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DETERMINING NODES OF THE GLOBAL ATTRACTOR FOR AN INCOMPRESSIBLE NON-NEWTONIAN FLUID*

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Abstract This paper estimates the finite number of the determining nodes to the equations for an incompressible non-Newtonian fluid with space-periodic or no-slip boundary conditions. The authors prove that, whenever the second order derivatives of two different solutions within the global attractor have the same time-asymptotic behavior at finite number of points in the physical space, then the two solutions possess the same time-asymptotic behavior at almost everywhere points of the physical space.

Keywords Incompressible non-Newtonian fluid, determining nodes, global attractor, asymptotic behavior.

MSC(2010) 35B40, 35Q35, 76D05.

1. Introduction

In this paper, we investigate the determining nodes of the global attractor for the following incompressible non-Newtonian fluid equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot\big(2\mu_0(\epsilon + |\mathbf{e}(\mathbf{u})|^2)^{-\alpha/2}\mathbf{e}(\mathbf{u}) - 2\mu_1\Delta\mathbf{e}(\mathbf{u})\big) + \nabla p = \mathbf{f}, \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1.2}$$

in $\Omega\times[0,+\infty)$ $(\Omega\subseteq\mathbb{R}^2)$, with initial value

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}),\ \ \mathbf{x} = (x_1,x_2)\in\Omega, \tag{1.3}$$

where the unknown vector function $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ and scalar function $p = p(\mathbf{x},t)$ stand for the velocity field and pressure of the fluid, respectively, and the given vector function $\mathbf{f} = \mathbf{f}(\mathbf{x},t)$ is the external force. In equation (1.1), $\mathbf{e}(\mathbf{u}) = (\mathbf{e}_{jk}(\mathbf{u}))_{2\times 2}$ is the symmetric deformations velocity tensor whose components are

$$\mathbf{e}_{jk}(\mathbf{u}) = \frac{1}{2}\Big(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\Big),\ \ j,k = 1,2,$$

and $|\mathbf{e}(\mathbf{u})|^2 = \sum\limits_{j,k=1}^{2}\mathbf{e}_{jk}^2(\mathbf{u})$. In addition, ϵ,μ_0,μ_1 and α are constitutive parameters.

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There are two boundary conditions are possible to equations (1.1)-(1.2). The first boundary condition is the space-periodic case. We assume that the fluid fills the entire space $\mathbb{R}^2$ but with the condition that $\mathbf{u}$, $\mathbf{f}$ and p are L_j-periodic in each variable $x_j, j = 1, 2$. In this case, we use Ω to denote the period $\Omega = (-L_1/2, L_1/2) \times (-L_2/2, L_2/2)$ and we consider the spatially periodic solutions of (1.1)-(1.3), with $\{\mathfrak{e}_1, \mathfrak{e}_2\}$ the natural basis of $\mathbb{R}^2$. The spatially periodic conditions associated to (1.1)-(1.3) are

$$u_k(\mathbf{x}, t) = u_k(\mathbf{x} + L_j\mathfrak{e}_j, t), \;\; j, k = 1, 2, \;\; t \geqslant 0, \tag{1.4}$$

$$\int_\Omega \mathbf{u}(\mathbf{x}, t)\mathrm{d}\mathbf{x} = 0. \tag{1.5}$$

The second boundary condition corresponds the case that $\Omega \subseteq \mathbb{R}^2$ is a bounded and suitable smooth domain. In this situation, (1.1)-(1.2) is supplemented by the boundary conditions

$$\mathbf{u} = \mathbf{0}, \;\; 2\mu_1 \frac{\partial \mathbf{e}_{jk}}{\partial x_k}\gamma_j\gamma_m = 0, \;\; \text{on } \partial\Omega \times [0, +\infty), \;\; j, k, m = 1, 2, \tag{1.6}$$

where $\vec{\gamma} = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ is the exterior unit normal to $\partial\Omega$. The first condition in (1.6) represents the usual no-slip condition associated with a viscous fluid; the second one expresses the fact that the first moments of the traction vanish on $\partial\Omega$.

The theory of multipolar material was firstly formulated by Green and Rivlin [9, 10]. Later, Bellout *et al.* [2] and Necǎs and Šilhavy [19] developed the mathematical theory of multipolar viscous fluids. We refer to [18] for the definition of non-Newtonian fluid, as well as for the physical background of the non-Newtonian fluid. There are many papers on the existence and uniqueness, regularity and long-time behavior of solutions to equations (1.1)-(1.4) and (1.1)-(1.3) with (1.5), or to the related versions (see e.g. [3–5, 11, 14, 18, 20, 22, 23, 25–27]. For example, Bloom and Hao in [5] proved the existence of a maximal attractor for equations (1.1)-(1.3) with (1.5) in two-dimensional (2D) unbounded channel like domains. Zhao and Li in [22] proved that the global attractor obtained by [5] is actually a H^2 global attractor. Zhao and Zhou in [23] investigated the existence and H^2-regularity of the pullback attractor in 2D bounded domains.

The motivation of this paper is to investigate the property of the solutions within the global attractor associated to equations (1.1)-(1.3). Recently, rather abstract, research on the asymptotic properties of this non-Newtonian fluid equations should prove valuable to the furtherance of the use of computers as experimental tools in the study of the dynamics of non-Newtonian fluids. We know that, in many practical situations, the experimental data are collected from measurements at a finite number of points in the physical space.

