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ON SOLVABILITY OF A CLASS OF NONLINEAR ELLIPTIC TYPE EQUATION WITH VARIABLE EXPONENT

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Abstract In this paper, we study the Dirichlet problem for the implicit degenerate nonlinear elliptic equation with variable exponent in a bounded domain $\Omega \subset \mathbb{R}^n$. We obtain sufficient conditions for the existence of a solution without regularization and any restriction between the exponents. Furthermore, we define the domain of the operator generated by posed problem and investigate its some properties and also its relations with known spaces that enable us to prove existence theorem.

Keywords PDEs with nonstandart nonlinearity, solvability theorem, variable exponent, implicit degenerate PDEs.

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1. Introduction

In this work, we investigate the Dirichlet problem for the nonlinear elliptic equation with variable nonlinearity

$$\begin{cases} -\Delta\left(|u|^{p(x)-2}u\right)+a(x,u)=h(x), \\ u\mid_{\partial\Omega}=0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is a bounded domain which has sufficiently smooth boundary (at least Lipschitz boundary) and $p : \Omega \longrightarrow \mathbb{R}$, $2 \leq p^- \leq p(x) \leq p^+ < \infty$, $p \in C^1(\bar{\Omega})$. Also $a : \Omega \times \mathbb{R} \to \mathbb{R}$, $a(x,\tau)$ is a function with variable nonlinearity in τ (for example $a(x,u) = a_0(x,u)|u|^{\xi(x)-1}u + a_1(x,u)$, see Section 4).

In recent years, there has been an increasing interest in the study of equations with variable exponents of nonlinearities. The interest in the study of differential equations that involves variable exponents is motivated by their applications to the theory of elasticity and hydrodynamics, in particular the models of electrorheological fluids [9, 28] in which a substantial part of viscous energy, the thermistor problem [38], image processing [8] and modeling of non-Newtonian fluids with thermo-convective effects [5] etc.

The main feature in the equation

$$-\ \Delta\left(|u|^{\alpha(x)-2}u\right)+a(x,u)=h(x) \tag{1.2}$$

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is clearly the exponential nonlinearity with respect to the solution that makes it implicit degenerate. Such equations may appear, for instance, in the mathematical description of the process of nonstable filtration of an ideal barotropic gas in a nonhomogeneous porous medium. The equation of state of the gas has the form $p = \rho^{\alpha(x)}$ where p is the pressure, ρ is the density, and the exponent $\alpha(x)$ is a given function then by using the known physical laws in that case, we obtain an equation in the form of (1.2) (for sample see [4]). For the several of the most important applications of nonlinear partial differential equations with variable exponent arise from mathematical modelling of suitable processes in mechanics, mathematical physics, image processings etc., we refer to [23] (see also [21, 22, 28]).

For some cases in gas dynamics as mentioned above, Lagrangian function f in the definition of integral functional

$$F_\Omega(u) = \int_\Omega f(x, u, Du)\, dx$$

may satisfy the general nonstandard growth condition of the type

$$c_0 |\tau|^{m(x)} - a |y|^{\xi(x)} - g(x) \leq f(x, y, \tau) \leq c_1 |\tau|^{m_0(x)} + a |y|^{\xi(x)} + g(x)$$

with $1 < m(x) \leq m_0(x)$ and $m(x) \leq \xi(x)$ where all exponents are continuous functions over $\bar{\Omega}$. In [40] Zhikov gives an example which shows that if f satisfies such type of inequality then appropriate functional defined by f may have the Lavrentiev phenomenon, the minimizer of the functional is irregular. As known, it has important applications in mechanics, there are too many papers in variational problems which has been devoted to the case that f holds this type condition [1, 16, 20, 40].

Also we note that the relation between the weak solutions of the class of elliptic equations $-\mathrm{div} A(x, u, Du) = B(x, u, Du)$ and minimizer of the functional F_Ω under the nonstandard growth condition given above was studied in [2, 13].

Recently, problems which are similar to (1.1) have been studied in a lot of papers [5–7, 14, 24, 25, 37]. In [39] Zhikov investigated elliptic problem such as

$$\begin{cases} \Delta_{p(x)} u = \mathrm{div} g, \\ u \mid_{\partial\Omega} = 0, \end{cases}$$

here $g \in (L^\infty(\Omega))^d$, $\Delta_{p(x)}$ is $p(x)$-Laplacian and $\Omega \subset \mathbb{R}^d$ is bounded Lipschitz domain. He established the weak solution of the considered problem with using the sequence of solutions of the problems which converges to considered problem.

