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ASYMPTOTIC AND CONVERGENT EXPANSIONS FOR SOLUTIONS OF THIRD-ORDER LINEAR DIFFERENTIAL EQUATIONS WITH A LARGE PARAMETER

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Abstract In previous papers [6–8,10], we derived convergent and asymptotic expansions of solutions of second order linear differential equations with a large parameter. In those papers we generalized and developed special cases not considered in Olver's theory [Olver, 1974]. In this paper we go one step forward and consider linear differential equations of the third order: $y''' + a\Lambda^2 y' + b\Lambda^3 y = f(x)y' + g(x)y$, with $a, b \in \mathbb{C}$ fixed, f' and g continuous, and Λ a large positive parameter. We propose two different techniques to handle the problem: (i) a generalization of Olver's method and (ii) the transformation of the differential problem into a fixed point problem from which we construct an asymptotic sequence of functions that converges to the unique solution of the problem. Moreover, we show that this second technique may also be applied to nonlinear differential equations with a large parameter. As an application of the theory, we obtain new convergent and asymptotic expansions of the Pearcey integral $P(x, y)$ for large $|x|$.

Keywords Third-order differential equations, asymptotic expansions, Green's functions, Banach's fixed point theorem, Pearcey integral.

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1. Introduction

The most famous asymptotic method for second-order linear differential equations containing a large parameter is, no doubt, Olver's method [14, Chaps. 10, 11, 12]. In [14, Chap. 10], Olver considers a differential equation without singular or transition points, an equation of the form

$$y'' - \Lambda^2 y = f(z)y, \qquad \Lambda \to \infty, \tag{1.1}$$

where Λ is a complex parameter, z is a complex variable and f is an analytic function in a certain region of the complex plane. Olver completes the theory developed in the well-known Liouville-Green approximation, giving a rigorous meaning to the approximation and providing error bounds for the expansions of solutions of (1.1).

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In summary, we have that two independent solutions of (1.1) have the form

$$y_1(z) = e^{\Lambda z}\left[\sum_{k=0}^{n-1}\frac{A_k(z)}{\Lambda^k} + R_{1,n}(z)\right], \quad y_2(z) = e^{-\Lambda z}\left[\sum_{k=0}^{n-1}(-1)^k\frac{A_k(z)}{\Lambda^k} + R_{2,n}(z)\right], \tag{1.2}$$

where $R_{1,n}(z)$, $R_{2,n}(z) = \mathcal{O}(\Lambda^{-n})$ uniformly for z in a certain region in the complex plane. The coefficients A_k are given by the following recurrence: $A_0(z) = 1$ and

$$A_{n+1}(z) = -\frac{1}{2}A'_n(z) + \frac{1}{2}\int f(z)A_n(z)dz, \qquad n = 0, 1, 2, \ldots.$$

The coefficients A_n are analytic at $z = 0$ when $f(z)$ is also analytic there. Olver's important contribution is the proof of the asymptotic character of the two expansions (1.2) and the derivation of error bounds for the remainders $R_{1,n}(z)$ and $R_{2,n}(z)$. Since Olver published his book [14], Olver's theory has been deeply studied, generalized and applied to specific problems. For example, Olver's theory has been recently revisited in [4] and [5], considering the presence of one or two turning points in the equation. In [18] it has been shown that Olver's expansion is valid in a larger domain than the one initially considered in Olver's theory. In [13] Olver's theory has been considered in relation to the eigenvalue problem of the Lamé and Mathieu equations.

In this paper we go one step forward and propose a different generalization of Olver's theory, we consider a third-order linear differential equation without singular or transition points. In any third-order linear differential equation, the term with the second order derivative may be removed by means of a Liouville transformation. Therefore, without loss of generality, we consider

$$y''' + a\Lambda^2 y' + b\Lambda^3 y = f(z)y' + g(z)y, \quad \Lambda \to \infty, \tag{1.3}$$

where a and b are fixed complex parameters with $ab \neq 0$, Λ is a large positive parameter and f' and g are continuous functions. The purpose of this paper is to analyze the asymptotic behavior of the solutions of this equation for large Λ. To this end, in the next section, we use a fixed point theorem and the Green function of an auxiliary initial value problem to derive an asymptotic as well as convergent expansion of any solution of the equation in terms of iterated integrals of $f(z)$ and $g(z)$; this technique is based on our previous investigations [10]. In Section 3, we generalize this technique to nonlinear problems, where we obtain an asymptotic expansion of the solution of an initial value problem for a nonlinear equation. In Section 4, we assume that f and g are infinitely differentiable and use Olver's techniques to derive asymptotic expansions, of Poincaré-type, of three independent solutions of the equation, different from those obtained in Section 2. Section 5 contains an example and some numerical experiments and Section 6 a few remarks and conclusions.

2. A fixed point method

In this section we assume that the functions $f'(z)$ and $g(z)$ are continuous in a star-like domain $\mathcal{D}$ (bounded or unbounded) in the complex plane centered at $z = 0$.

Consider the following initial value problem that selects one of the solutions of (1.3),

$$\begin{cases} y''' + a\Lambda^2 y' + b\Lambda^3 y - f(z)y' - g(z)y = 0 \quad \text{in } \mathcal{D}, \\ y(0) = \bar{y}_0, \quad y'(0) = \bar{y}'_0, \quad y''(0) = \bar{y}''_0, \end{cases} \tag{2.1}$$

with $\bar{y}_0 = \mathcal{O}(1)$, $\bar{y}'_0 = \mathcal{O}(\Lambda)$ and $\bar{y}''_0 = \mathcal{O}(\Lambda^2)$ as $\Lambda \to \infty$. In this section we derive a sequence of functions that converges, uniformly over compacts in $\mathcal{D}$, to the unique solution $y(z)$ of this problem. Moreover, we will show that this sequence may be rearranged in the form of a convergent expansion of $y(z)$ that is also an asymptotic expansion for large Λ. The first step in the analysis is the following auxiliary initial value problem

$$\begin{cases} \phi'''(z) + a\Lambda^2 \phi'(z) + b\Lambda^3 \phi(z) = 0 \ \text{ in } \mathcal{D}, \\ \phi(0) = \bar{y}_0, \quad \phi'(0) = \bar{y}'_0, \quad \phi''(0) = \bar{y}''_0. \end{cases} \tag{2.2}$$

The unique solution of this problem is

$$\begin{aligned} \phi(z) :=& \frac{\bar{y}''_0 - (\alpha_2+\alpha_3)\bar{y}'_0\Lambda + \alpha_2\alpha_3\bar{y}_0\Lambda^2}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\Lambda^2} e^{\alpha_1\Lambda z} + \frac{\bar{y}''_0 - (\alpha_1+\alpha_3)\bar{y}'_0\Lambda + \alpha_1\alpha_3\bar{y}_0\Lambda^2}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)\Lambda^2} e^{\alpha_2\Lambda z} \\ &+ \frac{\bar{y}''_0 - (\alpha_1+\alpha_2)\bar{y}'_0\Lambda + \alpha_1\alpha_2\bar{y}_0\Lambda^2}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)\Lambda^2} e^{\alpha_3\Lambda z}, \end{aligned} \tag{2.3}$$

where α_k, $k = 1, 2, 3$, are the roots of the equation $\alpha^3 + a\alpha + b = 0$. In the following, we assume that these roots are different.