The idea of the present paper originates from [7], in which the authors estimated the finite number of determining nodes for the Navier-Stokes equations. The issue of determining form for the Navier-Stokes equations was also investigated in [8, 13, 17]. In addition, the determining form for the nonlinear Schrödinger equations was investigated in [15], and the determining nodes for semi-linear parabolic equations was studied in [17].

Our goal is to estimate the finite number of determining nodes for the global attractor of equations (1.1)-(1.3) with space-periodic or no-slip boundary conditions. Our result reveals that the time-asymptotic behavior of solutions within the global attractor is completely determined by the time-asymptotic behavior of the solutions at finite number of points in the physical space.

Definition 1.1. Consider a set of N nodes or measurement points in the physical space Ω, denoted by $\Lambda = \{\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^N\}$. Let $\mathbf{u}(\mathbf{x}, t)$ and $\mathbf{v}(\mathbf{x}, t)$ be two solutions within the global attractor associated to equations (1.1)-(1.3). If the conditions

$$\max_{j=1,\cdots,N} |\Delta(\mathbf{u}(\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t))|_{\mathbf{x}=\mathbf{x}^j}| \longrightarrow 0 \quad \text{as } t \longrightarrow \infty, \tag{1.7}$$

implies

$$\int_\Omega |\Delta \mathbf{u}(\mathbf{x}, t) - \Delta \mathbf{v}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{1.8}$$

Then the set Λ is called a set of **determining nodes** for the global attractor associated to equations (1.1)-(1.3).

We want point out that above definition is different with the notion of *determining nodes* for the Navier-Stokes equations in [7]. The reason is that the non-Newtonian fluid equations addressed in this paper contain the term of fourth order derivative $\nabla \cdot (-2\mu_1 \Delta \mathbf{e}(\mathbf{u}))$. On the other hand, compared with the Navier-Stokes equations, this non-Newtonian fluid equations contain an additional nonlinear term $\nabla \cdot (\mu_0(\epsilon + |\mathbf{e}(\mathbf{u})|^2)^{-\alpha/2} \mathbf{e}(\mathbf{u}))$. We need do some technique estimation to handle with this nonlinear term when estimating the finite number of determining nodes.

The rest of this paper is organized as follows. In the next section, we introduce some notations and preliminary results. In Section 3, we first prove an estimation associated to the fourth order derivative $\nabla \cdot (-2\mu_1 \Delta \mathbf{e}(\mathbf{u}))$ and then estimate the finite number of determining nodes.

2. Notations and preliminary results

We first remark that, hereafter the notations and preliminary results are corresponding to equations (1.1)-(1.3) with space-periodic condition. But we can use similar notations and have similar preliminary results for equations (1.1)-(1.3) with no-slip boundary condition.

Throughout this article, we denote by $\mathbb{R}$ and $\mathbb{R}_+$ the sets of real and positive axis, respectively. We let $(L^q_{\text{per}}(\Omega))^2$ $(1 \leqslant q \leqslant +\infty)$ be the space of 2D vector functions $\mathbf{u} = \mathbf{u}(\mathbf{x})$ defined on $\mathbb{R}^2$ that are L_j-periodic in each variable $x_j (j = 1, 2)$, and which belong to $(L^q(\mathcal{O}))^2$ for every bounded open set $\mathcal{O} \subset \mathbb{R}^2$. Then we define the periodic Sobolev space as

$$(H^2_{\text{per}}(\Omega))^2 := \{\mathbf{u} \in (L^2_{\text{per}}(\Omega))^2 \ \big| \partial^m \mathbf{u} \in (L^2_{\text{per}}(\Omega))^2, \ |m| \leqslant 2\}$$

and endow the spaces $(L^2_{\text{per}}(\Omega))^2$ and $(H^2_{\text{per}}(\Omega))^2$ with norms $\|\cdot\|$ and $\|\cdot\|_{H^2_{\text{per}}}$ respectively (see e.g. [1,7]), where

$$\|\mathbf{u}\| := (\int_\Omega |\mathbf{u}|^2 \mathrm{d}\mathbf{x})^{1/2} \quad \text{and} \quad \|\mathbf{u}\|_{H^2_{\text{per}}} := (\sum_{|m| \leqslant 2} \int_\Omega |\partial^m \mathbf{u}|^2 \mathrm{d}\mathbf{x})^{1/2}.$$

Further, we set

$$(\mathcal{C}^\infty_{\text{per}})^2 := \text{the space of } \Omega\text{−periodic, 2D } \mathcal{C}^\infty \text{ vector fields defined on } \mathbb{R}^2,$$

$$\mathcal{V}_{\text{per}} := \big\{\mathbf{u} \in (\mathcal{C}^\infty_{\text{per}})^2, \int_\Omega \mathbf{u}(\mathbf{x}) \mathrm{d}\mathbf{x} = 0, \nabla \cdot \mathbf{u} = 0\big\},$$

$$H_{\rm per} := \text{closure of } \mathcal{V}_{\rm per} \text{ in } (L^2_{\rm per}(\Omega))^2 \text{ with norm } \|\cdot\|_{H_{\rm per}} = \|\cdot\| \text{ and dual space } H^*_{\rm per},$$
$$V_{\rm per} := \text{closure of } \mathcal{V}_{\rm per} \text{ in } (H^2_{\rm per}(\Omega))^2 \text{ with norm } \|\cdot\|_{V_{\rm per}} = \|\cdot\|_{H^2_{\rm per}} \text{ and dual space } V^*_{\rm per}.$$

