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CONSTRUCTION OF FULL H-MATRICES WITH THE GIVEN EIGENVALUES BASED ON THE GIVENS MATRICES*

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Abstract The inverse eigenvalue problem is about how to construct a desired matrix whose spectrum is the given number set. In this paper, in view of the Givens matrices, we prove that there exist three classes of full H-matrices which include strictly diagonally dominant full matrix, α-strictly diagonally dominant full matrix and α-double strictly diagonally dominant full matrix, and their spectrum are all the given number set. In addition, we design some numerical algorithms to explain how to construct the above-mentioned full H-matrices.

Keywords Full matrix, H-matrix, strictly diagonally dominant matrix, α-strictly diagonally dominant matrix, α-double strictly diagonally dominant matrix, givens matrix.

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1. Introduction

The inverse eigenvalue problem is about how to construct a matrix with desired structure whose spectrum is the given number set. It arises in many application areas such as system and control theory, structure analysis, geophysics, and so on [1]. In recent years, the question has become an active topic and attracted many researchers' considerable attention that is paid to the nonnegative matrix [8, 13], the Jacobi matrix [4, 9, 14–18], the Toeplitz matrix [2, 6] and the stochastic matrix [5] etc.. However, few researchers study the inverse eigenvalue problem on H-matrix. We all know that H-matrix plays an important role in numerical analysis, mathematical physics, control theory [3, 7] etc.. Therefore, it is worth following with interest the inverse eigenvalue problem on H-matrix. In this paper, motivated by the above facts and the properties of the full matrix [19], we focus on

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studying the inverse eigenvalue problem of the full H-matrix, i.e., given the number set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ satisfying some specified conditions, find or construct a $n \times n$ full H-matrix A whose spectrum is the given number set Λ. According to the equivalent conditions of the H-matrix, this paper mainly study the following three classes of full H-matrices including strictly diagonally dominant full matrix, α-strictly diagonally dominant full matrix and α-double strictly diagonally dominant full matrix.

The remainder of the paper is organized as follows. Some known definitions and conclusions are recalled in Section 2. In Section 3, based on the Givens matrices, we construct the three classes of full H-matrices whose spectrum are all the given number set. In Section 4, a numerical example is presented to illustrate the validity of the obtained results. A brief conclusion is given in Section 5.

2. Some definitions and conclusions

Let $\mathbb{R}^{n\times n}$ be the set of real $n \times n$ matrices. For $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, we denote

$$P_i(A) = \sum_{j=1,j\neq i}^{n} |a_{ij}|, \qquad Q_i(A) = \sum_{j=1,j\neq i}^{n} |a_{ji}|, \qquad i = 1, 2, \cdots, n.$$

In what follows, we use $\mathbb{Z}^+$ to denote the set of positive integers and I_m to denote the $m \times m$ identity matrix and $O_{p\times q}$ to denote the $p \times q$ zero matrix where $m, p, q \in \mathbb{Z}^+$.

The following definitions are useful in this paper.

Definition 2.1 ([19]). The matrix $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ is said to be a full matrix if $a_{ij} \neq 0$ for all $i, j = 1, 2, \cdots, n$.

Definition 2.2 ([3,7]). Let $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$,

(i) A is said to be a row (column) strictly diagonally dominant matrix if $|a_{ii}| > P_i(A)$ $(|a_{ii}| > Q_i(A))$ for all $i = 1, 2, \cdots, n$;

(ii) A is said to be an α-strictly diagonally dominant matrix, if there exists $\alpha \in [0, 1]$ such that $|a_{ii}| > \alpha P_i(A) + (1 - \alpha)Q_i(A)$ for all $i = 1, 2, \cdots, n$;

(iii) A is said to be an α-double strictly diagonally dominant matrix, if there exists $\alpha \in [0, 1]$ such that $|a_{ii}|\,|a_{jj}| > (\alpha P_i(A) + (1-\alpha)Q_i(A))\,(\alpha P_j(A) + (1-\alpha)Q_j(A))$ for all $i, j = 1, 2, \cdots, n$ with $i \neq j$.

As we all know that a matrix A is an H-matrix if A satisfies one of the following conditions: (i) A is a strictly diagonally dominant matrix; (ii) A is an α-strictly diagonally dominant matrix; (iii) A is an α-double strictly diagonally dominant matrix.

Definition 2.3 ([10]). $n \times n$ real matrix

$$R_{ij}^{(n)}(\theta) = \begin{pmatrix} I_{i-1} & & & & \\ & \cos\theta & \cdots & \sin\theta & \\ & \vdots & I_{j-i-1} & \vdots & \\ & -\sin\theta & \cdots & \cos\theta & \\ & & & & I_{n-j} \end{pmatrix}_{n\times n}, i, j = 1, 2, \cdots, n, i < j,$$

is called a Givens matrix where θ is the angle of the rotation.

Theorem 2.1 ([12]). *Given any* $n\in\mathbb{Z}^+$ *and any choice of real numbers* $a_0, a_1, \cdots, a_n$, *the polynomial equation*

$$a_n z^n + \cdots + a_1 z + a_0 = 0,$$

where the leading coefficient a_n *is not equal to zero has at most* n *distinct roots.*

Lemma 2.1 ([11]). *Let* $f(x)$ *be a real function with respect to variable* x. *If* $\lim\limits_{x\to 0^+} f(x) = a > 0$ ($\lim\limits_{x\to 0^+} f(x) = a < 0$), *then, there must exist a real number* $\delta > 0$, *such that* $f(x) > \frac{a}{2} > 0$ ($f(x) < \frac{a}{2} < 0$) *for any real number* $x \in (0, \delta)$.

3. The construction of the full H-matrices based on the Givens matrices

In this part, we make use of the Givens matrices to construct the three classes of full H-matrices whose spectrum are all the given number set. Firstly, we introduce some properties of the Givens matrix. In the following, in order to illustrate easily, $\cos\theta, \sin\theta$ are abbreviated to c, s, respectively. In addition, without loss of generality, we assume that $\theta \in \left(0, \frac{\pi}{2}\right]$.

3.1. Some properties of the Givens matrix

Theorem 3.1. *For any natural number* $n \geq 2$, *we have*

$$\prod_{i=1}^{n-1} R_{in}^{(n)}(\theta) = \begin{pmatrix} c & -s^2 & -cs^2 & \cdots & -c^{n-3}s^2 & c^{n-2}s \\ 0 & c & -s^2 & \cdots & -c^{n-4}s^2 & c^{n-3}s \\ 0 & 0 & c & \cdots & -c^{n-5}s^2 & c^{n-4}s \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c & s \\ -s & -cs & -c^2s & \cdots & -c^{n-2}s & c^{n-1} \end{pmatrix}_{n\times n}.$$

Proof. We will adopt mathematical induction to finish the proof.

(a) For $n = 2$, $R_{12}^{(2)}(\theta) = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$.

(b) Assume that

$$\prod_{i=1}^{n-2} R_{i,n-1}^{(n-1)}(\theta) = \begin{pmatrix} c & -s^2 & -cs^2 & \cdots & -c^{n-4}s^2 & c^{n-3}s \\ 0 & c & -s^2 & \cdots & -c^{n-5}s^2 & c^{n-4}s \\ 0 & 0 & c & \cdots & -c^{n-6}s^2 & c^{n-5}s \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c & s \\ -s & -cs & -c^2s & \cdots & -c^{n-3}s & c^{n-2} \end{pmatrix}_{(n-1)\times(n-1)},$$

then,

$$
\begin{aligned}
\prod_{i=1}^{n-1} R_{in}^{(n)}(\theta) &= R_{1n}^{(n)}(\theta)\left(\prod_{i=2}^{n-1} R_{in}^{(n)}(\theta)\right) \\
&= R_{1n}^{(n)}(\theta)\begin{pmatrix} 1 & O_{1\times(n-1)} \\ O_{(n-1)\times 1} & \prod_{i=1}^{n-2} R_{i,n-1}^{(n-1)}(\theta) \end{pmatrix} \\
&= \begin{pmatrix} c & 0 & 0 & 0 & \cdots & 0 & s \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ -s & 0 & 0 & 0 & \cdots & 0 & c \end{pmatrix}_{n\times n} \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & c & -s^2 & -cs^2 & \cdots & -c^{n-4}s^2 & c^{n-3}s \\ 0 & 0 & c & -s^2 & \cdots & -c^{n-5}s^2 & c^{n-4}s \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c & s \\ 0 & -s & -cs & -c^2s & \cdots & -c^{n-3}s & c^{n-2} \end{pmatrix}_{n\times n} \\
&= \begin{pmatrix} c & -s^2 & -cs^2 & \cdots & -c^{n-3}s^2 & c^{n-2}s \\ 0 & c & -s^2 & \cdots & -c^{n-4}s^2 & c^{n-3}s \\ 0 & 0 & c & \cdots & -c^{n-5}s^2 & c^{n-4}s \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c & s \\ -s & -cs & -c^2s & \cdots & -c^{n-2}s & c^{n-1} \end{pmatrix}_{n\times n} .
\end{aligned}
$$

$\square$

Theorem 3.2. *For the given real number set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ satisfying $\lambda_i \neq 0, i = 1, 2, \cdots, n, n \geq 2$ and $\lambda_1 \neq \lambda_2$. Let $R_n(\theta) = \prod_{j=2}^{n}\prod_{i=1}^{j-1} R_{ij}^{(n)}(\theta)$.*

(i) If $n \in \{2i | i \in \mathbb{Z}^+\}$, then we have

$$
\begin{aligned}
G_n(\theta) &= R_n^T(\theta) diag(\lambda_1, \lambda_2, \cdots, \lambda_n) R_n(\theta) \\
&= \begin{pmatrix} u_{11}^{(n)}(s) & cu_{12}^{(n)}(s) & u_{13}^{(n)}(s) & \cdots & u_{1,n-1}^{(n)}(s) & cu_{1n}^{(n)}(s) \\ cu_{21}^{(n)}(s) & u_{22}^{(n)}(s) & cu_{23}^{(n)}(s) & \cdots & cu_{2,n-1}^{(n)}(s) & u_{2n}^{(n)}(s) \\ u_{31}^{(n)}(s) & cu_{32}^{(n)}(s) & u_{33}^{(n)}(s) & \cdots & u_{3,n-1}^{(n)}(s) & cu_{3n}^{(n)}(s) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{n-1,1}^{(n)}(s) & cu_{n-1,2}^{(n)}(s) & u_{n-1,3}^{(n)}(s) & \cdots & u_{n-1,n-1}^{(n)}(s) & cu_{n-1,n}^{(n)}(s) \\ cu_{n1}^{(n)}(s) & u_{n2}^{(n)}(s) & cu_{n3}^{(n)}(s) & \cdots & cu_{n,n-1}^{(n)}(s) & u_{nn}^{(n)}(s) \end{pmatrix}_{n\times n} \\
&\triangleq \left(g_{ij}^{(n)}(s)\right)_{n\times n},
\end{aligned}
$$

where $u_{ii}^{(n)}(s), i = 1, 2, \cdots, n$ are polynomials with respect to s, whose constant terms are $\lambda_i, i = 1, 2, \cdots, n$ and leading coefficients are $\lambda_1 - \lambda_2$ or $\lambda_2 - \lambda_1$,

$u_{ij}^{(n)}(s), i \neq j$ are polynomials with respect to s without constant term, whose leading coefficients are $\lambda_1 - \lambda_2$ or $\lambda_2 - \lambda_1$;

(ii) If $n \in \{2i+1 | i \in \mathbb{Z}^+\}$, then we have

$$
\begin{aligned}
G_n(\theta) &= R_n^T(\theta) diag(\lambda_1, \lambda_2, \cdots, \lambda_n) R_n(\theta) \\
&= \begin{pmatrix}
v_{11}^{(n)}(s) & cv_{12}^{(n)}(s) & v_{13}^{(n)}(s) & \cdots & cv_{1,n-1}^{(n)}(s) & v_{1n}^{(n)}(s) \\
cv_{21}^{(n)}(s) & v_{22}^{(n)}(s) & cv_{23}^{(n)}(s) & \cdots & v_{2,n-1}^{(n)}(s) & cv_{2n}^{(n)}(s) \\
v_{31}^{(n)}(s) & cv_{32}^{(n)}(s) & v_{33}^{(n)}(s) & \cdots & cv_{3,n-1}^{(n)}(s) & v_{3n}^{(n)}(s) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
cv_{n-1,1}^{(n)}(s) & v_{n-1,2}^{(n)}(s) & cv_{n-1,3}^{(n)}(s) & \cdots & v_{n-1,n-1}^{(n)}(s) & cv_{n-1,n}^{(n)}(s) \\
v_{n1}^{(n)}(s) & cv_{n2}^{(n)}(s) & v_{n3}^{(n)}(s) & \cdots & cv_{n,n-1}^{(n)}(s) & v_{nn}^{(n)}(s)
\end{pmatrix}_{n \times n} \\
&\triangleq \left(g_{ij}^{(n)}(s) \right)_{n \times n},
\end{aligned}
$$

where $v_{ii}^{(n)}(s), i = 1, 2, \cdots, n$ are polynomials with respect to s, whose constant terms are $\lambda_i, i = 1, 2, \cdots, n$ and leading coefficients are $\lambda_1 - \lambda_2$ or $\lambda_2 - \lambda_1$, $v_{ij}^{(n)}(s), i \neq j$ are polynomials with respect to s without constant term, whose leading coefficients are $\lambda_1 - \lambda_2$ or $\lambda_2 - \lambda_1$.