Then, after the change of unknown $y(z) \to w(z) = y(z) - \phi(z)$, and using (2.2), we find that problem (2.1) for $y(z)$ is transformed into the following problem for $w(z)$:

$$\begin{cases} w'''(z) + a\Lambda^2 w'(z) + b\Lambda^3 w(z) = F(z, w) := f(z)[w'(z) + \phi'(z)] + g(z)[w(z) + \phi(z)] \\ w(0) = w'(0) = w''(0) = 0, \end{cases} \tag{2.4}$$

with $z \in \mathcal{D}$. Take any point $Z \in \mathcal{D}$ and the closed segment $\mathcal{L} := [0, Z] \subset \mathcal{D}$ that joins the origin $z = 0$ with the point $z = Z$ (see Fig. 1). For convenience, we restrict the differential equation in (2.1) (and hence (2.4)), to the segment $\mathcal{L}$.

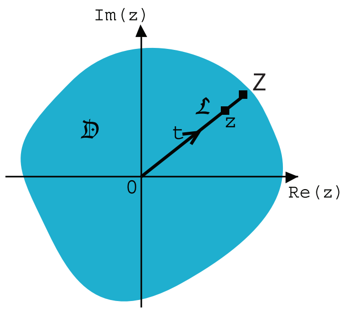

Figure 1. Domain $\mathcal{D}$ and integration path $\mathcal{L}$ associated to the problem (2.1).

At his point, we seek solutions of the equation $\mathbf{L}[w] := w''' + a\Lambda^2 w' + b\Lambda^3 w - F(z, w)$ in the Banach space $\mathcal{B} := \{w \in \mathcal{C}(\mathcal{L}),\ w(0) = 0\}$, equipped with the supremum norm

$$||w||_\infty := \sup_{z\in\mathcal{L}} |w(z)|.$$

We write the equation $\mathbf{L}[w]=0$ in the form $\mathbf{L}[w]=\mathbf{M}[w]-F(z,w)$, with $\mathbf{M}[w]:=w'''+a\Lambda^2w'+b\Lambda^3w$. Then we solve the equation $\mathbf{L}[w]=0$ for w using the Green function $G(z,t)$ of the operator $\mathbf{M}$ with the appropriate initial conditions [17]. $G(z,t)$ is the unique solution of the problem

$$\begin{cases} G_{zzz}+a\Lambda^2G_z+b\Lambda^3G=\delta(z-t) \ \text{ in } \mathcal{L}, \\ G(0,t)=G_z(0,t)=G_{zz}(0,t)=0, \ \ t\in\mathcal{L}, \end{cases}$$

and it is given by

$$G(z,t)=\frac{1}{\Lambda^2}K(z,t),$$

$$K(z,t):=\left[\frac{e^{\alpha_1\Lambda(z-t)}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}+\frac{e^{\alpha_2\Lambda(z-t)}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)}+\frac{e^{\alpha_3\Lambda(z-t)}}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}\right] \times \chi_{[0,z]}(t), \tag{2.5}$$

where $\chi_{[0,z]}(t)$ is the characteristic function of the interval $[0,z]$. Then, any solution $w(z)$ of (2.4) is a solution of the Volterra integral equation $w(z)=[\mathbf{T}w](z)$, where the integral operator $\mathbf{T}$ is defined by

$$[\mathbf{T}w](z):=\frac{1}{\Lambda^2}\int_0^z K(z,t)\left\{f(t)[w'(t)+\phi'(t)]+g(t)[w(t)+\phi(t)]\right\}dt.$$

Integrating by parts and using that $K(z,z)=w(0)=0$, we obtain that

$$[\mathbf{T}w](z):=\frac{1}{\Lambda^2}\int_0^z H(z,t)[w(t)+\phi(t)]dt,$$

with

$$H(z,t):=K(z,t)[g(t)-f'(t)]-K_t(z,t)f(t). \tag{2.6}$$

We define α^* as the root $\{\alpha_1,\, \alpha_2 \text{ or } \alpha_3\}$ for which $\Re[\alpha Z]$ is maximal. Then, we define

$$\tilde{w}(z)=e^{-\alpha^*\Lambda z}w(z), \qquad \tilde{\phi}(z)=e^{-\alpha^*\Lambda z}\phi(z),$$

and we find that, for any solution $w(z)=e^{\alpha^*\Lambda z}\tilde{w}(z)$ of (2.4), $\tilde{w}(z)$ is a solution of the integral equation

$$\tilde{w}(z)=[\tilde{\mathbf{T}}\tilde{w}](z),$$

where we have defined the operator

$$[\tilde{\mathbf{T}}\tilde{w}](z):=\frac{1}{\Lambda^2}\int_0^z \tilde{H}(z,t)[\tilde{w}(t)+\tilde{\phi}(t)]dt, \quad \tilde{H}(z,t):=e^{\alpha^*\Lambda(t-z)}H(z,t).$$

For any complex z in $\mathcal{L}$ and $\Lambda \geq \Lambda_0 > 0$, we have that

$$\begin{aligned}
|\tilde{H}(z,t)| \leq & \left|e^{\alpha^*\Lambda(t-z)}K(z,t)\right| |g(t)+f'(t)| + \left|e^{\alpha^*\Lambda(t-z)}K_t(z,t)\right| |f(t)| \\
\leq & \left|e^{\alpha^*\Lambda(t-z)}K(z,t)\right| (||g||_\infty + ||f'||_\infty) + \left|e^{\alpha^*\Lambda(t-z)}K_t(z,t)\right| ||f||_\infty \\
\leq & \left[\frac{e^{-\Re[(\alpha^*-\alpha_1)\Lambda(z-t)]}}{|\alpha_1-\alpha_2||\alpha_1-\alpha_3|} + \frac{e^{-\Re[(\alpha^*-\alpha_2)\Lambda(z-t)]}}{|\alpha_2-\alpha_1||\alpha_2-\alpha_3|} + \frac{e^{-\Re[(\alpha^*-\alpha_3)\Lambda(z-t)]}}{|\alpha_3-\alpha_1||\alpha_3-\alpha_2|}\right] \\
& \times (||g||_\infty + ||f'||_\infty) \\
& + \Lambda\left[\frac{|\alpha_1|e^{-\Re[(\alpha^*-\alpha_1)\Lambda(z-t)]}}{|\alpha_1-\alpha_2||\alpha_1-\alpha_3|} + \frac{|\alpha_2|e^{-\Re[(\alpha^*-\alpha_2)\Lambda(z-t)]}}{|\alpha_2-\alpha_1||\alpha_2-\alpha_3|}\right. \\
& \left. + \frac{|\alpha_3|e^{-\Re[(\alpha^*-\alpha_3)\Lambda(z-t)]}}{|\alpha_3-\alpha_1||\alpha_3-\alpha_2|}\right] ||f||_\infty \\
\leq & \left[\frac{1}{|\alpha_1-\alpha_2||\alpha_1-\alpha_3|} + \frac{1}{|\alpha_2-\alpha_1||\alpha_2-\alpha_3|} + \frac{1}{|\alpha_3-\alpha_1||\alpha_3-\alpha_2|}\right] \\
& \times (||g||_\infty + ||f'||_\infty) \\
& + \left[\frac{|\alpha_1|}{|\alpha_1-\alpha_2||\alpha_1-\alpha_3|} + \frac{|\alpha_2|}{|\alpha_2-\alpha_1||\alpha_2-\alpha_3|} + \frac{|\alpha_3|}{|\alpha_3-\alpha_1||\alpha_3-\alpha_2|}\right] \\
& \times ||f||_\infty \Lambda \\
\leq & C_1 + C_2\Lambda \leq C\Lambda,
\end{aligned}$$

with the evident values of C_1 and C_2. C is any constant independent of Λ (it may depend on Λ_0) such that $C_1 + C_2\Lambda \leq C\Lambda$ for $\Lambda \geq \Lambda_0$.