In [8] authors have studied the problem related to image recovery. To investigate that problem they first considered the elliptic part of the problem namely they investigated the minimization problem

$$\min_{u \in BV \cap L^2(\Omega)} \int_\Omega \phi(x, Du) + \frac{\lambda}{2}(u - I)^2,$$

and proved the existence of solution in more general class by using variational method. Here BV is the space of functions of bounded variation defined on Ω.

In [7] authors have considered the Dirichlet boundary value problem for the elliptic equation

$$-\sum_i D_i \left(a_i(x, u) |D_i u|^{p_i(x)-2} D_i u \right) + c(x, u) |u|^{\sigma(x)-2} u = f(x).$$

Under sufficient conditions they showed the existence of weak solution by using Browder-Minty theorem for the special case $a_i(x,u) \equiv A_i(x)$, $c(x,u) \equiv C(x)$. In the general case, the solution was constructed via Galerkin's method under additional conditions for $a_i(x,u)$ and $c(x,u)$.

In the most of these papers we mentioned above, authors have studied the problems which involves $p(.)$-Laplacian type equation by using monotonicity methods. To the best of our knowledge, by now there are no results on the existence of solutions to the elliptic equations of the type (1.1) with nonconstant exponents of nonlinearity. However similar type problem to (1.1) was studied in [6] and authors investigated the regularized problem to show the existence of weak solution. In the present paper, we investigate the problem (1.1) without regularization. We also note that earlier Dubinskii [10] (for details see [19]) investigated problems which are similar to (1.1) for constant exponents and obtained existence results. Afterward, Raviart [27] obtained some results on uniqueness of solution for this type problems.

Here we prove the existence of sufficiently smooth, in some sense, solution of the problem (1.1). Unlike the above papers, we investigate (1.1) without monotonicity type conditions. Since we consider the posed problem under more general (weak) conditions, in that case any method which is related to monotonicity can not be used. Therefore we use a different method to investigate the problem (1.1). We show that considered problem is homeomorphic to the following problem:

$$\begin{cases} \sum\limits_{i=1}^{n} -D_i\left(|u|^{p_0-2} D_i u\right) + c(x,u) = h(x), \\ u \mid_{\partial\Omega} = 0, \end{cases} \tag{1.3}$$

(see Section 3) and using this fact, we obtain existence of solution of problem (1.1) (Section 4).

Moreover we study the posed problem in the space, that generated by this problem. Investigating most of boundary value problem on its own space leads to obtain better results. Henceforth here considered problem is investigated on its own space. Unlike linear boundary value problems, the sets generated by nonlinear problems are subsets of linear spaces, but not possessing the linear structure [29–36].

This paper is organized as follows: In the next section, we recall some useful results on the generalized Orlicz-Lebesgue spaces (Subsec. 2.1) and results on nonlinear spaces (pn-spaces) (Subsec. 2.2). In Section 3, under the sufficient conditions we show the existence of weak solution for the problem (1.3). In Section 4, we give some additional results which are required for existence theorem (Subsec. 4.1) and prove existence of a generalized solution for the main problem (1.1) (Subsec. 4.2).

2. Preliminaries

2.1. Generalized Lebesgue spaces

In this subsection, some available facts from the theory of the generalized Lebesgue spaces also called Orlicz-Lebesgue spaces will be introduced. We present these facts without proofs which can be found in [11, 12, 17, 18].

Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$ such that $|\Omega| > 0$. (Throughout the paper, we denote by $|\Omega|$ the Lebesgue measure of Ω). By $P(\Omega)$ we denote the family of all measurable functions $p : \Omega \longrightarrow [1, \infty]$.

For $p \in P(\Omega)$, $\Omega_\infty^p \equiv \Omega_\infty \equiv \{x \in \Omega |\ p(x) = \infty\}$ then on the set of all functions on Ω define the functional σ_p and $\|.\|_p$ by

$$\sigma_p(u) \equiv \int_{\Omega \backslash \Omega_\infty} |u|^{p(x)}\, dx + \underset{\Omega_\infty}{ess}\sup |u(x)|$$

and

$$\|u\|_{L^{p(x)}(\Omega)} \equiv \inf\left\{\lambda > 0 |\ \sigma_p\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

Clearly if $p \in L^\infty(\Omega)$ then

$$1 \leq p^- \equiv \underset{\Omega}{ess}\inf |p(x)| \leq \underset{\Omega}{ess}\sup |p(x)| \equiv p^+ < \infty$$

in that case we have

$$\sigma_p(u) \equiv \int_\Omega |u|^{p(x)}\, dx.$$

The Generalized Lebesgue space is defined as follows:

$$L^{p(x)}(\Omega) := \{u : u \text{ is a measurable real-valued function such that } \sigma_p(u) < \infty\}.$$

