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THE VALUE DISTRIBUTION OF MEROMORPHIC SOLUTIONS OF SOME SECOND ORDER NONLINEAR DIFFERENCE EQUATION*

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Abstract In this paper, we investigate some properties of solutions for some nonlinear difference equation, and obtain some estimates of the exponent of convergence of poles and growth of its transcendental meromorphic solutions $f(z)$ and its difference $\Delta f(z)$. Moreover, we study the existence and forms of rational solutions. We also give some examples to support our theoretical discussion.

Keywords Difference equation, meromorphic solution, growth, rational solutions.

MSC(2010) 30D35, 39B12.

1. Introduction

In this paper, we assume that the reader is familiar with the standard notations and basic results of Nevanlinna's value distribution theory (see [14,19]). In addition, we use the notions $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$, $\lambda(f)$ and $\lambda(\frac{1}{f})$ to denote the exponents of convergence of zeros and poles of $f(z)$, respectively, $\tau(f)$ to denote the exponent of convergence of fixed points of $f(z)$. A meromorphic function α is a small function with respect to f, if $T(r,\alpha) = S(r,f)$, where $S(r,f)$ is used to denote any quantity that satisfies $S(r,f) = o(T(r,f))$, where $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure.

Let c be a fixed non-zero complex number. Then the forward difference $\Delta_c^n f$ for each $n \in \mathbb{N}$ is defined in the standard way [23] by

$$\Delta_c f(z) = f(z+c) - f(z),$$
$$\Delta_c^n f(z) = \Delta_c(\Delta_c^{n-1} f(z)) = \Delta_c^{n-1} f(z+c) - \Delta_c^{n-1} f(z), n \geq 2.$$

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In particular, if $c = 1$, we use the usual difference notation $\Delta_c f(z) = \Delta f(z)$.

The theory of difference equations, the methods used in their solutions, and their wide applications have advanced beyond their adolescent stage to occupy a central position in applicable analysis. In fact, in the last twenty years, the proliferation of the subject is witnessed by numerous research articles and several monographs, annual international conferences, and new journals devoted to the study of difference equations and their applications. In recent years, some papers [3,4,11] investigated the properties of the second order difference equations

$$x_{n+1} = \frac{ax_{n-1}}{b + cx_n x_{n-1}}. \tag{1.1}$$

And equation (1.1) is a special case of the family of rational difference equations

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + \gamma x_{n-1}}{A + Bx_n x_{n-1} + Cx_{n-1}},$$

recently investigated by Amleh, Camouzis and Ladas [2] for nonnegative values of both the parameters and the initial conditions. In [21] and [22], Suzuki have studied the equation of the form

$$u(t + 2) = f(u(t), u(t + 1))$$

of an economics and a population model, in which f is a rational function.

In 2000, Ablowitz, Halburd and Herbst [1] looked for a purely complex analytic analogue of the Painlevé property for discrete equations. Later on, as the difference analogues of Nevanlinna's theory are being investigated [10, 12], there are many interests in the complex analytic properties of meromorphic solutions of complex difference equations, and many results on the complex difference equations are got rapidly, such as [6–10, 13, 15–18, 20, 24]. In particular, In [16], Ishizaki gave some surveys of basic properties of difference Riccati equation

$$y(z + 1) = \frac{A(z) + y(z)}{1 - y(z)},$$

where $A(z)$ is a rational function, which are analogues in the differential case [5]. In [6], Chen investigated the growth and poles and fixed points of transcendental solutions, and they proved the following theorem.

Theorem 1.1. *Let $P(z)$, $Q(z)$, $R(z)$ be polynomials with $P(z)Q(z)R(z) \not\equiv 0$, and $y(z)$ is a finite order transcendental meromorphic solution of the Pielou logistic equation*

$$y(z + 1) = \frac{P(z)y(z)}{Q(z) + R(z)y(z)}.$$

Then

$$\lambda\left(\frac{1}{y}\right) = \sigma(y) \geq 1.$$

On the other hand, the theory of complex differences and difference equations has many backgrounds in physics. Many physical models can be abstracted as difference equations, and properties of some special functions can be illustrated by relative difference equations. In particular, difference Painlevé equations find a wide range of applications in quantum mechanics. Complex difference equation is the inevitable development from real domain to complex domain, and is also the inevitable development from discontinuous discrete equations to new continuous difference equation. So that, a natural question is, we can say what about the difference equation (1.1) in complex domain. The following examples maybe give us some ideas.