Thus, $|\tilde{H}(z,t)| \leq C\Lambda$, uniformly for $t, z \in [0, Z]$, for a certain positive constant C independent of Λ, t and z (it depends on f and g restricted to $\mathcal{L}$).

From the Banach fixed point theorem [1, pp. 26, Theorem 3.1], it is well known that, if any power of the operator $\tilde{\mathbf{T}}$ is contractive in $\mathcal{B}$, then the equation $\tilde{w}(z) = [\tilde{\mathbf{T}}\tilde{w}](z)$ has a unique solution $\tilde{w}(z)$ (fixed point of $\tilde{\mathbf{T}}$). Equivalently, the equation $w(z) = [\mathbf{T}w](z)$ has a unique solution $w(z)$ (fixed point of $\mathbf{T}$) and the sequence $w_{n+1} = [\mathbf{T}w_n]$, $w_0 = 0$, converges to that solution $w(z)$. We show below the contractive character of the operator $\tilde{\mathbf{T}}$. From its definition we have that, for any couple u, $v \in \mathcal{B}$,

$$|[\tilde{\mathbf{T}}u](z) - [\tilde{\mathbf{T}}v](z)| \leq \frac{C}{\Lambda}\int_0^z |u(t)-v(t)||dt| \leq C\frac{|z|}{\Lambda}||u-v||_\infty.$$

We also have

$$|[\tilde{\mathbf{T}}^2u](z) - [\tilde{\mathbf{T}}^2v](z)| \leq \frac{C}{\Lambda}\int_0^z |[\tilde{\mathbf{T}}u](t) - [\tilde{\mathbf{T}}v](t)||dt| \leq \frac{C^2}{2}\left(\frac{|z|}{\Lambda}\right)^2 ||u-v||_\infty$$

and

$$|[\tilde{\mathbf{T}}^3u](z) - [\tilde{\mathbf{T}}^3v](z)| \leq \frac{C}{\Lambda}\int_0^z |[\tilde{\mathbf{T}}^2u](t) - [\tilde{\mathbf{T}}^2v](t)||dt| \leq \frac{C^3}{3!}\left(\frac{|z|}{\Lambda}\right)^3 ||u-v||_\infty.$$

It is straightforward to prove, by means of induction over n that, for $n = 1, 2, 3, \ldots$,

$$|[\tilde{\mathbf{T}}^nu](z) - [\tilde{\mathbf{T}}^nv](z)| \leq \frac{C^n}{n!}\left(\frac{|z|}{\Lambda}\right)^n ||u-v||_\infty. \tag{2.7}$$

This means that, for bounded z, the operator $\tilde{\mathbf{T}}^n$ is contractive for large enough n, and then, we have that the sequence $w_{n+1} = [\mathbf{T}w_n]$, $n = 0, 1, 2, \ldots$, $w_0 = 0$, converges, for any $z \in \mathcal{L}$ bounded, to the unique solution $w(z)$ of problem (2.4). Or equivalently, the sequence $y_n := w_n + \phi$, that is,

$$y_{n+1}(z) = \phi(z) + \frac{z}{\Lambda^2}\int_0^1 H(z, zt) y_n(zt) dt, \qquad y_0(z) = \phi(z), \tag{2.8}$$

converges, for $z \in \mathcal{L}$ bounded, to the unique solution $y(z)$ of (2.1).

Let's define the remainder of the approximation by $R_n(z) := y(z) - y_n(z)$. Setting $v(z) = w(z)$ and $u(z) = w_0(z) = 0$ in (2.7) and using that $[\mathbf{T}^n w] = w$ and $[\mathbf{T}^n w_0] = w_n$ we find

$$|w(z) - w_n(z)| \leq \frac{C^n}{n!}\left(\frac{|z|}{\Lambda}\right)^n ||w||_\infty.$$

Using that $y(z) = w(z) + \phi(z)$ we get that the remainder $R_n(z)$ is bounded in the form

$$|R_n(z)| \leq \frac{C^n}{n!}\left(\frac{|z|}{\Lambda}\right)^n ||y - \phi||_\infty.$$

Moreover, we have that, for problem (2.1),

$$y_{n+1}(z) - y_n(z) = \frac{z}{\Lambda^2}\int_0^1 H(z, zt)[y_n(zt) - y_{n-1}(zt)]dt$$

and then

$$||y_{n+1} - y_n||_\infty \leq C\frac{|z|}{\Lambda^2} ||y_n - y_{n-1}||_\infty.$$

This means that the expansion

$$y(z) = \phi(z) + \sum_{k=0}^{n-1}[y_{k+1}(z) - y_k(z)] + R_n(z)$$

is an asymptotic expansion for large Λ and bounded $z \in \mathcal{L}$.

We see from (2.8) that the sequence $y_n(z)$ is a sequence of analytic functions in $\mathcal{D}$ that converges uniformly in any compact contained in $\mathcal{D}$, that is, the unique solution $y(z)$ of problem (2.1) is analytic in $\mathcal{D}$.

3. The nonlinear case

The technique used in the previous section may be generalized to nonlinear problems of the form

$$y''' + a\Lambda^2 y' + b\Lambda^3 y = F(z, y, y'), \quad \Lambda \to \infty, \tag{3.1}$$

where the function $F(z, y, y')$ is continuous for $(z, y, y') \in \mathcal{D} \times \mathbb{C} \times \mathbb{C}$ and satisfies the following Lipschitz condition in its second and third variables

$$|F(z, y, \bar{y}) - F(z, v, \bar{v})| \leq L|y - v| + J|\bar{y} - \bar{v}|, \quad \forall\, y, v, \bar{y}, \bar{v} \in \mathbb{C} \text{ and } z \in \mathcal{D}, \tag{3.2}$$

with L and J positive constants independent of $z, y, v, \bar{y}, \bar{v}$. A well-posed initial value problem for the differential equation (3.1) is

$$\begin{cases} y'''(z) + a\Lambda^2 y'(z) + b\Lambda^3 y(z) = F(z, y(z), y'(z)) \text{ in } \mathcal{D}, \\ y(0) = \bar{y}_0, \quad y'(0) = \bar{y}_0', \quad y''(0) = \bar{y}_0'', \end{cases} \tag{3.3}$$

where $\bar{y}_0 = \mathcal{O}(1)$, $\bar{y}_0' = \mathcal{O}(\Lambda)$ and $\bar{y}_0'' = \mathcal{O}(\Lambda^2)$ are complex numbers.

An appropriate modification of the analysis of the previous sections provides, for problem (3.3), the same conclusions that we derived for problem (2.1) and that we summarize in Theorem 1 below. We need to introduce the following norm in the Banach space $\mathcal{B}_1 := \{w \in \mathcal{C}^1(\mathcal{L}),\, w(0) = w'(0) = 0\}$,

$$||w||_1 := \sup_{z \in \mathcal{L}} \{(L + J|\alpha^*|\Lambda)|w(z)| + J|w'(z)|\}. \tag{3.4}$$

Theorem 3.1. *Let $F : \mathcal{D} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be continuous and satisfy* (3.2). *Then, problem* (3.3) *has a unique solution $y(z)$. Moreover:*

(i) For $n = 0, 1, 2, \ldots$, the sequence

$$y_{n+1}(z) = \phi(z) + \frac{z}{\Lambda^2} \int_0^1 K(z, zt) F\left(zt, y_n(zt), y_n'(zt)\right) dt, \quad y_0(z) = \phi(z), \tag{3.5}$$

with $\phi(z)$ defined in (2.3) *and $K(z,t)$ in* (2.5), *converges, for $z \in \mathcal{L}$ bounded, to the unique solution $y(z)$ of poblem* (3.3).