The space $L^{p(x)}(\Omega)$ becomes a Banach space under the norm $\|.\|_{L^{p(x)}(\Omega)}$which is so-called Luxemburg norm.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $p \in L^\infty(\Omega)$ then Generalized Sobolev space is defined as follows:

$$W^{m,p(x)}(\Omega) \equiv \left\{u \in L^{p(x)}(\Omega) \,|\, D^\alpha u \in L^{p(x)}(\Omega),\ \ |\alpha| \leq m\right\}$$

and this space is separable Banach space under the norm:

$$\|u\|_{W^{m,p(x)}(\Omega)} \equiv \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^{p(x)}(\Omega)}.$$

The following results are known for these spaces: [11, 18, 26].

Lemma 2.1. *Let* $0 < |\Omega| < \infty$, *and* $p_1, p_2 \in P(\Omega)$ *then*

$$L^{p_1(x)}(\Omega) \subset L^{p_2(x)}(\Omega) \iff p_2(x) \leq p_1(x) \ \textit{for a.e } x \in \Omega.$$

Lemma 2.2. *The dual space of* $L^{p(x)}(\Omega)$ *is* $L^{p^*(x)}(\Omega)$ *if and only if* $p \in L^\infty(\Omega)$. *The space* $L^{p(x)}(\Omega)$ *is reflexive if and only if*

$$1 < p^- \leq p^+ < \infty,$$

here

$$p^*(x) \equiv \begin{cases} \infty, & \textit{for } x \in \Omega_1^p, \\ 1, & \textit{for } x \in \Omega_\infty^p, \\ \frac{p(x)}{p(x)-1}, & \textit{for other } x \in \Omega. \end{cases}$$

Lemma 2.3. *Let* $p, q \in C(\bar{\Omega})$ *and* $p, q \in L^\infty(\Omega)$. *Assume that*

$$mp(x) < n,\ q(x) < \frac{np(x)}{n - mp(x)},\ \forall x \in \bar{\Omega}.$$

Then there is a continuous and compact embedding $W^{m,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

2.2. On pn-spaces

In this subsection, we introduce some function classes which are complete metric spaces and directly connected to the considered problem. Also we give some embedding results for these spaces [31,35] (see also [29,30,32,34,36]).

Definition 2.1. Let $\alpha \geq 0$, $\beta \geq 1$, $\varrho = (\varrho_{1,..,}\varrho_n)$ is multi-index, $m \in \mathbb{Z}^+$, $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is bounded domain with sufficiently smooth boundary.

$$S_{m,\alpha,\beta}(\Omega) \equiv \left\{ u \in L_1(\Omega) \mid [u]^{\alpha+\beta}_{S_{m,\alpha,\beta}(\Omega)} \equiv \sum_{0\leq|\varrho|\leq m} \left(\int_\Omega |u|^\alpha |D^\varrho u|^\beta \, dx \right) < \infty \right\}$$

in particularly,

$$\mathring{S}_{1,\alpha,\beta}(\Omega) \equiv \left\{ u \in L_1(\Omega) \mid [u]^{\alpha+\beta}_{S_{1,\alpha,\beta}(\Omega)} \equiv \sum_{i=1}^{n} \left(\int_\Omega |u|^\alpha |D_i u|^\beta \, dx \right) < \infty \right\} \cap \{ u \mid_{\partial\Omega} \equiv 0 \}.$$

These spaces are called pn-spaces.[‡]

Theorem 2.1. *Let $\alpha \geq 0$, $\beta \geq 1$ then $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, $\varphi(t) \equiv |t|^{\frac{\alpha}{\beta}} t$ is a homeomorphism between $S_{1,\alpha,\beta}(\Omega)$ and $W^{1,\beta}(\Omega)$.*

Theorem 2.2. *The following embeddings are satisfied:*

(i) Let α, $\alpha_1 \geq 0$ and $\beta_1 \geq 1$, $\beta \geq \beta_1$, $\frac{\alpha_1}{\beta_1} \geq \frac{\alpha}{\beta}$, $\alpha_1 + \beta_1 \leq \alpha + \beta$ then we have

$$\mathring{S}_{1,\alpha,\beta}(\Omega) \subseteq \mathring{S}_{1,\alpha_1,\beta_1}(\Omega).$$

(ii) Let $\alpha \geq 0$, $\beta \geq 1$, $n > \beta$ and $\frac{n(\alpha+\beta)}{n-\beta} \geq r$ then there is a continuous embedding

$$\mathring{S}_{1,\alpha,\beta}(\Omega) \subset L^r(\Omega).$$

Furthermore for $\frac{n(\alpha+\beta)}{n-\beta} > r$ the embedding is compact.