Example 1.1. The function $f(z) = \tan \frac{\pi}{2} z$ satisfies the second nonlinear difference equation

$$f(z+2) = \frac{-zf(z)}{z + 2zf(z)f(z+1)},$$

where $f(z)$ has two Borel exceptional values i and $-i$.

Example 1.2. The function $f(z) = \frac{\cot \frac{\pi}{2} z}{z}$ satisfies the second nonlinear difference equation

$$f(z+2) = \frac{-f(z)}{1 + 2(z+1)^2 f(z)f(z+1)},$$

where $f(z)$ has no Borel exceptional value.

Example 1.2 and Example 1.3 remind us to consider the value distributions of the solution $f(z)$ and its difference $\Delta f(z)$ of the equation

$$f(z+2) = \frac{P(z)f(z)}{Q(z) + R(z)f(z)f(z+1)}. \tag{1.2}$$

In this paper, we investigate the existence and forms of rational solutions, and α-points, poles and growth of transcendental meromorphic solutions $f(z)$ for the nonlinear difference equation (1.2), and obtain some estimates of exponents of convergence of poles, and growth of $f(z)$ and differences $\Delta f(z)$ of meromorphic solutions of (1.2). We prove the following theorems.

Theorem 1.2. *Let $P(z)$, $Q(z)$, $R(z)$ be polynomials such that*

$$P(z)Q(z)R(z) \not\equiv 0.$$

Then every finite order transcendental meromorphic solution $f(z)$ of the nonlinear difference equation (1.2) satisfies

(i) $\lambda(\frac{1}{f}) = \sigma(f) \geq 1$;

(ii) $\lambda\left(\frac{1}{\Delta f(z)}\right) = \sigma(\Delta f(z)) = \sigma(f)$.

Theorem 1.3. *Let $P(z)$, $Q(z)$, $R(z)$ be polynomials such that*

$$\deg R(z) > \max\{\deg P(z), \deg Q(z)\}. \tag{1.3}$$

(i) If the difference equation (1.2) has a irreducible rational solution $f(z) = \frac{S(z)}{T(z)}$ with $\deg S(z) = s$ and $\deg T(z) = t$, then

$$t - s = \frac{1}{2}(\deg R(z) - \deg P(z)) \quad or \quad t - s = \frac{1}{2}(\deg R(z) - \deg Q(z)).$$

(ii)If $f(z)$ is a finite order transcendental meromorphic solution of the nonlinear difference equation (1.2), then

(a) $f(z)$ *has at most one Nevanlinna exceptional value* 0*;*

(b) $\tau(f(z+n)) = \sigma(f(z)), n = 0, 1, 2, \cdots$.

Remark 1.1. Generally, for a constant $c \neq 0$, $\tau(f(z+c)) \neq \tau(f(z))$ for a meomorphic function $f(z)$ of finite order. For example, the function $f(z) = e^z + z - c$ satisfies

$$\tau(f(z+c)) = 0 \neq \tau(f(z)) = 1.$$

Example 1.3. The difference equation

$$f(z+2) = \frac{f(z)}{(z^2+3z+3) - z(z-1)(z^2+3z+3)f(z)f(z+1)}$$

has a rational solution $f(z) = \frac{1}{z^2-z+1}$, where $\deg S(z) = 0, \deg T(z) = 2$ such that $t - s = 2 = \frac{1}{2}(\deg R(z) - \deg P(z))$.

Example 1.4. The difference equation

$$f(z+2) = \frac{zf(z)}{z(z+2) - z(z^2-1)(z+2)f(z)f(z+1)}$$

has a rational solution $f(z) = \frac{1}{z}$, where $\deg S(z) = 0, \deg T(z) = 1$ such that $t - s = 1 = \frac{1}{2}(\deg R(z) - \deg Q(z))$.

2. Lemmas for proof of Theorems

First we need the following lemmas for the proof of Theorem 1.4.

