omega_2^{24}s_2 = \omega_1^{43}d_2s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\beta_1} - \delta^{-1}d_2M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}} + h.o.t., \\ \omega_2^{14}s_2 = -d_2s_1^{\frac{\lambda_1^2}{\lambda_1^1}} + h.o.t., \end{cases} \tag{4.9}$$

for $\mu \in L_{12}$ and $0 < |\mu| \ll 1$. By the second equation of (4.9), we know that $s_1 = 0 \Leftrightarrow s_2 = 0$. Then the heterodimensional cycle can not coexistence with the homoclinic orbit. Eliminating s_2 in (4.9), it follows that:

$$(\omega_2^{24})^{-1}\omega_1^{43}s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\beta_1} = -(\omega_2^{14})^{-1}s_1^{\frac{\lambda_1^2}{\lambda_1^1}} + \delta^{-1}(\omega_2^{24})^{-1}M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}} + h.o.t. \tag{4.10}$$

By $\lambda_1^2 < \lambda_1^3$, it is obvious that the heterodimensional cycle can not coexistence with the periodic orbit.

(2) When $\omega_2^{14} = 0, \omega_2^{24} \neq 0$, bifurcation equations (3.14) are changed into

$$\begin{cases} \omega_2^{24}s_2 = \omega_1^{43}d_2 s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\beta_1} - \delta^{-1}d_2M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}} + h.o.t., \\ -d_2s_1^{\frac{\lambda_1^2}{\lambda_1^1}} + \omega_2^{12}s_2^{\frac{\rho_2^2}{\rho_2^1}}[\omega_1^{44}s_1^{\beta_1} \\ -d_1^{-1}(\omega_1^{23}\omega_1^{34} - \omega_1^{24}\omega_1^{33})s_2^{\frac{1}{\beta_2}} + \delta^{-1}\omega_1^{44}M_1^4\mu] + h.o.t. = 0. \end{cases} \tag{4.11}$$

By the first equation of (4.11), we have that $s_2 = o(s_1^{\frac{\lambda_1^3}{\lambda_1^1}})$, then $s_2^{\frac{\rho_2^2}{\rho_2^1}} = o(s_1^{\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1}}) = o(s_1^{\frac{\lambda_1^3}{\lambda_1^1}})$. By the second equation, we know that system (4.11) has no nonnegative solutions except the solution $s_1 = 0, s_2 = 0$. Then the heterodimensional cycle can not coexistence with homoclinic orbit and periodic orbit. □

Theorem 4.7. *Suppose that hypotheses* (H_1)-(H_4) *are valid, Rank*$(M_2^1,\ M_2^2) = 2$, $\frac{1}{\beta_2} > \beta_1 > 1$, $\omega_2^{14} \neq 0, \omega_2^{24} = 0$, *then for* $\mu \in L_{12}$ *and* $0 < |\mu| \ll 1$, *the following results hold.*

(1) When $\omega_2^{14}d_2 > 0$, *then systen* (2.1) *has no periodic orbits near* Γ.

(2) When $\omega_2^{14}d_2 < 0, \omega_1^{43} \neq 0$ *and* $\frac{\lambda_1^3}{\lambda_1^1} + \beta_1 < \frac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1}$, *then there exists one unique periodic orbit near* Γ *as* $\omega_1^{43}M_1^3\mu > 0$; *System* (2.1) *has no periodic orbit near* Γ *as* $\omega_1^{43}M_1^3\mu < 0$.

(3) When $\omega_2^{14}d_2 < 0, \omega_1^{43} \neq 0$ *and* $\frac{\lambda_1^3}{\lambda_1^1} < \frac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1} < \frac{\lambda_1^3}{\lambda_1^1} + \beta_1$, *then*

(a) If $\omega_1^{43}M_1^3\mu < 0, d_2\omega_1^{43}\omega_1^{44}\omega_2^{22}M_1^4\mu > 0$, *system* (2.1) *has no periodic orbit near* Γ.

(b) If $\omega_1^{43}M_1^3\mu > 0, d_2\omega_1^{43}\omega_1^{44}\omega_2^{22}M_1^4\mu > 0$ *or* $d_2\omega_1^{43}\omega_1^{44}\omega_2^{22}M_1^4\mu < 0$, *there exists one unique periodic orbit near* Γ.