(iii) If $\alpha \geq 0$, $\beta \geq 1$ and $p \geq \alpha + \beta$ then

$$W_0^{1,p}(\Omega) \subset \mathring{S}_{1,\alpha,\beta}(\Omega)$$

is hold.

Now we present general solvability result [33] (see also for similar theorems [29, 30, 34, 36]) that will be used to show existence of weak solution of the posed problem (1.1).

Definition 2.2. Let X, Y be Banach spaces, Y^* is the dual space of Y and S_0 is a weakly complete pn-space. $f : S_0 \subset X \longrightarrow Y$ a nonlinear mapping. f is a "coercive" operator in a generalized sense if there exists a bounded operator

[‡] $S_{1,\alpha,\beta}(\Omega)$ is metric space with the following metric: $\forall u, v \in S_{1,\alpha,\beta}(\Omega)$,

$$d_{S_{1,\alpha,\beta}}(u, v) = \left\| |u|^{\frac{\alpha}{\beta}} u - |v|^{\frac{\alpha}{\beta}} v \right\|_{W^{1,\beta}(\Omega)}.$$

$g : X_0 \subseteq S_0 \longrightarrow Y^*$ that satisfies the conditions, $\overline{X_0} = S_0$, $\overline{Img} = Y^*$ and a continuous function $\mu : \mathbb{R}^+ \longrightarrow \mathbb{R}$ non-decreasing such that the following relation is valid for a dual form $\langle .,. \rangle$ with respect to the pair of spaces (Y, Y^*):

$$\langle f(x), g(x) \rangle \geq \mu\left([x]_{S_0}\right) \text{ for } x \in X_0 \text{ and } \exists r > 0 \ni \mu(r) \geq 0.$$

In this case it is said that the mappings f and g generate a "coercive pair " on X_0.

Definition 2.3. Let X_0 be a topological space such that $X_0 \subset S_0 \subset X$, and let f be a nonlinear mapping acting from X to Y where Y is a reflexive space such that both Y and Y^* are strictly convex. An element $x \in S_0$ satisfying

$$\langle f(x), y^* \rangle = \langle y, y^* \rangle, \ \forall y^* \in M^* \subseteq Y^*, \ y \in Y, \tag{2.1}$$

the equation (2.1) is called a M^*-solution of the equation $f(x) = y$.

We will consider the following conditions:

(a) $f : S_0 \longrightarrow Y$ is a weakly compact (weakly continuous) mapping and there exists a closed linear subspace Y_0 of Y such that $f : S_0 \longrightarrow Y_0 \subseteq Y$.

(b) There exist a mapping $g : X_0 \subset S_0 \longrightarrow Y^*$ such that $g(X_0)$ contains a linear manifold from Y^* which is dense in a closed linear subspace Y_0^* of Y^* and generates a "coercive pair" with f on X_0 in a generalized sense.

Moreover one of the following conditions (1) or (2) hold:

(1) If g is a linear continuous operator then S_0 is a "reflexive" space [30] and X_0 is a separable vector topological space which is dense in S_0 and $\ker g^* = \{0\}$(where g^* denotes the adjoint of the linear continuous operator g).

(2) If g is a nonlinear operator then Y_0^* is a separable subspace of Y^* and g^{-1} is weakly continuous from Y^* to S_0.

Theorem 2.3. *Let the conditions (a),(b) and either (1) or (2) hold. Furthermore, assume that a set $Y_0 \subseteq Y$ is given such that for each $y \in Y_0$ the following condition is satisfied: there exists $r = r(y) > 0$ such that*

$$\mu([x]_{S_0}) \geq \langle y, g(x) \rangle, \ \forall x \in X_0, [x]_{S_0} \geq r.$$

Then equation (2.1) is Y_0^*-solvable in S_0 for any y from the subset Y_0 of Y.

