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DYNAMICS OF A HIGHER-ORDER RATIONAL DIFFERENCE EQUATION*

Qi Wang[1,2] and Qinqin Zhang[1,†]

Abstract We consider a higher order rational difference equation. Firstly, we skillfully give a sufficient and necessary condition for the existence and uniqueness of the initial value problem. And then we investigate the local stability, asymptotic behavior, periodicity and oscillation of solutions for the difference equation. Finally, we give some numerical simulations to illustrate our results.

Keywords Difference equation, initial value problem, local stability, periodicity, oscillation.

MSC(2010) 39A10, 65Q10, 65Q30.

1. Introduction

Why are people interested in studying difference equations? We here would like to put forward two strong reasons as the impetus. First, it provides us some simple and useful mathematic models to help elucidate interesting phenomena in applications. And second, they can kind of display some surprising complicated dynamics comparing with its analogue differential equations. For example, for a single species, the simplest differential equations, with no time-delays, lead to very simple dynamics: a familiar example is the logistic,

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}),$$

with a globally stable equilibrium point at $N = K$ for all $r > 0$. But the corresponding simplest difference equation

$$N_{t+1} = N_t e^{r(1-\frac{N_t}{K})},$$

displays intricate dynamics behavior. The phenomenon of a threefold regime of a stable point, giving way to stable cycles of period 2^n, giving way to chaotic behavior, is a generic one which is liable to occur in any model for discrete generations with the possibility of strongly density-dependent population growth [17, 18].

[†]the corresponding author. Email address:qqzhang@gzhu.edu.cn(Q. Zhang)
[1]School of Mathematics and Information Science, Guangzhou University, 230 Guangzhou University City Outer Ring Road, 510006 Guangzhou, China
[2]College of Science, Guangxi University of Science and Technology, 268 East Ring Road, 545006 Liuzhou, China
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Considerable studies on difference equations have been directed to their various dynamic behaviors [1–34]. For example, we refer to [1, 2, 4–6, 8, 14, 22, 23, 27] on stability and global attractors, [1, 19, 20] on oscillation, [3, 7, 9, 11–13, 25, 26, 28, 29, 34] on boundary value problems and periodicity and [24, 30–33] on subharmonic solutions and homoclinic orbits.

Consider the difference equation

$$x_{n+1} = \frac{Ax_{n-k}}{B + C\prod_{i=0}^{k} x_{n-i}},\ n = 0, 1, 2, \ldots,$$

with the initial conditions $x_{-i} = b_{-i}, i = 0, 1, 2, \ldots, k$, where k is a nonnegative integer, $b_{-k}, b_{-k+1}, \ldots, b_0$ are given $k+1$ constants, A, B, C are positive constants. If we set $x_n = \sqrt[k+1]{\frac{B}{C}} y_n$, then the above initial value problem(IVP for short) is translated into

$$\begin{cases} y_{n+1} = \dfrac{\gamma y_{n-k}}{1 + \prod_{i=0}^{k} y_{n-i}},\ n = 0, 1, 2, \ldots, \\ y_{-i} = a_{-i},\ i = 0, 1, 2, \ldots, k, \end{cases} \tag{1.1}$$

where $\gamma = \frac{A}{B}$ and $a_{-i} = \sqrt[k+1]{\frac{C}{B}} b_{-i}, i = 0, 1, 2, \ldots, k$.

In this paper, we skillfully give a sufficient and necessary condition for the existence and uniqueness of solutions of IVP(1.1). And then we investigate the local stability, asymptotic behavior, periodicity and oscillation of solutions of IVP(1.1). Finally, we give some numerical simulations to illustrate our results.

For some special cases, IVP(1.1) has been studied in the literature, e.g., [1] discussed the case $k = 2$ and [7] discussed the case $k = 1$.

2. Preliminaries

We first introduce some notations, definitions, and preliminary facts that will come into play later on. Let $\mathbb{N}$ stand for the set of natural numbers, $f : J^{k+1} \to J$ be a continuous function, where $k \in \mathbb{N}$ and J is an interval of real numbers. Consider the difference equation

$$y_{n+1} = f(y_n, y_{n-1}, \ldots, y_{n-k}),\ n \in \mathbb{N} \tag{2.1}$$

with initial values $y_{-k}, y_{-k+1}, \ldots, y_0 \in J$.

We say that $\bar{y}$ is an equilibrium point of equation (2.1) if

$$\bar{y} = f(\bar{y}, \bar{y}, \ldots, \bar{y}).$$

Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of equation (2.1). $\{y_n\}_{n=-k}^{\infty}$ is called p-periodic solution of equation (2.1) if

$$y_{n-k+p} = y_{n-k}, n \in \mathbb{N}.$$

$\{y_n\}_{n=-k}^{\infty}$ is called eventually p-periodic solution of equation (2.1) if there exists $\bar{n} \in \mathbb{N}$ such that

$$y_{n-k+p} = y_{n-k}, n = \bar{n}, \bar{n}+1, \bar{n}+2, \ldots.$$

Now we suppose that function $f(u_0, u_1, \ldots, u_k)$ is continuously differentiable. The linearized equation of equation (2.1) about the equilibrium point $\bar{y}$ is

$$Z_{n+1} = a_0 Z_n + a_1 Z_{n-1} + \cdots + a_k Z_{n-k},\ n \in \mathbb{N}, \tag{2.2}$$

where

$$a_i = \frac{\partial f}{\partial u_i}(\bar{y}, \bar{y}, \ldots, \bar{y}),\ i = 0, 1, 2, \ldots, k.$$

The characteristic equation of equation (2.2) is

$$\lambda^{k+1} - a_0 \lambda^k - a_1 \lambda^{k-1} - \cdots - a_k = 0. \tag{2.3}$$

Definition 2.1. Let $\bar{y}$ be an equilibrium point of equation (2.1).

(a) $\bar{y}$ is called locally stable if, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $y_{-k}, y_{-k+1}, \ldots, y_0 \in J$ and

$$\sum_{i=-k}^{0} |y_i - \bar{y}| < \delta,$$

then

$$|y_n - \bar{y}| < \epsilon \text{ for } n \geq -k.$$

(b) $\bar{y}$ is called locally asymptotically stable if it is locally stable and if there exists $\eta > 0$ such that if $y_{-k}, y_{-k+1}, \ldots, y_0 \in J$ and

$$\sum_{i=-k}^{0} |y_i - \bar{y}| < \eta,$$

then

$$\lim_{n\to\infty} y_n = \bar{y}.$$

(c) $\bar{y}$ is called a global attractor if, for $y_{-k}, y_{-k+1}, \ldots, y_0 \in J$, we have

$$\lim_{n\to\infty} y_n = \bar{y}.$$

(d) $\bar{y}$ is called globally asymptotically stable if it is locally stable and a global attractor.

(e) $\bar{y}$ is called hyperbolic if no root of equation (2.3) has modulus equal to one. Otherwise, it is called nonhyperbolic.

