Journal of Applied Analysis and Computation
Volume 7, Number 3, August 2017, 1177–1184
Website:http://jaac-online.com/
DOI:10.11948/2017073

# EXACT STATIONARY-WAVE SOLUTIONS IN THE STANDARD MODEL OF THE KERR-NONLINEAR OPTICAL FIBER WITH THE BRAGG GRATING*

Yanggeng Fu[1] and Jibin Li[1,2,†]

**Abstract** By using dynamical system method to the standard model of the Kerr-Nonlinear optical fiber with the Bragg grating, under fixed parameter conditions, all possible exact parametric representations of the bounded stationary wave solutions are obtained from the double sine-Gordon equation.

**Keywords** Double sine-Gordon equation, Kerr-Nonlinear optical fiber, Bragg gratings, phase kink wave solution, periodic solution.

**MSC(2010)** 34C37, 34C23, 74J30, 58Z05.

## 1. Introduction

Merhasin, et al. [5] stated that periodic structures in optical waveguides known as Bragg gratings (BGs), which provide for resonant reflection of light, and thus strong linear coupling between counter-propagating waves, have been a subject of intensive theoretical and experimental research, due to their numerous applications to optical sensors and various telecommunication devices (such as add-drop multiplexers, dispersion compensators, narrowband filters, etc.), as well as their great potential as media for fundamental studies of nonlinear optical dynamics.

A standard theoretical model of a Kerr-nonlinear medium equipped with the BG is based on a system of coupled-mode equations for amplitudes of the counter-propagating waves, $U(x,t)$ and $V(x,t)$, which are coupled linearly by the BG reflection, and nonlinearly by the cross-phase modulation, and also take into account the self-phase modulation effect:

$$\begin{aligned} &iU_t + iU_x + \left(\tfrac{1}{2}|U|^2 + |V|^2\right)U + \kappa V = 0,\\ &iV_t - iV_x + \left(\tfrac{1}{2}|U|^2 + |V|^2\right)V + \kappa U = 0, \end{aligned} \tag{1.1}$$

where $U(x,t)$ and $V(x,t)$ are amplitudes of counter-propagating waves, $x$ and $t$ are the coordinates along the fiber and time, and $\kappa$ $(>0)$ is the Bragg reflectivity [2,3,6].

†the corresponding author. Email address: lijb@zjnu.cn (J. Li)
1School of Mathematical Sciences, Huaqiao University, Quanzhou, Fujian 362021, China
2Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China
*The authors were supported by National Natural Science Foundation of China (11401229, 11471289, 11571318).

Chow, et al. [2] looked for a general stationary-wave solution of equation (1.1) as

$$U(x,t) = u(x)\exp(-i\omega t), \quad V(x,t) = v(x)\exp(-i\omega t), \tag{1.2}$$

with the frequency $\omega$. Considering the reduction $v(x) = -u^*(x)$, then, complex function $u(x)$ is sought for the Madelung form

$$u(x) = R(x)\exp\left[\frac{1}{4}i\Psi(x)\right]. \tag{1.3}$$

Substituting (1.2) and (1.3) into (1.1), it is easy to see that the amplitude satisfies

$$R^2(x) = \frac{1}{6}\left[\Psi' - 4\omega + 4\kappa\cos\left(\frac{1}{2}\Psi\right)\right], \tag{1.4}$$

with $\Psi' \equiv \frac{d\Psi}{dx}$ and the phase obey the stationary version of the double sine-Gordon equation

$$\Psi'' = 8\kappa\omega\sin\left(\frac{1}{2}\Psi\right) - 4\kappa^2\sin(\Psi). \tag{1.5}$$

Two families of exact periodic solutions of equation (1.1) were constructed in [2]. The solutions are named *sn* and *cn* waves, according to the elliptic functions used in their analytical representation. Due to the dynamical behavior of equation (1.5) has not been discussed by the above reference. The results given by [2] are not complete, they did not find all exact solutions of equation (1.5). To realize the complete study, in this paper, we use the dynamical system method [4] to give all possible exact stationary-wave solutions of equation (1.5) depending on the changes of the parameter pair $(\omega, \kappa)$.

The paper is organized as follows. In section 2, we consider the dynamical behavior of solutions of equation (1.5). In section 3, under given parameter conditions, we investigate the exact explicit parametric representations for all solutions of (1.5). In section 4, we give the main conclusion of this paper.