3. Existence Results for Problem (1.3)

As mentioned in introduction, studying the existence of solution of the problem (1.1) requires to investigate problem (1.3) therefore firstly we give the existence results for problem

$$\begin{cases} \sum\limits_{i=1}^{n} -D_i\left(|u|^{p_0-2} D_i u\right) + c(x,u) = h(x), \\ u \mid_{\partial\Omega} = 0, \end{cases}$$

here $p_0 \geq 2$ and $c : \Omega \times \mathbb{R} \to \mathbb{R}$, $c(x,\tau)$ has a variable nonlinearity up to τ (see inequality (3.1)). Let the function $c(x,\tau)$ in problem (1.3) hold the following conditions:

(i) *$c : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Caratheodory function and for the measurable function $\alpha : \Omega \longrightarrow \mathbb{R}$ which satisfy $1 < \alpha^{-} \leq \alpha(x) \leq \alpha^{+} < \infty$, $c(x,\tau)$ holds the inequality*

$$|c(x,\tau)| \leq c_0(x)|\tau|^{\alpha(x)-1} + c_1(x), \ \ (x,\tau) \in \Omega \times \mathbb{R}, \tag{3.1}$$

here c_0, c_1 are nonnegative, measurable functions defined on Ω.

Since on different values of α^{+} depending on p_0 and n i.e. $\alpha^{+} < p_0$, $p_0 \leq \alpha^{+} < \tilde{p}$ and $\tilde{p} \leq \alpha^{+} < \infty$ where $\tilde{p}$ is critical exponent in Theorem 2.2 (ii) different conditions is required because of the circumstances appearing in the embedding theorems for these spaces hence we separate the domain Ω to three disjoint sets say Ω_1, Ω_2 and Ω_3 up to these cases. By doing this, we obtained more slightly sufficient conditions to show the existence of weak solution.[§]

Let $\eta \in (0,1)$ is sufficiently small and we define the sets

$$\begin{aligned} \Omega_1 &\equiv \{x \in \Omega | \ \alpha(x) \in [1, p_0 - \eta)\}, \\ \Omega_2 &\equiv \{x \in \Omega | \ \alpha(x) \in [p_0 - \eta, \tilde{p})\}, \\ \Omega_3 &\equiv \left\{x \in \Omega | \ \alpha(x) \in [\tilde{p}, \alpha^{+}]\right\}, \end{aligned}$$

here critical $\tilde{p} > p_0$ and will be defined later.

(ii) *There exists a measurable function $\alpha_1 : \Omega_2 \longrightarrow \mathbb{R}$ which satisfy $1 \leq \alpha_1^{-} \leq \alpha_1(x) \leq \alpha_1^{+} < p_0$, such that $c(x,\tau)$ holds the inequality*

$$c(x,\tau)\tau \geq -c_2(x)|\tau|^{\alpha_1(x)} - c_3(x), \ \ (x,\tau) \in \Omega_2 \times \mathbb{R}, \tag{3.2}$$

here c_2, c_3 are nonnegative, measurable functions defined on Ω_2.

(iii) *On $\Omega_3 \times \mathbb{R}$, $c(x,\tau)$ satisfies the inequality*

$$c(x,\tau)\tau \geq c_4(x)|\tau|^{\alpha(x)} - c_5(x), \ \ (x,\tau) \in \Omega_3 \times \mathbb{R}, \tag{3.3}$$

here α is the same function as in (3.1) *and $c_4(x) \geq \bar{C}_0 > 0$ a.e. $x \in \Omega_3$ and c_4, c_5 are nonnegative, measurable functions defined on Ω_3.*

We investigate the problem (1.3) for the functions $h \in W^{-1,q_0}(\Omega) + L^{\alpha^*(x)}(\Omega)$ where α^* is conjugate of α i.e. $\alpha^*(x) \equiv \frac{\alpha(x)}{\alpha(x)-1}$ and $q_0 \equiv \frac{p_0}{p_0-1}$.

Let us denote Q_0 by

$$Q_0 \equiv \mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega) \cap L^{\alpha(x)}(\Omega).$$

We understand the solution of the considered problem in the following sense.

Definition 3.1. A function $u \in Q_0$, is called the generalized solution (weak solution) of problem (1.3) if it satisfies the equality

$$\sum_{i=1}^{n} \int_{\Omega} \left(|u|^{p_0-2} D_i u\right) D_i w dx + \int_{\Omega} c(x,u) w dx = \int_{\Omega} h w dx$$

for all $w \in W_0^{1,p_0}(\Omega) \cap L^{\alpha(x)}(\Omega)$.

[§]Since Ω is separated to three disjoint subsets, in some sense, one can consider that problem as unity of three different problems.