Proof. We will adopt mathematical induction to finish the proof. Denote $\alpha_i^j = \lambda_i - \lambda_j, i, j = 1, 2, \cdots, n, i \neq j$.

(a) For $n = 2$,

$$
G_2(\theta) = R_2^T(\theta) \mathrm{diag}(\lambda_1, \lambda_2) R_2(\theta) = \begin{pmatrix} -\alpha_1^2 s^2 + \lambda_1 & c(\alpha_1^2 s) \\ c(\alpha_1^2 s) & \alpha_1^2 s^2 + \lambda_2 \end{pmatrix}.
$$

Obviously, $u_{11}^{(2)}(s) = -\alpha_1^2 s^2 + \lambda_1$, $u_{22}^{(2)}(s) = \alpha_1^2 s^2 + \lambda_2$, $u_{12}^{(2)}(s) = u_{21}^{(2)}(s) = \alpha_1^2 s$, so $u_{11}^{(2)}(s)$ and $u_{22}^{(2)}(s)$ are quadratic polynomials with respect to s with constant terms λ_1 and λ_2, respectively, whose leading coefficients are $\lambda_1 - \lambda_2$ or $\lambda_2 - \lambda_1$, $u_{12}^{(2)}(s)$ and $u_{21}^{(2)}(s)$ are one-order polynomials with respect to s without constant term, whose leading coefficients are $\lambda_1 - \lambda_2$.

(b) For $n = 3$,

$$
G_3(\theta) = R_3^T(\theta) \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) R_3(\theta) \triangleq \begin{pmatrix} v_{11}^{(3)}(s) & cv_{12}^{(3)}(s) & v_{13}^{(3)}(s) \\ cv_{21}^{(3)}(s) & v_{22}^{(3)}(s) & cv_{23}^{(3)}(s) \\ v_{31}^{(3)}(s) & cv_{32}^{(3)}(s) & v_{33}^{(3)}(s) \end{pmatrix},
$$

where

$$
\begin{aligned}
v_{11}^{(3)}(s) &= \alpha_1^2 s^4 - (\alpha_1^2 + \alpha_1^3) s^2 + \lambda_1, \\
v_{22}^{(3)}(s) &= -\alpha_1^2 s^6 + 2\alpha_1^2 s^5 + \alpha_2^3 s^4 - 2\alpha_1^2 s^3 + (\alpha_1^2 - \alpha_2^3) s^2 + \lambda_2, \\
v_{33}^{(3)}(s) &= \alpha_1^2 s^6 - 2\alpha_1^2 s^5 - \alpha_1^3 s^4 + 2\alpha_1^2 s^3 + (\alpha_1^3 + \alpha_2^3) s^2 + \lambda_3,
\end{aligned}
$$

$$v_{12}^{(3)}(s) = v_{21}^{(3)}(s) = \alpha_1^2 s^4 - \alpha_1^2 s^3 - \alpha_1^3 s^2 + \alpha_1^2 s,$$
$$v_{13}^{(3)}(s) = v_{31}^{(3)}(s) = \alpha_1^2 s^5 - \alpha_1^2 s^4 - (\alpha_1^2 + \alpha_1^3)s^3 + \alpha_1^2 s^2 + \alpha_1^3 s,$$
$$v_{23}^{(3)}(s) = v_{32}^{(3)}(s) = \alpha_1^2 s^5 - 2\alpha_1^2 s^4 - \alpha_2^3 s^3 + \alpha_1^2 s^2 + \alpha_2^3 s.$$

Obviously, $v_{11}^{(3)}(s)$, $v_{22}^{(3)}(s)$ and $v_{33}^{(3)}(s)$ are polynomials with respect to s, whose constant terms are λ_1, λ_2 and λ_3, respectively, and leading coefficients are $\lambda_1 - \lambda_2$ or $\lambda_2 - \lambda_1$, and $v_{12}^{(3)}(s)$, $v_{13}^{(3)}(s)$, $v_{21}^{(3)}(s)$, $v_{23}^{(3)}(s)$, $v_{31}^{(3)}(s)$ and $v_{32}^{(3)}(s)$ are polynomials with respect to s without constant term, whose leading coefficients are $\lambda_1 - \lambda_2$.

(c) If $n \in \{2i | i \in \mathbb{Z}^+\}$, then $n-1 \in \{2i+1 | i \in \mathbb{Z}^+\}$. Assume that

$$G_{n-1}(\theta)=\begin{pmatrix} v_{11}^{(n-1)}(s) & cv_{12}^{(n-1)}(s) & v_{13}^{(n-1)}(s) & \cdots & cv_{1,n-2}^{(n-1)}(s) & v_{1,n-1}^{(n-1)}(s) \\ cv_{21}^{(n-1)}(s) & v_{22}^{(n-1)}(s) & cv_{23}^{(n-1)}(s) & \cdots & v_{2,n-2}^{(n-1)}(s) & cv_{2,n-1}^{(n-1)}(s) \\ v_{31}^{(n-1)}(s) & cv_{32}^{(n-1)}(s) & v_{33}^{(n-1)}(s) & \cdots & cv_{3,n-2}^{(n-1)}(s) & v_{3,n-1}^{(n-1)}(s) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ cv_{n-2,1}^{(n-1)}(s) & v_{n-2,2}^{(n-1)}(s) & cv_{n-2,3}^{(n-1)}(s) & \cdots & v_{n-2,n-2}^{(n-1)}(s) & cv_{n-2,n-1}^{(n-1)}(s) \\ v_{n-1,1}^{(n-1)}(s) & cv_{n-1,2}^{(n-1)}(s) & v_{n-1,3}^{(n-1)}(s) & \cdots & cv_{n-1,n-2}^{(n-1)}(s) & v_{n-1,n-1}^{(n-1)}(s) \end{pmatrix}_{(n-1)\times(n-1)},$$

where $v_{ii}^{(n-1)}(s), i = 1, 2, \cdots, n-1$ are polynomials with respect to s, whose constant terms are $\lambda_i, i = 1, 2, \cdots, n-1$ and leading coefficients are $\lambda_1 - \lambda_2$ or $\lambda_2 - \lambda_1$, $v_{ij}^{(n-1)}(s), i \neq j$ are polynomials with respect to s without constant term, whose leading coefficients are $\lambda_1 - \lambda_2$ or $\lambda_2 - \lambda_1$. Then by Theorem 3.1, we obtain

$$\begin{aligned} G_n(\theta) =& R_n^T(\theta)\mathrm{diag}(\lambda_1, \cdots, \lambda_{n-1}, \lambda_n)R_n(\theta) \\ =& \left(\prod_{i=1}^{n-1} R_{in}^{(n)}(\theta)\right)^T \begin{pmatrix} R_{n-1}^T(\theta) & O_{(n-1)\times 1} \\ O_{1\times(n-1)} & 1 \end{pmatrix} \left(\begin{array}{ccc|c} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_{n-1} & \\ \hline & & & \lambda_n \end{array}\right) \begin{pmatrix} R_{n-1}(\theta) & O_{(n-1)\times 1} \\ O_{1\times(n-1)} & 1 \end{pmatrix} \prod_{i=1}^{n-1} R_{in}^{(n)}(\theta) \\ =& \left(\prod_{i=1}^{n-1} R_{in}^{(n)}(\theta)\right)^T \begin{pmatrix} G_{n-1}(\theta) & O_{(n-1)\times 1} \\ O_{1\times(n-1)} & \lambda_n \end{pmatrix} \prod_{i=1}^{n-1} R_{in}^{(n)}(\theta) \\ =& \begin{pmatrix} c & 0 & 0 & \cdots & 0 & -s \\ -s^2 & c & 0 & \cdots & 0 & -cs \\ -cs^2 & -s^2 & c & \cdots & 0 & -c^2 s \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -c^{n-3}s^2 & -c^{n-4}s^2 & -c^{n-5}s^2 & \cdots & c & -c^{n-2}s \\ c^{n-2}s & c^{n-3}s & c^{n-4}s & \cdots & s & c^{n-1} \end{pmatrix}_{n\times n} \end{aligned}$$

$$\times \begin{pmatrix} v_{11}^{(n-1)}(s) & cv_{12}^{(n-1)}(s) & \cdots & cv_{1,n-2}^{(n-1)}(s) & v_{1,n-1}^{(n-1)}(s) & 0 \\ cv_{21}^{(n-1)}(s) & v_{22}^{(n-1)}(s) & \cdots & v_{2,n-2}^{(n-1)}(s) & cv_{2,n-1}^{(n-1)}(s) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ cv_{n-2,1}^{(n-1)}(s) & v_{n-2,2}^{(n-1)}(s) & \cdots & v_{n-2,n-2}^{(n-1)}(s) & cv_{n-2,n-1}^{(n-1)}(s) & 0 \\ v_{n-1,1}^{(n-1)}(s) & cv_{n-1,2}^{(n-1)}(s) & \cdots & cv_{n-1,n-2}^{(n-1)}(s) & v_{n-1,n-1}^{(n-1)}(s) & 0 \\ 0 & 0 & \cdots & 0 & 0 & \lambda_n \end{pmatrix}_{n\times n}$$

$$\times \begin{pmatrix} c & -s^2 & -cs^2 & \cdots & -c^{n-3}s^2 & c^{n-2}s \\ 0 & c & -s^2 & \cdots & -c^{n-4}s^2 & c^{n-3}s \\ 0 & 0 & c & \cdots & -c^{n-5}s^2 & c^{n-4}s \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c & s \\ -s & -cs & -c^2s & \cdots & -c^{n-2}s & c^{n-1} \end{pmatrix}_{n\times n}$$

$$= \begin{pmatrix} u_{11}^{(n)}(s) & cu_{12}^{(n)}(s) & u_{13}^{(n)}(s) & \cdots & u_{1,n-1}^{(n)}(s) & cu_{1n}^{(n)}(s) \\ cu_{21}^{(n)}(s) & u_{22}^{(n)}(s) & cu_{23}^{(n)}(s) & \cdots & cu_{2,n-1}^{(n)}(s) & u_{2n}^{(n)}(s) \\ u_{31}^{(n)}(s) & cu_{32}^{(n)}(s) & u_{33}^{(n)}(s) & \cdots & u_{3,n-1}^{(n)}(s) & cu_{3n}^{(n)}(s) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{n-1,1}^{(n)}(s) & cu_{n-1,2}^{(n)}(s) & u_{n-1,3}^{(n)}(s) & \cdots & u_{n-1,n-1}^{(n)}(s) & cu_{n-1,n}^{(n)}(s) \\ cu_{n1}^{(n)}(s) & u_{n2}^{(n)}(s) & cu_{n3}^{(n)}(s) & \cdots & cu_{n,n-1}^{(n)}(s) & u_{nn}^{(n)}(s) \end{pmatrix}_{n\times n}.$$

It is easy to verify that $u_{ii}^{(n)}(s), i = 1, 2, \cdots, n$ are polynomials with respect to s, whose constant terms are $\lambda_i, i = 1, 2, \cdots, n$ and leading coefficients are $\lambda_1 - \lambda_2$ or $\lambda_2 - \lambda_1$, $u_{ij}^{(n)}(s), i \neq j$ are polynomials with respect to s without constant term, whose leading coefficients are $\lambda_1 - \lambda_2$ or $\lambda_2 - \lambda_1$. This shows that (ii) holds.