Let $(\cdot,\cdot)$ be the inner product in $(L^2_{\rm per}(\Omega))^2$ or $H_{\rm per}$, and $\langle\cdot,\cdot\rangle$ be the dual pairing between $V_{\rm per}$ and $V^*_{\rm per}$. Then, we introduce the following three operators:

$$\langle A\mathbf{u},\mathbf{v}\rangle := 2\sum_{j,k,m=1}^{2}\int_\Omega \frac{\partial \mathbf{e}_{jk}(\mathbf{u})}{\partial x_m}\frac{\partial \mathbf{e}_{jk}(\mathbf{v})}{\partial x_m}\mathrm{d}\mathbf{x},\ \ \forall\,\mathbf{u},\ \mathbf{v}\in V_{\rm per}, \tag{2.1}$$
$$b(\mathbf{u},\mathbf{v},\mathbf{w}) := \sum_{j,k=1}^{2}\int_\Omega u_j\frac{\partial v_k}{\partial x_j}w_k\,\mathrm{d}\mathbf{x},\ \ \forall\mathbf{u},\ \mathbf{v},\ \mathbf{w}\in V_{\rm per},$$
$$\langle N(\mathbf{u}),\mathbf{v}\rangle := \sum_{j,k=1}^{2}\int_\Omega \mu(\mathbf{u})\mathbf{e}_{jk}(\mathbf{u})\mathbf{e}_{jk}(\mathbf{v})\,\mathrm{d}\mathbf{x},\ \ \forall\,\mathbf{u},\ \mathbf{v}\in V_{\rm per}.$$

For any $\mathbf{u},\mathbf{v}\in V_{\rm per}$, we set $\langle B(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle := b(\mathbf{u},\mathbf{v},\mathbf{w}),\ \ \forall\,\mathbf{w}\in V_{\rm per}$, and especially,

$$B(\mathbf{u}) := B(\mathbf{u},\mathbf{u}),\ \ \mathbf{u}\in V_{\rm per}.$$

From the above definitions, one can check that the operators $B(\cdot)$ and $N(\cdot)$ are continuous from $V_{\rm per}$ to $V^*_{\rm per}$, and A is a linear continuous operator both from $V_{\rm per}$ to $V^*_{\rm per}$ and from $D(A)$ to $H_{\rm per}$, where

$$D(A) := \{\mathbf{u}\in V_{\rm per}\ \big|\ \langle A\mathbf{u},\mathbf{v}\rangle = (\mathbf{g},\mathbf{v})\ \text{ with some }\ \mathbf{g}\in H_{\rm per},\forall\,\mathbf{v}\in V_{\rm per}\}.$$

In addition, if $\mathbf{u}\in D(A)$, then $N(\mathbf{u})$ can be extended to $H_{\rm per}$ via

$$\langle N(\mathbf{u}),\mathbf{v}\rangle = -\int_\Omega \big\{\nabla\cdot\big(\mu(\mathbf{u})\mathbf{e}(\mathbf{u})\big)\big\}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x},\ \ \forall\,\mathbf{v}\in H_{\rm per}. \tag{2.2}$$

We next present some useful estimations and properties for the operators A and $b(\cdot,\cdot,\cdot)$.

Lemma 2.1. *There are some positive constants c_i $(i=1,2,3)$ depending only on Ω such that*

$$c_1\|\mathbf{u}\|^2_{V_{\rm per}} \leqslant \langle A\mathbf{u},\mathbf{u}\rangle \leqslant c_2\|\mathbf{u}\|^2_{V_{\rm per}},\ \ \forall\,\mathbf{u}\in V_{\rm per}, \tag{2.3}$$
$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{v}),\ \ b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0,\ \ \forall\,\mathbf{u},\mathbf{v},\mathbf{w}\in V_{\rm per}, \tag{2.4}$$
$$|b(\mathbf{u},\mathbf{v},A\mathbf{w})| \leqslant c_3\|\mathbf{u}\|_{V_{\rm per}}\|\mathbf{v}\|_{V_{\rm per}}\|A\mathbf{w}\|,\ \ \forall\,\mathbf{u},\mathbf{v}\in V_{\rm per},\mathbf{w}\in D(A). \tag{2.5}$$

Proof. The proof of (2.3) can be found in [4]. The relations in (2.4) are now classical results which can be found in [7]. We next prove (2.5). In fact, since $(L^\infty_{\rm per}(\Omega))^2 \hookrightarrow (H^2_{\rm per}(\Omega))^2$, we obtain, using the Hölder inequality and the fact $\|\nabla\cdot\| \leqslant \|\cdot\|_{V_{\rm per}}$,

$$\begin{aligned}|b(\mathbf{u},\mathbf{v},A\mathbf{w})| &= |\sum_{i,j=1}^{2}\int_\Omega u_i\frac{\partial v_j}{\partial x_i}Aw_j\mathrm{d}\mathbf{x}|\\ &\leqslant c\|\mathbf{u}\|_{(L^\infty_{\rm per}(\Omega))^2}\|\nabla\mathbf{v}\|\|A\mathbf{w}\|\\ &\leqslant c\|\mathbf{u}\|_{V_{\rm per}}\|\mathbf{v}\|_{V_{\rm per}}\|A\mathbf{w}\|,\end{aligned}$$

where c is a constant depending only on Ω. $\square$

Using the notations and operators introduced above, we can express the weak version of equations (1.1)-(1.5) in the solenoidal vector field as the following

$$\frac{\partial \mathbf{u}}{\partial t} + \mu_1 A\mathbf{u} + B(\mathbf{u}) + N(\mathbf{u}) = \mathbf{f}(t) \ \text{ in } \ \mathcal{D}'(0,+\infty; V_{\text{per}}^*), \tag{2.6}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0. \tag{2.7}$$

We next specify the definition of solutions to equations (2.6)-(2.7).