(c) If $\omega_1^{43}M_1^3\mu<0, d_2\omega_1^{43}\omega_1^{44}\omega_2^{22}M_1^4\mu<0$ *and* $\omega_1^{43}\Delta_1<0$ *(resp.,* $\Delta_1=0$ *or* $M_1^3\mu\Delta_1> 0$*), then system* (2.1) *has exactly two periodic orbits (resp., has a unique double periodic orbit, or has no periodic orbit), where*

$$\Delta_1 = -\delta^{-1}M_1^3\mu + (1 - \frac{1}{\alpha_1})\delta^{-1}d_2^{-1}\omega_2^{22}\omega_1^{44}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu(-\frac{\omega_2^{22}\omega_1^{44}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu}{\delta\alpha_1\omega_1^{43}d_2})^{\frac{1}{\alpha_1}},$$

$$\alpha_1 = \frac{\lambda_1^1\rho_2^1\beta_1}{\lambda_1^2\rho_2^2 - \lambda_1^3\rho_2^1} > 1.$$

(4) When $\omega_2^{14}d_2 < 0, \omega_1^{43} \neq 0$ *and* $\frac{\lambda_1^3}{\lambda_1^1} > \frac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1}$, *then*

(a) If $M_1^4\mu > 0,\ \omega_1^{44}\omega_2^{22}d_2M_1^3\mu M_1^4\mu < 0$*, system* (2.1) *has no periodic orbit near* Γ*.*

(b) If $M_1^4\mu < 0,\ d_2\omega_1^{44}\omega_2^{22}M_1^3\mu > 0$ *or* $d_2\omega_1^{44}\omega_2^{22}M_1^3\mu < 0$*, system* (2.1) *has a unique periodic orbit near* Γ*.*

(c) If $M_1^4\mu>0, \omega_1^{44}\omega_2^{22}d_2M_1^3\mu>0$ *and* $\omega_1^{44}\omega_2^{22}\Delta_2<0$ *(resp.,* $\Delta_2=0$*, or* $\omega_1^{44}\omega_2^{22}\Delta_2>0$*), then system* (2.1) *has exactly two periodic orbits (resp., has a unique double periodic orbit, or has no periodic orbit), where*

$$\Delta_2 = \omega_1^{44}\omega_2^{22}\nu_1^{\frac{\rho_2^2}{\rho_2^1}} M_1^4\mu + (1-\frac{1}{\alpha_2})d_2M_1^3\mu(\frac{d_2M_1^3\mu}{\alpha_2\delta\omega_2^{22}\omega_1^{44}\nu_1^{\rho_2^2/\rho_2^1}})^{\frac{1}{\alpha_2-1}},$$

$$\alpha_2 = \frac{\lambda_1^1\rho_2^1\beta_1}{\lambda_1^3\rho_2^1-\lambda_1^2\rho_2^2} > 1.$$

(5) When $\omega_2^{14}d_2 < 0, \omega_1^{43} = 0$ *and* $\omega_1^{23} \neq 0$*, then*

(a) For $\frac{\lambda_1^3}{\lambda_1^1} + \frac{\lambda_1^2}{\lambda_1^1\beta_1} < \frac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1}$*, system* (2.1) *has a unique periodic orbit near* Γ *as* $\omega_1^{23}d_1M_1^3\mu < 0$*; system* (2.1) *has no periodic orbit near* Γ *as* $\omega_1^{23}d_1M_1^3\mu > 0$*.*

(b) For $\frac{\lambda_1^3}{\lambda_1^1} < \frac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1} < \frac{\lambda_1^3}{\lambda_1^1} + \frac{\lambda_1^2}{\lambda_1^1\beta_1} < \frac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1} + \beta_1$*, system* (2.1) *has no periodic orbit near* Γ *as* $\omega_1^{23}\omega_1^{44}\omega_2^{22}d_1d_2M_1^4\mu < 0, \omega_1^{23}d_1M_1^3\mu > 0$*; system* (2.1) *has a unique periodic orbit as* $\omega_1^{23}d_1M_1^3\mu > 0,\ \omega_1^{23}\omega_1^{44}\omega_2^{22}d_1d_2M_1^4\mu < 0$ *or* $\omega_1^{23}\omega_1^{44}\omega_2^{22}d_1d_2M_1^4\mu > 0$*; system* (2.1) *has exactly two periodic orbits (resp., has a unique double periodic orbit, or has no periodic orbit) as* $\omega_1^{23}\omega_1^{44}\omega_2^{22}d_1d_2M_1^4\mu > 0, \omega_1^{23}d_1M_1^3\mu > 0$ *and* $d_1\omega_1^{23}\Delta_3 < 0$ *(resp.,* $\Delta_3 = 0$*, or* $\omega_1^{23}d_1\Delta_3 > 0$*), where*

$$\Delta_3 = -\delta^{-1}d_2^{-1}\omega_2^{22}\omega_1^{44}M_1^4\mu\nu_1^{\frac{\rho_2^2}{\rho_2^1}} + (1-\frac{1}{\alpha_3})\delta^{-1}M_1^3\mu(-\frac{M_1^3\mu}{\delta d_1\alpha_3\omega_1^{33}\nu_1^{1/\beta_2}})^{\frac{1}{\alpha_3}},$$

$$\alpha_3 = \frac{\lambda_1^1\rho_2^1}{(\lambda_1^3\rho_2^1-\lambda_1^2\rho_2^2)\beta_2} > 1.$$