(d) If $n \in \{2i+1 | i \in \mathbb{Z}^+\}$, then $n - 1 \in \{2i | i \in \mathbb{Z}^+\}$. The following proof is similar to (c), here it is omitted. In this case, we can prove that (i) holds. $\square$

Theorem 3.3. *For the given real number set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ satisfying $\lambda_i \neq 0, i = 1, 2, \cdots, n, n \geq 2$ and $\lambda_1 \neq \lambda_2$, we have*

- *(i) The spectrum of the matrix $G_n(\theta)$ is the given set Λ;*
- *(ii) There exists a real number $\delta \in (0, \frac{\pi}{2}]$, such that for any real number $\theta \in (0, \delta)$, $n \times n$ real matrix $G_n(\theta)$ is a full matrix;*
- *(iii)* $\lim\limits_{\theta\to 0^+} \left|g_{ij}^{(n)}(s)\right| = 0, i \neq j, \quad \lim\limits_{\theta\to 0^+} \left|g_{ij}^{(n)}(s)\right| = |\lambda_i|, i = j.$

Proof.

(1) Since the Givens matrices $R_{ij}^{(n)}(\theta), i, j = 1, 2, \cdots, n, i < j$ are orthogonal matrices, i.e., $\left(R_{ij}^{(n)}(\theta)\right)^T = \left(R_{ij}^{(n)}(\theta)\right)^{-1}, i, j = 1, 2, \cdots, n, i < j$, so $R_n^T(\theta) = (R_n(\theta))^{-1}$.

Therefore, $G_n(\theta) = R_n^T(\theta)\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)R_n(\theta)$ is similar to the $n \times n$ diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$, therefore, the spectrum of the matrix $G_n(\theta)$ is the given set Λ. This shows that (i) holds.

(2) By Theorem 3.2, we only need to prove that there exists a real number $\delta \in (0, \frac{\pi}{2}]$, such that for any real number $\theta \in (0, \delta)$, $g_{ij}^{(n)}(s) \neq 0, i, j = 1, 2, \cdots, n$. Note that

$$g_{ij}^{(n)}(s) = \begin{cases} c^{\frac{1-(-1)^{i+j}}{2}} u_{ij}^{(n)}(s) = (-s^2+1)^{\frac{1-(-1)^{i+j}}{4}} u_{ij}^{(n)}(s), \ n \in \{2i | i \in \mathbb{Z}^+\}, \\ c^{\frac{1-(-1)^{i+j}}{2}} v_{ij}^{(n)}(s) = (-s^2+1)^{\frac{1-(-1)^{i+j}}{4}} v_{ij}^{(n)}(s), \ n \in \{2i+1 | i \in \mathbb{Z}^+\}. \end{cases}$$

Since $\lambda_1 \neq \lambda_2$, so the leading coefficients of all $(g_{ij}^{(n)}(s))^2, i, j = 1, 2, \cdots, n$ are not equal to zero. By Theorem 2.1, there exist at most $m_{ij} \in \mathbb{Z}^+, i, j = 1, 2, \cdots, n$ roots $x_1^{ij}, x_2^{ij}, \cdots, x_{m_{ij}}^{ij}$, $i, j = 1, 2, \cdots, n$ for the equations $\left(g_{ij}^{(n)}(x)\right)^2 = 0, i, j = 1, 2, \cdots, n$. We denote $S = \{x_s^{ij} | s = 1, 2, \cdots, m_{ij}, \ i, j = 1, 2, \cdots, n\}$. If $S \cap (0, 1] = \emptyset$, we adopt $\delta = \frac{\pi}{2}$, then for any $\theta \in (0, \delta)$, $\left(g_{ij}^{(n)}(s)\right)^2 \neq 0, i, j = 1, 2, \cdots, n$, and for any $\theta \in (0, \delta)$, $g_{ij}^{(n)}(s) \neq 0, i, j = 1, 2, \cdots, n$ further. If $S \cap (0, 1] \neq \emptyset$, let $a = \min\{y | y \in S \cap (0, 1]\}$, then there exists a real number $\delta \in (0, \frac{\pi}{2}]$ such that $\sin\delta = a$. Thus, for any real number $\theta \in (0, \delta)$, $\left(g_{ij}^{(n)}(s)\right)^2 \neq 0, i, j = 1, 2, \cdots, n$, and for any real number $\theta \in (0, \delta)$, $g_{ij}^{(n)}(s) \neq 0, i, j = 1, 2, \cdots, n$ further. This shows that (ii) holds.

(3) If $n \in \{2i | i \in \mathbb{Z}^+\}$, $u_{ii}^{(n)}(s), i = 1, 2, \cdots, n$ are polynomials with respect to s with constant term $\lambda_i, i = 1, 2, \cdots, n$, $u_{ij}^{(n)}(s), i \neq j$ are polynomials with respect to s without constant term. Note that $\theta \to 0^+$ leads to $\sin\theta \to 0$, hence

$$\lim_{\theta \to 0^+} \left| g_{ii}^{(n)}(s) \right| = \lim_{\theta \to 0^+} \left| u_{ii}^{(n)}(s) \right| = |\lambda_i|, \quad i = 1, 2, \cdots, n,$$
$$\lim_{\theta \to 0^+} \left| g_{ij}^{(n)}(s) \right| = \lim_{\theta \to 0^+} \left| (-s^2+1)^{\frac{1-(-1)^{i+j}}{4}} u_{ij}^{(n)}(s) \right| = 0, \quad i \neq j.$$

If $n \in \{2i+1 | i \in \mathbb{Z}^+\}$, $v_{ii}^{(n)}(s), i = 1, 2, \cdots, n$ are polynomials with respect to s with constant term $\lambda_i, i = 1, 2, \cdots, n$, $v_{ij}^{(n)}(s), i \neq j$ are polynomials with respect to s without constant term. Note that $\theta \to 0^+$ leads to $\sin\theta \to 0$, hence

$$\lim_{\theta \to 0^+} \left| g_{ii}^{(n)}(s) \right| = \lim_{\theta \to 0^+} \left| v_{ii}^{(n)}(s) \right| = |\lambda_i|, \quad i = 1, 2, \cdots, n,$$
$$\lim_{\theta \to 0^+} \left| g_{ij}^{(n)}(s) \right| = \lim_{\theta \to 0^+} \left| (-s^2+1)^{\frac{1-(-1)^{i+j}}{4}} v_{ij}^{(n)}(s) \right| = 0, \quad i \neq j.$$

This shows that (iii) holds. □

Theorem 3.4. *For the given real number set* $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ *satisfying* $\lambda_i \neq 0, i = 1, 2, \cdots, n, n \geq 2, \lambda_1 \neq \lambda_2$ *and* $k \in \mathbb{R} \setminus \{0\}$*, there exists a real number* $\delta \in (0, \frac{\pi}{2}]$*, such that for any real number* $\theta \in (0, \delta)$*, the* $n \times n$ *real matrix* $D_n(\theta) = E^{(n)}(2,1(k))G_n(\theta)E^{(n)}(2,1(-k))$ *is a full matrix, where* $E^{(n)}(2, 1(t))$ *denotes the* $n \times n$ *elementary matrix derived by adding* t *(*$t = k$ *or* $-k$*) times the first row to the second row of* I_n*.*

Proof. For any real number $k \in \mathbb{R} \setminus \{0\}$, if $n \in \{2i | i \in \mathbb{Z}^+\}$, let

$$\begin{aligned} D_n(\theta) &= E^{(n)}(2,1(k))G_n(\theta)E^{(n)}(2,1(-k)) \\ &= \begin{pmatrix} u_{11}^{(n)}(s) - kcu_{12}^{(n)}(s) & cu_{12}^{(n)}(s) & \cdots & cu_{1n}^{(n)}(s) \\ cu_{21}^{(n)}(s) + ku_{11}^{(n)}(s) - ku_{22}^{(n)}(s) - k^2cu_{12}^{(n)}(s) & u_{22}^{(n)}(s) + kcu_{12}^{(n)}(s) & \cdots & u_{2n}^{(n)}(s) + kcu_{1n}^{(n)}(s) \\ u_{31}^{(n)}(s) - kcu_{32}^{(n)}(s) & u_{32}^{(n)}(s) & \cdots & cu_{3,n}^{(n)}(s) \\ \vdots & \vdots & \ddots & \vdots \\ cu_{n1}^{(n)}(s) - ku_{n2}^{(n)}(s) & u_{n2}^{(n)}(s) & \cdots & u_{nn}^{(n)}(s) \end{pmatrix} \\ &\triangleq \left(d_{ij}^{(n)}(s)\right)_{n \times n}. \end{aligned}$$

It is easy to verify that all $d_{ij}^{(n)}(s), i, j = 1, 2, \cdots, n$ are not identically zero for any $k \in \mathbb{R} \setminus \{0\}$ and $\delta \in (0, \frac{\pi}{2}]$ by the proof of Theorem 3.2. Since

$$\begin{aligned} \left\{x | d_{11}^{(n)}(x) = 0\right\} &= \left\{x | u_{11}^{(n)}(x) = k\sqrt{1-x^2}u_{12}^{(n)}(x)\right\} \\ &\subseteq \left\{x | \left(u_{11}^{(n)}(x)\right)^2 - k^2\left(1-x^2\right)\left(u_{12}^{(n)}(x)\right)^2 = 0\right\}. \end{aligned}$$

By Theorem 2.1, there exist at most $m_{11} \in \mathbb{Z}^+$ roots for the equation

$$\left(u_{11}^{(n)}(x)\right)^2 - k^2(1-x^2)\left(u_{12}^{(n)}(x)\right)^2 = 0.$$

So, there exist at most m_{11} roots for $d_{11}^{(n)}(x) = 0$. Similarly, there exist at most $m_{i1} \in \mathbb{Z}^+, i = 3, 4, \cdots, n$ roots for $d_{i1}^{(n)}(x) = 0, i = 3, 4, \cdots, n$, and there exist at most $m_{2j} \in \mathbb{Z}^+, j = 2, 3, \cdots, n$ roots for $d_{2j}^{(n)}(x) = 0, j = 2, 3, \cdots, n$. Since

$$\begin{aligned} \left\{x | d_{21}^{(n)}(x) = 0\right\} &= \left\{x | \sqrt{1-x^2}u_{21}^{(n)}(x) - \sqrt{1-x^2}k^2u_{12}^{(n)}(x) = -k(u_{11}^{(n)}(x) - u_{22}^{(n)}(x))\right\} \\ &\subseteq \left\{x | \left(\sqrt{1-x^2}u_{21}^{(n)}(x) - \sqrt{1-x^2}k^2u_{12}^{(n)}(x)\right)^2 - k^2\left(u_{11}^{(n)}(x) - u_{22}^{(n)}(x)\right)^2 = 0\right\}. \end{aligned}$$

By Theorem 2.1, there exist at most $m_{21} \in \mathbb{Z}^+$ roots for the equation

$$(1-x^2)\left(u_{21}^{(n)}(x) - k^2u_{12}^{(n)}(x)\right)^2 - k^2\left(u_{11}^{(n)}(x) - u_{22}^{(n)}(x)\right)^2 = 0.$$

So, there exist at most $m_{21} \in \mathbb{Z}^+$ roots for $d_{21}^{(n)}(x) = 0$. Next, the proof is similar to the proof of Theorem 3.3(ii), so there exists a real number $\delta \in (0, \frac{\pi}{2}]$, such that for any real number $\theta \in (0, \delta)$, $n \times n$ real matrix $D_n(\theta) = E^{(n)}(2,1(k))G_n(\theta)E^{(n)}(2,1(-k))$ is a full matrix.

If $n \in \{2i+1 | i \in \mathbb{Z}^+\}$, the proof is similar, here, it is omitted. □

Theorem 3.5. *For any $k \in \mathbb{R} \setminus \{0\}$, the spectrum of $D_n(\theta)$ is $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$.*

Proof. By the properties of the elementary matrix, $\left(E^{(n)}(2,1(k))\right)^{-1} = E^{(n)}(2,1(-k))$, we know that $D_n(\theta) = E^{(n)}(2,1(k))G_n(s)E^{(n)}(2,1(-k))$ is similar to $G_n(\theta)$. By Theorem 3.3(i), the spectrum of $G_n(\theta)$ is $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, so the spectrum of $D_n(\theta)$ is also the set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. □

3.2. Main results for construction of the full H-matrices based on the Givens matrices

The following Theorem 3.6 shows that for the given real number set $\Lambda=\{\lambda_1,\lambda_2,\cdots,\lambda_n\}$ satisfying some specified conditions, there exists a row and column strictly diagonally dominant full matrix whose spectrum is the given set Λ.