(f) $\bar{y}$ is called unstable if it is not locally stable.

Definition 2.2. Let $\bar{y}$ be an equilibrium point of equation (2.1).

(a) A positive semicycle of the solution $\{y_n\}_{n=-k}^{\infty}$ of equation (2.1) consists of a string of terms $y_l, y_{l+1}, \ldots, y_m$, all greater than or equal to $\bar{y}$ with $l \geq -k$ and $m \leq +\infty$ such that either $l = -k$ or $l > -k$ and $y_{l-1} < \bar{y}$ and either $m = +\infty$ or $m < +\infty$ and $y_{m+1} < \bar{y}$.

(b) A negative semicycle of the solution $\{y_n\}_{n=-k}^{\infty}$ of equation (2.1) consists of a string of terms $y_l, y_{l+1}, \ldots, y_m$, all less than $\bar{y}$ with $l \geq -k$ and $m \leq +\infty$ such that either $l = -k$ or $l > -k$ and $y_{l-1} \geq \bar{y}$ and either $m = +\infty$ or $m < +\infty$ and $y_{m+1} \geq \bar{y}$.

Definition 2.3. Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of equation (2.1). $\{y_n\}_{n=-k}^{\infty}$ is called nonoscillatory if there exists $\bar{n}(\bar{n} \geq -k)$ such that either $y_n > \bar{y}$ or $y_n < \bar{y}$ for $n \geq \bar{n}$ and it is called oscillatory if it is not nonoscillatory.

The following result is useful in determining the local stability of the equilibrium point $\bar{y}$ of equation (2.1).

Lemma 2.1 (The Linearized Stability Theorem)**.**

(i) If every root of equation (2.3) *has absolute value less than one, then the equilibrium point* $\bar{y}$ *of equation* (2.1) *is locally asymptotically stable.*

(ii) If at least one of the roots of equation (2.3) *has absolute value greater than one, then the equilibrium point* $\bar{y}$ *of equation* (2.1) *is unstable.*

3. A sufficient and necessary condition on the existence and uniqueness of solutions for IVP(1.1)

The next result provides a sufficient and necessary condition on the existence and uniqueness of solutions for IVP(1.1).

Theorem 3.1. *A sufficient and necessary condition on the existence and uniqueness of solutions for IVP*(1.1) *is*

$$\prod_{i=0}^{k} a_{-i} \neq -\frac{1}{\sum_{i=0}^{n} \gamma^i}, \ n \in \mathbb{N}. \tag{3.1}$$

Proof. Firstly, we will prove that if the solution of IVP(1.1) exists and is unique, denotes by $\{y_n\}_{n=-k}^{\infty}$, then (3.1) is satisfied. Actually, if (3.1) is not satisfied, then there exists $\bar{n} \in \mathbb{N}$ such that

$$\prod_{i=0}^{k} a_{-i} = -\frac{1}{\sum_{i=0}^{\bar{n}} \gamma^i}.$$

It follows from (1.1) that

$$\prod_{i=0}^{k} y_{1-i} = \frac{y_1}{y_{-k}} \prod_{i=0}^{k} a_{-i} = \frac{\gamma}{1 - \frac{1}{\sum_{i=0}^{\bar{n}} \gamma^i}} (-\frac{1}{\sum_{i=0}^{\bar{n}} \gamma^i}) = -\frac{1}{\sum_{i=0}^{\bar{n}-1} \gamma^i}.$$

Similarly, we can get

$$\prod_{i=0}^{k} y_{2-i} = -\frac{1}{\sum_{i=0}^{\bar{n}-2} \gamma^i}.$$

By induction, we generally obtain

$$\prod_{i=0}^{k} y_{m-i} = -\frac{1}{\sum_{i=0}^{\bar{n}-m} \gamma^i}, m = 0, 1, 2, \ldots, \bar{n}.$$

And so

$$\prod_{i=0}^{k} y_{\bar{n}-i} = -1.$$

This implies by (1.1) that $y_{\bar{n}+1}$ doesn't exist, which is a contradiction. Hence (3.1) must be satisfied.

Conversely, we will prove that if (3.1) is satisfied, then the solution of IVP(1.1) exists and is unique. By way of contradiction, assume that the solution of IVP(1.1) doesn't exist, then there exists $\bar{m} \in \mathbb{N}$ such that

$$\prod_{i=0}^{k} y_{\bar{m}-i} = -1, \tag{3.2}$$

but

$$\prod_{i=0}^{k} y_{n-i} \neq -1, n = 0, 1, 2, \dots, \bar{m} - 1.$$

By (1.1) and (3.2), we get

$$\prod_{i=0}^{k} y_{\bar{m}-i-1} = \frac{y_{\bar{m}-k-1}}{y_{\bar{m}}} \prod_{i=0}^{k} y_{\bar{m}-i} = -\frac{1 + \prod_{i=0}^{k} y_{\bar{m}-i-1}}{\gamma},$$

which yields

$$\prod_{i=0}^{k} y_{\bar{m}-i-1} = -\frac{1}{1+\gamma}.$$

Similarly, we can get

$$\prod_{i=0}^{k} y_{\bar{m}-i-2} = -\frac{1}{1+\gamma+\gamma^2}.$$

By induction, we generally obtain

$$\prod_{i=0}^{k} y_{\bar{m}-i-n} = -\frac{1}{\sum_{i=0}^{n} \gamma^i}, n = 0, 1, 2, \dots, \bar{m}.$$

And so

$$\prod_{i=0}^{k} a_{-i} = \prod_{i=0}^{k} y_{-i} = -\frac{1}{\sum_{i=0}^{\bar{m}} \gamma^i},$$

which is in contradiction with (3.1). Hence the proof is finished. □

Remark 3.1. Throughout this paper, we assume that (3.1) is always satisfied. And for convenience, set $\alpha = \prod_{i=0}^{k} a_{-i}$ and

$$I_n = \begin{cases} 1 - \gamma + \alpha(1 - \gamma^n) \text{ for } n \in \mathbb{N}, \text{when } \gamma \neq 1; \\ 1 + n\alpha \text{ for } n \in \mathbb{N}, \text{when } \gamma = 1. \end{cases}$$

Then $I_n \neq 0$ for $n \in \mathbb{N}$.