(ii) The remainder $R_n(z) := y(z) - y_n(z)$ is bounded by

$$|R_n(z)| \le \frac{C^n}{n!} \left(\frac{|z|}{\Lambda}\right)^n ||y - \phi||_1$$

and the expansion

$$y(z) = \phi + \sum_{k=0}^{n-1} [y_{k+1}(z) - y_k(z)] + R_n(z)$$

is an asymptotic expansion for large Λ and bounded $z \in \mathcal{L}$.

Proof. After the change of unknown $y(z) \to w(z) := y(z) - \phi(z)$, problem (3.3) reads

$$\begin{cases} w'''(z) + a\Lambda^2 w'(z) + b\Lambda^3 w(z) = \bar{F}(z, w, w') := F(z, w(z) + \phi(z), w'(z) + \phi'(z)), \\ w(0) = w'(0) = w''(0) = 0, \end{cases} \tag{3.6}$$

with $z \in \mathcal{D}$. The solution of this problem satisfies the Volterra integral equation of the second kind $w(z) = [\mathbf{T}w](z)$ where now, the operator $\mathbf{T}$ is nonlinear, and defined by

$$[\mathbf{T}w](z) := \frac{1}{\Lambda^2} \int_0^z K(z,t) F\left(t, w(t) + \phi(t), w'(t) + \phi'(t)\right) dt,$$

with $K(z,t)$ given in (2.5).

As in the linear case, we define α^* as the root $\{\alpha_1,\ \alpha_2$ or $\alpha_3\}$ for which $\Re[\alpha Z]$ is maximal, $\tilde{w}(z)=e^{-\alpha^*\Lambda z}w(z)$, $\tilde{\phi}(z)=e^{-\alpha^*\Lambda z}\phi(z)$ and then, we find that, for any solution $w(z)=e^{\alpha^*\Lambda z}\tilde{w}(z)$ of (3.6), $\tilde{w}(z)$ is a solution of the integral equation

$$\tilde{w}(z)=[\tilde{\mathbf{T}}\tilde{w}](z),$$

where we have defined the operator

$$[\tilde{\mathbf{T}}\tilde{w}](z):=\frac{1}{\Lambda^2}\int_0^z \tilde{K}(z,t)\tilde{F}\left(t,\tilde{w}(t),\tilde{w}'(t)\right)dt,\quad \tilde{K}(z,t):=e^{\alpha^*\Lambda(t-z)}K(z,t),$$

and

$$\tilde{F}\left(t,\tilde{w}(t),\tilde{w}'(t)\right)=e^{-\alpha^*\Lambda t}F\left(t,e^{\alpha^*\Lambda t}(\tilde{w}(t)+\tilde{\phi}(t)),e^{\alpha^*\Lambda t}(\alpha^*\Lambda\tilde{w}(t)+\tilde{w}'(t)+\tilde{\phi}'(t))\right).$$

For any complex z in $\mathcal{L}$, we have that

$$\begin{aligned}|\tilde{K}(z,t)|\leq&\frac{e^{-\Re[(\alpha^*-\alpha_1)\Lambda(z-t)]}}{|\alpha_1-\alpha_2||\alpha_1-\alpha_3|}+\frac{e^{-\Re[(\alpha^*-\alpha_2)\Lambda(z-t)]}}{|\alpha_2-\alpha_1||\alpha_2-\alpha_3|}+\frac{e^{-\Re[(\alpha^*-\alpha_3)\Lambda(z-t)]}}{|\alpha_3-\alpha_1||\alpha_3-\alpha_2|}\\ \leq&\frac{1}{|\alpha_1-\alpha_2||\alpha_1-\alpha_3|}+\frac{1}{|\alpha_2-\alpha_1||\alpha_2-\alpha_3|}+\frac{1}{|\alpha_3-\alpha_1||\alpha_3-\alpha_2|}\equiv C_1,\end{aligned}$$

with C_1 independent of Λ, z and t. We show below the contractive character of the operator $\tilde{\mathbf{T}}$. From its definition we have that, for any couple $\tilde{u}$, $\tilde{v}\in\mathcal{B}_1$,

$$\begin{aligned}|[\tilde{\mathbf{T}}\tilde{u}](z)-[\tilde{\mathbf{T}}\tilde{v}](z)|\leq&\frac{1}{\Lambda^2}\left|\int_0^z|\tilde{K}(z,t)||\tilde{F}(t,\tilde{u}(t),\tilde{u}'(t))-\tilde{F}(t,\tilde{v}(t),\tilde{v}'(t))||dt\right|\\ \leq&\frac{C_1}{\Lambda^2}\int_0^z\left((L+J|\alpha^*|\Lambda)|\tilde{u}(t)-\tilde{v}(t)|+J|\tilde{u}'(t)-\tilde{v}'(t)|\right)|dt\\ \leq&C_1\frac{|z|}{\Lambda^2}||\tilde{u}-\tilde{v}||_1.\end{aligned}\tag{3.7}$$

By means of similar arguments we also have that

$$\left|\frac{d}{dz}\left[[\tilde{\mathbf{T}}\tilde{u}](z)-[\tilde{\mathbf{T}}\tilde{v}](z)\right]\right|\leq\frac{1}{\Lambda^2}\int_0^z|\tilde{K}_z(z,t)||\tilde{F}(t,\tilde{u}(t),\tilde{u}'(t))-\tilde{F}(t,\tilde{v}(t),\tilde{v}'(t))||dt|.$$

For any complex z in $\mathcal{L}$, we find that

$$\begin{aligned}|\tilde{K}_z(z,t)|\leq&|\alpha^*|\Lambda|\tilde{K}(z,t)|+\Lambda\left[\frac{|\alpha_1|}{|\alpha_1-\alpha_2||\alpha_1-\alpha_3|}+\frac{|\alpha_2|}{|\alpha_2-\alpha_1||\alpha_2-\alpha_3|}\right.\\ &\left.+\frac{|\alpha_3|}{|\alpha_3-\alpha_1||\alpha_3-\alpha_2|}\right]\\ \leq&\Lambda\left[\frac{|\alpha^*|+|\alpha_1|}{|\alpha_1-\alpha_2||\alpha_1-\alpha_3|}+\frac{|\alpha^*|+|\alpha_2|}{|\alpha_2-\alpha_1||\alpha_2-\alpha_3|}+\frac{|\alpha^*|+|\alpha_3|}{|\alpha_3-\alpha_1||\alpha_3-\alpha_2|}\right]\equiv C_2\Lambda,\end{aligned}$$

with C_2 independent of Λ, z and t. Thus

$$\begin{aligned}\left|\frac{d}{dz}\left[[\tilde{\mathbf{T}}\tilde{u}](z)-[\tilde{\mathbf{T}}\tilde{v}](z)\right]\right|\leq&\frac{C_2}{\Lambda}\int_0^z\left((L+J|\alpha^*|\Lambda)|\tilde{u}(t)-\tilde{v}(t)|+J|\tilde{u}'(t)-\tilde{v}'(t)|\right)|dt\\ \leq&C_2\frac{|z|}{\Lambda}||\tilde{u}-\tilde{v}||_1.\end{aligned}\tag{3.8}$$