Proof. For any $\mu \in L_{12}$ and $0 < |\mu| \ll 1$, when $\omega_2^{14} \neq 0, \omega_2^{24} = 0$, then bifurcation equations (3.14) are changed into

$$\begin{cases} \omega_1^{43}d_2s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\beta_1} - \delta^{-1}d_2M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}} - d_1^{-1}d_2\omega_2^{23}s_1^{\frac{\lambda_1^3}{\lambda_1^1}}s_2^{\frac{1}{\beta_2}} + \omega_2^{22}s_2^{\frac{\rho_2^2}{\rho_2^1}}[\omega_1^{44}s_1^{\beta_1} \\ - d_1^{-1}(\omega_1^{23}\omega_1^{34} - \omega_1^{24}\omega_1^{33})s_2^{\frac{1}{\beta_2}} + \delta^{-1}\omega_1^{44}M_1^4\mu] + h.o.t. = 0, \\ \omega_2^{14}s_2 = -d_2s_1^{\frac{\lambda_1^2}{\lambda_1^1}} + h.o.t.. \end{cases} \tag{4.12}$$

By $d_2 = \begin{vmatrix} \omega_2^{12} & \omega_2^{22} \\ \omega_2^{14} & \omega_2^{24} \end{vmatrix} \neq 0$, and $\omega_2^{24} = 0$, we have $\omega_2^{22} \neq 0$. By the second equation of (4.12), we have $s_2 = -(\omega_2^{14})^{-1}d_2s_1^{\frac{\lambda_1^2}{\lambda_1^1}} + h.o.t. = O(s_1^{\frac{\lambda_1^2}{\lambda_1^1}}) > 0$ as $\omega_2^{14}d_2 < 0, s_1 > 0$. Put the expression of s_2 into the first equation and take $\nu_1 = -(\omega_2^{14})^{-1}d_2$, we obtain

$$\begin{aligned} &\omega_1^{43}s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\beta_1} - \delta^{-1}M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}} - d_1^{-1}\omega_2^{23}\nu_1^{\frac{1}{\beta_2}}s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\frac{\lambda_1^2}{\lambda_1^1\beta_2}} + \omega_2^{22}d_2^{-1}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}s_1^{\frac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1}}[\omega_1^{44}s_1^{\beta_1} \\ &+ \delta^{-1}\omega_1^{44}M_1^4\mu] + h.o.t. = 0. \end{aligned} \tag{4.13}$$

(1) If $\omega_2^{14}d_2 > 0$, then by the expression of s_2, (4.12) has no positive solution, that is, system (2.1) has no periodic orbit except the persistent heterodimensional cycle.

Next we consider the bifurcation problem under the case $\omega_2^{14}d_2 < 0$.

(2) If $\omega_1^{43} \neq 0$ and $\frac{\lambda_1^3}{\lambda_1^1} + \beta_1 < \frac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1}$, equation (4.13) turns out to be

$$\omega_1^{43}s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\beta_1} = \delta^{-1}M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}} + h.o.t.. \tag{4.14}$$

It is obvious that (4.14) has only one positive solution $s_1 = [\delta^{-1}(\omega_1^{43})^{-1}M_1^3\mu]^{\frac{1}{\beta_1}} + h.o.t.$ as $\omega_1^{43}M_1^3\mu > 0$. That is, system (2.1) has a unique periodic orbit near Γ.

(3) If $\omega_1^{43} \neq 0$ and $\frac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1} < \frac{\lambda_1^3}{\lambda_1^1} + \beta_1 < \frac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1} + \beta_1$, equation (4.13) is equivalent to

$$\omega_1^{43}s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\beta_1} - \delta^{-1}M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}} + \delta^{-1}\omega_2^{22}\omega_1^{44}d_2^{-1}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu s_1^{\frac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1}} + h.o.t. = 0, \tag{4.15}$$

which can be simplified to

$$\omega_1^{43}s_1^{\beta_1} = \delta^{-1}M_1^3\mu - \delta^{-1}\omega_2^{22}\omega_1^{44}d_2^{-1}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu s_1^{\frac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1}-\frac{\lambda_1^3}{\lambda_1^1}} + h.o.t..$$

Setting $t_1 = s_1^{\frac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1}-\frac{\lambda_1^3}{\lambda_1^1}}$, $\alpha_1 = \frac{\lambda_1^1\rho_2^1\beta_1}{\lambda_1^2\rho_2^2 - \lambda_1^3\rho_2^1} > 1$, then the above equation becomes

$$\omega_1^{43}t_1^{\alpha_1} = \delta^{-1}M_1^3\mu - \delta^{-1}\omega_2^{22}\omega_1^{44}d_2^{-1}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu t_1 + h.o.t.. \tag{4.16}$$