4. Local stability of the equilibrium points of IVP(1.1)

In this section, we investigate the local stability of the equilibrium points of IVP(1.1). Note that IVP(1.1) has the equilibrium points $\bar{y} = 0$ and $\bar{y} = \sqrt[k+1]{\gamma - 1}$ ($\gamma \geq 1$ when

k is odd). Then for the equilibrium point $\bar{y}=0$, the linearized equation associated with IVP(1.1) is

$$Z_{n+1}=\gamma Z_{n-k}, \tag{4.1}$$

whose characteristic equation is

$$\lambda^{k+1}-\gamma=0. \tag{4.2}$$

For the equilibrium point $\bar{y}=\sqrt[k+1]{\gamma-1}$($\gamma\geq 1$ when k is odd). The linearized equation associated with IVP(1.1) is

$$Z_{n+1}=\sum_{i=0}^{k-1}(\frac{1}{\gamma}-1)Z_{n-i}+\frac{1}{\gamma}Z_{n-k}, \tag{4.3}$$

whose characteristic equation is

$$\lambda^{k+1}-\sum_{i=0}^{k-1}(\frac{1}{\gamma}-1)\lambda^{k-i}-\frac{1}{\gamma}=0. \tag{4.4}$$

Theorem 4.1. *For the equilibrium point $\bar{y}=0$, the following statements are true.*

(i) Suppose $\gamma<1$, then $\bar{y}=0$ is locally asymptotically stable.

(ii) Suppose $\gamma=1$, then $\bar{y}=0$ is nonhyperbolic.

(iii) Suppose $\gamma>1$, then $\bar{y}=0$ is unstable.

Proof. Note that equation (4.2) has the roots $\lambda_m=\sqrt[k+1]{\gamma}e^{\frac{2m\pi}{k+1}i}$, $m=0,1,2,\ldots,k$, where $i=\sqrt{-1}$, hence the proof follows by Definition 2.1 and Lemma 2.1. □

Theorem 4.2. *For the equilibrium point $\bar{y}=\sqrt[k+1]{\gamma-1}$($\gamma\geq 1$ when k is odd), the following statements are true.*

(i) Suppose $\gamma<1$, then $\bar{y}=\sqrt[k+1]{\gamma-1}$ is unstable.

(ii) Suppose $\gamma\geq 1$, then $\bar{y}=\sqrt[k+1]{\gamma-1}$ is nonhyperbolic.

Proof. Note that

$$\lambda^{k+1}-\sum_{i=0}^{k-1}(\frac{1}{\gamma}-1)\lambda^{k-i}-\frac{1}{\gamma}=(\lambda-\frac{1}{\gamma})\sum_{i=0}^{k}\lambda^{k-i},$$

it is obviously that equation (4.4) has the roots $\lambda_0=\frac{1}{\gamma}$, $\lambda_m=e^{\frac{2m\pi}{k+1}i}$, $m=1,2,3,\ldots,k$, where $i=\sqrt{-1}$. The proof follows by Definition 2.1 and Lemma 2.1. □

5. The closed solution of IVP(1.1) and it's convergence

In this section, we seek the closed solution of IVP(1.1) and investigate it's convergence.

Theorem 5.1. *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of IVP(1.1). Then*

$$y_{m(k+1)+j+1} = y_{-k+j}\gamma^{m+1}\prod_{i=0}^{m}\frac{I_{i(k+1)+j}}{I_{i(k+1)+j+1}}, \tag{5.1}$$

for $j = 0, 1, 2, \dots, k$; $m \in \mathbb{N}$.

Proof. We will prove the conclusion by induction. By IVP(1.1), note that $I_n + \alpha\gamma^n I_0 = I_{n+1}$ and $I_n \neq 0$, $n \in \mathbb{N}$, we have

$$\begin{aligned}
y_1 &= \frac{\gamma y_{-k}}{1+\prod_{i=0}^{k} y_{-i}} = y_{-k}\gamma\frac{1}{1+\alpha} = y_{-k}\gamma\frac{I_0}{I_1},\\
y_2 &= \frac{\gamma y_{-k+1}}{1+\prod_{i=0}^{k} y_{-i+1}} = \frac{y_{-k+1}\gamma}{1+\alpha\frac{y_1}{y_{-k}}} = \frac{y_{-k+1}\gamma}{1+\alpha\gamma\frac{I_0}{I_1}} = y_{-k+1}\gamma\frac{I_1}{I_2},\\
y_3 &= \frac{\gamma y_{-k+2}}{1+\prod_{i=0}^{k} y_{-i+2}} = \frac{y_{-k+2}\gamma}{1+\alpha\frac{y_1}{y_{-k}}\frac{y_2}{y_{-k+1}}} = \frac{y_{-k+2}\gamma}{1+\alpha\gamma^2\frac{I_0}{I_1}\frac{I_1}{I_2}} = y_{-k+2}\gamma\frac{I_2}{I_3}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
y_{j+1} &= \frac{\gamma y_{-k+j}}{1+\prod_{i=0}^{k} y_{-i+j}} = \frac{y_{-k+j}\gamma}{1+\alpha\prod_{i=0}^{j-1}\frac{y_{i+1}}{y_{-k+i}}} = \frac{y_{-k+j}\gamma}{1+\alpha\gamma^j\prod_{i=0}^{j-1}\frac{I_i}{I_{i+1}}}\\
&= \frac{y_{-k+j}\gamma}{1+\alpha\gamma^j\frac{I_0}{I_j}} = y_{-k+j}\gamma\frac{I_j}{I_{j+1}},
\end{aligned}$$

for $j = 1, 2, 3, \dots, k$. Hence, the conclusion holds for $m = 0$.

Suppose the conclusion holds for $m \le n$, then for $m = n+1$,

$$\begin{aligned}
y_{(n+1)(k+1)+1} &= \frac{\gamma y_{n(k+1)+1}}{1+\prod_{l=0}^{k} y_{(n+1)(k+1)-l}} = \frac{\gamma y_{n(k+1)+1}}{1+\prod_{i=0}^{k} y_{n(k+1)+i+1}}\\
&= \frac{y_{n(k+1)+1}\gamma}{1+\alpha\prod_{i=0}^{k}\frac{y_{n(k+1)+i+1}}{y_{i-k}}} = \frac{y_{n(k+1)+1}\gamma}{1+\alpha\prod_{i=0}^{k}(\prod_{l=0}^{n}\frac{y_{n(k+1)+i+1-l(k+1)}}{y_{n(k+1)+i+1-(l+1)(k+1)}})}\\
&= \frac{y_{n(k+1)+1}\gamma}{1+\alpha\prod_{i=0}^{k}(\gamma^{n+1}\prod_{l=0}^{n}\frac{I_{l(k+1)+i}}{I_{l(k+1)+i+1}})} = \frac{y_{n(k+1)+1}\gamma}{1+\alpha\gamma^{(n+1)(k+1)}\frac{I_0}{I_{(n+1)(k+1)}}}\\
&= y_{n(k+1)+1}\gamma\frac{I_{(n+1)(k+1)}}{I_{(n+1)(k+1)+1}} = y_{-k}\gamma^{n+2}\prod_{i=0}^{n+1}\frac{I_{i(k+1)}}{I_{i(k+1)+1}},\\
y_{(n+1)(k+1)+2} &= \frac{\gamma y_{n(k+1)+2}}{1+\prod_{l=0}^{k} y_{(n+1)(k+1)-l+1}} = \frac{\gamma y_{n(k+1)+2}}{1+\prod_{i=0}^{k} y_{n(k+1)+i+2}}\\
&= \frac{y_{n(k+1)+2}\gamma}{1+\alpha\prod_{i=0}^{k}\frac{y_{n(k+1)+i+2}}{y_{i-k}}}\\
&= \frac{y_{n(k+1)+2}\gamma}{1+\alpha\frac{y_{(n+1)(k+1)+1}}{y_{n(k+1)+1}}\prod_{i=0}^{k}(\prod_{l=0}^{n}\frac{y_{n(k+1)+i+1-l(k+1)}}{y_{n(k+1)+i+1-(l+1)(k+1)}})}\\
&= \frac{y_{n(k+1)+2}\gamma}{1+\alpha\gamma\frac{I_{(n+1)(k+1)}}{I_{(n+1)(k+1)+1}}\prod_{i=0}^{k}(\gamma^{n+1}\prod_{l=0}^{n}\frac{I_{l(k+1)+i}}{I_{l(k+1)+i+1}})}\\
&= \frac{y_{n(k+1)+2}\gamma}{1+\alpha\gamma^{(n+1)(k+1)+1}\frac{I_0}{I_{(n+1)(k+1)+1}}}
\end{aligned}$$