Lemma 2.1 ([10]). *Let* $f(z)$ *be a meromorphic function with order* $\sigma = \sigma(f), \sigma < +\infty$, *and let* η *be a fixed non zero complex number, then for each* $\varepsilon > 0$, *we have*

$$T(r, f(z+\eta)) = T(r, f) + O\left(r^{\sigma-1+\varepsilon}\right) + O(\log r).$$

Lemma 2.2 ([10]). *Let* $f(z)$ *be a meromorphic function of finite order and let* c *be a nonzero complex constant. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f).$$

Lemma 2.3 ([10]). *Let* $f(z)$ *be a meromorphic function with exponent of convergence of poles* $\lambda(\frac{1}{f}) = \lambda < +\infty$, *and let* $\eta \neq 0$ *be a fixed complex number, then for each* $\varepsilon\,(0 < \varepsilon < 1)$,

$$N(r, f(z+\eta)) = N(r, f(z)) + O\left(r^{\lambda-1+\varepsilon}\right) + O(\log r).$$

Lemma 2.4 (Valiron-Mohonko [19]). *Let* $f(z)$ *be a meromorphic function. Then for all irreducible rational functions in* f

$$R(z, f(z)) = \frac{a_n(z)f(z)^n + \cdots + a_0(z)f(z)}{b_m(z)f(z)^m + \cdots + b_0(z)f(z)}$$

with meromorphic coefficients $a_i(z), b_j(z)$ *being small with respect to* f, *the characteristic function of* $R(z, f(z))$ *satisfies*

$$T(r, R(z, f(z))) = \max\{n, m\}T(r, f) + S(r, f).$$

Lemma 2.5 ([12,20]). *Let $w(z)$ is a nonconstant finite order transcendental meromorphic solution of the difference equation of*

$$P(z,w)=0,$$

where $P(z,w)$ is a difference polynomial in $w(z)$. If $P(z,\alpha)\not\equiv 0$ for a meromorphic function $\alpha(z)$ satisfying $T(r,\alpha)=S(r,w)$, then

$$m\left(r,\frac{1}{w-\alpha}\right)=S(r,w)$$

holds for all r outside of a possible exceptional set with finite logarithmic measure.

Lemma 2.6 ([20]). *Let $f(z)$ be a transcendental meromorphic solution with finite order σ of a difference equation of the form*

$$U(z,f)P(z,f)=Q(z,f),$$

where $U(z,f)$, $P(z,w)$, $Q(z,f)$ are difference polynomials such that the total degree $\deg_f U(z,f)=n$ in $f(z)$ and its shifts and $\deg_f Q(z,f)\le n$. If $U(z,f)$ contains just one term of maximal total degree in $f(z)$ and its shifts, then then for each $\varepsilon>0$,

$$m(r,P(z,f))=O\left(r^{\sigma-1+\varepsilon}\right)+S(r,f).$$

3. Proof of Theorems

Proof of Theorem 1.2.

Suppose that $f(z)$ is a finite order transcendental meromorphic solution of (1.2).

(i) First, we prove that $\sigma(f)\ge 1$. Conversely, we suppose that $\sigma(f)<1$. Set $y(z)=f(z)f(z+1)$, if $y(z)$ is also transcendental, by Lemma 2.1, we obtain that $T(r,y)\le 2T(r,f)+O(r^{\sigma-1+\varepsilon})+O(\log r)$. Thus,

$$\sigma(y)\le\sigma(f)<1. \tag{3.1}$$

On the other hand, by (1.2), the function $y(z)=f(z)f(z+1)$ solves the Pielou logistic equation

$$y(z+1)=\frac{P(z)y(z)}{Q(z)+R(z)y(z)}.$$

By Theorem 1.1, we see that $\sigma(y)\ge 1$, which contradicts (3.1). If $y(z)=f(z)f(z+1)$ is a rational function, then the equation (1.2) degenerate into the linear difference equation as

$$f(z+2)=\frac{A(z)}{B(z)}f(z),$$

where $A(z)$ and $B(z)$ are nonzero polynomials. We rewrite the above equation as

$$B(z)f(z+2)-A(z)f(z)=0.$$

Noticing that $f(z)$ is a transcendental meromorphic function with $\sigma(f)<1$, by the Lemma 2.1 in [6], we obtain that $A(z)\equiv B(z)\equiv 0$, a contradiction. Hence, $\sigma(f)\ge 1$.

Secondly, we prove that $\lambda(\frac{1}{f}) = \sigma(f)$. Set $y_1(z) = \frac{1}{f(z)}$, then $y_1(z)$ is transcendental, $T(r, y_1) = T(r, f) + O(\log r)$, and $S(r, y_1) = S(r, f)$. Substituting $\frac{1}{y_1(z)}$ into (1.2), we obtain

$$E(z, y_1) = Q(z)y_1(z)y_1(z+1) + R(z) - P(z)y_1(z+2)y_1(z+1) = 0.$$

Combining with the condition $P(z)Q(z)R(z) \not\equiv 0$, we have

$$E(z, 0) = R(z) \not\equiv 0. \tag{3.2}$$

Thus, by Lemma 2.5,

$$N\left(r, \frac{1}{y_1}\right) = T(r, y_1) + S(r, y_1)$$

holds for all r outside of a possible exceptional set with finite logarithmic measure, that is,