Take

$$h(t_1,\mu) = \omega_1^{43}t_1^{\alpha_1} - \delta^{-1}M_1^3\mu + \delta^{-1}\omega_2^{22}\omega_1^{44}d_2^{-1}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu t_1 + h.o.t..$$

With the analysis above, we know that each positive zero point t_1 of $h(t_1,\mu)$ corresponds to a unique pair of positive solutions $(s_1; s_2)$ of (4.12). Thus, in the following, we focus our attention on seeking the positive zero point of $h(t_1,\mu)$. Let

$$L(t_1,\mu) = -\delta^{-1}\omega_2^{22}\omega_1^{44}d_2^{-1}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu t_1 + \delta^{-1}M_1^3\mu + h.o.t.,$$

$$N(t_1,\mu) = \omega_1^{43}t_1^{\alpha_1} + h.o.t.,$$

then we have $h(t_1,\mu) = N(t_1,\mu) - L(t_1,\mu)$. Note that

$$h(0,\mu) = -\delta^{-1}M_1^3\mu,\quad h'_{t_1}(t_1,\mu) = \alpha_1\omega_1^{43}t_1^{\alpha_1-1} + \delta^{-1}d_2^{-1}\omega_2^{22}\omega_1^{44}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu + h.o.t..$$

Then if $d_2\omega_1^{43}\omega_2^{22}\omega_1^{44}M_1^4\mu < 0$, $h'_{t_1}(t_1,\mu) = 0$ has a unique sufficiently small positive zero point

$$\bar{t} = \left(-\frac{\omega_2^{22}\omega_1^{44}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu}{\delta\alpha_1\omega_1^{43}d_2}\right)^{\frac{1}{\alpha_1-1}} + h.o.t..$$

While, it has no small positive zero point if $d_2\omega_1^{43}\omega_2^{22}\omega_1^{44}M_1^4\mu > 0$.

(a) If $\omega_1^{43}M_1^3\mu < 0, d_2\omega_1^{43}\omega_2^{22}\omega_1^{44}M_1^4\mu > 0$, then the curve $N(t_1,\mu)$ and the straight-line $L(t_1,\mu)$ cannot intersect in the right half-plane, thus system (4.16) has no non-negative solutions.

(b) If $\omega_1^{43}M_1^3\mu > 0, d_2\omega_1^{43}\omega_2^{22}\omega_1^{44}M_1^4\mu > 0$, then the curve $N(t_1,\mu)$ and the straight-line $L(t_1,\mu)$ intersect at a unique positive point, that is, $h(t_1,\mu) = 0$ has a unique sufficiently small positive zero point. Next we show that the positive zero point is sufficiently small.

Without loss of the generality, take $\omega_1^{43} < 0, M_1^3\mu < 0, d_2\omega_2^{22}\omega_1^{44}M_1^4\mu < 0$, we have $h(0,\mu) = -\delta^{-1}M_1^3\mu > 0, h'_{t_1}(t_1,\mu) < 0, h(\tilde{t},\mu) = \delta^{-1}d_2^{-1}\omega_2^{22}\omega_1^{44}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu\tilde{t} + h.o.t. < 0$, where

$$\tilde{t} = [\delta^{-1}(\omega_1^{43})^{-1}M_1^3\mu]^{\frac{1}{\alpha_1}} + h.o.t. \ll 1.$$

Then there is a unique small t_1 satisfying $0 < t_1 < \tilde{t} \ll 1$, such that $h(t_1,\mu) = 0$.

If $\omega_1^{43}M_1^3\mu > 0$, $d_2\omega_1^{43}\omega_1^{44}\omega_2^{22}M_1^4\mu < 0$, we also have that there exists one unique periodic orbit near Γ.

(c) If $\omega_1^{43}M_1^3\mu < 0, d_2\omega_2^{2}\omega_1^{43}\omega_1^{44}M_1^4\mu < 0$, without loss of generality, take $\omega_1^{43} > 0, M_1^3\mu < 0, d_2\omega_2^{22}\omega_1^{44}M_1^4\mu < 0$, then we have $h(0,\mu) > 0, h_{t_1t_1}(t_1,\mu) > 0$. Take

$$\begin{aligned} h(\bar{t},\mu) &= \omega_1^{43}\bar{t}_1^{\alpha_1} - \delta^{-1}M_1^3\mu + \delta^{-1}\omega_2^{22}\omega_1^{44}d_2^{-1}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}\omega_1^{44}M_1^4\mu\bar{t}_1 + h.o.t, \\ &= -\delta^{-1}M_1^3\mu + (1-\frac{1}{\alpha_1})\delta^{-1}d_2^{-1}\omega_2^{22}\omega_1^{44}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu\bar{t} + h.o.t. \\ &= \Delta_1. \end{aligned}$$