$$= y_{n(k+1)+2}\gamma \frac{I_{(n+1)(k+1)+1}}{I_{(n+1)(k+1)+2}} = y_{-k+1}\gamma^{n+2} \prod_{i=0}^{n+1} \frac{I_{i(k+1)+1}}{I_{i(k+1)+2}}.$$

Similarly,

$$\begin{aligned}
y_{(n+1)(k+1)+j} &= \frac{\gamma y_{n(k+1)+j}}{1+\prod_{l=0}^{k} y_{(n+1)(k+1)-l+j-1}} = \frac{\gamma y_{n(k+1)+j}}{1+\prod_{i=0}^{k} y_{n(k+1)+i+j}} \\
&= \frac{y_{n(k+1)+j}\gamma}{1+\alpha\prod_{i=0}^{k} \frac{y_{n(k+1)+i+j}}{y_{i-k}}} = \frac{y_{n(k+1)+j}\gamma}{1+\alpha\prod_{s=0}^{j-2} \frac{y_{(n+1)(k+1)+s+1}}{y_{n(k+1)+s+1}} \prod_{i=0}^{k} \frac{y_{n(k+1)+i+1}}{y_{i-k}}} \\
&= \frac{y_{n(k+1)+j}\gamma}{1+\alpha\prod_{s=0}^{j-2} \gamma\frac{I_{(n+1)(k+1)+s}}{I_{(n+1)(k+1)+s+1}} \prod_{i=0}^{k} (\gamma^{n+1}\prod_{l=0}^{n} \frac{I_{l(k+1)+i}}{I_{l(k+1)+i+1}})} \\
&= \frac{y_{n(k+1)+j}\gamma}{1+\alpha\gamma^{(n+1)(k+1)+j-1}\frac{I_0}{I_{(n+1)(k+1)+j-1}}} = y_{n(k+1)+j}\gamma\frac{I_{(n+1)(k+1)+j-1}}{I_{(n+1)(k+1)+j}} \\
&= y_{-k+j-1}\gamma^{n+2}\prod_{i=0}^{n+1}\frac{I_{i(k+1)+j-1}}{I_{i(k+1)+j}},
\end{aligned}$$

for $j = 2, 3, \dots, k+1$. Hence the conclusion holds for $m = n+1$, from which we get the conclusion. □

The proof of the next lemma follows by simple computations and will be omitted.

Lemma 5.1. *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of IVP*(1.1)*, then the following statements are true.*

(i) Suppose $\alpha = 0$, then

$$I_n = \begin{cases} 1-\gamma \;\; for \;\; n \in \mathbb{N}, when \;\; \gamma \neq 1; \\ 1 \;\; for \;\; n \in \mathbb{N}, when \;\; \gamma = 1. \end{cases}$$

(ii) Suppose $\alpha > 0$, then

$$\begin{cases} I_{n+1} < I_n < 0 \;\; for \;\; n \in \mathbb{N}, when \;\; \gamma > 1; \\ 0 < I_n < I_{n+1} \;\; for \;\; n \in \mathbb{N}, when \;\; \gamma \leq 1. \end{cases}$$

(iii) Suppose $\alpha < 0$.

(a) If $\gamma > 1$, then there exists $\bar{n} \in \mathbb{N}$ such that

$$\begin{cases} 0 < I_n < I_{n+1} \;\; for \;\; n > \bar{n}; \\ I_{n-1} < I_n < 0 \;\; for \;\; n \leq \bar{n}. \end{cases}$$

(b) If $\gamma = 1$, then there exists $\bar{n} \in \mathbb{N}$ such that

$$\begin{cases} I_{n+1} < I_n < 0 \;\; for \;\; n > \bar{n}; \\ 0 < I_n < I_{n-1} \;\; for \;\; n \leq \bar{n}. \end{cases}$$

(c) If $\alpha < \gamma - 1 < 0$, then there exists $\bar{n} \in \mathbb{N}$ such that

$$\begin{cases} I_{n+1} < I_n < 0 \;\; for \;\; n > \bar{n}; \\ 0 < I_n < I_{n-1} \;\; for \;\; n \leq \bar{n}. \end{cases}$$

(d) If $\gamma - 1 \leq \alpha$, then $0 < I_{n+1} < I_n$ *for* $n \in \mathbb{N}$.

Theorem 5.2. *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of IVP(1.1). Then the following statements are true.*

(i) *Suppose $\gamma < 1$, then $\lim_{n\to\infty} y_n = 0$.*

(ii) *Suppose $\gamma = 1$.*

  (a) *If $\alpha = 0$, then $\{y_n\}_{n=-k}^{\infty}$ is a $(k+1)$-periodic solution of IVP(1.1).*

  (b) *If $\alpha \neq 0$, then $\lim_{n\to\infty} y_n = 0$.*

(iii) *Suppose $\gamma > 1$.*

  (a) *If $\alpha = 0$, then*

$$\begin{cases} \lim\limits_{m\to\infty} y_{m(k+1)+j+1} = \infty, \ \text{when } y_{-k+j} \neq 0; \\ y_{m(k+1)+j+1} = 0 \ for \ m \in \mathbb{N}, \ \text{when } y_{-k+j} = 0. \end{cases}$$

  (b) *If $\alpha < 0$, then*

$$\lim_{m\to\infty} y_{m(k+1)+j+1} = \begin{cases} -\,sgn\,(y_{-k+j})\, exp(\varphi_j), \ \text{when } j = j_0; \\ sgn\,(y_{-k+j})\, exp(\varphi_j), \ \text{when } j \neq j_0, \end{cases}$$

  where $\varphi_j = ln\,(|y_{-k+j}|) + \sum\limits_{i=0}^{\infty} ln\left(\frac{I_{i(k+1)+j}}{I_{i(k+1)+j+1}}\right), j = 0, 1, 2, \ldots, k; j_0 = \bar{n} \bmod (k+1)$, $\bar{n}$ *satisfies* $I_{\bar{n}} I_{\bar{n}+1} < 0$.