## 2. Bifurcations of phase portraits of the double sine-Gordon equation (5)

Equation (1.5) is equivalent to the system

$$\frac{d\Psi}{dx} = y, \quad \frac{dy}{dx} = 8\kappa\left(\omega - \kappa\cos\left(\frac{1}{2}\Psi\right)\right)\sin\left(\frac{1}{2}\Psi\right). \tag{2.1}$$

System (2.1) is a planar Hamiltonian system with the Hamiltonian quality

$$H(\Psi, y) = \frac{1}{2}y^2 + 4\kappa[4\omega\cos\frac{1}{2}\Psi - \kappa\cos\Psi] = h. \tag{2.2}$$

Clearly, system (2.1) is $4\pi-$periodic with respect to $\Psi$. Hence, the state $(\Psi, y)$ can be viewed on a phase cylinder $S^1 \times R$, where $S^1 = [-\Psi_a, 4\pi - \Psi_a]$ with $\Psi_a, 4\pi - \Psi_a$ identified (as an example, see Fig. 1, where $\Psi_a = 0$ for $\omega > 0$ and $\Psi_a = -2\pi$ for $\omega < 0$.

When $|\omega| \geq \kappa$, there exist two equilibrium points of (2.1) with $O(0,0)$ and $E_1(2\pi, 0)$. For $\omega > 0$ $(< 0)$, $O(0,0)$ is a saddle point (a center), $E_1$ is a center (a

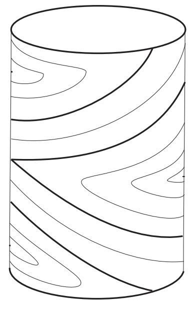
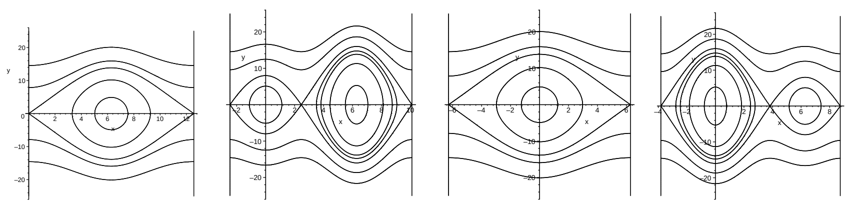


**Figure 1.** The phase cylinder of system (2.1).

saddle) (see Figs. 2(a) and 2(c)). When $\omega > 0$, $\Psi = 0$ and $\Psi = 4\pi$ are identified, while when $\omega < 0$, $\Psi = -2\pi$ and $\Psi = 2\pi$ are identified.

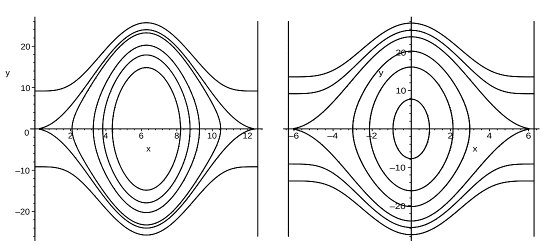
When $0 < |\omega| < \kappa$, there exist four equilibrium points of system (2.1) with $E_{\pm}(\pm\Psi_0, 0)$, $O(0,0)$, $E_1(2\pi, 0)$, where $\Psi_0 = 2\arccos(\omega/\kappa)$, $\Psi = -\Psi_0$ and $\Psi = 4\pi - \Psi_0$ are identified. The equilibrium points $O$ and $E_1$ are centers, $E_{\pm}$ are saddle points (see Figs. 2(b), 2(d)).



(a) $\omega > \kappa$ (b) $0 < \omega < \kappa$ (c) $\omega < -\kappa$ (d) $-\kappa < \omega < 0$

**Figure 2.** The phase portraits of system (2.1).

Especially, when $|\omega| = \kappa$, for $\omega < 0$ $(> 0)$, equilibrium point $O(0,0)$ is a center (a two-order saddle point), while equilibrium point $E_1(2\pi, 0)$ is a two-order saddle point (a center), we have the phase portraits of system (2.1) shown in Fig. 3.