Definition 2.1. A global weak solution of equations (2.6)-(2.7) is a function

$$\mathbf{u} \in L^2(0,+\infty; H_{\text{per}}) \cap L^2(0,+\infty; V_{\text{per}}) \cap L^\infty(0,+\infty; H_{\text{per}})$$

with $\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0$, such that (2.6) holds in the distribution sense $\mathcal{D}'(0,+\infty; V_{\text{per}}^*)$. If $\mathbf{u}$ is a global weak solution and $\mathbf{u} \in L^2(0,+\infty;V_{\text{per}}) \cap L^2(0,+\infty;D(A)) \cap L^\infty(0,+\infty;V_{\text{per}})$, then $\mathbf{u}$ is called a global strong solution.

For the existence and uniqueness of global solutions to equations (2.6)-(2.7), as well as the existence and H^2 regularity of the global attractors for the associated solution semigroup, we have the following result.

Theorem 2.1. *Assume $\epsilon > 0$, $\mu_0 > 0$, $\mu_1 > 0$ and $\alpha \in (0,1)$.*

(I) *If $\mathbf{f} \in L^2(0,+\infty; H_{\text{per}})$. Then for any given $\mathbf{u}_0 \in H_{\text{per}}$, equations (2.6)-(2.7) possess a unique global weak solution; for any given $\mathbf{u}_0 \in V_{\text{per}}$, equations (2.6)-(2.7) possess a unique global strong solution.*

(II) *If $\mathbf{f} \in H_{\text{per}}$ is independent of time t. Then the solution operators*

$$S(t) : \begin{cases} \mathbf{u}_0 \in H_{\text{per}} \mapsto S(t)\mathbf{u}_0 = \mathbf{u}(t) \in H_{\text{per}}, \ \ \forall\, t \in \mathbb{R}_+, \\ \mathbf{u}_0 \in V_{\text{per}} \mapsto S(t)\mathbf{u}_0 = \mathbf{u}(t) \in V_{\text{per}}, \ \ \forall\, t \in \mathbb{R}_+, \end{cases}$$

generate a continuous semigroup $\{S(t)\}_{t\geqslant 0}$ in spaces H_{per} and V_{per}, respectively, and the semigroup $\{S(t)\}_{t\geqslant 0}$ possesses a global attractor $\mathcal{A}^H$ satisfying
(a) *$\mathcal{A}^H$ is compact in H_{per};* (b) *$S(t)\mathcal{A}^H = \mathcal{A}^H$, $\forall\, t \in \mathbb{R}_+$;*
(c) *for any bounded set $\mathcal{B}^H \subset H_{\text{per}}$, $\lim_{t\to+\infty} \operatorname{dist}_{H_{\text{per}}}(S(t)\mathcal{B}^H, \mathcal{A}^H) = 0$.*

Also $\{S(t)\}_{t\geqslant 0}$ possesses a global attractor $\mathcal{A}^V$ satisfying
(i) *$\mathcal{A}^V$ is compact in V_{per};* (ii) *$S(t)\mathcal{A}^V = \mathcal{A}^V$, $\forall\, t \in \mathbb{R}_+$;*
(iii) *for any bounded set $\mathcal{B}^V \subset V_{\text{per}}$, $\lim_{t\to+\infty} \operatorname{dist}_{V_{\text{per}}}(S(t)\mathcal{B}^V, \mathcal{A}^V) = 0$.*

Furthermore,

$$\mathcal{A}^H = \mathcal{A}^V. \tag{2.8}$$

Proof. The assertion (I) can be proved by the similar arguments of Bloom and Hao [4,5], and the assertion (II) can be established by the analogous approaches of Zhao and Li [21,22], with the spaces H and V replaced by H_{per} and V_{per}, respectively. $\square$

From (2.8) we see that the global attractors $\mathcal{A}^H$ and $\mathcal{A}^V$ coincide with each other. Thus we denote them by the same notation $\mathcal{A}$ in the rest of the paper. From (II)(a,b), (II)(i,ii) and (2.8) we see there exists some fixed $\rho > 0$ such that

$$\|\mathbf{u}(t)\|^2_{V_{\text{per}}} \leqslant \rho^2, \ \ \forall\, t \in \mathbb{R}_+, \ \ \forall\, \mathbf{u} \in \mathcal{A}. \tag{2.9}$$

In fact, ρ is the diameter of the global attractor $\mathcal{A}$ which depends on the constitutive parameters and the external force $\mathbf{f}$ of the equations. We end this section with a useful lemma.