  (c) *If $\alpha > 0$, then*

$$\lim_{m\to\infty} y_{m(k+1)+j+1} = sgn\,(y_{-k+j})\, exp(\varphi_j), \ j = 0, 1, 2, \ldots, k,$$

$$\text{where } \varphi_j = ln\,(|y_{-k+j}|) + \sum_{i=0}^{\infty} ln\left(\frac{I_{i(k+1)+j}}{I_{i(k+1)+j+1}}\right), j = 0, 1, 2, \ldots, k.$$

Proof. (i) Suppose $\gamma < 1$. Consider the following three cases.

Case 1: $\alpha = 0$. By Theorem 5.1 and Lemma 5.1, we have

$$y_{m(k+1)+j+1} = y_{-k+j}\gamma^{m+1} \prod_{i=0}^{m} \frac{I_{i(k+1)+j}}{I_{i(k+1)+j+1}} = y_{-k+j}\gamma^{m+1}, j = 0, 1, 2, \ldots, k.$$

Note that $\gamma < 1$, we have

$$\lim_{m\to\infty} y_{m(k+1)+j+1} = 0, \ j = 0, 1, 2, \ldots, k,$$

hence

$$\lim_{n\to\infty} y_n = 0.$$

Case 2: Either $\alpha > 0$ or $\alpha < \gamma - 1$. By Lemma 5.1, there exists $\bar{n} \in \mathbb{N}$, such that $0 < \frac{I_n}{I_{n+1}} < 1$ for all $m(k+1) + j + 1 > \bar{n}$, hence there exists $M > 0$ such that

$$|y_{m(k+1)+j+1}| = |y_{-k+j}|\gamma^{m+1} \prod_{i=0}^{m} \frac{I_{i(k+1)+j}}{I_{i(k+1)+j+1}} < M\gamma^{m+1}, \ j = 0, 1, 2, \ldots, k.$$

Since

$$\lim_{m\to\infty} M\gamma^{m+1} = 0,$$

we have

$$\lim_{m\to\infty} y_{m(k+1)+j+1} = 0,\ \ j = 0, 1, 2, \ldots, k,$$

hence

$$\lim_{n\to\infty} y_n = 0.$$

Case 3: $\gamma - 1 \le \alpha < 0$. By Lemma 5.1, we have $0 < I_{n+1} < I_n$ for all $n \in \mathbb{N}$, note that

$$\gamma I_n - I_{n+1} = (\gamma - 1)(1 - \gamma + \alpha) \le 0,\ \ n \in \mathbb{N}$$

and

$$\lim_{n\to\infty} \ln\left(\frac{\gamma I_n}{I_{n+1}}\right) = \ln(\gamma) < 0.$$

By (5.1) we have

$$\lim_{m\to\infty} \ln\left(|y_{m(k+1)+j+1}|\right) = \ln(|y_{-k+j}|) + \sum_{i=0}^{\infty} \ln\left(\frac{\gamma I_{i(k+1)+j}}{I_{i(k+1)+j+1}}\right) = -\infty, j = 0, 1, 2, \ldots, k,$$

then

$$\lim_{m\to\infty} y_{m(k+1)+j+1} = 0,\ \ j = 0, 1, 2, \ldots, k,$$

hence

$$\lim_{n\to\infty} y_n = 0.$$

(ii) Suppose $\gamma = 1$.

(a) If $\alpha = 0$, by Theorem 5.1 and Lemma 5.1, we have

$$y_{m(k+1)+j+1} = y_{-k+j} \prod_{i=0}^{m} \frac{I_{i(k+1)+j}}{I_{i(k+1)+j+1}} = y_{-k+j}, j = 0, 1, 2, \ldots, k,$$

namely, $\{y_n\}_{n=-k}^{\infty}$ is a $(k+1)$-periodic solution of IVP(1.1).

(b) If $\alpha \ne 0$, by Lemma 5.1, we have

$$ln\left(\frac{I_n}{I_{n+1}}\right) \sim \frac{-\alpha}{1 + (n+1)\alpha}, \quad \text{as } n \to \infty$$

and

$$\sum_{n=0}^{\infty} \frac{-\alpha}{1 + (n+1)\alpha} = -\infty.$$

By (5.1) we have

$$\lim_{m\to\infty} \ln\left(|y_{m(k+1)+j+1}|\right) = \ln\left(|y_{-k+j}|\right) + \sum_{i=0}^{\infty} \ln\left(\frac{I_{i(k+1)+j}}{I_{i(k+1)+j+1}}\right) = -\infty, j = 0, 1, 2, \ldots, k,$$

then

$$\lim_{m\to\infty} y_{m(k+1)+j+1} = 0,\ \ j = 0, 1, 2, \ldots, k,$$

hence

$$\lim_{n\to\infty} y_n = 0.$$

(iii) Suppose $\gamma > 1$.

(a) If $\alpha = 0$, by (5.1) and Lemma 5.1, we have

$$y_{m(k+1)+j+1} = y_{-k+j}\gamma^{m+1}\prod_{i=0}^{m}\frac{I_{i(k+1)+j}}{I_{i(k+1)+j+1}} = y_{-k+j}\gamma^{m+1}, j = 0, 1, 2, \ldots, k.$$

Note that $\gamma > 1$, we have

$$\begin{cases} \lim\limits_{m\to\infty} y_{m(k+1)+j+1} = \infty, \text{ when } y_{-k+j} \neq 0; \\ y_{m(k+1)+j+1} = 0 \text{ for } m \in \mathbb{N}, \text{ when } y_{-k+j} = 0. \end{cases}$$

(b) If $\alpha < 0$, note that

$$\ln\left(\frac{\gamma I_n}{I_{n+1}}\right) \sim \frac{(\gamma-1)(1-\gamma+\alpha)}{1-\gamma+\alpha(1-\gamma^{n+1})}, \quad \text{as } n \to \infty$$

and the series $\sum\limits_{n=0}^{\infty}\frac{(\gamma-1)(1-\gamma+\alpha)}{1-\gamma+\alpha(1-\gamma^{n+1})}$ is convergent, hence $\sum\limits_{n=0}^{\infty}\ln\left(\frac{\gamma I_n}{I_{n+1}}\right)$ is convergent. Set $\varphi_j = \ln(|y_{-k+j}|) + \sum\limits_{i=0}^{\infty}\ln\left(\frac{\gamma I_{i(k+1)+j}}{I_{i(k+1)+j+1}}\right)$, $j = 0, 1, 2, \ldots, k$. By (5.1) we have

$$\lim_{m\to\infty}\ln\left(|y_{m(k+1)+j+1}|\right) = \ln(|y_{-k+j}|) + \sum_{i=0}^{\infty}\ln\left(\frac{\gamma I_{i(k+1)+j}}{I_{i(k+1)+j+1}}\right) = \varphi_j. \tag{5.2}$$