Theorem 3.6. *For the given real number set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ satisfying $\lambda_i \neq 0, i = 1, 2, \cdots, n, n \geq 2$ and $\lambda_1 \neq \lambda_2$, there must exist a real number $\delta_1 \in (0, \frac{\pi}{2}]$, such that for any real number $\theta \in (0, \delta_1)$, $G_n(\theta)$ is a row and column strictly diagonally dominant full matrix whose spectrum is the given set Λ.*

Proof. Under the assumptions, by Theorem 3.3, the spectrum of $G_n(\theta)$ is the given set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, for $\theta \in (0, \frac{\pi}{2}]$, there must exist a real number $\delta_{11} \in (0, \frac{\pi}{2}]$, such that $G_n(\theta)$ is a full matrix for any real number $\theta \in (0, \delta_{11})$, and

$$\lim_{\theta\to 0^+} \left|g_{ij}^{(n)}(s)\right| = 0, i \neq j, \quad \lim_{\theta\to 0^+} \left|g_{ij}^{(n)}(s)\right| = |\lambda_i|, i = j.$$

Hence,

$$\lim_{\theta\to 0^+} \left(\left|g_{ii}^{(n)}(s)\right| - P_i(G_n(\theta))\right) = |\lambda_i| > 0, i = 1, 2, \cdots, n,$$
$$\lim_{\theta\to 0^+} \left(\left|g_{ii}^{(n)}(s)\right| - Q_i(G_n(\theta))\right) = |\lambda_i| > 0, i = 1, 2, \cdots, n.$$

By Lemma 2.1, there must exist a common real number $\delta_{12} \in (0, \frac{\pi}{2}]$, such that for any real number $\theta \in (0, \delta_{12})$,

$$\left|g_{ii}^{(n)}(s)\right| - P_i(G_n(\theta)) > \frac{|\lambda_i|}{2} > 0, i = 1, 2, \cdots, n,$$
$$\left|g_{ii}^{(n)}(s)\right| - Q_i(G_n(\theta)) > \frac{|\lambda_i|}{2} > 0, i = 1, 2, \cdots, n.$$

Let $\delta_1 = \min\{\delta_{11}, \delta_{12}\}$, then for any real number $\theta \in (0, \delta_1)$, $G_n(\theta)$ is a row and column strictly diagonally dominant full matrix whose spectrum is the given set Λ. □

Immediately, we have the following Algorithm 3.1 which illustrates how to construct a row and column strictly diagonally dominant full matrix whose spectrum is the given set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ satisfying some specified conditions.

Algorithm 3.1:

Step 1 Input the real number set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ satisfying $\lambda_i \neq 0, i = 1, 2, \cdots, n$, $n \geq 2$ and $\lambda_1 \neq \lambda_2$, and θ_0 satisfying $\theta_0 \in (0, \delta_1)$, where the real number δ_1 is defined by Theorem 3.6;

Step 2 Compute

$$R_n(\theta_0) = \prod_{j=2}^{n}\prod_{i=1}^{j-1} R_{ij}^{(n)}(\theta_0),$$

where $R_{ij}^{(n)}(\theta_0), i, j = 1, 2, \cdots, n, i < j$ is defined by Definition 2.3;

Step 3 Compute

$$G_n(\theta_0) = R_n^T(\theta_0)\text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)R_n(\theta_0);$$

Step 4 Output $G_n(\theta_0)$.

The following Theorem 3.7 shows that for the given set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\} \subseteq \mathbb{R}$ satisfying some specified conditions, there exists a $n \times n$ real matrix whose spectrum is the given set Λ, such that it is a column strictly diagonally dominant full matrix rather than a row strictly diagonally dominant full matrix.

Theorem 3.7. *For the given real number set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ satisfying $\lambda_i \neq 0, i = 1, 2, \cdots, n, n \geq 2$ and $|\lambda_1| > |\lambda_2|$, the real number k satisfying $\frac{|\lambda_2|}{|\lambda_1 - \lambda_2|} < |k| < \frac{|\lambda_1|}{|\lambda_1 - \lambda_2|}$, there exists a real number $\delta_2 \in (0, \frac{\pi}{2}]$, such that for any real number $\theta \in (0, \delta_2)$, $n \times n$ real matrix $D_n(\theta) = E^{(n)}(2, 1(k))G_n(\theta)E^{(n)}(2, 1(-k))$ whose spectrum is the given set Λ is a column strictly diagonally dominant full matrix rather than a row strictly diagonally dominant full matrix.*

Proof. By Theorem 3.3(iii), we get

$$\lim_{\theta \to 0^+} \left| g_{ij}^{(n)}(s) \right| = 0, i \neq j, \quad \lim_{\theta \to 0^+} \left| g_{ij}^{(n)}(s) \right| = |\lambda_i|, i = j,$$

thus,

$$\begin{aligned}
&\lim_{\theta \to 0^+} \left[\left| d_{11}^{(n)}(s) \right| - Q_1(D_n(\theta)) \right] = |\lambda_1| - |k||\lambda_1 - \lambda_2|, \\
&\lim_{\theta \to 0^+} \left[\left| d_{22}^{(n)}(s) \right| - P_2(D_n(\theta)) \right] = |\lambda_2| - |k||\lambda_1 - \lambda_2|, \\
&\lim_{\theta \to 0^+} \left[\left| d_{ii}^{(n)}(s) \right| - Q_i(D_n(\theta)) \right] = |\lambda_i| > 0, i = 2, 3, \cdots, n.
\end{aligned}$$

When the real number k satisfies $\frac{|\lambda_2|}{|\lambda_1 - \lambda_2|} < |k| < \frac{|\lambda_1|}{|\lambda_1 - \lambda_2|}$, we have

$$|\lambda_1| - |k||\lambda_1 - \lambda_2| > 0, \quad |\lambda_2| - |k||\lambda_1 - \lambda_2| < 0,$$

thus,

$$\begin{aligned}
&\lim_{\theta \to 0^+} \left[\left| d_{11}^{(n)}(s) \right| - Q_1(D_n(\theta)) \right] = |\lambda_1| - |k||\lambda_1 - \lambda_2| > 0, \\
&\lim_{\theta \to 0^+} \left[\left| d_{22}^{(n)}(s) \right| - P_2(D_n(\theta)) \right] = |\lambda_2| - |k||\lambda_1 - \lambda_2| < 0.
\end{aligned}$$

By Lemma 2.1, there exists a common real number $\delta_{21} \in (0, \frac{\pi}{2}]$, such that for any real number $\theta \in (0, \delta_{21})$,

$$\begin{aligned}
&\left| d_{11}^{(n)}(s) \right| - Q_1(D_n(\theta)) > \frac{|\lambda_1| - |k||\lambda_1 - \lambda_2|}{2} > 0, \\
&\left| d_{22}^{(n)}(s) \right| - P_2(D_n(\theta)) < \frac{|\lambda_2| - |k||\lambda_1 - \lambda_2|}{2} < 0, \\
&\left| d_{ii}^{(n)}(s) \right| - Q_i(D_n(\theta)) > \frac{|\lambda_i|}{2} > 0, i = 2, 3, \cdots, n.
\end{aligned}$$

By Theorem 3.4, there exists a real number $\delta_{22} \in (0, \frac{\pi}{2}]$, such that for any real number $\theta \in (0, \delta_{22})$, $D_n(\theta)$ is a full matrix. So for any real number k satisfying $\frac{|\lambda_2|}{|\lambda_1 - \lambda_2|} < |k| < \frac{|\lambda_1|}{|\lambda_1 - \lambda_2|}$, we adopt $\delta_2 = \min\{\delta_{21}, \delta_{22}\}$, then for any real number $\theta \in (0, \delta_2)$, $D_n(\theta)$ is a column strictly diagonally dominant full matrix rather than a row strictly diagonally dominant full matrix. By Theorem 3.5, the spectrum of $D_n(\theta)$ is the given set Λ. □

Immediately, we have the following Algorithm 3.2 which illustrates how to construct a matrix whose spectrum is the given set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ satisfying some specified conditions, such that it is a column strictly diagonally dominant full matrix rather than a row strictly diagonally dominant full matrix.

Algorithm 3.2:

Step 1 Input the real number set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ satisfying $\lambda_i \neq 0, i = 1, 2, \cdots, n$, $n \geq 2$, $|\lambda_1| > |\lambda_2|$ and the real number k_0 satisfying $\frac{|\lambda_2|}{|\lambda_1 - \lambda_2|} < |k_0| < \frac{|\lambda_1|}{|\lambda_1 - \lambda_2|}$, and the real number θ_0 satisfying $\theta_0 \in (0, \delta_2)$, where the real number δ_2 is defined by Theorem 3.7;

Step 2 Compute

$$R_n(\theta_0) = \prod_{j=2}^{n} \prod_{i=1}^{j-1} R_{ij}^{(n)}(\theta_0),$$

where $R_{ij}^{(n)}(\theta_0), i, j = 1, 2, \cdots, n, i < j$ is defined by Definition 2.3;

Step 3 Compute

$$G_n(\theta_0) = R_n^T(\theta_0) \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) R_n(\theta_0);$$

Step 4 Output $D_n(\theta_0) = E^{(n)}(2, 1(k_0)) G_n(\theta_0) E^{(n)}(2, 1(-k_0))$.

The following Theorem 3.8 shows that for the given set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\} \subseteq \mathbb{R}$ satisfying some specified conditions, there exists a $n \times n$ real matrix whose spectrum is the given set Λ, such that it is a row strictly diagonally dominant full matrix rather than a column strictly diagonally dominant full matrix.

Theorem 3.8. *For the given real number set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ satisfying $\lambda_i \neq 0, i = 1, 2, \cdots, n, n \geq 2$, $|\lambda_1| < |\lambda_2|$ and the real number k satisfying $\frac{|\lambda_1|}{|\lambda_1 - \lambda_2|} < |k| < \frac{|\lambda_2|}{|\lambda_1 - \lambda_2|}$, there exists a real number $\delta_3 \in (0, \frac{\pi}{2}]$, such that for any real number $\theta \in (0, \delta_3)$, $n \times n$ real matrix $D_n(\theta) = E^{(n)}(2, 1(k)) G_n(\theta) E^{(n)}(2, 1(-k))$ whose spectrum is the given set Λ is a row strictly diagonally dominant full matrix rather than a column strictly diagonally dominant full matrix.*

Proof. The proof is similar to the proof of Theorem 3.7. Here, it is omitted. □

Immediately, we have the following Algorithm 3.3 which illustrates how to construct a matrix whose spectrum is the given set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ satisfying some specified conditions, such that it is a row strictly diagonally dominant full matrix rather than a column strictly diagonally dominant full matrix.

Algorithm 3.3:

Step 1 Input the real number set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ satisfying $\lambda_i \neq 0, i = 1, 2, \cdots, n$, $n \geq 2$, $|\lambda_1| < |\lambda_2|$ and the real number k_0 satisfying $\frac{|\lambda_1|}{|\lambda_1 - \lambda_2|} < |k_0| < \frac{|\lambda_2|}{|\lambda_1 - \lambda_2|}$, and the real number θ_0 satisfying $\theta_0 \in (0, \delta_3)$, where δ_3 is defined by Theorem 3.8.

Step 2 Compute

$$R_n(\theta_0) = \prod_{j=2}^{n} \prod_{i=1}^{j-1} R_{ij}^{(n)}(\theta_0),$$

where $R_{ij}^{(n)}(\theta_0), i,j=1,2,\cdots,n, i<j$ is defined by Definition 2.3;

Step 3 Compute

$$G_n(\theta_0)=R_n^T(\theta_0)\mathrm{diag}(\lambda_1,\lambda_2,\cdots,\lambda_n)R_n(\theta_0);$$

Step 4 Output $D_n(\theta_0)=E^{(n)}(2,1(k_0))G_n(\theta_0)E^{(n)}(2,1(-k_0))$.

As we all know that if a matrix is a strictly diagonally dominant full matrix, then it must be an α-strictly diagonally dominant full matrix. The following Theorem 3.9 and Theorem 3.10 show that for the given number set $\Lambda=\{\lambda_1,\lambda_2,\cdots,\lambda_n\}\subseteq\mathbb{R}$ satisfying some specified conditions, there exists a $n\times n$ real matrix whose spectrum is the given set Λ, such that it is an α-strictly diagonally dominant full matrix rather than a strictly diagonally dominant full matrix.