Lemma 2.2 ([7]). *Let $\phi = \phi(t)$ and $\beta = \beta(t)$ be locally integrable real-valued functions on $[0; +\infty)$ that satisfy the following conditions for some $T > 0$:*

$$\liminf_{t\to+\infty} \frac{1}{T}\int_t^{t+T} \phi(\tau)\mathrm{d}\tau > 0, \tag{2.10}$$

$$\limsup_{t\to+\infty} \frac{1}{T}\int_t^{t+T} \phi^-(\tau)\mathrm{d}\tau < \infty, \tag{2.11}$$

$$\lim_{t\to+\infty} \frac{1}{T}\int_t^{t+T} \beta^+(\tau)\mathrm{d}\tau = 0, \tag{2.12}$$

where $\phi^-(t) = \max\{-\phi(t), 0\}$ and $\beta^+(t) = \max\{\beta(t), 0\}$. Suppose that $\xi = \xi(t)$ is an absolutely continuous nonnegative function on $[0, +\infty)$ that satisfies the following inequality almost everywhere $[0, +\infty)$:

$$\frac{\mathrm{d}\xi(t)}{\mathrm{d}t} + \phi(t)\xi(t) \leqslant \beta(t). \tag{2.13}$$

Then $\xi(t) \longrightarrow 0$ as $t \longrightarrow +\infty$.

3. Determining nodes

Consider a set of N nodes or measurement points in the physical domain Ω, denoted by $\Lambda = \{\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^N\}$. We assume that the points in Λ are uniformly distributed within the physical domain Ω in the sense that Ω can be covered by N identical squares such that each square contains one and only one of the given points. Let $\mathbf{u}(\mathbf{x}, t)$ and $\mathbf{v}(\mathbf{x}, t)$ be two solutions within the global attractor $\mathcal{A}$. To measure the difference between these two solution throughout the set Λ, we set $\mathbf{w}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t)$ and denote

$$\eta(\mathbf{w}) = \max_{1\leq j\leq N} \Big|\Delta\mathbf{w}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}^j}\Big|. \tag{3.1}$$

The following lemma, although it is a slight modification of [7, Lemma 2.1], is one of the key ingredients when we estimate the finite number of determining nodes for the global attractor $\mathcal{A}$ in the sense of Definition 1.1.

Lemma 3.1. *Let the physical domain Ω be covered by N identical squares and the points $\Lambda = \{\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^N\}$ are uniformly distributed within Ω. Then, for each vector field $\mathbf{w} \in D(A)$, there holds*

$$N^2\|\mathbf{w}\|^2_{V_{\mathrm{per}}} \leqslant c_4\|A\mathbf{w}\|^2 + c_4N^2\eta(\mathbf{w})^2, \tag{3.2}$$

where c_4 is a constant depends only on the shape of the domain Ω.

Proof. Let $\mathbb{P}_L$ be the Leray projector (see e.g. [7, page 121]). By [7, Lemma 2.1] we see that for any $\mathbf{w} \in D(-\mathbb{P}_L\Delta)$ (the domain of $-\mathbb{P}_L\Delta$), there holds

$$\|\mathbf{w}\|^2 \leqslant \frac{c_5}{\lambda_1}\big(\max_{1\leqslant j\leqslant N}|\mathbf{w}(\mathbf{x}^j)|\big)^2 + \frac{c_5}{\lambda_1^2N^2}\|\mathbb{P}_L\Delta\mathbf{w}\|^2, \tag{3.3}$$

where λ_1 is the first eigenvalue of the operator $-\mathbb{P}_L\Delta$ and c_5 is a constant depends only on the shape of the domain Ω. By the definition of the operator A (see (2.1)), we find that $A = -\mathbb{P}_L\Delta^2$. Thus, for any $\mathbf{w} \in D(A)$, we have $\Delta\mathbf{w} \in D(-\mathbb{P}_L\Delta)$. So (3.3) gives

$$\begin{aligned}\|\Delta\mathbf{w}\|^2 &\leqslant \frac{c_5}{\lambda_1}\big(\max_{1\leqslant j\leqslant N}|\Delta\mathbf{w}(\mathbf{x}^j)|\big)^2 + \frac{c_5}{\lambda_1^2N^2}\|\mathbb{P}_L\Delta^2\mathbf{w}\|^2,\\ &= \frac{c_5}{\lambda_1}\eta(\mathbf{w})^2 + \frac{c_5}{\lambda_1^2N^2}\|A\mathbf{w}\|^2, \quad \forall\,\mathbf{w} \in D(A).\end{aligned} \tag{3.4}$$

By the definition of the norm $\|\cdot\|_{V_{\mathrm{per}}}$ and the Poincaré inequality, we conclude the fact that the norm $\|\Delta\cdot\|$ is equivalent to $\|\cdot\|_{V_{\mathrm{per}}}$. This fact and (3.4) imply that (3.2) holds true. $\square$

We next prove that the global attractor $\mathcal{A}$ obtained in Section 2 has finite number of determining nodes in the sense of Definition 1.1.

Theorem 3.1. *Let the physical domain Ω be covered by N identical squares and the points $\Lambda = \{\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^N\}$ are uniformly distributed within Ω. Suppose ϵ, μ_0, μ_1 are positive parameters and $\alpha \in (0,1)$. Then there exists a constant $C = C(\|\mathbf{f}\|, \Omega, \epsilon, \mu_0, \mu_1, \alpha)$ such that, if*

$$N \geqslant C(\|\mathbf{f}\|, \Omega, \epsilon, \mu_0, \mu_1, \alpha),$$

then the set Λ is a set of determining nodes for the global attractor $\mathcal{A}$ in the sense of Definition 1.1.