(a) $\omega = \kappa$ (b) $\omega = -\kappa$

**Figure 3.** The phase portraits of system (2.1) when $|\omega| = \kappa$.

Let

$$h_0 = H(0,0) = 4\kappa(4\omega - \kappa), \quad h_1 = H(2\pi, 0) = -4\kappa(4\omega + \kappa),$$
$$h_2 = H(\pm\Psi_0, 0) = 4(2\omega^2 + \kappa^2).$$

Thus, we see from Fig. 2 and Fig. 3 that system (2.1) has the following dynamics.

**1. The case** $|\omega| \geq \kappa$.

When $\omega > 0$ ($< 0$), for $h \in [h_1, h_0)$ ($h \in [h_0, h_1)$), the level curve defined by $H(\Psi, y) = h$ gives rise to a family $\Gamma_o^h$ of oscillating periodic orbits of system (2.1) enclosing the center $E_1(2\pi, 0)$ ($O(0,0)$);

For $h = h_0$ ($h = h_1$), the level curves defined by $H(\Psi, y) = h_0$ ($h_1$) determine two homoclinic orbits to the saddle point $O(0,0)$ ($E_1(2\pi, 0)$) in the phase cylinder.

For $h \in (h_0, \infty)$ ($h \in (h_1, \infty)$), the level curves defined by $H(\Psi, y) = h$ give rise to two families $\Gamma_{r1}^h$ and $\Gamma_{r2}^h$ of rotating periodic orbits of system (2.1) in the phase cylinder.

**2. The case** $0 < |\omega| < \kappa$.

For $h \in (h_0, h_2)$ and $h \in (h_1, h_2)$ the level curves defined by $H(\Psi, y) = h$ give rise to two families $\Gamma_{o1}^h$ and $\Gamma_{o2}^h$ of oscillating periodic orbits of system (2.1) enclosing the center $O(0,0)$ and $E_1(2\pi, 0)$, respectively.

For $h = h_2$, the level curves defined by $H(\Psi, y) = h_2$ determine four heteroclinic orbits to the saddle point $E_\pm(0,0)$ in the phase cylinder.

For $h \in (h_2, \infty)$, the level curves defined by $H(\Psi, y) = h$ give rise to two families $\Gamma_{r1}^h$ and $\Gamma_{r2}^h$ of rotating periodic orbits of system (2.1) in the phase cylinder.

# 3. The parametric representations of all orbits of system (2.1)

In this section, we use the Hamiltonian (2.2) and the first equation of system (2.1) to calculate the parametric representations of all bounded orbits of system (2.1). In fact, we have

$$x = \int \frac{d\Psi}{y} = \int \frac{d\Psi}{\sqrt{2h - [32\kappa\cos(\Psi/2) + 8\kappa^2\cos\Psi]}} = \int \frac{4dw}{\sqrt{a(h) + c(h)w^2 + e(h)w^4}}, \tag{3.1}$$

where $a(h) = 2h - 32\kappa\omega + 8\kappa^2$, $c(h) = 4[h - 12\kappa^2]$, $e(h) = 2h + 32\kappa\omega + 8\kappa^2$. Obviously,

$$\begin{aligned}
&a(h_0) = 0,\ c(h_0) = 64\kappa(\omega - \kappa),\ e(h_0) = 64\omega\kappa;\\
&a(h_1) = -64\omega,\ c(h_1) = -64\kappa(\omega + \kappa),\ e(h_1) = 0;\\
&a(h_2) = 16(\omega - \kappa)^2,\ c(h_2) = 16(\omega^2 - \kappa^2),\ e(h_2) = 16(\omega + \kappa)^2.
\end{aligned}$$

Let $\Delta = c^2 - 4ae = 512(8\kappa^2\omega^2 + 4\kappa^4 - h\kappa^2)$ and

$$\alpha^2 = \frac{1}{2e}(-c + \sqrt{\Delta}),\ \beta^2 = \frac{1}{2e}(-c - \sqrt{\Delta}),\ \alpha_1^2 = -\alpha^2,\ \beta_1^2 = -\beta^2. \tag{3.2}$$

Notice that for $h = h_2 = 4(2\omega^2 + \kappa^2)$, we have $\Delta = 0$.

By using the above results, under the different parameter conditions and corresponding to the different Hamiltonian $h$, we calculate the integral given by the right hand of (3.1). It follows the following conclusions.

**1. The case** $|\omega| \geq \kappa$.

(i) First, we assume that $\omega > 0$. In this case, if $\omega > \kappa$, we have $h_1 < 0 < h_0 < h_2$. If $\omega = \kappa$, then, $h_0 = h_2 = 12\omega^2$.