Theorem 3.9. *For the given real number set $\Lambda=\{\lambda_1,\lambda_2,\cdots,\lambda_n\}$ satisfying $\lambda_i\neq 0, i=1,2,\cdots,n, n\geq 2$ and $\lambda_1\neq\lambda_2$, adjust the order of $\lambda_1,\lambda_2,\cdots,\lambda_n$ such that the new sequence $\lambda_{l_1},\lambda_{l_2},\cdots,\lambda_{l_n}$ satisfies that $\frac{|\lambda_{l_2}|}{|\lambda_{l_1}|}=\max\left\{\frac{|\lambda_{l_i}|}{|\lambda_{l_j}|}\,\middle|\,|\lambda_{l_j}|\geq|\lambda_{l_i}|, \lambda_{l_j}\neq\lambda_{l_i}\right\}$. Without loss of generality, we still use $\lambda_1,\lambda_2,\cdots,\lambda_n$ to denote the new sequence where $|\lambda_1|\geq|\lambda_2|$ and $\lambda_1\neq\lambda_2$.*

(i) If $\alpha\in\left(0,\frac{|\lambda_2|}{|\lambda_1|+|\lambda_2|}\right]$, then for any real number k satisfying $\frac{|\lambda_1|}{|\lambda_1-\lambda_2|}<|k|<\frac{|\lambda_1|}{(1-\alpha)|\lambda_1-\lambda_2|}$, there exists a real number $\delta_4\in(0,\frac{\pi}{2}]$, such that for any real number $\theta\in(0,\delta_4)$, $n\times n$ real matrix $D_n(\theta)$ whose spectrum is the given set Λ is an α-strictly diagonally dominant full matrix rather than a strictly diagonally dominant full matrix;

(ii) If $\alpha\in\left(\frac{|\lambda_2|}{|\lambda_1|+|\lambda_2|},\frac{|\lambda_2|}{|\lambda_1|}\right)$, then for any real number k satisfying $\frac{|\lambda_1|}{|\lambda_1-\lambda_2|}<|k|<\frac{|\lambda_2|}{\alpha|\lambda_1-\lambda_2|}$, there exists a real number $\delta_5\in(0,\frac{\pi}{2}]$, such that for any real number $\theta\in(0,\delta_5)$, $n\times n$ real matrix $D_n(\theta)$ whose spectrum is the given set Λ is an α-strictly diagonally dominant full matrix rather than a strictly diagonally dominant full matrix.

Proof. Since the proofs of case (i) and case (ii) are similar, we only prove case (i) below. By Theorem 3.5, the spectrum of $D_n(\theta)$ is the given set Λ. By Theorem 3.3, we obtain

$$\lim_{\theta\to 0^+}\left|g_{ij}^{(n)}(s)\right|=0, i\neq j,\quad \lim_{\theta\to 0^+}\left|g_{ij}^{(n)}(s)\right|=|\lambda_i|, i=j,$$

thus,

$$\begin{aligned}
&\lim_{\theta\to 0^+}\left(\left|d_{11}^{(n)}(s)\right|-Q_1(D_n(\theta))\right)=|\lambda_1|-|k||\lambda_1-\lambda_2|,\\
&\lim_{\theta\to 0^+}\left(\left|d_{22}^{(n)}(s)\right|-P_2(D_n(\theta))\right)=|\lambda_2|-|k||\lambda_1-\lambda_2|,\\
&\lim_{\theta\to 0^+}\left(\left|d_{11}^{(n)}(s)\right|-\alpha P_1(D_n(\theta))-(1-\alpha)Q_1(D_n(\theta))\right)=|\lambda_1|-(1-\alpha)|k||\lambda_1-\lambda_2|,\\
&\lim_{\theta\to 0^+}\left(\left|d_{22}^{(n)}(s)\right|-\alpha P_2(D_n(\theta))-(1-\alpha)Q_2(D_n(\theta))\right)=|\lambda_2|-\alpha|k||\lambda_1-\lambda_2|,\\
&\lim_{\theta\to 0^+}\left(\left|d_{ii}^{(n)}(s)\right|-\alpha P_i(D_n(\theta))-(1-\alpha)Q_i(D_n(\theta))\right)=|\lambda_i|>0, i=3,4,\cdots,n.
\end{aligned}$$

When $|\lambda_1| \geq |\lambda_2|, \alpha \in \left(0, \frac{|\lambda_2|}{|\lambda_1|+|\lambda_2|}\right], |k| \in \left(\frac{|\lambda_1|}{|\lambda_1-\lambda_2|}, \frac{|\lambda_1|}{(1-\alpha)|\lambda_1-\lambda_2|}\right)$, we have

$$|\lambda_1| - |k||\lambda_1 - \lambda_2| < 0, \quad |\lambda_2| - |k||\lambda_1 - \lambda_2| < 0,$$
$$|\lambda_2| - \alpha|k||\lambda_1 - \lambda_2| > 0, \ |\lambda_1| - (1-\alpha)|k||\lambda_1 - \lambda_2| > 0.$$

Hence,

$$\lim_{\theta\to 0^+}\left(\left|d_{11}^{(n)}(s)\right| - Q_1(D_n(\theta))\right) = |\lambda_1| - |k||\lambda_1-\lambda_2| < 0,$$
$$\lim_{\theta\to 0^+}\left(\left|d_{22}^{(n)}(s)\right| - P_2(D_n(\theta))\right) = |\lambda_2| - |k||\lambda_1-\lambda_2| < 0,$$
$$\lim_{\theta\to 0^+}\left(\left|d_{11}^{(n)}(s)\right| - \alpha P_1(D_n(\theta)) - (1-\alpha)Q_1(D_n(\theta))\right) = |\lambda_1| - (1-\alpha)|k||\lambda_1-\lambda_2| > 0,$$
$$\lim_{\theta\to 0^+}\left(\left|d_{22}^{(n)}(s)\right| - \alpha P_2(D_n(\theta)) - (1-\alpha)Q_2(D_n(\theta))\right) = |\lambda_2| - \alpha|k||\lambda_1-\lambda_2| > 0.$$

By Lemma 2.1, there exists a common real number $\delta_{41} \in (0, \frac{\pi}{2}]$, such that for any real number $\theta \in (0, \delta_{41})$,

$$\left|d_{11}^{(n)}(s)\right| - Q_1(D_n(\theta)) < \frac{|\lambda_1|-|k||\lambda_1-\lambda_2|}{2} < 0,$$
$$\left|d_{22}^{(n)}(s)\right| - P_2(D_n(\theta)) < \frac{|\lambda_2|-|k||\lambda_1-\lambda_2|}{2} < 0,$$
$$\left|d_{11}^{(n)}(s)\right| - \alpha P_1(D_n(\theta)) - (1-\alpha)Q_1(D_n(\theta)) > \frac{|\lambda_1|-(1-\alpha)|k||\lambda_1-\lambda_2|}{2} > 0,$$
$$\left|d_{22}^{(n)}(s)\right| - \alpha P_2(D_n(\theta)) - (1-\alpha)Q_2(D_n(\theta)) > \frac{|\lambda_2|-\alpha|k||\lambda_1-\lambda_2|}{2} > 0,$$
$$\left|d_{ii}^{(n)}(s)\right| - \alpha P_i(D_n(\theta)) - (1-\alpha)Q_i(D_n(\theta)) > \frac{|\lambda_i|}{2} > 0, i = 3, 4, \cdots, n,$$

which shows that $D_n(\theta)$ is an α-strictly diagonally dominant matrix rather than a strictly diagonally dominant matrix. By Theorem 3.4, there exists a real number $\delta_{42} \in (0, \frac{\pi}{2}]$, such that for any real number $\theta \in (0, \delta_{42})$, $D_n(\theta)$ is a full matrix. Let $\delta_4 = \min\{\delta_{41}, \delta_{42}\}$, then for any real number $\theta \in (0, \delta_4)$, $D_n(\theta)$ is an α-strictly diagonally dominant full matrix rather than a strictly diagonally dominant full matrix. □

Theorem 3.10. *For the given real number set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ satisfying $\lambda_i \neq 0, i = 1, 2, \cdots, n, n \geq 2$ and $\lambda_1 \neq \lambda_2$, adjust the order of $\lambda_1, \lambda_2, \cdots, \lambda_n$, such that the new sequence $\lambda_{l_1}, \lambda_{l_2}, \cdots, \lambda_{l_n}$ satisfies that $\frac{|\lambda_{l_2}|-|\lambda_{l_1}|}{|\lambda_{l_2}|} = \min\left\{\frac{|\lambda_{l_j}|-|\lambda_{l_i}|}{|\lambda_{l_j}|} \,\middle|\, |\lambda_{l_i}| < |\lambda_{l_j}|\right\}$. Without loss of generality, we still use $\lambda_1, \lambda_2, \cdots, \lambda_n$ to denote the new sequence where $|\lambda_1| < |\lambda_2|$.*

(i) If $\alpha \in \left(\frac{|\lambda_2|-|\lambda_1|}{|\lambda_2|}, \frac{|\lambda_2|}{|\lambda_1|+|\lambda_2|}\right]$, then for any real number k satisfying $\frac{|\lambda_2|}{|\lambda_1-\lambda_2|} < |k| < \frac{|\lambda_1|}{(1-\alpha)|\lambda_1-\lambda_2|}$, there exists a real number $\delta_6 \in (0, \frac{\pi}{2}]$, such that for any real number $\theta \in (0, \delta_6)$, $n \times n$ real matrix $D_n(\theta)$ whose spectrum is the given set Λ is an α-strictly diagonally dominant full matrix rather than a strictly diagonally dominant full matrix;

(ii) If $\alpha \in \left(\frac{|\lambda_2|}{|\lambda_1|+|\lambda_2|}, 1\right)$, then for any real number k satisfying $\frac{|\lambda_2|}{|\lambda_1-\lambda_2|} < |k| < \frac{|\lambda_2|}{\alpha|\lambda_1-\lambda_2|}$, there exists a real number $\delta_7 \in (0, \frac{\pi}{2}]$, such that for any real number $\theta \in (0, \delta_7)$, $n \times n$ real matrix $D_n(\theta)$ whose spectrum is the given set Λ is

an α-strictly diagonally dominant full matrix rather than a strictly diagonally dominant full matrix.

Proof. The proof is similar to the proof of Theorem 3.9. Here, it is omitted. □

Immediately, we have the following Algorithm 3.4 which illustrates how to construct a matrix whose spectrum is the given set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ satisfying some specified conditions, such that it is an α-strictly diagonally dominant full matrix rather than a strictly diagonally dominant full matrix.

Algorithm 3.4:

Step 1 Input the real number set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, $n \geq 2$ satisfying the assumptions of Theorem 3.9 and Theorem 3.10, and the real number $\alpha_0 \in (0, 1)$.

Step 2 Adjust the order of $\lambda_1,\lambda_2,\cdots,\lambda_n$ such that the new sequence $\lambda_{l_1},\lambda_{l_2},\cdots,\lambda_{l_n}$ satisfies $\frac{|\lambda_{l_2}|}{|\lambda_{l_1}|} = \max\left\{\frac{|\lambda_{l_i}|}{|\lambda_{l_j}|}||\lambda_{l_j}| \geq |\lambda_{l_i}|, \lambda_{l_j} \neq \lambda_{l_i}\right\}$.

If $\alpha_0 \in \left(0, \frac{|\lambda_{l_2}|}{|\lambda_{l_1}|+|\lambda_{l_2}|}\right]$, choose the real number k_0 satisfying $\frac{|\lambda_{l_1}|}{|\lambda_{l_1}-\lambda_{l_2}|} < |k_0| < \frac{|\lambda_{l_1}|}{(1-\alpha)|\lambda_{l_1}-\lambda_{l_2}|}$, $\theta_0 \in (0, \delta_4)$, where δ_4 is defined by Theorem 3.9(i) and turn to Step 4.

If $\alpha_0 \in \left(\frac{|\lambda_{l_2}|}{|\lambda_{l_1}|+|\lambda_{l_2}|}, \frac{|\lambda_{l_2}|}{|\lambda_{l_1}|}\right)$, choose the real number k_0 satisfying $\frac{|\lambda_{l_1}|}{|\lambda_{l_1}-\lambda_{l_2}|} < |k_0| < \frac{|\lambda_{l_2}|}{\alpha|\lambda_{l_1}-\lambda_{l_2}|}$, $\theta_0 \in (0, \delta_5)$, where δ_5 is defined by Theorem 3.9(ii) and turn to Step 4.

If $\alpha_0 \in \left[\frac{|\lambda_{l_2}|}{|\lambda_{l_1}|}, 1\right)$, turn to Step 3.

Step 3 Adjust the order of $\lambda_1,\lambda_2,\cdots,\lambda_n$ such that the new sequence $\lambda_{l_1},\lambda_{l_2},\cdots,\lambda_{l_n}$ satisfies $\frac{|\lambda_{l_2}|-|\lambda_{l_1}|}{|\lambda_{l_2}|} = \min\left\{\frac{|\lambda_{l_j}|-|\lambda_{l_i}|}{|\lambda_{l_j}|}|\,|\lambda_{l_i}| < |\lambda_{l_j}|\right\}$.