Proof. Since we estimate the determining nodes for the global attractor $\mathcal{A}$, we assume that the external force $\mathbf{f} \in H_{\mathrm{per}}$ is independent of time t. Consider two solutions $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ within the global attractor $\mathcal{A}$. Then we have

$$\frac{\partial\mathbf{u}}{\partial t} + \mu_1 A\mathbf{u} + B(\mathbf{u}) + N(\mathbf{u}) = \mathbf{f} \ \text{ in } \ \mathcal{D}'(0, +\infty; H_{\mathrm{per}}), \tag{3.5}$$
$$\frac{\partial\mathbf{v}}{\partial t} + \mu_1 A\mathbf{v} + B(\mathbf{v}) + N(\mathbf{v}) = \mathbf{f} \ \text{ in } \ \mathcal{D}'(0, +\infty; H_{\mathrm{per}}). \tag{3.6}$$

Let $\mathbf{w}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t)$ and $\eta(\mathbf{w})$ be defined by (3.1). Then we assume

$$\lim_{t\to+\infty}\eta(\mathbf{w}(t)) = 0. \tag{3.7}$$

We need to show that

$$\lim_{t\to+\infty}\int_\Omega |\Delta\mathbf{w}(\mathbf{x}, t)|^2\mathrm{d}\mathbf{x} = 0. \tag{3.8}$$

Actually, we will prove

$$\lim_{t\to+\infty}\|\mathbf{w}(\mathbf{x}, t)\|^2_{V_{\mathrm{per}}} = 0. \tag{3.9}$$

Note that equations (3.5)-(3.6) give

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} + \mu_1 A\mathbf{w} + B(\mathbf{w}, \mathbf{u}) + B(\mathbf{v}, \mathbf{w}) + N(\mathbf{u}) - N(\mathbf{v}) = \mathbf{0}. \tag{3.10}$$

Taking the inner product of (3.10) with $A\mathbf{w}$ in H and using the relation (2.4) yield

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}(t)\|^2_{V_{\text{per}}}+\mu_1\|A\mathbf{w}\|^2+b(\mathbf{w},\mathbf{u},A\mathbf{w})\\&+b(\mathbf{v},\mathbf{w},A\mathbf{w})+(N(\mathbf{u})-(N(\mathbf{v}),A\mathbf{w}))=0.\end{aligned}\tag{3.11}$$

Now, using (2.5) and Cauchy inequality, we have

$$\begin{aligned}|b(\mathbf{w},\mathbf{u},A\mathbf{w})|&\leqslant c_3\|\mathbf{w}\|_{V_{\text{per}}}\|\mathbf{u}\|_{V_{\text{per}}}\|A\mathbf{w}\|\\&\leqslant\frac{2c_3^2}{\mu_1}\|\mathbf{w}\|^2_{V_{\text{per}}}\|\mathbf{u}\|^2_{V_{\text{per}}}+\frac{\mu_1}{8}\|A\mathbf{w}\|^2.\end{aligned}\tag{3.12}$$

Similarly,

$$|b(\mathbf{v},\mathbf{w},A\mathbf{w})|\leqslant\frac{2c_3^2}{\mu_1}\|\mathbf{w}\|^2_{V_{\text{per}}}\|\mathbf{v}\|^2_{V_{\text{per}}}+\frac{\mu_1}{8}\|A\mathbf{w}\|^2.\tag{3.13}$$

To estimate the nonlinear term $(N(\mathbf{u})-(N(\mathbf{v}),A\mathbf{w}))$ in equation (3.11), we set $\mathcal{F}(\mathbb{S})=2\mu_0(\epsilon+|\mathbb{S}|^2)^{-\alpha/2}\mathbb{S}$, where ($\mathbb{R}^{2\times2}_{\text{sym}}$ denotes the set of symmetric matrix of order 2×2)

$$\mathbb{S}=\begin{pmatrix}s_{11}&s_{12}\\s_{21}&s_{22}\end{pmatrix}\in\mathbb{R}^{2\times2}_{\text{sym}},\quad|\mathbb{S}|^2=\sum_{j,k=1}^{2}s_{jk}^2,\quad s_{jk}\in\mathbb{R},\quad j,k=1,2.$$

Then the first and second order Fréchet derivatives of $\mathcal{F}(\mathbb{S})$ satisfy (see [23, (3.10)])

$$\|D\mathcal{F}(\mathbb{S})\|+\|D^2\mathcal{F}(\mathbb{S})\|\leqslant c_6=c_6(\mu_0,\varepsilon,\alpha),\quad\forall\,\mathbb{S}\in\mathbb{R}^{2\times2}_{\text{sym}}.\tag{3.14}$$

For any $\mathbb{S}_1,\mathbb{S}_2\in\mathbb{R}^{2\times2}_{\text{sym}}$, we have

$$\mathcal{F}(\mathbb{S}_2)-\mathcal{F}(\mathbb{S}_1)=\int_0^1 D\mathcal{F}(\mathbb{S}_1+\tau(\mathbb{S}_2-\mathbb{S}_1))(\mathbb{S}_2-\mathbb{S}_1)\,\mathrm{d}\tau.\tag{3.15}$$