Hence, if $h(\bar{t},\mu) = \Delta_1 = 0$, straight-line L is tangent to the curve N at point $t = \bar{t}$, that is, $t = \bar{t}$ is the double positive zero point of $h(t_1,\mu) = 0$; if $h(\bar{t},\mu) = \Delta_1 > 0$, straight-line L does not intersect the curve N, which implies $h(t_1,\mu) = 0$ has no positive solution; if $h(\bar{t},\mu) = \Delta_1 < 0$, then the straight-line L intersects the curve N at exact two points $0 < t_1' < \bar{t} < t_1''$, which means $h(t_1,\mu) = 0$ has two positive solutions.

(4) If $\omega_1^{43} \neq 0$ and $\dfrac{\lambda_1^3}{\lambda_1^1} > \dfrac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1}$, we have $s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\beta_1} = o(s_1^{\frac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1}+\beta_1})$, now system (4.13) is reduced to

$$\omega_2^{22}d_2^{-1}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}s_1^{\frac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1}}[\omega_1^{44}s_1^{\beta_1} + \delta^{-1}\omega_1^{44}M_1^4\mu] - \delta^{-1}M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}} + h.o.t. = 0. \tag{4.17}$$

By eliminating the $s_1^{\frac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1}}$ from both sides of (4.17) and setting $t_2 = s_1^{\frac{\lambda_1^3\rho_2^1-\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1}}$, $\alpha_2 = \dfrac{\lambda_1^1\rho_2^1\beta_1}{\lambda_1^3\rho_2^1-\lambda_1^2\rho_2^2} > 1$, system (4.17) is transformed into

$$\delta\omega_2^{22}\omega_1^{44}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}t_2^{\alpha_2} = -\omega_1^{44}\omega_2^{22}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu + d_2M_1^3\mu t_2 + h.o.t.. \tag{4.18}$$

Then taking similar techniques to (4.16), we obtain the conclusion.

(5) If $\omega_1^{43}=0$ and $\omega_1^{23}\neq 0$, then system (4.13) becomes

$$\begin{aligned} & d_1^{-1}\omega_1^{23}\nu_1^{\frac{1}{\beta_2}} s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\frac{\lambda_1^2}{\lambda_1^1\beta_2}} + \delta^{-1}M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}} \\ & -\omega_2^{22}d_2^{-1}\nu_1^{\frac{\rho_2^2}{\rho_2^1}} s_1^{\frac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1}}[\omega_1^{44}s_1^{\beta_1}+\delta_{-1}\omega_1^{44}M_1^4\mu] + h.o.t. = 0. \end{aligned} \tag{4.19}$$

(a) If $\dfrac{\lambda_1^3}{\lambda_1^1}+\dfrac{\lambda_1^2}{\lambda_1^1\beta_2}<\dfrac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1}$, then (4.19) is simplified to

$$d_1^{-1}\omega_1^{23}\nu_1^{\frac{1}{\beta_2}} s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\frac{\lambda_1^2}{\lambda_1^1\beta_2}} = -\delta^{-1}M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}} + h.o.t. = 0. \tag{4.20}$$

Obviously, (4.20) has a unique sufficiently small positive solution

$$0<s_1=(-\delta^{-1}d_1(\omega_1^{23})^{-1}\nu_1^{-\frac{1}{\beta_2}}M_1^3\mu)^{\frac{\lambda_1^1\beta_2}{\lambda_1^2}}+h.o.t.$$

as $\omega_1^{23}d_1M_1^3\mu<0$, which corresponds to a unique pair of positive solutions $(s_1;s_2)$ of (4.13). Then system (2.1) has one unique periodic orbit.

(b) If $\dfrac{\lambda_1^3}{\lambda_1^1}<\dfrac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1}<\dfrac{\lambda_1^3}{\lambda_1^1}+\dfrac{\lambda_1^2}{\lambda_1^1\beta_2}<\dfrac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1}+\beta_1$, we obtain the following equation from system (4.19)

$$d_1^{-1}\omega_1^{23}\nu_1^{\frac{1}{\beta_2}} s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\frac{\lambda_1^2}{\lambda_1^1\beta_2}} + \delta^{-1}M_1^3\mu s_1^{\frac{\lambda_1^3}{\lambda_1^1}} - \delta^{-1}\omega_2^{22}\omega_1^{44}d_2^{-1}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu s_1^{\frac{\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1}} + h.o.t. = 0.$$