By Lemma 5.1, there exists $\bar{n} \in \mathbb{N}$ such that

$$\begin{cases} 0 < I_n < I_{n+1} \text{ for } n > \bar{n}; \\ I_{n-1} < I_n < 0 \text{ for } n \leq \bar{n}. \end{cases}$$

Let $\bar{n} = i_0(k+1) + j_0,\ 0 \leq j_0 \leq k$, then by (5.1) we know

$$\begin{aligned} sgn\left(y_{m(k+1)+j+1}\right) &= sgn\left(y_{-k+j}\gamma^{m+1}\prod_{i=0}^{m}\frac{I_{i(k+1)+j}}{I_{i(k+1)+j+1}}\right) \\ &= \begin{cases} -\,sgn\left(y_{-k+j}\right) \text{ for } m \geq i_0, \text{ when } j = j_0; \\ sgn\left(y_{-k+j}\right) \text{ for } m \geq i_0, \text{ when } j \neq j_0, \end{cases} \end{aligned}$$

from which and (5.2) we get

$$\lim_{m\to\infty} y_{m(k+1)+j+1} = \begin{cases} -\,sgn\left(y_{-k+j}\right) exp(\varphi_j), \text{ when } j = j_0; \\ sgn\left(y_{-k+j}\right) exp(\varphi_j), \text{ when } j \neq j_0. \end{cases}$$

(c) If $\alpha > 0$, the method is similar to (b), by Lemma 5.1 we get

$$\lim_{m\to\infty} y_{m(k+1)+j+1} = sgn\left(y_{-k+j}\right) exp(\varphi_j),\ j = 0, 1, 2, \ldots, k,$$

where $\varphi_j = \ln(|y_{-k+j}|) + \sum\limits_{i=0}^{\infty}\ln\left(\frac{\gamma I_{i(k+1)+j}}{I_{i(k+1)+j+1}}\right)$, $j = 0, 1, 2, \ldots, k$. □

6. Periodicity of IVP(1.1)

Theorem 6.1. *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of IVP(1.1). Then $\{y_n\}_{n=-k}^{\infty}$ is eventually $(k+1)$-periodic solution of IVP(1.1) if and only if*

$$\prod_{i=0}^{k} a_{-i} = \gamma - 1. \tag{6.1}$$

In fact, eventually $(k+1)$-periodic solution of IVP(1.1) must be $(k+1)$-periodic solution of IVP(1.1).

Proof. Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of IVP(1.1). Firstly, we will prove that if $\{y_n\}_{n=-k}^{\infty}$ is eventually $(k+1)$-periodic solution of IVP(1.1), then (6.1) is satisfied. Actually, if $\prod_{i=0}^{k} a_{-i} = 0$, then $I_n = I_{n+1},\ n \in \mathbb{N}$, by Theorem 5.1 we know $y_{n+1} = \gamma y_{n-k},\ n \in \mathbb{N}$, note that $\{y_n\}_{n=-k}^{\infty}$ is $(k+1)$-periodic solution of IVP(1.1), hence $\gamma = 1$, (6.1) is satisfied. Suppose that $\prod_{i=0}^{k} a_{-i} \neq 0$, then $y_n \neq 0,\ n \in \mathbb{N}$. By IVP(1.1) we know

$$(\gamma - \prod_{i=0}^{k} y_{n-i+1}) y_{n-k} = y_{n+1},\ n \in \mathbb{N}. \tag{6.2}$$

Since $\{y_n\}_{n=-k}^{\infty}$ is eventually $(k+1)$-periodic solution of IVP(1.1), there exists $\bar{n} \in \mathbb{N}$ such that

$$y_{n+1} = y_{n-k}, n = \bar{n}, \bar{n}+1, \bar{n}+2, \ldots, \tag{6.3}$$

and so

$$y_{\bar{n}+1} = y_{\bar{n}-k}. \tag{6.4}$$

Combing (6.2) with (6.4), we get

$$\prod_{i=0}^{k} y_{\bar{n}-i+1} = \gamma - 1. \tag{6.5}$$

Combing (6.5) with (6.4), we get

$$\prod_{i=0}^{k} y_{\bar{n}-i} = \gamma - 1. \tag{6.6}$$

Substituting (6.6) into (6.2) we get

$$y_{\bar{n}} = y_{\bar{n}-k-1}. \tag{6.7}$$

Similarly, from (6.7) we can get

$$y_{\bar{n}-1} = y_{\bar{n}-k-2}.$$

By induction, we generally obtain

$$y_{n+1} = y_{n-k}, n = 0, 1, 2, \ldots, \bar{n} - 1. \tag{6.8}$$

From (6.8) we know $y_1 = y_{-k}$, combing with (1.1) we get

$$\prod_{i=0}^{k} a_{-i} = \prod_{i=0}^{k} y_{-i} = \gamma - 1,$$

namely (6.1) is satisfied.

Conversely, we will prove that if (6.1) is satisfied, then the solution $\{y_n\}_{n=-k}^{\infty}$ of IVP(1.1) is eventually $(k+1)$-periodic. By IVP(1.1) we know

$$y_1 = \frac{\gamma y_{-k}}{1 + \prod_{i=0}^{k} a_{-i}} = y_{-k}.$$

Note that $\prod_{i=0}^{k} y_{-i+1} = \prod_{i=0}^{k} y_{-i} = \gamma - 1$, by IVP(1.1) we can also get $y_2 = y_{-k+1}$. By induction, one can get $y_{n+1} = y_{n-k},\ n \in \mathbb{N}$. Hence $\{y_n\}_{n=-k}^{\infty}$ is a $(k+1)$-periodic solution of IVP(1.1). Of course $\{y_n\}_{n=-k}^{\infty}$ is eventually $(k+1)$-periodic solution of IVP(1.1). □

7. Oscillation behavior

In this section, we investigate the oscillation of the solution of IVP(1.1).

Theorem 7.1. *Suppose $\alpha = 0$, then the solution $\{y_n\}_{n=-k}^{\infty}$ of IVP(1.1) is oscillatory about $\bar{y} = 0$.*

Proof. Suppose $\alpha = 0$, then by Lemma 5.1 we know $\frac{I_n}{I_{n+1}} = 1,\ n \in \mathbb{N}$, hence by Lemma 5.1 we have

$$y_{m(k+1)+j+1} = y_{-k+j}\gamma^{m+1}, \tag{7.1}$$

for $j = 0, 1, 2, \ldots, k;\ m \in \mathbb{N}$.