Applying (2.2) and (3.14)-(3.15), as well as Hölder and Cauchy inequalities, we have

$$\begin{aligned}|(N(\mathbf{u})-(N(\mathbf{v}),A\mathbf{w})|&=\Big|\int_\Omega\big\{\nabla\cdot\big[\mathcal{F}(\mathbf{e}(\mathbf{u}))-\mathcal{F}(\mathbf{e}(\mathbf{v}))\big]\big\}\cdot A\mathbf{w}\mathrm{d}\mathbf{x}\Big|\\&=\Big|\int_\Omega\big\{\nabla\cdot\int_0^1 D\mathcal{F}\big(\mathbf{e}(\mathbf{u})+\tau\mathbf{e}(\mathbf{w})\big)\mathbf{e}(\mathbf{w})\mathrm{d}\tau\big\}\cdot A\mathbf{w}\mathrm{d}\mathbf{x}\Big|\\&\leqslant\int_\Omega\int_0^1\|D^2\mathcal{F}\big(\mathbf{e}(\mathbf{u})+\tau\mathbf{e}(\mathbf{w})\big)\|\mathrm{d}\tau\ \ |\mathbf{e}(\mathbf{w})||A\mathbf{w}|\mathrm{d}\mathbf{x}\\&\quad+\int_\Omega\int_0^1\|D\mathcal{F}\big(\mathbf{e}(\mathbf{u})+\tau\mathbf{e}(\mathbf{w})\big)\|\mathrm{d}\tau\ \ |\nabla\mathbf{e}(\mathbf{w})||A\mathbf{w}|\mathrm{d}\mathbf{x}\\&\leqslant c_6(\|\nabla\mathbf{w}\|+\|\triangle\mathbf{w}\|)\|A\mathbf{w}\|\\&\leqslant\frac{16c_6^2}{\mu_1}\|\mathbf{w}\|^2_{V_{\text{per}}}+\frac{\mu_1}{4}\|A\mathbf{w}\|^2.\end{aligned}\tag{3.16}$$

It then follows from (3.11)-(3.13) and (3.16) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|^2_{V_{\text{per}}}+\mu_1\|A\mathbf{w}\|^2\leqslant\frac{4c_3^2}{\mu_1}\|\mathbf{w}\|^2_{V_{\text{per}}}(\|\mathbf{u}\|^2_{V_{\text{per}}}+\|\mathbf{v}\|^2_{V_{\text{per}}})+\frac{32c_6^2}{\mu_1}\|\mathbf{w}\|^2_{V_{\text{per}}}.\tag{3.17}$$

Now from (3.2) we see $\|A\mathbf{w}\|^2 \geqslant \frac{N^2}{c_4}\|\mathbf{w}\|^2_{V_{\text{per}}} - N^2\eta(\mathbf{w})^2$, and inserting which into (3.17) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|^2_{V_{\text{per}}} + \|\mathbf{w}\|^2_{V_{\text{per}}}\Big(\frac{\mu_1 N^2}{c_4} - \frac{4c_3^2}{\mu_1}(\|\mathbf{u}\|^2_{V_{\text{per}}} + \|\mathbf{v}\|^2_{V_{\text{per}}}) - \frac{32c_6^2}{\mu_1}\Big) \leqslant \mu_1 N\eta(\mathbf{w})^2. \tag{3.18}$$

We now write

$$\xi(t) = \|\mathbf{w}(t)\|^2_{V_{\text{per}}}, \tag{3.19}$$

$$\phi(t) = \frac{\mu_1 N^2}{c_4} - \frac{4c_3^2}{\mu_1}(\|\mathbf{u}(t)\|^2_{V_{\text{per}}} + \|\mathbf{v}(t)\|^2_{V_{\text{per}}}) - \frac{32c_6^2}{\mu_1}, \tag{3.20}$$

$$\beta(t) = \mu_1 N\eta(\mathbf{w}(t))^2. \tag{3.21}$$

Then inequality (3.18) is of the form (2.13). We next check that the functions defined by (3.19)-(3.21) satisfy the conditions of Lemma 2.2. In fact, by (2.9) we have $\|\mathbf{u}(t)\|^2_{V_{\text{per}}} + \|\mathbf{v}(t)\|^2_{V_{\text{per}}} \leqslant 2\rho^2$ for any $t \in \mathbb{R}_+$. Hence, if we pick N large enough such that

$$N > \frac{(8c_3^2c_4\rho^2 + 32c_4c_6^2)^{1/2}}{\mu_1}, \tag{3.22}$$

then for any $T > 0$,

$$\liminf_{t\to+\infty}\frac{1}{T}\int_t^{t+T}\phi(\tau)\mathrm{d}\tau \geqslant \frac{\mu_1 N^2}{c_4} - \frac{8c_3^2\rho^2}{\mu_1} - \frac{32c_6^2}{\mu_1} > 0. \tag{3.23}$$

On the other hand, by (3.20) we have for any $T > 0$ that

$$\liminf_{t\to+\infty}\frac{1}{T}\int_t^{t+T} -\phi(\tau)\mathrm{d}\tau \leqslant \frac{8c_3^2\rho^2}{\mu_1} + \frac{32c_6^2}{\mu_1} - \frac{\mu_1 N^2}{c_4} < +\infty. \tag{3.24}$$

The relations (3.23)-(3.24) show that the function $\beta(t)$ defined by (3.20) satisfies the conditions (2.10)-(2.11). Clearly, (3.7) implies that the function $\beta(t)$ defined by (3.21) satisfies the inequality (2.12). Therefore, by Lemma 2.2 we conclude that if N satisfies (3.22), then (3.9) holds true. The proof is complete. □

Remark 3.1. When the two-dimensional non-Newtonian fluid equations (1.1)-(1.3) are supplemented with no-slip boundary condition (1.6). We also can prove the well-posedness and establish that the semigroup associated with the solution operators possesses a global attractor. Similar to Theorem 3.1, we can also establish that this global attractor has finite number of determining nodes. We want to point out that, for the periodic case, the estimates are usually better than in the non-slip case due to $b(w, w, Aw) = 0$, cf. [7, p_{135}].