Eliminating the common factor $s_1^{\frac{\lambda_1^3}{\lambda_1^1}}$. Take $t_2=s_1^{\frac{-\lambda_1^3\rho_2^1+\lambda_1^2\rho_2^2}{\lambda_1^1\rho_2^1}}$, $\alpha_3=\dfrac{\lambda_1^1\rho_2^1}{(\lambda_1^3\rho_2^1-\lambda_1^2\rho_2^2)\beta_2}$, which mean $\alpha_3>1$. Then we have

$$d_1^{-1}\omega_1^{23}\nu_1^{\frac{1}{\beta_2}}t_2^{\alpha_3} = -\delta^{-1}M_1^3\mu+\delta^{-1}\omega_2^{22}\omega_1^{44}d_2^{-1}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu t_2+h.o.t. \tag{4.21}$$

Applying analogous techniques used for (4.16) to the above equation, one can complete the proof. □

5. Example

In this section we shall present an example to illustrate our results and eliminate doubts about the existence of system which has a heterodimensional cycle with both orbit flip and inclination flip.

Take into account the following 4-dimensional system

$$\dot z = f(z)+g(z,\ \mu), \tag{5.1}$$

and its unperturbed system

$$\dot z = f(z), \tag{5.2}$$

where $z=(z_1,z_2,z_3,z_4)^* \in \mathbb{R}^4, \mu=(\mu_1,\mu_2,\mu_3)^* \in \mathbb{R}^3,\ g(z,0)=0,\ 0<|\mu|\ll 1$, and

$$f(z)=\begin{pmatrix} -(z_1-1)(z_1+1)+3(z_1^2+z_2^2-1)+z_1z_4 \\ -z_1z_2 \\ \frac{1}{3}z_3(20+19z_1) \\ -3z_1z_4 \end{pmatrix},$$

$$g(z,\mu)=\begin{pmatrix} (z_1+1)(z_1-1)\mu_1 \\ (z_1+1)^{\frac{1}{2}}(z_1-1)^2\mu_2 \\ (z_1+1)(z_1-1)^2\mu_1 \\ (z_1+1)(z_1-1)^2\mu_3 \end{pmatrix}.$$

For $\mu=0$, system (5.2) has two equilibria

$$p_1=(-1,0,0,0),\ p_2=(1,0,0,0),$$

which are joined by a heteroclinic cycle $\Gamma=\Gamma_1\cup\Gamma_2$. And the heteroclinic orbit $\Gamma_i=\{z=r_i(t),t\in R\}$, $i=1,2$ are expressed by

$$\Gamma_1=\{z=r_1(t)=(\frac{1-e^{-2t}}{1+e^{-2t}},\ 2\sqrt{\frac{1}{2+e^{2t}+e^{-2t}}},\ 0,\ 0)^*,\ t\in\mathbb{R}\},$$
$$\Gamma_2=\{z=r_2(t)=(\frac{1-e^{4t}}{1+e^{4t}},\ 0,\ 0,\ 0)^*,\ t\in\mathbb{R}\},$$

which satisfies $r_1(-\infty)=r_2(+\infty)=p_1$, $r_1(+\infty)=r_2(-\infty)=p_2$.

Note that

$$Df(z)=\begin{pmatrix} 4z_1+z_4 & 6z_2 & 0 & z_1 \\ -z_2 & -z_1 & 0 & 0 \\ \frac{19}{3}z_3 & 0 & \frac{1}{3}(20+19z_1) & 0 \\ -3z_4 & 0 & 0 & -3z_1 \end{pmatrix},$$

then we have

$$Df(p_1)=diag(-4,1,\frac{1}{3},3),\ Df(p_2)=diag(4,-1,13,-3),$$

which means $\Gamma=\Gamma_1\cup\Gamma_2$ is a heterodimensional cycle. Notice that Γ_1 tends to the equilibrium point p_1 along the strong unstable direction z_1 as $t\to-\infty$. Since the plane z_1z_3 is invariant, $T_{r_2(t)}W^u_{p_2}\to span\{(1,0,0,0)^*,\ (0,0,1,0)^*\}$, as $t\to+\infty$, where $(0,0,1,0)^*$ is the unit eigenvector of p_1 corresponding to the positive eigenvalue $\frac{1}{3}$, so $W^u_{p_2}$ undergoes strong inclination flip as $t\to+\infty$ (see Figure 3)

Let $0<\delta\ll 1$ and $T_i (i=1,2)$ be large enough such that

$$r_1(-T_1)=(-\sqrt{1-\delta^2},\delta,0,0)^*,\quad r_1(T_1)=(\sqrt{1-\delta^2},\delta,0,0)^*,$$
$$r_2(-T_2)=(1-\delta,0,0,0)^*,\qquad r_2(T_2)=(-1+\delta,0,0,0)^*,$$