When $h \in (h_1, h_0)$, the family $\Gamma_o^h$ of the oscillating orbits of (3.3) enclosing the critical point $E_1(2\pi, 0)$ has the following parametric representation [1]:

$$\Psi(x) = 4\arctan\left(\frac{\beta_1 \mathrm{dn}(\Omega x, k)}{k\mathrm{sn}(\Omega x, k)}\right), \tag{3.3}$$

where $\Omega = \Delta^{\frac{1}{4}},\ k = \frac{\beta_1}{\sqrt{\alpha^2+\beta_1^2}}$.

When $h = h_0 = 4\kappa(4\omega - \kappa)$, the two homoclinic orbits have the following parametric representations: for $\omega > \kappa$,

$$\Psi(x) = 2\pi \pm 4\arctan\left[\sqrt{1-\frac{\kappa}{\omega}}\sinh\left(2\sqrt{\kappa(\omega-\kappa)}x\right)\right] \tag{3.4}$$

and for $\omega = \kappa$,

$$\Psi(x) = \pm 4\arctan\left(\frac{1}{2\omega x}\right). \tag{3.5}$$

(3.4) and (3.5) are called phase solitary waves.

When $h \in (h_0, h_2)$, two families $\Gamma_{r1}^h,\ \Gamma_{r2}^h$ of the rotating orbits of system (2.1) have the following parametric representations:

$$\Psi(x) = \pm 4\arctan\left(\frac{\alpha_1 \mathrm{cn}(\Omega_r x, k_r)}{\mathrm{sn}(\Omega_r x, k_r)}\right), \tag{3.6}$$

where $k_r = \frac{\sqrt{\alpha_1^2-\beta_1^2}}{\alpha_1},\ \Omega_r = \alpha_1\sqrt{e}$.

When $h = h_2$, two rotating orbits $\Gamma_{r1}^{h_2},\ \Gamma_{r2}^{h_2}$ of (2.1) have the following parametric representations:

$$\Psi(x) = 2\pi \pm 4\arctan\left(\sqrt{\frac{\omega-\kappa}{\omega+\kappa}}\tan\left(\sqrt{\omega^2-\kappa^2}x\right)\right). \tag{3.7}$$

When $h \in (h_2, \infty)$, we have $\Delta < 0$. Write that

$$\rho^2 = \frac{1}{2e}\left(-c + i\sqrt{|\Delta|}\right),\quad \bar{\rho}^2 = \frac{1}{2e}\left(-c - i\sqrt{|\Delta|}\right).$$

Clearly, $\rho^2\bar{\rho}^2 = \frac{a(h)}{e(h)},\ \rho^2 + \bar{\rho}^2 = \frac{c(h)}{e(h)}$. Thus, two families $\Gamma_{r1}^h,\ \Gamma_{r2}^h$ of the rotating orbits of system (2.1) have the following parametric representations:

$$\Psi(x) = \pm 4\arctan\left(\sqrt{\frac{\rho\bar{\rho}(1+\mathrm{cn}(\Omega_{r0}\xi, k_{r0}))}{1-\mathrm{cn}(\Omega_{r0}\xi, k_{r0})}}\right), \tag{3.8}$$

where $\Omega_{r0} = \sqrt{a(h)},\ k_{r0}^2 = -\frac{(\rho-\bar{\rho})^2}{4\rho\bar{\rho}}$.

(ii) Second, we assume that $\omega < 0$. In this case, if $|\omega| > \kappa$, we have $h_0 = -4\kappa(4|\omega| + \kappa) < 0 < 4\kappa(4|\omega| - \kappa) = h_1 < h_2$. If $|\omega| = \kappa$, then, $h_1 = h_2$.

When $h \in (h_0, h_1)$, the family $\Gamma_o^h$ of the oscillating orbits of system (2.1) enclosing the critical point $O(0, 0)$ has the following parametric representation:

$$\Psi(x) = 4\arctan\left(\frac{\beta_1 k\mathrm{sn}(\Omega\xi, k)}{\mathrm{dn}(\Omega x, k)}\right), \tag{3.9}$$

where $\Omega = \Delta^{\frac{1}{4}},\ k = \frac{\alpha}{\sqrt{\alpha^2+\beta_1^2}}$.