Then, from (3.4), (3.7) and (3.8),

$$||\tilde{\mathbf{T}}\tilde{u}-\tilde{\mathbf{T}}\tilde{v}||_1 \leq \frac{|z|}{\Lambda}\left[\frac{C_1 L}{\Lambda}+J(C_1|\alpha^*|+C_2)\right]||\tilde{u}-\tilde{v}||_1 \leq C\frac{|z|}{\Lambda}||\tilde{u}-\tilde{v}||_1,$$

where C is a positive constant independent of Λ for $\Lambda \geq \Lambda_0$. From here, the proof of the contractive character of the operator $\tilde{\mathbf{T}}$ and the convergence of the sequence (3.5) to the unique solution $y(z)$ of (3.3) is similar to the linear case. In particular we get

$$||[\tilde{\mathbf{T}}^n\tilde{u}](z)-[\tilde{\mathbf{T}}^n\tilde{v}](z)||_1 \leq \frac{C^n}{n!}\left(\frac{|z|}{\Lambda}\right)^n ||\tilde{u}-\tilde{v}||_1. \tag{3.9}$$

Setting $\tilde{v}(z)=w(z)$ and $\tilde{u}(z)=w_0(z)=0$ in (3.9) and using that $[\tilde{\mathbf{T}}^n w]=w$ and $[\tilde{\mathbf{T}}^n w_0]=w_n$ we find

$$||w(z)-w_n(z)||_1 \leq \frac{C^n}{n!}\left(\frac{|z|}{\Lambda}\right)^n ||w||_1.$$

Using that $y(z)=w(z)+\phi(z)$ we find that the remainder $R_n(z):=y(z)-y_n(z)$ is bounded in the form

$$|R_n(z)| \leq \frac{C^n}{n!}\left(\frac{|z|}{\Lambda}\right)^n ||y-\phi||_1.$$

Moreover, we have that, for problem (3.3),

$$y_{n+1}(z)-y_n(z)=\frac{z}{\Lambda^2}\int_0^1 K(z,zt)[F(zt,y_n(zt),y_n'(zt))-F(zt,y_{n-1}(zt),y_{n-1}'(zt))]dt,$$

$$y_{n+1}'(z)-y_n'(z)=\frac{z}{\Lambda^2}\int_0^1 K_z(z,zt)[F(zt,y_n(zt),y_n'(zt))-F(zt,y_{n-1}(zt),y_{n-1}'(zt))]dt$$

and then

$$||y_{n+1}-y_n||_1 \leq C\frac{|z|}{\Lambda}||y_n-y_{n-1}||_1.$$

This means that the expansion

$$y(z)=\phi(z)+\sum_{k=0}^{n-1}[y_{k+1}(z)-y_k(z)]+R_n(z)$$

is an asymptotic expansion for large Λ and bounded $z \in \mathcal{L}$. □

Example 3.1. Consider the initial value problem

$$\begin{cases} y'''+\Lambda^2 y' = -y^2, \quad z \in [0,Z], \\ y(0)=1, \quad y'(0)=y''(0)=0. \end{cases} \tag{3.10}$$

This problem is of the form considered in Theorem 1 with $a=1$, $b=0$, $F(z,y,y')=-y^2$, and then $|F(z,y,y')-F(z,v,v')| \leq |y+v||y-v|$. This function is not Lipschitz continuous $\forall\, y,v \in \mathbb{C}$, but it is Lipschitz continuous for $y,v \in D \subset \mathbb{C}$, D compact. When all the terms of the sequence $y_n(z)$ defined by (3.5) are uniformly bounded in $z \in [0, Z]$ and $n \in \mathbb{N}$, then the Lipschitz condition holds for $y_n(z)$, $n = 0, 1, 2, \ldots$, and Theorem 1 applies. From Theorem 1 we have that, for $n = 0, 1, 2, \ldots$,

$$\begin{cases} y_0(z) = \phi(z) = 1, \\ y_{n+1}(z) = \phi(z) - \dfrac{2z}{\Lambda^2} \displaystyle\int_0^1 \sin^2\left(\frac{\Lambda z(1-t)}{2}\right) y_n^2(zt)dt. \end{cases}$$

When all the functions $y_n(z)$ are uniformly bounded in $z \in [0, Z]$, this sequence converges uniformly and absolutely to the unique solution of (3.10). Fig. 2 illustrates the approximation of the unique solution of (3.10) supplied by this approximation.

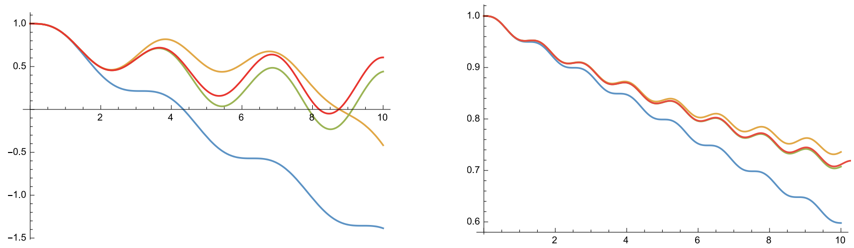

Figure 2. These graphs show the exact solution $y(x)$ (red) for $x \in [0, 10]$, and the approximations $y_1(x)$ (blue), $y_2(x)$ (gold) and $y_3(x)$ (green) for $\Lambda = 5$ (left picture) and $\Lambda = 10$ (right picture).

4. Olver's method for equation (1.3)

In this section we assume that the functions $f(z)$ and $g(z)$ are infinitely differentiable in the star-like domain $\mathcal{D}$. Consider three (at this moment unknown) independent solutions $Y_1(z)$, $Y_2(z)$ and $Y_3(z)$ of (1.3). We propose the following representations in the form of formal asymptotic expansions for large Λ,

$$Y_j(z) = Y_{j,n}(z) + R_{j,n}(z), \quad j = 1, 2, 3, \tag{4.1}$$

with

$$Y_{j,n}(z) := e^{\alpha_j \Lambda z} \sum_{k=0}^{n-1} \frac{A_{j,k}(z)}{\Lambda^k}, \tag{4.2}$$

and the obvious definition of $R_{j,n}(z)$. When we introduce (4.1) and (4.2) in the equation $y''' + a\Lambda^2 y' + b\Lambda^3 y = fy' + gy$ we find that the three functions, $Y_j(z)$, $j = 1, 2, 3$, formally satisfy the differential equation, term-wise in Λ^k if, for $n = 0, 1, 2, \ldots$,

$$\begin{aligned} A_{j,n+2}(z) = &\frac{1}{3\alpha_j^2 + a} \left\{ \int_0^z [f(t)(\alpha_j A_{j,n+1}(t) + A'_{j,n}(t)) + g(t)A_{j,n}(t)]dt \right. \\ &\left. -3\alpha_j A'_{j,n+1}(z) - A''_{j,n}(z) \right\}, \end{aligned} \tag{4.3}$$

$$R'''_{j,n}(z) + a\Lambda^2 R'_{j,n}(z) + b\Lambda^3 R_{j,n}(z) = f(z)R'_{j,n}(z) + g(z)R_{j,n}(z) + \Psi_{j,n}(z),$$

with

$$\Psi_{j,n}(z) := \frac{e^{\alpha_j \Lambda z}}{\Lambda^{n-1}} \left[(3\alpha_j^2 + a) \left(A'_{n+1}(z) + \frac{A'_n(z)}{\Lambda} \right) + 3\alpha_j A''_n(z) - \alpha_j f(z) A_n(z) \right]$$

and

$$3\alpha_j^2 A'_{j,0}(z) = 0, \quad \alpha_j(3\alpha_j A'_{j,1}(z) + 3A''_{j,0}(z) - f(z)A_{j,0}(z)) = 0.$$

Then, we may fix, for example, $A_{j,0}(z) = 0$ and $A_{j,1}(z) = 1$.