Theorem 3.1. *Let (i)-(iii) hold. If* $c_2 \in L^{\frac{p_0}{p_0-\alpha_1(x)}}(\Omega_2)$, $c_3 \in L^1(\Omega_2)$, $c_5 \in L^1(\Omega_3)$, $c_4 \in L^\infty(\Omega_3)$, $c_1 \in L^{\beta_1(x)}(\Omega)$, $c_0 \in L^{\beta(x)}(\Omega)$ *where*

$$\beta_1(x) \equiv \begin{cases} \alpha^*(x), \text{ if } x \in \Omega_1, \\ q_0, \qquad \text{if } x \in \Omega_2 \cup \Omega_3, \end{cases}$$

and

$$\beta(x) \equiv \begin{cases} \frac{p_0\alpha^*(x)}{p_0-\alpha(x)}, \text{ if } x \in \Omega_1, \\ \frac{\tilde{p}\alpha^*(x)}{\tilde{p}-\alpha(x)}, \text{ if } x \in \Omega_2, \\ \infty, \qquad \text{if } x \in \Omega_3, \end{cases} \qquad \tilde{p} \equiv \frac{np_0}{n-q_0},$$

then $\forall h \in W^{-1,q_0}(\Omega) + L^{\alpha^*(x)}(\Omega)$ *problem* (1.3) *has a generalized solution in the space* Q_0.

The proof is based on Theorem 2.3. To use this, we introduce the following spaces and mappings in order to apply Theorem 2.3 to prove Theorem 3.1.

$$S_0 \equiv \mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega) \cap L^{\alpha(x)}(\Omega), \; Y \equiv W^{-1,q_0}(\Omega) + L^{\alpha^*(x)}(\Omega),$$
$$X_0 \equiv W_0^{1,p_0}(\Omega) \cap L^{\alpha(x)}(\Omega)$$

and

$$Y_0^* \equiv Y^* \equiv X_0,$$
$$f : S_0 \longrightarrow Y,$$
$$f(u) \equiv \sum_{i=1}^{n} -D_i\left(|u|^{p_0-2} D_i u\right) + c(x,u), \tag{3.4}$$
$$g : X_0 \subset S_0 \longrightarrow Y^*,$$
$$g \equiv Id. \tag{3.5}$$

We prove some lemmas to show that all conditions of Theorem 2.3 are fulfilled under the conditions of Theorem 3.1.

Lemma 3.1. *Under the conditions of Theorem 3.1, the mappings f and g defined by* (3.4) *and* (3.5) *respectively generate a "coercive pair" on* $W_0^{1,p_0}(\Omega) \cap L^{\alpha(x)}(\Omega)$.

Proof. Since $g \equiv Id$, being "coercive pair" equals to order coercivity of f on the space $W_0^{1,p_0}(\Omega) \cap L^{\alpha(x)}(\Omega)$. For $u \in W_0^{1,p_0}(\Omega) \cap L^{\alpha(x)}(\Omega)$

$$\begin{aligned}\langle f(u), u\rangle &= \sum_{i=1}^{n}\left(\int_\Omega |u|^{p_0-2}|D_i u|^2 dx\right) + \int_\Omega c(x,u)\,u dx \\ &= \sum_{i=1}^{n}\left(\int_\Omega |u|^{p_0-2}|D_i u|^2 dx\right) + \int_{\Omega_1} c(x,u)\,u dx \\ &\quad + \int_{\Omega_2} c(x,u)\,u dx + \int_{\Omega_3} c(x,u)\,u dx.\end{aligned}$$

Using (3.1), (3.2), (3.3), we obtain

$$\begin{aligned}\langle f(u), u\rangle \geq & \sum_{i=1}^{n}\left(\int_{\Omega}|u|^{p_0-2}\left|D_i u\right|^2 dx\right)-\int_{\Omega_1}\left|c_0(x)\right||u|^{\alpha(x)} dx-\int_{\Omega_1}\left|c_1(x)\right||u| dx \\ & -\int_{\Omega_2}\left|c_2(x)\right||u|^{\alpha_1(x)} dx-\int_{\Omega_2}\left|c_3(x)\right| dx \\ & +\int_{\Omega_3}\left|c_4(x)\right||u|^{\alpha(x)} dx-\int_{\Omega_3}\left|c_5(x)\right| dx. \end{aligned} \tag{3.6}$$

Let estimate the second, third and fourth integrals in (3.6) respectively. For arbitrary $\epsilon_i > 0$ $(i = 1, 2, 3)$ by using Young's inequality, we get

$$\begin{aligned}&\int_{\Omega_1}\left|c_0(x)\right||u|^{\alpha(x)} dx \\ \leq & \epsilon_1 \int_{\Omega_1}\left(\frac{\alpha(x)}{p_0}\right)|u|^{p_0} dx+\int_{\Omega_1}\left(\frac{1}{\epsilon_1}\right)^{\frac{\alpha(x)}{p_0-\alpha(x)}}\left(\frac{p_0-\alpha(x)}{p_0}\right)\left|c_0(x)\right|^{\frac{p_0}{p_0-\alpha(x)}} dx \\ \leq & \epsilon_1 \int_{\Omega_1}|u|^{p_0} dx+\left(\frac{1}{\epsilon_1}\right)^{\frac{p_0}{\eta}} \int_{\Omega_1}\left|c_0(x)\right|^{\frac{p_0}{p_0-\alpha(x)}} dx .\end{aligned}$$