When $h = h_1 = -4\kappa(4\omega + \kappa)$, the two homoclinic orbits have the following parametric representations: for $|\omega| > \kappa$,

$$\Psi(x) = \pm 4 \arctan\left(\sqrt{\frac{|\omega|}{|\omega| - \kappa}} \sinh\left(2\sqrt{\kappa(|\omega| - \kappa)}x\right)\right). \tag{3.10}$$

For $|\omega| = \kappa$,

$$\Psi(x) = \pm 4 \arctan\left(2\sqrt{|\omega|\kappa}x\right). \tag{3.11}$$

When $h \in (h_1, h_2)$, two families $\Gamma_{r1}^h,\ \Gamma_{r2}^h$ of the rotating orbits of system (2.1) have the following parametric representations:

$$\Psi(x) = \pm 4 \arctan\left(\frac{\beta_1 \mathrm{sn}(\Omega_r x, k_r)}{\mathrm{cn}(\Omega_r x, k_r)}\right), \tag{3.12}$$

where $\Omega_r = \frac{\alpha\sqrt{e(h)}}{4},\ k_r^2 = \frac{\alpha^2-\beta_1^2}{\alpha^2}$.

When $h = h_2$, two rotating orbits $\Gamma_{r1}^{h_2},\ \Gamma_{r2}^{h_2}$ of system (2.1) has the following parametric representations:

$$\Psi(x) = \pm 4 \arctan\left(\sqrt{\frac{|\omega| + \kappa}{|\omega| - \kappa}} \tan\left(\sqrt{\omega^2 - \kappa^2}x\right)\right). \tag{3.13}$$

When $h \in (h_2, \infty)$, two families $\Gamma_{r1}^h,\ \Gamma_{r2}^h$ of the rotating orbits of system (2.1) has the following parametric representation:

$$\Psi(x) = 2\pi \pm 4 \arctan\left(\sqrt{\frac{\rho\bar{\rho}(1 + \mathrm{cn}(\Omega_{r0}x, k_{r0}))}{1 - \mathrm{cn}(\Omega_{r0}x, k_{r0})}}\right), \tag{3.14}$$

where $\Omega_{r0},\ k_{r0}$ and $\rho,\ \bar{\rho}$ are the same as (3.8).

**2. The case** $|\omega| < \kappa$.

(i) When $\omega > 0,\ h_1 < h_0 < h_2$.

When $h \in (h_1, h_0)$, the family $\Gamma_{o2}^h$ of the oscillating orbits of system (2.1) enclosing the critical point $E_1(2\pi, 0)$ has the same parametric representation as (3.3).

When $h = h_0$, the oscillating orbit $\Gamma_{o2}^{h_0}$ of system (2.1) enclosing the critical point $E_1(2\pi, 0)$ has the parametric representation:

$$\Psi(x) = 4 \arctan\left(\sqrt{\frac{\kappa - \omega}{\omega}} \csc\left(2\sqrt{\kappa(\kappa - \omega)}x\right)\right). \tag{3.15}$$

When $h \in (h_0, h_2)$, two families $\Gamma_{o1}^h$ and $\Gamma_{o2}^h$ of the oscillating orbits of system (2.1), enclosing respectively the critical point $O(0, 0)$ and $E_1(2\pi, 0)$ have the parametric representations:

$$\Psi(x) = 4 \arctan(\beta \mathrm{sn}(\Omega_{o1}x, k_{o1})), \tag{3.16}$$

where $\Omega_{o1} = \frac{\alpha\sqrt{e(h)}}{4},\ k_{o1}^2 = \frac{\beta^2}{\alpha^2}$ and

$$\Psi(x) = 4 \arctan\left(\frac{\alpha}{\mathrm{sn}(\Omega_{o2}x, k_{o2})}\right), \tag{3.17}$$

where $\Omega_{o2} = \Omega_{o1},\ k_{o2}^2 = k_{o1}^2$.

When $h = h_2$, four heteroclinic orbits $\Gamma_{o1}^{h_2}$ and $\Gamma_{o2}^{h_2}$ have the parametric representations:

$$\Psi(x) = \pm 4 \arctan\left(\sqrt{\frac{\kappa - \omega}{\omega + \kappa}} \tanh\left(\sqrt{\kappa^2 - \omega^2} x\right)\right), \tag{3.18}$$

and

$$\Psi(x) = 2\pi \pm 4 \arctan\left(\sqrt{\frac{\kappa - \omega}{\omega + \kappa}} \tanh\left(\sqrt{\kappa^2 - \omega^2} x\right)\right). \tag{3.19}$$

(3.18) and (3.19) are called phase kink waves and phase ani-kink waves.