The solutions $Y_j(z)$ are regular at $z = 0$. Therefore, without loss of generality, we may set $R_{j,n}(0) = R'_{j,n}(0) = R''_{j,n}(0) = 0$. Then, the remainder $R_n(z)$ is a solution of the initial value problem

$$\begin{cases} R'''_n(z) + a\Lambda^2 R'_{j,n}(z) + b\Lambda^3 R_{j,n}(z) = f(z)R'_{j,n}(z) + g(z)R_{j,n}(z) + \Psi_{j,n}(z) \ \text{ in } \mathcal{D}, \\ R_{j,n}(0) = 0, \quad R'_{j,n}(0) = 0, \quad R''_{j,n}(0) = 0. \end{cases} \tag{4.4}$$

This problem for $R_{j,n}(z)$ is similar to problem (2.4) for $w(z)$, with the exception that the term $f(z)\phi'(z)+g(z)\phi(z)$ on the right-hand side of the differential equation is replaced by the term $\Psi_{j,n}(z)$. Therefore, proceeding as in Section 2 we find that $R_{j,n}(z)$ is a solution of the Volterra integral equation

$$R_{j,n}(z) = \frac{1}{\Lambda^2}\int_0^z H(z,t)R_{j,n}(t)dt + \frac{1}{\Lambda^2}\int_0^z K(z,t)\Psi_{j,n}(t)dt,$$

with $K(z,t)$ and $H(z,t)$ defined in (2.5) and (2.6) respectively. Then, we define

$$\tilde{R}_{j,n}(z) = e^{-\alpha^*\Lambda z} R_{j,n}(z), \quad \tilde{\Psi}_{j,n}(z) = e^{-\alpha^*\Lambda z}\Psi_{j,n}(z),$$

and find that for any solution $R_{j,n}(z) = e^{\alpha^*\Lambda z}\tilde{R}_{j,n}(z)$ of (4.4), $\tilde{R}_{j,n}(z)$ is a solution of the integral equation

$$\tilde{R}_{j,n}(z) = [\tilde{\mathbf{T}}\tilde{R}_{j,n}](z),$$

where we have defined the operator

$$[\tilde{\mathbf{T}}\tilde{R}_{j,n}](z) := \frac{1}{\Lambda^2}\int_0^z \tilde{H}(z,t)\tilde{R}_{j,n}(t)dt + \frac{1}{\Lambda^2}\int_0^z \tilde{K}(z,t)\tilde{\Psi}_{j,n}(t)dt,$$

where $\tilde{H}(z,t) := e^{\alpha^*\Lambda(t-z)}H(z,t)$ and $\tilde{K}(z,t) := e^{\alpha^*\Lambda(t-z)}K(z,t)$.

Using that $|\tilde{H}(z,t)| \le C\Lambda$ and $|\tilde{K}(z,t)| \le C$ for $t, z \in \mathcal{L}$, with C a positive constant independent of z, t and $\Lambda \ge \Lambda_0$, we derive the bound

$$|\tilde{R}_{j,n}(z)| \le \frac{C}{\Lambda}\int_0^z |\tilde{R}_{j,n}(t)||dt| + \frac{C}{\Lambda^2}\int_0^z \left|\tilde{\Psi}_{j,n}(t)\right| |dt|.$$

Applying Gronwall's lemma [3] we obtain

$$|\tilde{R}_{j,n}(z)| \le \frac{C}{\Lambda^2}\left[1 + \frac{C|z|}{\Lambda}e^{\frac{C|z|}{\Lambda}}\right]\int_0^z \left|\tilde{\Psi}_{j,n}(t)\right| |dt|.$$

When $A''_n(t)$ and $f(t)$ are integrable in $\mathcal{L}$ (this is granted when $\mathcal{L}$ is bounded), we also have the bounds

$$\begin{aligned} |\tilde{R}_{j,n}(z)| \le & \frac{C}{\Lambda^{n+1}}\left[1 + \frac{C|z|}{\Lambda}e^{\frac{C|z|}{\Lambda}}\right]\int_0^z \left|(3\alpha_j^2 + a)\left(A'_{n+1}(t) + \frac{A'_n(t)}{\Lambda}\right)\right. \\ & \left. + 3\alpha_j A''_n(t) - \alpha_j f(z)A_n(t)\right| |dt| \\ \le & \frac{C}{\Lambda^{n+1}}\left[1 + \frac{C|z|}{\Lambda}e^{\frac{C|z|}{\Lambda}}\right]\left[|3\alpha_j^2 + a|\left(||A'_{n+1}||_{\mathcal{L}} + \frac{||A'_n||_{\mathcal{L}}}{|\Lambda|}\right)\right. \\ & \left. + 3|\alpha_j|||A''_n||_{\mathcal{L}} + |\alpha_j|||fA_n||_{\mathcal{L}}\right] \end{aligned} \tag{4.5}$$

with

$$||F||_{\mathcal{L}} := \int_{\mathcal{L}} |F(t)||dt|.$$

This bound shows the asymptotic character of the expansions (4.1).

Remark 4.1. The unique solution $y(z)$ of problem (2.1) is approximated by $y_n(z) := c_{1,n}Y_{1,n}(z)+c_{2,n}Y_{2,n}(z)+c_{3,n}Y_{3,n}(z)$, where the coefficients $c_{j,n}$, $j = 1, 2, 3$, must be approximated at any order n of the approximation by using the conditions $y(0) = \bar{y}_0$, $y'(0) = \bar{y}'_0$ and $y''(0) = \bar{y}''_0$.

Remark 4.2. We see from (4.3) that the coefficients $A_{j,n}(z)$, $j = 1, 2, 3$, $n = 0, 1, 2, \ldots$, are infinitely differentiable in $\mathcal{D}$. Moreover, when $f'(z)$ and $g(z)$ are analytic in $\mathcal{D}$, the coefficients $A_{j,n}(z)$, $j = 1, 2, 3$, $n = 0, 1, 2, \ldots$, are analytic in $\mathcal{D}$ too.

5. Example and numerical experiments

Consider the initial value problem

$$\begin{cases} u'''(y) - \dfrac{x}{2}u'(y) - \dfrac{y}{4}u(y) = 0, \quad y \in [0, Y] \subset \mathcal{D}, \\ u(0) = P(x, 0), \quad u'(0) = 0, \quad u''(0) = P_{yy}(x, 0), \end{cases} \tag{5.1}$$

where $P(x, y)$ is the Pearcey integral [2]

$$P(x, y) := \int_0^\infty e^{-t^4-xt^2}\cos(yt)dt, \tag{5.2}$$

that is the unique solution of (5.1). It has recently been shown that the Pearcey integral and the Pearcey kernel have applications in probability (see [9] and references there in for details).