Since $\alpha = \prod_{i=0}^{k} y_{-i} = 0$, there exists $i_0 (0 \le i_0 \le k)$ such that $y_{-k+i_0} = 0$, by (7.1) we have

$$y_{m(k+1)+i_0+1} = y_{-k+i_0}\gamma^{m+1} = 0,\ m = 0, 1, 2, \ldots,$$

hence $\{y_n\}_{n=-k}^{\infty}$ is oscillatory about $\bar{y} = 0$. □

Theorem 7.2. *The following statements are true.*

(i) Suppose $\alpha > 0$, then $\{y_n\}_{n=-k}^{\infty}$ is positive(negative when k is odd) or $\{y_n\}_{n=-k}^{\infty}$ oscillates about $\bar{y} = 0$ with negative semicycles of length at most $2\lfloor \frac{k}{2} \rfloor$, where $\lfloor \cdot \rfloor$ is floor function.

(ii) Suppose $\gamma - 1 \le \alpha < 0$, then $\{y_n\}_{n=-k}^{\infty}$ is negative(when k is even) or $\{y_n\}_{n=-k}^{\infty}$ oscillates about $\bar{y} = 0$ with negative semicycles of length at most $2\lfloor \frac{k+1}{2} \rfloor - 1$, where $\lfloor \cdot \rfloor$ is floor function.

Proof. Suppose either $\alpha > 0$ or $\gamma - 1 \le \alpha < 0$, by Lemma 5.1, we know $\frac{I_n}{I_{n+1}} > 0$, $n \in \mathbb{N}$, hence by (5.1) we have

$$sgn(y_{m(k+1)+j+1}) = sgn\left(y_{-k+j}\gamma^{m+1}\prod_{i=0}^{m}\frac{I_{i(k+1)+j}}{I_{i(k+1)+j+1}}\right) = sgn(y_{-k+j}),$$

for $j = 0, 1, 2, \dots, k;\ m \in \mathbb{N}$.

That is, each subsequence $\{y_{m(k+1)+j+1}\}_{m=-1}^{\infty}$, $j = 0, 1, 2, \dots, k$ preserves sign. It follows that, if $y_{-k+j} > 0$(respectively, $y_{-k+j} < 0$), $j = 0, 1, 2, \dots, k$, then $\{y_{m(k+1)+j+1}\}_{m=0}^{\infty}$ is positive(respectively, negative).

(i) If $\alpha > 0$, then either $\{y_n\}_{n=-k}^{\infty}$ is positive(negative when k is odd) or $\{y_n\}_{n=-k}^{\infty}$ oscillates about $\bar{y} = 0$ with negative semicycles of length at most $k - 1$ when k is odd, $\{y_n\}_{n=-k}^{\infty}$ oscillates about $\bar{y} = 0$ with negative semicycles of length at most k when k is even, that means that $\{y_n\}_{n=-k}^{\infty}$ oscillates about $\bar{y} = 0$ with negative semicycles of length at most τ, where $\tau = 2\lfloor \frac{k}{2} \rfloor$, $\lfloor \cdot \rfloor$ is floor function.

(ii) If $\gamma - 1 \leq \alpha < 0$, then $\{y_n\}_{n=-k}^{\infty}$ is negative(when k is even) or $\{y_n\}_{n=-k}^{\infty}$ oscillates about $\bar{y} = 0$ with negative semicycles of length at most k when k is odd, $\{y_n\}_{n=-k}^{\infty}$ oscillates about $\bar{y} = 0$ with negative semicycles of length at most $k - 1$ when k is even, that means that $\{y_n\}_{n=-k}^{\infty}$ oscillates about $\bar{y} = 0$ with negative semicycles of length at most τ, where $\tau = 2\lfloor \frac{k+1}{2} \rfloor - 1$, $\lfloor \cdot \rfloor$ is floor function. □

Theorem 7.3. *Suppose either $\alpha < 0 \leq \gamma - 1$ or $\alpha < \gamma - 1 < 0$, if there exist $j_0 (0 \leq j_0 \leq k)$ and $\bar{n} \in \mathbb{N}$ such that*

$$I_n = \begin{cases} 1 - \gamma + \alpha(1 - \gamma^n),\ n \in \mathbb{N}, when\ \gamma \neq 1, \\ 1 + n\alpha,\ n \in \mathbb{N}, when\ \gamma = 1, \end{cases}$$

$$y_{-k+j} \begin{cases} < 0\ for\ j = j_0, \\ > 0\ for\ j \neq j_0, \end{cases}$$

and

$$\frac{I_n}{I_{n+1}} \begin{cases} < 0\ for\ n = \bar{n}, \\ > 0\ for\ n \neq \bar{n}, \end{cases}$$

where j_0 and $\bar{n}$ satisfy $(\bar{n} - j_0 - 1) \mod (k+1) = 0$. Then $\{y_n\}_{n=-k}^{\infty}$ is eventually positive; Otherwise, $\{y_n\}_{n=-k}^{\infty}$ is oscillatory about $\bar{y} = 0$.

Proof. Suppose either $\alpha < 0 \leq \gamma - 1$ or $\alpha < \gamma - 1 < 0$, then by Lemma 5.1 there exists $\bar{n} \in \mathbb{N}$ such that $\frac{I_{\bar{n}}}{I_{\bar{n}+1}} < 0$, $\frac{I_n}{I_{n+1}} > 0$ for $n \neq \bar{n}$. Hence

$$\begin{aligned} & sgn\left(y_{m(k+1)+j+1}\right) \\ = & sgn\left(y_{-k+j}\gamma^{m+1}\prod_{i=0}^{m}\frac{I_{i(k+1)+j}}{I_{i(k+1)+j+1}}\right) \\ = & \begin{cases} -sgn\left(y_{-k+j}\right),\ \text{when}\ \left(m(k+1)+j+1-\bar{n}\right) \mod (k+1) = 0, \\ sgn\left(y_{-k+j}\right),\ \text{when}\ \left(m(k+1)+j+1-\bar{n}\right) \mod (k+1) \neq 0, \end{cases} \end{aligned}$$

for $m(k+1) + j + 1 \geq \bar{n}$.

If there exists $j_0 (0 \leq j_0 \leq k)$ such that

$$y_{-k+j} \begin{cases} < 0 \text{ for } j = j_0, \\ > 0 \text{ for } j \neq j_0, \end{cases}$$

and $(\bar{n} - j_0 - 1) \mod (k+1) = 0$.

Then

$$sgn\left(y_{m(k+1)+j+1}\right) = \begin{cases} -sgn\left(y_{-k+j}\right),\ \text{when}\ j = j_0, \\ sgn\left(y_{-k+j}\right),\ \text{when}\ j \neq j_0, \end{cases} = 1$$

for $m(k+1)+j+1 \geq \bar{n}$. Hence $\{y_n\}_{n=-k}^{\infty}$ is eventually positive; Otherwise, $\{y_n\}_{n=-k}^{\infty}$ is oscillatory about $\bar{y} = 0$. $\square$

8. Numerical results

In this section, we give a few numerical results for some special values of the parameters. We take $k = 3$ and $k = 4$ in IVP(1.1).

Case 1: $\gamma < 1$.