If $\alpha_0 \in \left(\frac{|\lambda_{l_2}|-|\lambda_{l_1}|}{|\lambda_{l_2}|}, \frac{|\lambda_{l_2}|}{|\lambda_{l_1}|+|\lambda_{l_2}|}\right]$, choose the real number k_0 satisfying $\frac{|\lambda_{l_2}|}{|\lambda_{l_1}-\lambda_{l_2}|} < |k_0| < \frac{|\lambda_{l_1}|}{(1-\alpha)|\lambda_{l_1}-\lambda_{l_2}|}$, $\theta_0 \in (0, \delta_6)$, where δ_6 is defined by Theorem 3.10(i) and turn to Step 4.

If $\alpha_0 \in \left(\frac{|\lambda_{l_2}|}{|\lambda_{l_1}|+|\lambda_{l_2}|}, 1\right)$, choose the real number k_0 satisfying $\frac{|\lambda_{l_2}|}{|\lambda_{l_1}-\lambda_{l_2}|} < |k_0| < \frac{|\lambda_{l_2}|}{\alpha|\lambda_{l_1}-\lambda_{l_2}|}$, $\theta_0 \in (0, \delta_7)$, where $\theta_0 \in (0, \delta_7)$ is defined by Theorem 3.10(ii) and turn to Step 4.

If $\alpha_0 \in \left(0, \frac{|\lambda_{l_2}|-|\lambda_{l_1}|}{|\lambda_{l_2}|}\right]$, output "error".

Step 4 Compute

$$R_n(\theta_0) = \prod_{j=2}^{n}\prod_{i=1}^{j-1} R_{ij}^{(n)}(\theta_0),$$

where $R_{ij}^{(n)}(\theta_0), i, j = 1, 2, \cdots, n, i < j$ is defined by Definition 2.3;

Step 5 Compute

$$G_n(\theta_0) = R_n^T(\theta_0)\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)R_n(\theta_0);$$

Step 6 Output $D_n(\theta_0) = E^{(n)}(2, 1(k_0))G_n(\theta_0)E^{(n)}(2, 1(-k_0))$.

It is well known that if a matrix is an α-strictly diagonally dominant full matrix, then it must be an α-double strictly diagonally dominant full matrix. The following Theorem 3.11 shows that for the given set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\} \subseteq \mathbb{R}$ satisfying some specified conditions, there exists a $n \times n$ real matrix whose spectrum is the given set Λ, such that it is an α-double strictly diagonally dominant full matrix rather than an α-strictly diagonally dominant full matrix.

Theorem 3.11. *For the given real number set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ satisfying $\lambda_i \neq 0, i = 1, 2, \cdots, n, n \geq 2$ and $\lambda_1 \neq \lambda_2$, adjust the order of $\lambda_1, \lambda_2, \cdots, \lambda_n$, such that the new sequence $\lambda_{l_1}, \lambda_{l_2}, \cdots, \lambda_{l_n}$ satisfies $\frac{|\lambda_{l_2}|}{|\lambda_{l_1}|+|\lambda_{l_2}|} = \min\left\{\frac{|\lambda_{l_j}|}{|\lambda_{l_i}|+|\lambda_{l_j}|} | \lambda_{l_i} \neq \lambda_{l_j}\right\}$. Without loss of generality, we still use $\lambda_1, \lambda_2, \cdots, \lambda_n$ to denote the new sequence where $\lambda_1 \neq \lambda_2$. If $\alpha \in \left(\frac{|\lambda_2|}{|\lambda_1|+|\lambda_2|}, 1\right]$, for any real number k satisfying $\frac{|\lambda_2|}{\alpha|\lambda_1-\lambda_2|} < |k| < \frac{\sqrt{|\lambda_1\lambda_2|}}{\sqrt{\alpha(1-\alpha)}|\lambda_1-\lambda_2|}$, there exists a real number $\delta_8 \in (0, \frac{\pi}{2}]$, such that for any real number $\theta \in (0, \delta_8)$, $n \times n$ real matrix $D_n(\theta)$ whose spectrum is the given set Λ is an α-double strictly diagonally dominant full matrix rather than an α-strictly diagonally dominant full matrix.*

Proof. By Theorem 3.3,

$$\lim_{\theta\to 0^+}\left|g_{ij}^{(n)}(s)\right| = 0, i \neq j, \quad \lim_{\theta\to 0^+}\left|g_{ij}^{(n)}(s)\right| = |\lambda_i|, i = j,$$

then

$$\begin{aligned}
&\lim_{\theta\to 0^+}\Big\{\left|d_{ii}^{(n)}(s)\right|\left|d_{jj}^{(n)}(s)\right| \\
&\qquad -[\alpha P_i(D_n(\theta))+(1-\alpha)Q_i(D_n(\theta))][\alpha P_j(D_n(\theta))+(1-\alpha)Q_j(D_n(\theta))]\Big\} \\
&\quad = |\lambda_i||\lambda_j| > 0, i, j = 3, 4, \cdots, n, \\
&\lim_{\theta\to 0^+}\Big\{\left|d_{11}^{(n)}(s)\right|\left|d_{22}^{(n)}(s)\right| \\
&\qquad -[\alpha P_1(D_n(\theta))+(1-\alpha)Q_1(D_n(\theta))][\alpha P_2(D_n(\theta))+(1-\alpha)Q_2(D_n(\theta))]\Big\} \\
&\quad = |\lambda_1||\lambda_2| - \alpha(1-\alpha)|k|^2|\lambda_1-\lambda_2|^2, \\
&\lim_{\theta\to 0^+}\Big\{\left|d_{11}^{(n)}(s)\right|\left|d_{ii}^{(n)}(s)\right| \\
&\qquad -[\alpha P_1(D_n(\theta))+(1-\alpha)Q_1(D_n(\theta))][\alpha P_i(D_n(\theta))+(1-\alpha)Q_i(D_n(\theta))]\Big\} \\
&\quad = |\lambda_1||\lambda_i| > 0, i = 3, 4, \cdots, n, \\
&\lim_{\theta\to 0^+}\Big\{\left|d_{22}^{(n)}(s)\right|\left|d_{ii}^{(n)}(s)\right| \\
&\qquad -[\alpha P_2(D_n(\theta))+(1-\alpha)Q_2(D_n(\theta))][\alpha P_i(D_n(\theta))+(1-\alpha)Q_i(D_n(\theta))]\Big\} \\
&\quad = |\lambda_2||\lambda_i| > 0, i = 3, 4, \cdots, n, \\
&\lim_{\theta\to 0^+}\Big\{\left|d_{22}^{(n)}(s)\right| - [\alpha P_2(D_n(\theta))+(1-\alpha)Q_2(D_n(\theta))]\Big\} \\
&\quad = |\lambda_2| - \alpha|k||\lambda_1-\lambda_2|.
\end{aligned}$$

When $\alpha \in \left(\frac{|\lambda_2|}{|\lambda_1|+|\lambda_2|}, 1\right]$, $|k| \in \left(\frac{|\lambda_2|}{\alpha|\lambda_1-\lambda_2|}, \frac{\sqrt{|\lambda_1\lambda_2|}}{\sqrt{\alpha(1-\alpha)}|\lambda_1-\lambda_2|}\right)$, we have

$$|\lambda_1||\lambda_2| > \alpha(1-\alpha)|k|^2|\lambda_1-\lambda_2|^2, \quad |\lambda_2| < \alpha|k||\lambda_1-\lambda_2|.$$

Hence,

$$\begin{aligned}
&\lim_{\theta\to 0^+}\Big\{\left|d_{11}^{(n)}(s)\right|\left|d_{22}^{(n)}(s)\right| \\
&\qquad -[\alpha P_1(D_n(\theta))+(1-\alpha)Q_1(D_n(\theta))][\alpha P_2(D_n(\theta))+(1-\alpha)Q_2(D_n(\theta))]\Big\} \\
&\quad =|\lambda_1||\lambda_2|-\alpha(1-\alpha)k^2(\lambda_1-\lambda_2)^2>0, \\
&\lim_{\theta\to 0^+}\Big\{\left|d_{22}^{(n)}(s)\right|-[\alpha P_2(D_n(\theta))+(1-\alpha)Q_2(D_n(\theta))]\Big\} \\
&\quad =|\lambda_2|-\alpha|k||\lambda_1-\lambda_2|<0.
\end{aligned}$$

By Lemma 2.1, there exists a common real number $\delta_{81}\in(0,\frac{\pi}{2}]$, such that for any real number $\theta\in(0,\delta_{81})$,

$$\begin{aligned}
&\left|d_{ii}^{(n)}(s)\right|\left|d_{jj}^{(n)}(s)\right| \\
&\qquad -[\alpha P_i(D_n(\theta))+(1-\alpha)Q_i(D_n(\theta))][\alpha P_j(D_n(\theta))+(1-\alpha)Q_j(D_n(\theta))] \\
&\quad >\frac{|\lambda_i||\lambda_j|}{2}>0, i,j=3,4,\cdots,n, \\
&\left|d_{11}^{(n)}(s)\right|\left|d_{22}^{(n)}(s)\right| \\
&\qquad -[\alpha P_1(D_n(\theta))+(1-\alpha)Q_1(D_n(\theta))][\alpha P_2(D_n(\theta))+(1-\alpha)Q_2(D_n(\theta))] \\
&\quad >\frac{|\lambda_1||\lambda_2|-\alpha(1-\alpha)|k|^2|\lambda_1-\lambda_2|^2}{2}>0, \\
&\left|d_{11}^{(n)}(s)\right|\left|d_{ii}^{(n)}(s)\right| \\
&\qquad -[\alpha P_1(D_n(\theta))+(1-\alpha)Q_1(D_n(\theta))][\alpha P_i(D_n(\theta))+(1-\alpha)Q_i(D_n(\theta))] \\
&\quad >\frac{|\lambda_1||\lambda_i|}{2}>0, i=3,4,\cdots,n, \\
&\left|d_{22}^{(n)}(s)\right|\left|d_{ii}^{(n)}(s)\right| \\
&\qquad -[\alpha P_2(D_n(\theta))+(1-\alpha)Q_2(D_n(\theta))][\alpha P_i(D_n(\theta))+(1-\alpha)Q_i(D_n(\theta))] \\
&\quad >\frac{|\lambda_2||\lambda_i|}{2}>0, i=3,4,\cdots,n, \\
&\left|d_{22}^{(n)}(s)\right|-[\alpha P_2(D_n(\theta))+(1-\alpha)Q_2(D_n(\theta))] \\
&\quad <\frac{|\lambda_2|-\alpha|k||\lambda_1-\lambda_2|}{2}<0,
\end{aligned}$$

which implies that $D_n(\theta)$ is an α-double strictly diagonally dominant matrix rather than an α-strictly diagonally dominant matrix. By Theorem 3.4, there exists a real number $\delta_{82}\in(0,\frac{\pi}{2}]$, such that for any real number $\theta\in(0,\delta_{82})$, $D_n(\theta)$ is a full matrix. Let $\delta_8=\min\{\delta_{81},\delta_{82}\}$, then for any real number $\theta\in(0,\delta_8)$, $D_n(\theta)$ is an α-double strictly diagonally dominant full matrix rather than an α-strictly diagonally dominant full matrix. By Theorem 3.5, the spectrum of $D_n(\theta)$ is the given set Λ. □

Immediately, we have the following Algorithm 3.5 which illustrates how to construct a matrix whose spectrum is the given set $\Lambda=\{\lambda_1,\lambda_2,\cdots,\lambda_n\}$ satisfying some specified conditions, such that it is an α-double strictly diagonally dominant full matrix rather than an α-strictly diagonally dominant full matrix.

Algorithm 3.5:

Step 1 Input the real number set $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, $n \geq 2$, satisfying the assumptions of Theorem 3.11, and the real number $\alpha_0 \in (0, 1)$.