A convergent expansion of $P(x, y)$ is given by [16]

$$P(x, y) = \sum_{n=0}^{\infty} \frac{(-y^2)^n}{(2n)!}P_n(x), \quad (x, y) \in \mathbb{C}^2, \tag{5.3}$$

with

$$P_n(x) := \begin{cases} \dfrac{1}{2^{n+3/2}}\Gamma\left(n + \dfrac{1}{2}\right) U\left(\dfrac{n}{2} + \dfrac{1}{4}, \dfrac{1}{2}; \dfrac{x^2}{4}\right), & \text{if } \Re x \geq 0, \\ \dfrac{1}{4}\Gamma\left(\dfrac{n}{2} + \dfrac{1}{4}\right) M\left(\dfrac{n}{2} + \dfrac{1}{4}, \dfrac{1}{2}; \dfrac{x^2}{4}\right) - \dfrac{x}{4}\Gamma\left(\dfrac{n}{2} + \dfrac{3}{4}\right) M\left(\dfrac{n}{2} + \dfrac{3}{4}, \dfrac{3}{2}; \dfrac{x^2}{4}\right), & \text{if } \Re x < 0, \end{cases} \tag{5.4}$$

where $U(a, b; z)$ and $M(a, b; z)$ are confluent hypergeometric functions. From the asymptotic formula [12, eq.13.7.3], it follows that (5.3) is also an asymptotic expansion of $P(x, y)$ for large $|x|$ if $\Re x \geq 0$. It can be easily checked that $P(x, 0) = P_0(x)$ and $P_{yy}(x, 0) = -P_1(x)$.

A complete asymptotic expansion of the Pearcey integral (5.2) for large $|x|$, valid in a certain sector of the complex $x-$plane, can be derived by using Watson's lemma.

Moreover, a complete asymptotic expansion for large $|x|$, valid in a wider sector, was derived in [15] from a contour integral representation of the Pearcey integral. Using the theory developed in Section 2, we may derive an expansion of the Pearcey integral that is, not only asymptotic for large $|x|$, but also convergent: problem (5.1) is of the form (2.1) with $a=-e^{i\arg(x)}$, $b=0$, $\Lambda^2=|x|/2$, $f(y)=0$ and $g(y)=y/4$. The iterative method introduced in Section 2 provides a convergent as well as an asymptotic (for large $|x|$) sequence of elementary functions $u_n(x,y)$ that converges, uniformly in $[0,Y]$, to the unique solution $P(x,y)$ of problem (5.1): $u_n(x,y)\to P(x,y)$. The functions $u_n(x,y)$ are computed from the recurrence relation (2.8) particularized to the data of problem (5.1):

$$\begin{aligned} &u_{n+1}(x,y)=\phi(x,y)+\frac{y^2}{2x}\int_0^1\left[\cosh\left(y\sqrt{\frac{x}{2}}(1-t)\right)-1\right]tu_n(x,yt)dt,\\ &u_0(x,y)=\phi(x,y), \end{aligned} \tag{5.5}$$

with

$$\phi(x,y):=P_0(x)-P_1(x)\frac{2}{x}\left[\cosh\left(y\sqrt{\frac{x}{2}}\right)-1\right]. \tag{5.6}$$

The function $u_1(x,y)$, if $\Re x\geq 0$, is given by

$$\begin{aligned} u_1(x,y)=&\frac{1}{4x}\sqrt{\frac{\pi}{2}}\left[1-\cosh\left(\frac{\sqrt{x}y}{\sqrt{2}}\right)\right]U\left(\frac{3}{4},\frac{1}{2};\frac{x^2}{4}\right)\\ &+\frac{\sqrt{\pi}y}{64x^{5/2}}\left[6\sinh\left(\frac{\sqrt{x}y}{\sqrt{2}}\right)-\sqrt{2x}y\left(2+\cosh\left(\frac{\sqrt{x}y}{\sqrt{2}}\right)\right)\right]U\left(\frac{3}{4},\frac{1}{2};\frac{x^2}{4}\right)\\ &+\frac{\sqrt{x}}{4}e^{\frac{x^2}{8}}\left[1-\frac{1}{4x^2}\left(4+xy^2-4\cosh\left(\frac{\sqrt{x}y}{\sqrt{2}}\right)\right)\right]K_{1/4}\left(\frac{x^2}{8}\right) \end{aligned}$$

and, if $\Re x\leq 0$,

$$\begin{aligned} u_1(x,y)=&\frac{\left(2\sqrt{x}\left(-4x+y^2\right)+\sqrt{x}\left(8x+y^2\right)\cosh\left(\frac{\sqrt{x}y}{\sqrt{2}}\right)-3\sqrt{2}y\sinh\left(\frac{\sqrt{x}y}{\sqrt{2}}\right)\right)}{16x^{3/2}}\\ &\times\left[\Gamma\left(\frac{5}{4}\right)M\left(\frac{5}{4},\frac{3}{2};\frac{x^2}{4}\right)-\frac{1}{x}\Gamma\left(\frac{3}{4}\right)M\left(\frac{3}{4},\frac{1}{2};\frac{x^2}{4}\right)\right]\\ &+\frac{e^{\frac{x^2}{8}}\left(4-4x^2+xy^2-4\cosh\left(\frac{\sqrt{x}y}{\sqrt{2}}\right)\right)}{8x^{3/2}}\\ &\times\left[\frac{1}{16}\Gamma\left(-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)J_{-1/4}\left(\frac{x^2}{8}\right)+\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right)J_{1/4}\left(\frac{x^2}{8}\right)\right], \end{aligned}$$

where $J_\nu(z)$ and $K_\nu(z)$ are Bessel functions. In general, functions $u_n(x,y)$ are combination of confluent, Bessel and elementary functions. An equivalent iterative approximation to (5.5) and (5.6) of the Pearcy integral (5.2) has been given in [11]; the algorithm given in [11] is not general, but only valid for the particular example of the Pearcy integral.

On the other hand, applying Olver's method as it is specified in Section 4, we know that an asymptotic approximation of the order n of the unique solution of problem (5.1) is

$$u_n(y) = c_{1,n}\sum_{k=0}^{n-1} 2^{k/2}\frac{A_{1,k}(y)}{x^{k/2}} + c_{2,n}e^{\sqrt{\frac{x}{2}}y}\sum_{k=0}^{n-1} 2^{k/2}\frac{A_{2,k}(y)}{x^{k/2}} + c_{3,n}e^{-\sqrt{\frac{x}{2}}y}\sum_{k=0}^{n-1} 2^{k/2}\frac{A_{3,k}(y)}{x^{k/2}}, \tag{5.7}$$

where

$$A_{1,n+2}(y) = A''_{1,n}(y) - \frac{1}{4}\int_0^y tA_{1,n}(t)dt, \quad A_{1,0}(y) = 0, \quad A_{1,1}(y) = 1,$$

$$A_{2,n+2}(y) = \frac{1}{8}\int_0^y tA_{2,n}(t)dt - \frac{3}{2}A'_{2,n+1}(y) - \frac{1}{2}A''_{2,n}(y), \quad A_{2,0}(y) = 0, \quad A_{2,1}(y) = 1,$$

$$A_{3,n+2}(y) = \frac{1}{8}\int_0^y tA_{3,n}(t)dt + \frac{3}{2}A'_{3,n+1}(y) - \frac{1}{2}A''_{3,n}(y), \quad A_{3,0}(y) = 0, \quad A_{3,1}(y) = 1,$$

and the coefficients $c_{j,n}$, $j = 1,2,3$ in (5.7) must be computed at any order n of the approximation by using the conditions $u(0) = P(x,0)$, $u'(0) = 0$ and $u''(0) = P''(x,0)$. Coefficients $A_{j,n}$ are polynomials in the variable y. In Table 1, we show the first coefficients $A_{j,n}(y)$, $j = 1,2,3$, $n = 0,1,2,\ldots,10$, of the expansion (5.7).