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References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] H. Bellout, F. Bloom and J. Necǎs, *Phenomenological behavior of muitipolar viscous fluids*, Quart. Appl. Math., 1992, 50, 559–583.

[3] H. Bellout, F. Bloom and J. Nečas, *Young measure value solutions for non-Newtonian incompressible fluids*, Comm.Partial Differential Equations, 1994, 19, 1768–1803.

[4] F. Bloom and W. Hao, *Regularization of a non-Newtonian system in an unbounded channel: Existence and uniqueness of solutions*, Nonlinear Anal., 2001, 44, 281–309.

[5] F. Bloom and W. Hao, *Regularization of a non-Newtonian system in an unbounded channel: Existence of a maximal compact attractor*, Nonlinear Anal., 2001, 43, 743–766.

[6] C. Foias and R. Temam, *Determination of the solutions of the Navier-Stokes equations by a set of nodal values*, Math. Comp., 1984, 43, 117–133.

[7] C. Foias, O. Manley, R. Rosa and R. Temam, *Navier-Stokes equations and turbulence*, Cambridge University Press, Cambridge, 2004.

[8] C. Foias, M. Jolly, R. Kravchenko and E. S. Titi, *A determining form for the 2D Navier-Stokes equaitons-the Fourier modes case*, J. Math. Phys., 2012, 53, 115623.

[9] A. E. Green and R. S. Riviin, *Simple force and stress multipoles*, Arch. Rational Mech. Anal., 1964, 16, 325–353.

[10] A. E. Green and R. S. Riviin, *Multipolar continuum mechanics*, Arch. Rational Mech. Anal., 1964, 17, 113–147.

[11] B. Guo and P. Zhu, *Partial regularity of suitable weak solution to the system of the incompressible non-Newtonian fluids*, J. Differential Equations, 2002, 178, 281–297.

[12] B. Guo, G. Lin and Y. Shang, *Non-Newtonian fluids Dynamical Systems* (in Chinses), National Defence Industry Press, Beijing, 2006.

[13] D. A. Jones and E. S. Titi, *Upper bounds on the number of determining modes, and volum elements for the Navier-Stokes equations*, Indiana Univ. Math. J., 1993, 42, 875–887.

[14] N. Ju, *Existence of global attractor for the three-dimensional modified Navier-Stokes equations*, Nonlinearity, 2001, 14, 777–786.

[15] M. Jolly, T.Sadigov and E.S.Titi, *A determining form for the damped driven nonlinear Schröinger equation-ourier modes case*, J. Differential Equations, 2015, 258, 2711–2744.

[16] P. Korn, *On degrees of freedom of certain concervation turbulence models for the Navier-Stokes equations*, J. Math. Anal. Appl., 2011, 378, 49–63.

[17] R. Kakizawa, *Determining nodes for semilinear parabolic equations*, J. Math. Anal. Appl., 2011, 378, 375–386.

[18] J. Málek, J. Nečas, M. Rokyta and M. Rǔžičk, *Weak and measure-valued solutions to evolutionary PDE*, Champman-Hall, New York, 1996.

[19] J. Necăs and M. Šilhavy, *Multipolar viscous fluids*, Quart. Appl. Math., 1991, 49, 247–263.

[20] M. Pokorný, *Cauchy problem for non-Newtonian incompressible fluids*, Appl. Math., 1996, 41, 169–201.

[21] C. Zhao and Y. Li, *H^2-compact attractor for a non-Newtonian system in two-dimensional unbounded domains*, Nonlinear Anal., 2004, 56, 1091–1103.

[22] C. Zhao and Y. Li, *A note on the asymptotic smoothing effect of solutions to a non-Newtonian system in 2-D unbounded domains*, Nonlinear Anal., 2005, 60, 475–483.

[23] C. Zhao and S. Zhou, *Pullback attractors for nonautonomous incompressible non-Newtonian fluid*, J. Differential Equations, 2007, 238, 394–425.

[24] C. Zhao, Y. Li and S. Zhou, *Regularity of trajectory attractor and upper semi-continuity of global attractor for a 2D non-Newtonian fluid*, J. Differential Equations, 2009, 247, 2331–2363.

[25] C. Zhao, *Pullback asymptotic behavior of solutions for a non-autonomous non-Newtonian fluid on 2D unbounded domains*, J. Math. Phys., 2012, 12, 1–21.

[26] C. Zhao, G. Liu and W. Wang, *Smooth pullback attractors for a non-autonomous 2D non-Newtonian fluid and their tempered behaviors*, J. Math. Fluid Mech., 2014, 16, 243–262.

[27] C. Zhao, Y. Li, L. Yang and M. Zhang, *Pullback attractor and invariant measure for the equations of Ladyzhenskaya fluid*, Acta Mathemstica Sinica (Chinese series), 2018, 61, 1–12.