Step 2 Adjust the order of $\lambda_1,\lambda_2,\cdots,\lambda_n$ such that the new sequence $\lambda_{l_1},\lambda_{l_2},\cdots,\lambda_{l_n}$ satisfies that $\frac{|\lambda_{l_2}|}{|\lambda_{l_1}|+|\lambda_{l_2}|} = \min\left\{\frac{|\lambda_{l_j}|}{|\lambda_{l_i}|+|\lambda_{l_j}|} | \lambda_{l_i} \neq \lambda_{l_j}\right\}$,

If $\alpha_0 \in \left(\frac{|\lambda_{l_2}|}{|\lambda_{l_1}|+|\lambda_{l_2}|}, 1\right]$, choose the real number k_0 satisfying $\frac{|\lambda_{l_2}|}{\alpha_0|\lambda_{l_1}-\lambda_{l_2}|} < |k_0| < \frac{\sqrt{|\lambda_{l_1}\lambda_{l_2}|}}{\sqrt{\alpha_0(1-\alpha_0)}|\lambda_{l_1}-\lambda_{l_2}|}$, $\theta_0 \in (0, \delta_8)$, where δ_8 is defined by Theorem 3.11 and turn to Step 3.

If $\alpha_0 \in \left(0, \frac{|\lambda_{l_2}|}{|\lambda_{l_1}|+|\lambda_{l_2}|}\right]$, output "error".

Step 3 Compute

$$R_n(\theta_0) = \prod_{j=2}^{n}\prod_{i=1}^{j-1} R_{ij}^{(n)}(\theta_0),$$

where $R_{ij}^{(n)}(\theta_0), i, j = 1, 2, \cdots, n, i < j$ is defined by Definition 2.3;

Step 4 Compute

$$G_n(\theta_0) = R_n^T(\theta_0)\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)R_n(\theta_0);$$

Step 5 Output $D_n(\theta_0) = E^{(n)}(2, 1(k_0))G_n(\theta_0)E^{(n)}(2, 1(-k_0))$.

Remark 3.1. Theorems 3.6-3.11 can be generalized to the complex number field, i.e., the given number set Λ can be a complex set.

Remark 3.2. In the above, we provide the Theorems 3.6-3.11 for construction of the three classes of full H-matrices based on the Givens matrices where $\theta \in (0, \delta)$ and $\delta \in (0, \frac{\pi}{2}]$. Analogously, we can obtain the similar Theorems when we take $\theta \in (-\delta, 0)$ in the Givens matrices where $\delta \in (0, \frac{\pi}{2}]$.

4. Numerical examples

In this section, we first use a simple Example 4.1 to construct the three classes of full H-matrices according to the Algorithms 3.1-3.5.

Example 4.1. For the given real number set $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\} = \{1, 2, 3\}$ and real numbers $\frac{1}{5}$, $\frac{13}{15}$ and $\frac{4}{5}$.

(1) Let $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1$. By Algorithm 3.1, we choose $\theta_0 = \frac{\pi}{3}$, then

$$A = \begin{pmatrix} 1.3125 & -0.3605 & 0.4581 \\ -0.3605 & 1.8159 & 0.1643 \\ 0.4581 & 0.1643 & 2.8716 \end{pmatrix}.$$

So, A is obviously a row and column strictly diagonally dominant full matrix whose spectrum is the given set $\Lambda = \{1, 2, 3\}$.

(2) Let $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1$. By Algorithm 3.2, we choose $k_0 = \frac{5}{2}$ and $\theta_0 = \frac{\pi}{90}$, then

$$B = \begin{pmatrix} 2.9153 & 0.0324 & 0.0709 \\ 2.3210 & 2.0809 & 0.2133 \\ -0.0192 & 0.0361 & 1.0037 \end{pmatrix}.$$

So, B whose spectrum is the given set $\Lambda = \{1, 2, 3\}$ is obviously a column strictly diagonally dominant full matrix rather than a row strictly diagonally dominant full matrix.

(3) Let $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$. By Algorithm 3.3, we choose $k_0 = \frac{3}{2}$ and $\theta_0 = \frac{\pi}{60}$, then

$$C = \begin{pmatrix} 1.0782 & -0.0467 & -0.1070 \\ -1.4298 & 1.9303 & -0.2153 \\ -0.0247 & -0.0548 & 2.9915 \end{pmatrix}.$$

So, C whose spectrum is the given set $\Lambda = \{1, 2, 3\}$ is obviously a row strictly diagonally dominant full matrix rather than a column strictly diagonally dominant full matrix.

(4) Let $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1$. By Algorithm 3.4, we choose $k_0 = \frac{7}{2}$, $\alpha_0 = \frac{1}{5}$ and $\theta_0 = \frac{\pi}{120}$, then

$$D = \begin{pmatrix} 2.9112 & 0.0248 & 0.0530 \\ 3.2141 & 2.0867 & 0.2123 \\ -0.0409 & 0.0268 & 1.0021 \end{pmatrix} \triangleq (d_{ij})_{3\times 3},$$

where

$$\begin{aligned}
&|d_{11}| - Q_1(D) = -0.3438 < 0, \\
&|d_{22}| - P_2(D) = -1.3397 < 0, \\
&|d_{11}| - [\tfrac{1}{5} P_1(D) + (1 - \tfrac{1}{5}) Q_1(D)] = 0.2916 > 0, \\
&|d_{22}| - [\tfrac{1}{5} P_2(D) + (1 - \tfrac{1}{5}) Q_2(D)] = 1.3601 > 0, \\
&|d_{33}| - [\tfrac{1}{5} P_3(D) + (1 - \tfrac{1}{5}) Q_3(D)] = 0.7763 > 0.
\end{aligned}$$

So, D whose spectrum is the given set $\Lambda = \{1, 2, 3\}$ is a $\frac{1}{5}$-strictly diagonally dominant full matrix rather than a strictly diagonally dominant full matrix.

(5) Let $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 1$. By Algorithm 3.4, we choose $k_0 = \frac{40}{13}$, $\alpha_0 = \frac{13}{15}$ and $\theta_0 = \frac{\pi}{90}$, then

$$E = \begin{pmatrix} 2.1109 & -0.0361 & 0.0337 \\ -2.7607 & 2.8855 & 0.1721 \\ -0.1770 & 0.0685 & 1.0036 \end{pmatrix} \triangleq (e_{ij})_{3\times 3},$$

where

$$\begin{aligned}
&|e_{11}| - Q_1(E) = -0.8268 < 0,\\
&|e_{22}| - P_2(E) = -0.0473 < 0,\\
&|e_{11}| - [\tfrac{13}{15}P_1(E) + (1-\tfrac{13}{15})Q_1(E)] = 1.6587 > 0,\\
&|e_{22}| - [\tfrac{13}{15}P_2(E) + (1-\tfrac{13}{15})Q_2(E)] = 0.3298 > 0,\\
&|e_{33}| - [\tfrac{13}{15}P_3(E) + (1-\tfrac{13}{15})Q_3(E)] = 0.7634 > 0.
\end{aligned}$$

So, E whose spectrum is the given set $\Lambda = \{1,2,3\}$ is a $\frac{13}{15}$-strictly diagonally dominant full matrix rather than a strictly diagonally dominant full matrix.

(6) Let $\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 2$. By algorithm 3.5, we choose $k_0 = \frac{5}{4}, \alpha_0 = \frac{4}{5}$ and $\theta_0 = \frac{\pi}{90}$, then

$$F = \begin{pmatrix} 2.9108 & 0.0685 & 0.0372 \\ 2.4526 & 1.0891 & 0.0141 \\ 0.0777 & -0.0324 & 2.0002 \end{pmatrix} \triangleq (f_{ij})_{3\times 3},$$

where

$$\begin{aligned}
&|f_{22}| - [\tfrac{4}{5}P_2(F) + (1-\tfrac{4}{5})Q_2(F)] = -0.9044 < 0,\\
&|f_{11}||f_{22}| - [\tfrac{4}{5}P_1(F) + (1-\tfrac{4}{5})Q_1(F)][\tfrac{4}{5}P_2(F) + (1-\tfrac{4}{5})Q_2(F)] = 1.9927 > 0,\\
&|f_{11}||f_{33}| - [\tfrac{4}{5}P_1(F) + (1-\tfrac{4}{5})Q_1(F)][\tfrac{4}{5}P_3(F) + (1-\tfrac{4}{5})Q_3(F)] = 5.7641 > 0,\\
&|f_{22}||f_{33}| - [\tfrac{4}{5}P_2(F) + (1-\tfrac{4}{5})Q_2(F)][\tfrac{4}{5}P_3(F) + (1-\tfrac{4}{5})Q_3(F)] = 1.9824 > 0.
\end{aligned}$$

So, F whose spectrum is the given set $\Lambda = \{1,2,3\}$ is a $\frac{4}{5}$-double strictly diagonally dominant full matrix rather than a $\frac{4}{5}$-strictly diagonally dominant full matrix.

Next, we use another Example 4.2 to construct the three classes of full H-matrices according to the Algorithms 3.1-3.5.

Example 4.2. For the given real number set $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\} = \{4,5,6,7,8\}$ and real numbers $\frac{1}{5}$, $\frac{18}{25}$ and $\frac{4}{5}$.

(1) Let $\lambda_1 = 8, \lambda_2 = 7, \lambda_3 = 6, \lambda_4 = 5, \lambda_5 = 4$. By Algorithm 3.1, we choose $\theta_0 = \frac{\pi}{3}$, then

$$\hat{A} = \begin{pmatrix} 4.3320 & -0.6436 & -0.0298 & 0.0489 & 0.1233 \\ -0.6436 & 5.7995 & -0.9141 & -0.2560 & -0.0612 \\ -0.0298 & -0.9141 & 6.2829 & -0.4411 & 0.3988 \\ 0.0489 & -0.2560 & -0.4411 & 5.8661 & 0.6086 \\ 0.1233 & -0.0612 & 0.3988 & 0.6086 & 7.7195 \end{pmatrix}.$$

So, $\hat{A}$ is obviously a row and column strictly diagonally dominant full matrix whose spectrum is the given set $\Lambda = \{4,5,6,7,8\}$.

(2) Let $\lambda_1 = 8, \lambda_2 = 7, \lambda_3 = 6, \lambda_4 = 5, \lambda_5 = 4$. By Algorithm 3.2, we choose $k_0 = \frac{15}{2}$ and $\theta_0 = \frac{\pi}{360}$, then

$$\hat{B} = \begin{pmatrix} 7.9389 & 0.0080 & 0.0170 & 0.0261 & 0.0354 \\ 7.0530 & 7.0599 & 0.1358 & 0.2131 & 0.2916 \\ -0.0461 & 0.0084 & 6.0000 & 0.0088 & 0.0178 \\ -0.1042 & 0.0174 & 0.0088 & 5.0004 & 0.0092 \\ -0.1632 & 0.0265 & 0.0178 & 0.0092 & 4.0008 \end{pmatrix}.$$

So, $\hat{B}$ whose spectrum is the given set $\Lambda = \{4, 5, 6, 7, 8\}$ is obviously a column strictly diagonally dominant full matrix rather than a row strictly diagonally dominant full matrix.

(3) Let $\lambda_1 = 4, \lambda_2 = 5, \lambda_3 = 6, \lambda_4 = 7, \lambda_5 = 8$. By Algorithm 3.3, we choose $k_0 = \frac{9}{2}$ and $\theta_0 = \frac{\pi}{360}$, then

$$\hat{C} = \begin{pmatrix} 4.0369 & -0.0080 & -0.0170 & -0.0261 & -0.0354 \\ -4.3436 & 4.9642 & -0.0849 & -0.1348 & -0.1856 \\ 0.0209 & -0.0084 & 6.0000 & -0.0088 & -0.0178 \\ 0.0521 & -0.0174 & -0.0088 & 6.9996 & -0.0092 \\ 0.0838 & -0.0265 & -0.0178 & -0.0092 & 7.9992 \end{pmatrix}.$$

So, $\hat{C}$ whose spectrum is the given set $\Lambda = \{4, 5, 6, 7, 8\}$ is obviously a row strictly diagonally dominant full matrix rather than a column strictly diagonally dominant full matrix.