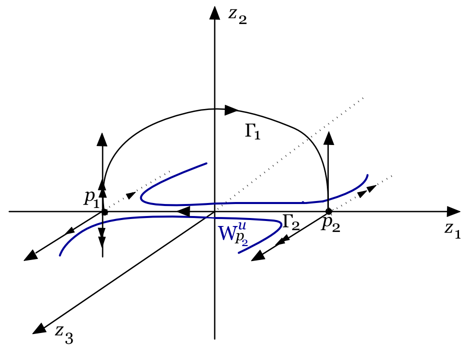

Figure 3. Heterodimensional cycle with orbit flip and inclination flip.

then we have

$$T_1 = \ln \frac{\delta}{1-\sqrt{1-\delta^2}} = \ln \frac{2}{\delta(1+O(\delta^2))}, \quad T_2 = \frac{1}{4}(\ln(2-\delta) - \ln\delta).$$

Now we consider the linear variational system of unperturbed system (5.2) along $\Gamma_i (i = 1, 2)$:

$$\dot{z} = Df(r_i(t))z, \tag{5.3}$$

and its adjoint system

$$\dot{\phi} = -(Df(r_i(t)))^* \phi, \tag{5.4}$$

where

$$Df(r_1(t)) = \begin{pmatrix} \frac{4(1-e^{-2t})}{1+e^{-2t}} & 12\sqrt{\frac{1}{2+e^{2t}+e^{-2t}}} & 0 & \frac{(1-e^{-2t})}{1+e^{-2t}} \\ -2\sqrt{\frac{1}{2+e^{2t}+e^{-2t}}} & -\frac{1-e^{-2t}}{1+e^{-2t}} & 0 & 0 \\ 0 & 0 & \frac{20}{3}+\frac{19}{3}\frac{1-e^{-2t}}{1+e^{-2t}} & 0 \\ 0 & 0 & 0 & -\frac{3(1-e^{-2t})}{1+e^{-2t}} \end{pmatrix},$$

$$Df(r_2(t)) = \begin{pmatrix} \dfrac{4(1-e^{4t})}{1+e^{4t}} & 0 & 0 & \dfrac{(1-e^{4t})}{1+e^{4t}} \\ 0 & -\dfrac{1-e^{4t}}{1+e^{4t}} & 0 & 0 \\ 0 & 0 & \dfrac{20}{3}+\dfrac{19}{3}\dfrac{1-e^{4t}}{1+e^{4t}} & 0 \\ 0 & 0 & 0 & -\dfrac{3(1-e^{4t})}{1+e^{4t}} \end{pmatrix}.$$

Next we discuss the persistent of the heterodimensional cycle of (5.2), by a similar computation given in section 2, we know that the persistent of the heterodimensional cycle is only related with elements in $Z_2(T_2)$, $Z_2(-T_2)$ as well as M_2^1, M_2^2. So, we only care about the fundamental solution matrix $Z_2(t)$ and $\Phi_2(t)$.

One fundamental solution matrix for (5.3) is

$$\hat{Z}_2(t)=\begin{pmatrix} C_1e^{4t}(1+e^{4t})^{-2} & 0 & 0 & C_1(t)e^{4t}(1+e^{4t})^{-2} \\ 0 & C_2e^{-t}(1+e^{4t})^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & C_3e^{13t}(1+e^{4t})^{-\frac{19}{6}} & \\ 0 & 0 & 0 & C_4e^{-3t}(1+e^{4t})^{\frac{3}{2}} \end{pmatrix}.$$

One fundamental solution matrix for (5.4) is

$$\hat{\Phi}_2(t) = (\hat{Z}_2^*(t))^{-1}$$
$$=\begin{pmatrix} C_1^{-1}e^{-4t}(1+e^{4t})^2 & 0 & 0 & 0 \\ 0 & C_2^{-1}e^{t}(1+e^{4t})^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & C_3^{-1}e^{-13t}(1+e^{4t})^{\frac{19}{6}} & 0 \\ d & 0 & 0 & C_4^{-1}e^{3t}(1+e^{4t})^{-\frac{3}{2}} \end{pmatrix},$$

where $d = -C_1(t)C_1^{-1}C_4^{-1}e^{3t}(1+e^{4t})^{-\frac{3}{2}}$, $C_1(t) = \int_0^t e^{-7s}(1-e^{4s})(1+e^{4s})^{\frac{5}{2}}ds + c_5$, c_1, c_2, c_3, c_4, c_5 are constants to be determined.

Note that we should perform the coordinates transformation by

$$z_1 \to y,\ z_2 \to \omega,\ z_3 \to x,\ z_4 \to u$$

in the small neighborhood of P_1 and perform the coordinates transformation by

$$z_1 \to x,\ z_2 \to y,\ z_3 \to u,\ z_4 \to v$$

in the small neighborhood of P_2 so as to match well with the system (3.1), (3.2) given in Section 2.