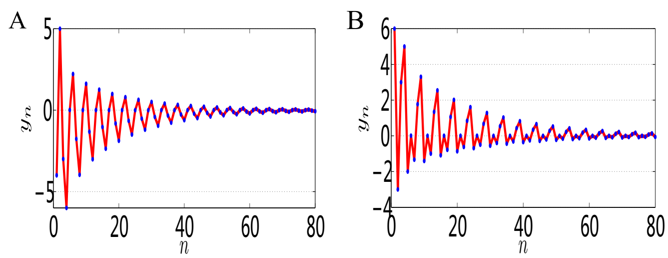

Figure 1. The case $\gamma < 1$. (A) $y_{n+1} = \frac{0.8y_{n-3}}{1+\prod_{i=0}^{3} y_{n-i}}$, $(y_1, y_2, y_3, y_4) = (-4, 5, -3, -6)$. (B) $y_{n+1} = \frac{0.85y_{n-4}}{1+\prod_{i=0}^{4} y_{n-i}}$, $(y_1, y_2, y_3, y_4, y_5) = (6, -3, 3, 5, -2)$.

Case 2: $\gamma = 1, \alpha = 0$.

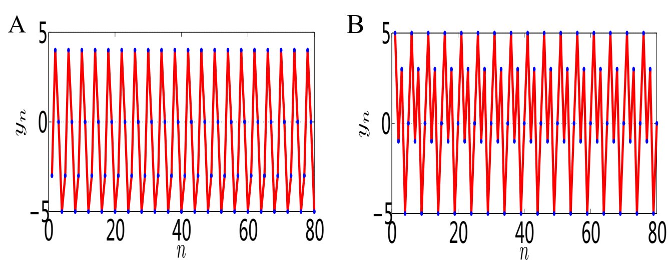

Figure 2. The case $\gamma = 1, \alpha = 0$. (A) $y_{n+1} = \frac{y_{n-3}}{1+\prod_{i=0}^{3} y_{n-i}}$, $(y_1, y_2, y_3, y_4) = (-3, 4, 0, -5)$. (B) $y_{n+1} = \frac{y_{n-4}}{1+\prod_{i=0}^{4} y_{n-i}}$, $(y_1, y_2, y_3, y_4, y_5) = (5, -1, 3, -5, 0)$.

Case 3: $\gamma = 1, \alpha \neq 0$.

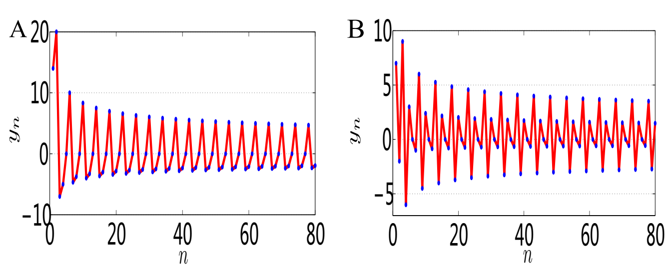

Figure 3. The case $\gamma = 1, \alpha \neq 0$. (A) $y_{n+1} = \frac{y_{n-3}}{1+\prod_{i=0}^{3} y_{n-i}}$, $(y_1, y_2, y_3, y_4) = (14, 20, -7, -5)$. (B) $y_{n+1} = \frac{y_{n-4}}{1+\prod_{i=0}^{4} y_{n-i}}$, $(y_1, y_2, y_3, y_4, y_5) = (7, -2, 9, -6, 3)$.

Case 4: $\gamma > 1, \alpha = 0$.

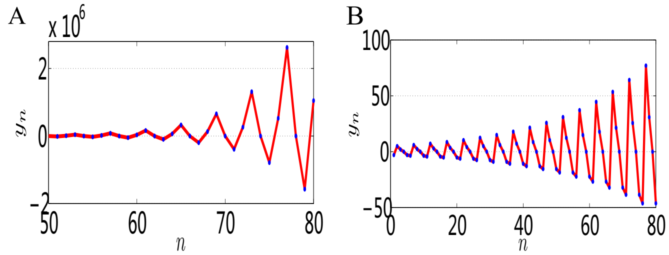

Figure 4. The case $\gamma > 1, \alpha = 0$. (A) $y_{n+1} = \frac{2y_{n-3}}{1+\prod_{i=0}^{3} y_{n-i}}, (y_1, y_2, y_3, y_4) = (5, 0, -3, 2)$. (B) $1.2y_{n+1} = \frac{y_{n-4}}{1+\prod_{i=0}^{4} y_{n-i}}, (y_1, y_2, y_3, y_4, y_5) = (-3, 5, 2, 0, -3)$.

Case 5: $\gamma > 1, \alpha < 0$.

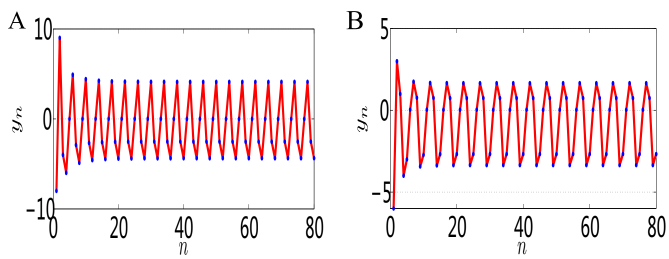

Figure 5. The case $\gamma > 1, \alpha < 0$. (A) $y_{n+1} = \frac{1.4y_{n-3}}{1+\prod_{i=0}^{3} y_{n-i}}, (y_1, y_2, y_3, y_4) = (-8, 9, -4, -6)$. (B) $y_{n+1} = \frac{1.2y_{n-4}}{1+\prod_{i=0}^{4} y_{n-i}}, (y_1, y_2, y_3, y_4, y_5) = (-6, 3, 1, -4, -3)$.

Case 6: $\gamma > 1, \alpha > 0$.

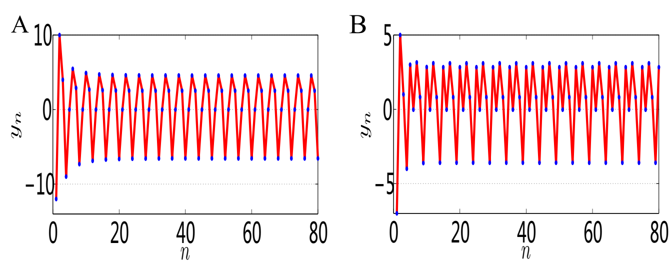

Figure 6. The case $\gamma > 1, \alpha > 0$. (A) $y_{n+1} = \frac{1.2y_{n-3}}{1+\prod_{i=0}^{3} y_{n-i}}, (y_1, y_2, y_3, y_4) = (-12, 10, 4, -9)$. (B) $y_{n+1} = \frac{1.7y_{n-4}}{1+\prod_{i=0}^{4} y_{n-i}}, (y_1, y_2, y_3, y_4, y_5) = (-7, 5, 1, -4, 3)$.

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