(4) Let $\lambda_1 = 8, \lambda_2 = 7, \lambda_3 = 6, \lambda_4 = 5, \lambda_5 = 4$. By Algorithm 3.4, we choose $k_0 = 9$, $\alpha_0 = \frac{1}{5}$ and $\theta_0 = \frac{\pi}{720}$, then

$$\hat{D} = \begin{pmatrix} 7.9621 & 0.0042 & 0.0086 & 0.0131 & 0.0176 \\ 8.6638 & 7.0376 & 0.0818 & 0.1263 & 0.1713 \\ -0.0300 & 0.0043 & 6.0000 & 0.0044 & 0.0088 \\ -0.0653 & 0.0087 & 0.0044 & 5.0001 & 0.0045 \\ -0.1009 & 0.0132 & 0.0088 & 0.0045 & 4.0002 \end{pmatrix} \triangleq (\hat{d}_{ij})_{5\times 5},$$

where

$$\begin{aligned}
&|\hat{d}_{11}| - Q_1(\hat{D}) = -0.8979 < 0, \\
&|\hat{d}_{22}| - P_2(\hat{D}) = -2.0056 < 0, \\
&|\hat{d}_{11}| - [\tfrac{1}{5}P_1(\hat{D}) + (1-\tfrac{1}{5})Q_1(\hat{D})] = 0.8654 > 0, \\
&|\hat{d}_{22}| - [\tfrac{1}{5}P_2(\hat{D}) + (1-\tfrac{1}{5})Q_2(\hat{D})] = 5.2047 > 0, \\
&|\hat{d}_{33}| - [\tfrac{1}{5}P_3(\hat{D}) + (1-\tfrac{1}{5})Q_3(\hat{D})] = 5.9076 > 0, \\
&|\hat{d}_{44}| - [\tfrac{1}{5}P_4(\hat{D}) + (1-\tfrac{1}{5})Q_4(\hat{D})] = 4.8649 > 0, \\
&|\hat{d}_{55}| - [\tfrac{1}{5}P_5(\hat{D}) + (1-\tfrac{1}{5})Q_5(\hat{D})] = 3.8130 > 0.
\end{aligned}$$

So, $\hat{D}$ whose spectrum is the given set $\Lambda = \{4,5,6,7,8\}$ is a $\frac{1}{5}$-strictly diagonally dominant full matrix rather than a strictly diagonally dominant full matrix.

(5) Let $\lambda_1 = 7, \lambda_2 = 8, \lambda_3 = 6, \lambda_4 = 5, \lambda_5 = 4$. By Algorithm 3.4, we choose $k_0 = 10$, $\alpha_0 = \frac{18}{25}$ and $\theta_0 = \frac{\pi}{720}$, then

$$\hat{E} = \begin{pmatrix} 7.0447 & -0.0045 & 0.0042 & 0.0087 & 0.0131 \\ -9.5558 & 7.9550 & 0.0511 & 0.0997 & 0.1488 \\ -0.0815 & 0.0086 & 6.0000 & 0.0044 & 0.0088 \\ -0.1216 & 0.0130 & 0.0044 & 5.0001 & 0.0045 \\ -0.1621 & 0.0175 & 0.0088 & 0.0045 & 4.0002 \end{pmatrix} \triangleq (\hat{e}_{ij})_{5\times 5},$$

where

$$\begin{aligned} &|\hat{e}_{11}| - Q_1(\hat{E}) = -2.8764 < 0, \\ &|\hat{e}_{22}| - P_2(\hat{E}) = -1.9003 < 0, \\ &|\hat{e}_{11}| - [\tfrac{18}{25}P_1(\hat{E}) + (1-\tfrac{18}{25})Q_1(\hat{E})] = 4.2448 > 0, \\ &|\hat{e}_{22}| - [\tfrac{18}{25}P_2(\hat{E}) + (1-\tfrac{18}{25})Q_2(\hat{E})] = 0.8469 > 0, \\ &|\hat{e}_{33}| - [\tfrac{18}{25}P_3(\hat{E}) + (1-\tfrac{18}{25})Q_3(\hat{E})] = 5.9065 > 0, \\ &|\hat{e}_{44}| - [\tfrac{18}{25}P_4(\hat{E}) + (1-\tfrac{18}{25})Q_4(\hat{E})] = 4.8639 > 0, \\ &|\hat{e}_{55}| - [\tfrac{18}{25}P_5(\hat{E}) + (1-\tfrac{18}{25})Q_5(\hat{E})] = 3.8122 > 0. \end{aligned}$$

So, $\hat{E}$ whose spectrum is the given set $\Lambda = \{4,5,6,7,8\}$ is a $\frac{18}{25}$-strictly diagonally dominant full matrix rather than a strictly diagonally dominant full matrix.

(6) Let $\lambda_1 = 8, \lambda_2 = 4, \lambda_3 = 7, \lambda_4 = 6, \lambda_5 = 5$. By algorithm 3.5, we choose $k_0 = \frac{5}{2}, \alpha_0 = \frac{4}{5}$ and $\theta_0 = \frac{\pi}{360}$, then

$$\hat{F} = \begin{pmatrix} 7.9131 & 0.0344 & 0.0086 & 0.0176 & 0.0267 \\ 9.8154 & 4.0868 & -0.0040 & 0.0267 & 0.0580 \\ 0.0728 & -0.0256 & 6.9996 & 0.0084 & 0.0174 \\ 0.0608 & -0.0173 & 0.0084 & 6.0000 & 0.0088 \\ 0.0487 & -0.0088 & 0.0174 & 0.0088 & 5.0004 \end{pmatrix} \triangleq (\hat{f}_{ij})_{5\times 5},$$

where

$$\begin{aligned} &|\hat{f}_{22}| - [\frac{4}{5}P_2(\hat{F}) + (1-\frac{4}{5})Q_2(\hat{F})] = -3.8537 < 0, \\ &|\hat{f}_{11}||\hat{f}_{22}| - [\frac{4}{5}P_1(\hat{F}) + (1-\frac{4}{5})Q_1(\hat{F})][\frac{4}{5}P_2(\hat{F}) + (1-\frac{4}{5})Q_2(\hat{F})] = 15.9072 > 0, \\ &|\hat{f}_{11}||\hat{f}_{33}| - [\frac{4}{5}P_1(\hat{F}) + (1-\frac{4}{5})Q_1(\hat{F})][\frac{4}{5}P_3(\hat{F}) + (1-\frac{4}{5})Q_3(\hat{F})] = 55.1675 > 0, \\ &|\hat{f}_{11}||\hat{f}_{44}| - [\frac{4}{5}P_1(\hat{F}) + (1-\frac{4}{5})Q_1(\hat{F})][\frac{4}{5}P_4(\hat{F}) + (1-\frac{4}{5})Q_4(\hat{F})] = 47.2955 > 0, \\ &|\hat{f}_{11}||\hat{f}_{55}| - [\frac{4}{5}P_1(\hat{F}) + (1-\frac{4}{5})Q_1(\hat{F})][\frac{4}{5}P_5(\hat{F}) + (1-\frac{4}{5})Q_5(\hat{F})] = 39.3843 > 0, \end{aligned}$$

$$|\hat{f}_{22}||\hat{f}_{33}| - [\frac{4}{5}P_2(\hat{F}) + (1-\frac{4}{5})Q_2(\hat{F})][\frac{4}{5}P_3(\hat{F}) + (1-\frac{4}{5})Q_3(\hat{F})] = 27.7563 > 0,$$
$$|\hat{f}_{22}||\hat{f}_{44}| - [\frac{4}{5}P_2(\hat{F}) + (1-\frac{4}{5})Q_2(\hat{F})][\frac{4}{5}P_4(\hat{F}) + (1-\frac{4}{5})Q_4(\hat{F})] = 23.8176 > 0,$$
$$|\hat{f}_{22}||\hat{f}_{55}| - [\frac{4}{5}P_2(\hat{F}) + (1-\frac{4}{5})Q_2(\hat{F})][\frac{4}{5}P_5(\hat{F}) + (1-\frac{4}{5})Q_5(\hat{F})] = 19.7282 > 0,$$
$$|\hat{f}_{33}||\hat{f}_{44}| - [\frac{4}{5}P_3(\hat{F}) + (1-\frac{4}{5})Q_3(\hat{F})][\frac{4}{5}P_4(\hat{F}) + (1-\frac{4}{5})Q_4(\hat{F})] = 41.9883 > 0,$$
$$|\hat{f}_{33}||\hat{f}_{55}| - [\frac{4}{5}P_3(\hat{F}) + (1-\frac{4}{5})Q_3(\hat{F})][\frac{4}{5}P_5(\hat{F}) + (1-\frac{4}{5})Q_5(\hat{F})] = 34.9913 > 0,$$
$$|\hat{f}_{44}||\hat{f}_{55}| - [\frac{4}{5}P_4(\hat{F}) + (1-\frac{4}{5})Q_4(\hat{F})][\frac{4}{5}P_5(\hat{F}) + (1-\frac{4}{5})Q_5(\hat{F})] = 29.9944 > 0.$$

So, $\hat{F}$ whose spectrum is the given set $\Lambda = \{4, 5, 6, 7, 8\}$ is a $\frac{4}{5}$-double strictly diagonally dominant full matrix rather than a $\frac{4}{5}$-strictly diagonally dominant full matrix.

5. Conclusions

In this paper, we mainly study the inverse eigenvalue problem of the three classes of full H-matrices including strictly diagonally dominant full matrix, α-strictly diagonally dominant full matrix and α-double strictly diagonally dominant full matrix. By using the Givens matrices. we prove that there exist three classes of full H-matrices whose spectrum are all the given number set. Moreover, we design some algorithms to implement the construction of the above three classes of full H-matrices. Finally, by the algorithms we design, we construct the above-mentioned H-matrices such that their spectrum are all the given number set in the numerical experiment. It is worth noting that the methods we propose only illustrate the existence of the three classes of full H-matrices, which don't mean that all of these three classes of full H-matrices whose spectrum is the given set must have the same structure as the matrices constructed by the algorithms we design.

References

[1] M. T. Chu and G. H. Golub, *Inverse Eigenvalue Problems: Theory, Algorithms, and Applications*, Oxford University Press, USA, 2005.

[2] M. T. Chu, F. Diele and S. Ragni, *On the inverse problem of constructing symmetric pentadiagonal Toeplitz matrices from their three largest eigenvalues*, Inverse Probl., 2005, 21(6), 1879–1894.

[3] T. B. Gan and T. Z. Huang, *Simple criteria for nonsingular H-matrices*, Linear Algebra Appl., 2003. DOI: 10.1016/S0024-3795(03)00646-3.

[4] K. Ghanbari, *A survey on inverse and generalized inverse eigenvalue problems for Jacobi matrices*, Appl. Math. Comput., 2008, 195(2), 355–363.

[5] S. G. Hwang and S. S. Pyo, *The inverse eigenvalue problem for symmetric doubly stochastic matrices*, Linear Algebra Appl., 2004. DOI: 10.1016/S0024-3795(03)00366-5.

[6] Z. Y. Liu, Y. L. Zhang, C. Ferreira and R. Ralha, *On inverse eigenvalue problems for block Toeplitz matrices with Toeplitz blocks*, Appl. Math. Comput., 2010, 216(6), 1819–1830.

[7] J. Z. Liu and Y. Q. Huang, *Some properties on Schur complements of H-matrices and diagonally dominant matrices*, Linear Algebra Appl., 2004. DOI: 10.1016/j.laa.2004.04.012.

[8] A. M. Nazari and F. Sherafat, *On the inverse eigenvalue problem for nonnegative matrices of order two to five*, Linear Algebra Appl., 2012, 436(7), 1771–1790.

[9] J. Peng, X. Y. Hu and L. Zhang, *A kind of inverse eigenvalue problems of Jacobi matrix*, Appl. Math. Comput., 2006, 175(2), 1543–1555.

[10] P. Pango and B. Champagne, *On the efficient use of Givens rotations in SVD-based subspace tracking algorithms*, Signal Processing, 1999, 74(3), 253–277.

[11] W. Rudin, *Principles of mathematical analysis*, McGraw-Hill Education, USA, 1976.

[12] J. J. Rotman, *Advanced Modern Algebra*, Prentice Hall, USA, 2003.

[13] H. Šmigoc, *The inverse eigenvalue problem for nonnegative matrices*, Linear Algebra Appl., 2004. DOI: 10.1016/j.laa.2004.03.036.

[14] X. Q. Wu and E. X. Jiang, *A new algorithm on the inverse eigenvalue problem for double dimensional Jacobi matrices*, Linear Algebra Appl., 2012, 437(7), 1760–1770.

[15] X. Q. Wu, *A divide and conquer algorithm on the double dimensional inverse eigenvalue problem for Jacobi matrices*, Appl. Math. Comput., 2012, 219(8), 3840–3846.

[16] Y. Wei and H. Dai, *An inverse eigenvalue problem for Jacobi matrix*, Appl. Math. Comput., 2015. DOI: 10.1016/j.amc.2014.11.101.

[17] Y. Wei, *A Jacobi matrix inverse eigenvalue problem with mixed date*, Linear Algebra Appl., 2013, 439(10), 2774–2783.

[18] Y. Wei, *Inverse eigenvalue problem of Jacobi matrix with mixed data*, Linear Algebra Appl., 2015. DOI: 10.1016/j.laa.2014.09.031.

[19] R. Wituła, D. Słota and M. Kampik, *Some properties of the full matrices*, Appl. Math. Comput., 2012, 219(3), 1222–1231.