Table 1. Coefficients $A_{j,n}$, $j = 1,2,3$, $n = 0,1,2,\ldots,10$ in (5.7).

n	$A_{1,n}$	$A_{2,n}$	$A_{3,n}$
0	0	0	0
1	1	1	1
2	0	0	0
3	$-\frac{y^2}{8}$	$\frac{y^2}{16}$	$\frac{y^2}{16}$
4	0	$-\frac{3y}{16}$	$\frac{3y}{16}$
5	$\frac{y^4}{128} - \frac{1}{4}$	$\frac{y^4+112}{512}$	$\frac{y^4+112}{512}$
6	0	$-\frac{5y^3}{256}$	$\frac{5y^3}{256}$
7	$-\frac{y^6}{3072} + \frac{y^2}{8}$	$\frac{y^6}{24576} + \frac{23y^2}{256}$	$\frac{y^6}{24576} + \frac{23y^2}{256}$
8	0	$-\frac{7y^5}{8192} - \frac{27y}{128}$	$\frac{7y^5}{8192} + \frac{27y}{128}$
9	$\frac{y^8}{98304} - \frac{9y^4}{512} + \frac{1}{4}$	$\frac{y^8}{1572864} + \frac{141y^4}{16384} + \frac{29}{128}$	$\frac{y^8}{1572864} + \frac{141y^4}{16384} + \frac{29}{128}$
10	0	$-\frac{3y^7}{131072} - \frac{425y^3}{8192}$	$\frac{3y^7}{131072} + \frac{425y^3}{8192}$

Thus, Olver's method gives an asymptotic expansion of the unique solution $u(y)$ of (5.1) for large $|x|$ in terms of elementary functions of y.

Table 2 shows some numerical approximations, for different values of x and y, of the solutions of (5.1) supplied by the iterative algorithm compared with the approximation given by Olver's method and with the convergent expansion (5.3). In these computations, we have considered as exact value of the Pearcy integral the approximation (5.3) with $n = 100$, that contains more than one hundred correct decimal digits.

Table 2. Numerical experiments about the relative errors in the approximation of the solution of problem (5.1) using the iterative method (5.5), Olver's method (5.7) and the convergent expansion (5.3) for different values of x, y and n.

$(x,y)=(1,1)$			
n	Formula (5.5)	Formula (5.7)	Formula (5.3)
1	$8.74e{-6}$	0.01165001	0.00607138
5	$9.28e{-23}$	0.21841101	$4.30e{-10}$
10	$7.67e{-49}$	0.22282390	$3.10e{-21}$
$(x,y)=(10,1)$			
n	Formula (5.5)	Formula (5.7)	Formula (5.3)
1	$9.05e{-6}$	0.01256326	0.00029499
5	$9.00e{-23}$	$3.95e{-5}$	$2.28e{-13}$
10	$7.26e{-49}$	$2.13e{-8}$	$1.91e{-26}$

$(x,y)=(100,I)$			
n	Formula (5.5)	Formula (5.7)	Formula (5.3)
1	$2.07e{-6}$	0.00246674	$3.12e{-6}$
5	$4.12e{-23}$	$7.62e{-10}$	$3.37e{-19}$
10	$4.42e{-49}$	$8.57e{-18}$	$5.87e{-37}$
$(x,y)=(1000,I)$			
n	Formula (5.5)	Formula (5.7)	Formula (5.3)
1	$3.04e{-8}$	0.00024804	$3.12e{-8}$
5	$1.30e{-25}$	$7.65e{-15}$	$3.39e{-25}$
10	$8.76e{-51}$	$1.09e{-27}$	$5.97e{-48}$
$(x,y)=(1+I,1)$			
n	Formula (5.5)	Formula (5.7)	Formula (5.3)
1	$8.66e{-6}$	0.01154975	0.00574029
5	$9.19e{-23}$	1.83839862	$3.98e{-10}$
10	$7.59e{-49}$	0.20691047	$2.85e{-21}$
$(x,y)=(10+10I,1)$			
n	Formula (5.5)	Formula (5.7)	Formula (5.3)
1	$8.95e{-6}$	0.01243848	0.00015730
5	$8.90e{-23}$	$9.65e{-6}$	$4.20e{-14}$
10	$7.18e{-49}$	$9.19e{-10}$	$1.22e{-27}$

$(x,y)=(I,I)$			
n	Formula (5.5)	Formula (5.7)	Formula (5.3)
1	$6.77e{-6}$	0.00925402	0.00887361
5	$7.10e{-23}$	0.20226875	$1.21e{-9}$
10	$5.86e{-49}$	0.14788278	$1.58e{-20}$
$(x,y)=(100I,I)$			
n	Formula (5.5)	Formula (5.7)	Formula (5.3)
1	$7.83e{-6}$	0.00982593	$3.12e{-6}$
5	$8.25e{-23}$	$3.04e{-9}$	$3.41e{-19}$
10	$6.80e{-49}$	$1.69e{-17}$	$6.06e{-37}$

6. Final remarks

We have extended to third-order equations both, our iterative method [10] and Olver's method [14, Chap. 10], designed, in principle, for second order linear differential equations. The iterative method can be also applied to nonlinear differential equations.

The approximations $y_n(z)$ of problem (2.1), derived with either, the fixed point method of Section 2, or Olver's method of Section 4, are analytic in $\mathcal{D}$ when $f(z)$ and $g(z)$ are analytic. In fact, when $f(z)$ and $g(z)$ are analytic in $\mathcal{D}$, the solution $y(z)$ of (2.1) is analytic in $\mathcal{D}$. The difference between the approximations given by Olver's method and the approximations given by the fixed point method is that the later are convergent, whereas the former, in general, are not. Then, the analytic properties of the solution are the same as the analytic properties of the approximants in both methods. Also, in Olver's method, the remainder $R_{j,n}(z)$ is analytic in $\mathcal{D}$. Another difference between the approximations supplied by the iterative and Olver's technique is the following. The iterative technique gives the approximations $y_n(z)$ to the unique solution of the problem (2.1) directly, from algorithm (2.8). On the other hand, Olver's technique gives, in a first instance, $Y_{j,n}(z)$, $j = 1, 2, 3$, from (4.2); then, we must compute the coefficients $c_{j,n}$, $j = 1, 2, 3$, at every step n of the approximation to obtain y_n as the linear combination $y_n(z) = c_{1,n}Y_{1,n}(z) + c_{2,n}Y_{2,n}(z) + c_{3,n}Y_{3,n}(z)$.

The error bound (2.7) is not uniform in z. This means that the convergent and asymptotic character of the expansion of Section 2 for the unique solutions of the initial value problem (2.1) is proved only over bounded subsets of $\mathcal{D}$. On the other hand, when Ψ_n, f' and g are integrable in unbounded paths $\mathcal{L}$, the bound (4.5) shows the uniform character of Olver's asymptotic expansions of Section 5 for three independent solutions of the differential equation $y''' + a\Lambda^2 y' + b\Lambda^3 y = f(z)y' + g(z)y$.

It would be worth considering the possibility of using the approximations given in Section 5 to implement computational algorithms of the Pearcey integral in *Mathematica* or other computer algebra systems. Olver's approximation performs well for large $|x|$ and moderate values of $|y|$, whereas the iterative algorithm seems to perform well in the whole complex x plane and moderate values of $|y|$.

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