$$N(r, f) = T(r, f) + S(r, f)$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. Thus,

$$\lambda\left(\frac{1}{f}\right) = \sigma(f). \tag{3.3}$$

(ii) By (1.2), we have

$$f(z+2)Q(z) + R(z)f(z)f(z+1)f(z+2) - P(z)f(z) = 0. \tag{3.4}$$

By the definition of the forward difference $\Delta_c^n f$, we see that

$$f(z+1) = \Delta f(z) + f(z), f(z+2) = \Delta f(z+1) + \Delta f(z) + f(z). \tag{3.5}$$

Substituting (3.5) into (3.4) and rearranging them, we obtain

$$R(z)f(z)^3 = [A(z, f) + B(z, f)f(z)]f(z) - Q(z)[\Delta f(z+1) + \Delta f(z)], \tag{3.6}$$

where

$$\begin{cases} A(z, f) = P(z) - Q(z) - R(z)[\Delta f(z)\Delta f(z+1) + (\Delta f(z))^2], \\ B(z, f) = [2\Delta f(z) + \Delta f(z+1)]R(z). \end{cases} \tag{3.7}$$

Since

$$N(r, \Delta f(z+1)) \leq N(r+1, \Delta f(z)) + o(N(r+1, \Delta f(z))),$$

there exist a set E having finite logarithmic measure such that for a large $r \notin E$,

$$N(r, \Delta f(z+1)) \leq N(r, \Delta f(z)) + o(N(r, \Delta f(z))), \tag{3.8}$$

By (3.7) and (3.8), we have that

$$\begin{aligned} N(r, A(z, f)) &\leq N(r, \Delta f(z)\Delta f(z+1)) + N(r, \Delta f(z)^2) + O(\log r) \\ &= 3N(r, \Delta f(z)) + N(r, \Delta f(z+1)) + O(\log r) \\ &\leq 4N(r, \Delta f(z)) + o(N(r, \Delta f(z))), \end{aligned} \tag{3.9}$$

and

$$N(r, B(z, f)) \leq 2N(r, \Delta f(z)) + o(N(r, \Delta f(z))). \tag{3.10}$$

Thus, by (3.6) and (3.8)-(3.10), we obtain

$$\begin{aligned} 3N(r, f) &= N(r, [A(z, f) + B(z, f)f(z)]f(z) - Q(z)[\Delta f(z+1) + \Delta f(z)]) \\ &\leq N(r, A(z, f)) + N(r, B(z, f)f(z)) + N(r, f) \\ &\quad + N(r, Q(z)[\Delta f(z+1) + \Delta f(z)]) + O(1) \\ &\leq 8N(r, \Delta f(z)) + 2N(r, f) + o(N(r, \Delta f(z))). \end{aligned}$$

Hence,

$$N(r, f) \leq 8N(r, \Delta f(z)) + o(N(r, \Delta f(z))).$$

So that, we see that

$$\lambda\left(\frac{1}{\Delta f}\right) \geq \lambda\left(\frac{1}{f}\right). \tag{3.11}$$

On the other hand, by Lemma 2.1 and Lemma 2.3, we easily have that

$$\sigma(\Delta f) \leq \sigma(f). \tag{3.12}$$

$$\lambda\left(\frac{1}{\Delta f}\right) \leq \sigma(\Delta f). \tag{3.13}$$

By (3.3) and (3.11)-(3.13), we see that

$$\lambda\left(\frac{1}{\Delta f}\right) = \sigma(\Delta f) = \sigma(f).$$

Thus, Theorem 1.2 is proved. □

Proof of Theorem 1.3. (i) Suppose that $f(z) = \frac{S(z)}{T(z)}$ is a irreducible rational solution of (1.2). Substituting $f(z) = \frac{S(z)}{T(z)}$ into (1.2), we see that

$$Q(z)S(z+2)T(z+1)T(z) + R(z)S(z+2)S(z+1)S(z) - P(z)T(z+2)T(z+1)S(z) = 0. \tag{3.14}$$

If $s \geq t$, then $\deg R(z)S(z+2)S(z+1)S(z)$ is the only maximal degree in (3.14), it is impossible.