Similarly, by **(ii)** and Young's inequality, we have the following estimate for the fourth integral,

$$\begin{aligned}&\int_{\Omega_2}\left|c_2(x)\right||u|^{\alpha_1(x)} dx \\ \leq & \epsilon_2 \int_{\Omega_2}\left(\frac{\alpha_1(x)}{p_0}\right)|u|^{p_0} dx+\int_{\Omega_2}\left(\frac{1}{\epsilon_2}\right)^{\frac{\alpha_1(x)}{p_0-\alpha_1(x)}}\left(\frac{p_0-\alpha_1(x)}{p_0}\right)\left|c_2(x)\right|^{\frac{p_0}{p_0-\alpha_1(x)}} dx \\ \leq & \epsilon_2 \int_{\Omega_2}|u|^{p_0} dx+\left(\frac{1}{\epsilon_2}\right)^{\frac{\alpha_1^+}{p_0-\alpha_1^+}} \int_{\Omega_2}\left|c_2(x)\right|^{\frac{p_0}{p_0-\alpha_1(x)}} dx\end{aligned}$$

and for the second one by applying Hölder-Young inequality, we get

$$\int_{\Omega_1}\left|c_1(x)\right||u| dx \leq \epsilon_3 \int_{\Omega_1}|u|^{p_0} dx+\left(\frac{1}{\epsilon_3}\right)^{p_0 q_0} \int_{\Omega_1}\left|c_1(x)\right|^{q_0} dx .$$

If we use these inequalities and condition **(iii)** in (3.6), we obtain

$$\begin{aligned}\langle f(u), u\rangle \geq & [u]_{\mathring{S}_{1,(p_0-2),2}(\Omega)}^{p_0}-\epsilon_4\|u\|_{L^{p_0}(\Omega_1 \cup \Omega_2)}^{p_0}+\bar{C}_0 \int_{\Omega_3}|u|^{\alpha(x)} dx \\ & -C_1(\epsilon_1)-C_2(\epsilon_2)-C_3(\epsilon_3)-\left\|c_3\right\|_{L^1(\Omega_2)}-\left\|c_5\right\|_{L^1(\Omega_3)} .\end{aligned}$$

Estimating the first and second terms on the right hand side of last inequality by Theorem 2.2, we obtain

$$\begin{aligned}\langle f(u), u\rangle \geq & \tilde{C}[u]_{\mathring{S}_{1,(p_0-2) q_0, q_0}(\Omega)}^{p_0}-\epsilon_4 C_4[u]_{\mathring{S}_{1,(p_0-2) q_0, q_0}(\Omega)}^{p_0}+\bar{C}_0 \int_{\Omega_3}|u|^{\alpha(x)} dx-K \\ \geq & C_5[u]_{\mathring{S}_{1,(p_0-2) q_0, q_0}(\Omega)}^{p_0}+\bar{C}_0 \int_{\Omega_3}|u|^{\alpha(x)} dx-K .\end{aligned}$$

Here, $K \equiv K\left(C_1(\epsilon_1), C_2(\epsilon_2), C_3(\epsilon_3), \|c_3\|_{L^1(\Omega_2)}, \|c_5\|_{L^1(\Omega_3)}\right)$, $\tilde{C} \equiv \tilde{C}\,(p_0, |\Omega|)$, $C_5 \equiv C_5\,(p_0, |\Omega|)$, $C_1 \equiv C_1\,(\sigma_\beta\,(c_0), \epsilon_1, p_0)$, $C_2 \equiv C_2\left(\sigma_{\frac{p_0}{p_0-\alpha_1(x)}}\,(c_2), \epsilon_2, p_0, \alpha_1^+\right)$, $C_3 \equiv C_3\,(\sigma_{\beta_1}\,(c_1), \epsilon_3, p_0)$ are positive constants.