When $h \in (h_2, \infty)$, two families $\Gamma_{r1}^h,\ \Gamma_{r2}^h$ of the rotating orbits of system (2.1) have the same parametric representations as (3.14).

(ii) When $\omega < 0,\ h_0 < h_1 < h_2$.

When $h \in (h_0, h_1)$, the family $\Gamma_{o1}^h$ of the oscillating orbits of system (2.1) enclosing the critical point $O(0,0)$ has the same parametric representation as (3.9).

When $h = h_1$, the oscillating orbit $\Gamma_{o1}^{h_1}$ of system (2.1) enclosing the critical point $O(0,0)$ has the parametric representation:

$$\Psi(x) = 4 \arctan\left(\sqrt{\frac{|\omega|}{\kappa - |\omega|}} \sin\left(2\sqrt{\kappa(\kappa - |\omega|)} x\right)\right). \tag{3.20}$$

When $h \in (h_1, h_2)$, two families $\Gamma_{o1}^h$ and $\Gamma_{o2}^h$ of the oscillating orbits of system (2.1) enclosing respectively the critical point $O(0,0)$ and $E_1(2\pi, 0)$ have the same parametric representations as (3.16) and (3.17), respectively.

When $h = h_2$, four heteroclinic orbits $\Gamma_{o1}^{h_2}$ and $\Gamma_{o2}^{h_2}$ have the parametric representations:

$$\Psi(x) = \pm 4 \arctan\left(\sqrt{\frac{|\omega| + \kappa}{\kappa - |\omega|}} \tanh\left(\sqrt{\kappa^2 - \omega^2} x\right)\right) \tag{3.21}$$

and

$$\Psi(x) = 2\pi \pm 4 \arctan\left(\sqrt{\frac{|\omega| + \kappa}{\kappa - |\omega|}} \tanh\left(\sqrt{\kappa^2 - \omega^2} x\right)\right). \tag{3.22}$$

When $h \in (h_2, \infty)$, two families $\Gamma_{r1}^h,\ \Gamma_{r2}^h$ of the rotating orbits of system (2.1) have the same parametric representations as (3.14).

## 4. Conclusion

To sum up, we see from the above discussions that the following result holds.

**Theorem 4.1.** *By taking* $\Psi(x)$ *as one of* (3.3)–(3.22) *and letting* $R(x)$ *given by* (1.4)*, i.e.,*

$$R(x) = \sqrt{\frac{1}{6}\left[\Psi' - 4\omega + 4\kappa \cos\left(\frac{1}{2}\Psi\right)\right]},$$

*then, system (1) has 20 exact stationary-wave solutions*

$$U(x,t) = R(x) \exp\left[\frac{1}{4} i \Psi(x)\right] \exp(-i\omega t), \tag{4.1}$$

*which depend on the parameter pair* $(\omega, \kappa)$ *and the Hamiltonian* $h$*.*

## Acknowledgements

We thank the anonymous referee for their valuable comments and suggestions.


## References


[1] P. F. Byrd and M. D. Fridman, *Handbook of Elliptic Integrals for Engineers and Scientists*, Springer, Berlin, 1971.

[2] K. W. Chow, I. M. Merhasin, B. A. Malomed, K. Nakkeeran, K. Senthilnathan and P. K. A. Wai, *Periodic waves in fiber Bragg gratings*, Phys. Rev. E, 2008, 77, 026602–1–8.

[3] Y. S. Kivshar and D. Luther-Davies, *Dark optical solitons: Physics and applications*, Phys. Rep., 1998, 298, 81–197.

[4] J. Li and H. Dai, *On the Study of Singular Travelling Wave Equations: Dynamical System Approach*, Science Press, Beijing, 2007.

[5] I. M. Merhasin, B. A. Malomed, K. Senthilnathan, K. Nakkeeran, P. K. A. Wai and K. W. Chow, *Solitons in Bragg gratings with saturable nonlinearities*, J. Opt. Soc. Am., 2007, 24(7), 1458–1468.

[6] D. E. Pelinovsky and P. G. Kevrekidis, *Dark solitons in external potentials*, Z. angew. Math. Phys., 2008, 59, 559–599.