Thus, we obtain

$$Z_2(t)=\begin{pmatrix} C_1(t)e^{4t}(1+e^{4t})^{-2} & 0 & C_1e^{4t}(1+e^{4t})^{-2} & 0 \\ 0 & C_2e^{-t}(1+e^{4t})^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & C_3e^{13t}(1+e^{4t})^{-\frac{19}{6}} \\ C_4e^{-3t}(1+e^{4t})^{\frac{3}{2}} & 0 & 0 & 0 \end{pmatrix}$$

for $t \in (-\infty,\ -T_2]$, and

$$Z_2(t)=\begin{pmatrix} 0 & 0 & 0 & C_3e^{13t}(1+e^{4t})^{-\frac{19}{6}} \\ C_1(t)e^{4t}(1+e^{4t})^{-2} & 0 & C_1e^{4t}(1+e^{4t})^{-2} & 0 \\ C_4e^{-3t}(1+e^{4t})^{\frac{3}{2}} & 0 & 0 & 0 \\ 0 & C_2e^{-t}(1+e^{4t})^{\frac{1}{2}} & 0 & 0 \end{pmatrix}$$

for $t \in [T_2, +\infty)$. By the initial values

$$Z_2(-T_2) = \begin{pmatrix} \omega_2^{11} & \omega_2^{21} & 1 & 0 \\ \omega_2^{12} & \omega_2^{22} & 0 & 0 \\ \omega_2^{13} & \omega_2^{23} & 0 & 1 \\ \omega_2^{14} & \omega_2^{24} & 0 & 0 \end{pmatrix}, \quad Z_2(T_2) = \begin{pmatrix} 0 & 0 & 0 & \omega_2^{41} \\ 0 & 0 & \omega_2^{32} & \omega_2^{42} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

we obtain that

$$C_1 = \tfrac{4}{\delta(2-\delta)}, \quad C_2 = (\tfrac{\delta(2-\delta)}{4})^{\frac{1}{2}}, \quad C_3 = (\tfrac{\delta}{2-\delta})^{-\frac{13}{4}}[1 + (\tfrac{\delta}{2-\delta})]^{\frac{19}{6}}, \quad C_4 = (\tfrac{\delta(2-\delta)}{4})^{\frac{3}{4}},$$
$$\omega_2^{11} = \omega_2^{21} = \omega_2^{13} = \omega_2^{23} = \omega_2^{24} = \omega_2^{42} = 0, \ \omega_2^{22} = \omega_2^{14} = 1, \omega_2^{32} = (\tfrac{2-\delta}{\delta})^2, \ \omega_2^{41} = (\tfrac{\delta}{2-\delta})^{-\frac{10}{3}}.$$

Accordingly, we have

$$\Phi_2(t) = \begin{pmatrix} 0 & 0 & C_1^{-1}e^{-4t}(1+e^{4t})^2 & 0 \\ 0 & C_2^{-1}e^{t}(1+e^{4t})^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & C_3^{-1}e^{-13t}(1+e^{4t})^{\frac{19}{6}} \\ C_4^{-1}e^{3t}(1+e^{4t})^{-\frac{3}{2}} & 0 & d & 0 \end{pmatrix}$$

for $t \in \mathbb{R}$. Note that

$$g_\mu(r_2(t),\ 0) = \begin{pmatrix} -\dfrac{4e^{4t}}{(1+e^{4t})^2} & 0 & 0 \\ 0 & \sqrt{\dfrac{2}{1+e^{4t}}}(-\dfrac{2e^{4t}}{1+e^{4t}})^2 & 0 \\ \dfrac{8e^{8t}}{(1+e^{4t})^3} & 0 & 0 \\ 0 & 0 & \dfrac{8e^{8t}}{(1+e^{4t})^3} \end{pmatrix},$$

then we have

$$M_2^1 = (\ 0,\ 0,\ \frac{2}{C_4}\int_0^{+\infty} \frac{x^{7/4}}{(1+x)^{9/2}}\ \mathrm{d}x\ ),$$
$$M_2^2 = (\ 0,\ \frac{\sqrt{2}}{C_2}\int_0^{+\infty} \frac{x^{5/4}}{(1+x)^3}\ \mathrm{d}x,\ 0\ ).$$

With M_2^1, M_2^2 being specifically given above, then by Theorem 1, the system (5.1) has a unique heterocdimensional loop $\Gamma^\mu = \Gamma_1^\mu \cup \Gamma_2^\mu$ as $\mu \in L_{12}$ and $0 < |\mu| \ll 1$. To illustrate other results concerning homoclinic bifurcation, periodic bifurcation, we need more information, which will cause much more complicated computation. However, the idea and procedure are more or less the same as this one.

Acknowledgments

We are grateful for many useful comments and suggestions from the reviewers.

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