If $s < t$, by (3.14), we have

$$\deg R(z)S(z+2)S(z+1)S(z) = \deg Q(z)S(z+2)T(z+1)T(z)$$

or

$$\deg R(z)S(z+2)S(z+1)S(z) = \deg P(z)T(z+2)T(z+1)S(z).$$

Hence we have that

$$t - s = \frac{1}{2}(\deg R(z) - \deg Q(z)) \quad \text{or} \quad t - s = \frac{1}{2}(\deg R(z) - \deg P(z)).$$

(ii) Suppose that $f(z)$ is a finite-order transcendental meromorphic solution of (1.2). First, we prove that α is not a Nevanlinna exceptional value of $f(z)$ for all nonzero complex number α. Set $y(z) = f(z) - \alpha$, then $y(z)$ is transcendental,

$T(r,y)=T(r,f)+O(\log r)$, and $S(r,y)=S(r,f)$. Substituting $y(z)+\alpha$ into (1.2), we obtain

$$E(z,y)=[y(z+2)+\alpha]\{Q(z)+R(z)[y(z)+\alpha][y(z+1)+\alpha]\}-P(z)[y(z)+\alpha]=0.$$

Thus,

$$E(z,0)=\alpha Q(z)+\alpha^3 R(z)-\alpha P(z). \tag{3.15}$$

By the assumed condition (1.3), (3.15) and $\alpha\neq 0$, we have that $E(z,0)\not\equiv 0$. Thus, by Lemma 2.5,

$$N\left(r,\frac{1}{y}\right)=T(r,y)+S(r,y)$$

holds for all r outside of a possible exceptional set with finite logarithmic measure, that is,

$$N\left(r,\frac{1}{f(z)-\alpha}\right)=T(r,f)+S(r,f).$$

So that

$$\delta(\alpha,f)=1-\overline{\lim_{r\to\infty}}\frac{N(r\frac{1}{f-\alpha})}{T(r,f)}=0.$$

Hence, α is not a Nevanlinna exceptional value of $f(z)$.

Secondly, we prove that ∞ is not a Nevanlinna exceptional value of $f(z)$ too. Set $y(z)=\frac{1}{f(z)}$, then $y(z)$ is transcendental, $T(r,y)=T(r,f)+O(\log r)$, and $S(r,y)=S(r,f)$. Substituting $\frac{1}{y(z)}$ into (1.2), Using the same method as above, we can prove the conclusion. Thus, $f(z)$ only has zero as its possible Nevanlinna exceptional value. Hence, $f(z)$ has at most one Nevanlinna exceptional value 0.

Lastly, we prove that $\tau(f(z))=\sigma(f(z))$. Set $g(z)=f(z)-z$, then $g(z)$ is transcendental, $T(r,g)=T(r,f)+O(\log r)$, and $S(r,g)=S(r,f)$. Substituting $f(z)=g(z)+z, f(z+1)=g(z+1)+z+1$ and $f(z+2)=g(z+2)+z+2$ into (1.2), we have that

$$E_0(z,g)=[g(z+2)+z+2]\{Q(z)+R(z)[g(z)+z][g(z+1)+z+1]\}-P(z)[g(z)+z].$$

Thus, $E_0(z,0)=(z+2)Q(z)+z(z+1)(z+2)R(z)-zP(z)$. By (1.3), we see that

$$\deg[z(z+1)(z+2)R(z)]>\max\{\deg[zP(z)],\deg[(z+2)Q(z)]\}.$$

Thus, $E_0(z,0)\not\equiv 0$, combining Lemma 2.5, we obtain

$$N\left(r,\frac{1}{g}\right)=T(r,g)+S(r,y)$$

holds for all r outside of a possible exceptional set with finite logarithmic measure, that is,

$$N\left(r,\frac{1}{f(z)-z}\right)=T(r,f)+S(r,f).$$

It gives that $\tau(f(z))=\sigma(f(z))$.

For $n=1,2,\cdots$, we set $g_n(z)=f(z+n)-z$, then $g_n(z)$ is transcendental, $T(r,g_n)=T(r,f)+O\left(r^{\sigma-1+\varepsilon}\right)+O(\log r)$, and $S(r,g_n)=S(r,f)$. Substituting

$f(z+n-1) = g_n(z-1)+z-1, f(z+n) = g_n(z)+z$ and $f(z+n+2) = g_n(z+1)+z+1$ into

$$f(z+n+1) = \frac{P(z+n-1)f(z+n-1)}{Q(z+n-1)+R(z+n-1)f(z+n-1)f(z+n)},$$

and using the same method as above, we can prove that $\tau(f(z+n)) = \sigma(f(z)), n = 1, 2, \cdots$.

Thus, Theorem 1.3 is proved. $\square$

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