From the last inequality we get

$$\langle f\,(u)\,, u\rangle \geq C_5\,[u]^{p_0}_{\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)} + \bar{C}_0 \int_{\Omega_3} |u|^{\alpha(x)}\,dx - K. \tag{3.7}$$

If we take account the following inequalities

$$\begin{aligned}
&\int_{\Omega_3} |u|^{\alpha(x)}\,dx \geq \|u\|^{p_0}_{L^{\alpha(x)}(\Omega_3)} - 1,\\
&[u]^{p_0}_{\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega_2)} \geq C_6\,\|u\|^{p_0}_{L^{\alpha(x)}(\Omega_2)}\,,\\
&[u]^{p_0}_{\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega_1)} \geq C_7\,\|u\|^{p_0}_{L^{\alpha(x)}(\Omega_1)}\,,
\end{aligned}$$

where $C_6 \equiv C_6\,(p_0, |\Omega|)$, $C_7 \equiv C_7\,(p_0, |\Omega|) > 0$ (comes from Theorem 2.2 and Lemma 2.1 into (3.7), we obtain

$$\langle f\,(u)\,, u\rangle \geq C_8\left([u]^{p_0}_{\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)} + \|u\|^{p_0}_{L^{\alpha(x)}(\Omega)}\right) - \tilde{K}.$$

So the proof is completed. □

Lemma 3.2. *Under the conditions of Theorem 3.1, the mapping f defined by* (3.4) *is bounded from $\mathring{S}_{1,(p_0-2)q_0,q_0}\,(\Omega) \cap L^{\alpha(x)}\,(\Omega)$ into $W^{-1,q_0}\,(\Omega) + L^{\alpha^*(x)}\,(\Omega)$.*

Proof. Firstly we define the mappings

$$\begin{aligned}
f_1\,(u) &\equiv \sum_{i=1}^{n} -D_i\left(|u|^{p_0-2}\,D_i u\right),\\
f_2\,(u) &\equiv c\,(x, u)\,.
\end{aligned}$$

We need to show that, these mappings are bounded from $\mathring{S}_{1,(p_0-2)q_0,q_0}\,(\Omega) \cap L^{\alpha(x)}\,(\Omega)$ to $W^{-1,q_0}\,(\Omega) + L^{\alpha^*(x)}\,(\Omega)$.

Let's show that f_1 is bounded: For $u \in \mathring{S}_{1,(p_0-2)q_0,q_0}\,(\Omega)$ and $v \in W_0^{1,p_0}\,(\Omega)$,

$$|\langle f_1\,(u)\,, v\rangle| \leq \sum_{i=1}^{n}\left(\int_\Omega |u|^{p_0-2}\,|D_i u|\,|D_i v|\,dx\right).$$

Using Hölder's inequality we get

$$\begin{aligned}
&\leq \left[\sum_{i=1}^{n}\left(\int_\Omega |u|^{(p_0-2)q_0}\,|D_i u|^{q_0}\,dx\right)\right]^{\frac{1}{q_0}}\left[\sum_{i=1}^{n}\left(\int_\Omega |D_i v|^{p_0}\,dx\right)\right]^{\frac{1}{p_0}}\\
&= [u]^{p_0-1}_{\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)}\,\|v\|_{W_0^{1,p_0}(\Omega)}\,.
\end{aligned}$$

Thus by the last inequality we obtain the boundness of f_1.

Similarly by using (3.1) and Theorem 2.2, $\forall\ u \in \mathring{S}_{1,(p_0-2)q_0,q_0}\,(\Omega) \cap L^{\alpha(x)}\,(\Omega)$, we have the following estimate

$$\sigma_{\alpha^*}\,(f_2\,(u)) = \sigma_{\alpha^*}\,(c\,(x, u)) \leq C_9\left(\sigma_\alpha\,(u) + [u]^{\tilde{p}}_{\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)}\right) + C_{10},$$

here $C_9 = C_9(\alpha^+, \alpha^-) > 0$, $C_{10} = C_{10}(\sigma_\beta(c_0), \sigma_{\beta_1}(c_1), |\Omega|) > 0$ are constants. So we prove that $f_2 : \mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega) \cap L^{\alpha(x)}(\Omega) \to L^{\alpha^*(x)}(\Omega)$ is bounded. $\square$

Lemma 3.3. *Under the conditions of Theorem 3.1, the mapping f defined by* (3.4) *is weakly compact from $\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega) \cap L^{\alpha(x)}(\Omega)$ into $W^{-1,q_0}(\Omega) + L^{\alpha^*(x)}(\Omega)$.*

Proof. Firstly we want to see the weak compactness of f_1. For $\{u_m\}_{m=1}^\infty \subset \mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega) \cap L^{\alpha(x)}(\Omega)$ bounded and $u_m \xrightarrow{S_0} u_0$ it is sufficient to show a subsequence of $\{u_{m_j}\}_{m=1}^\infty \subset \{u_m\}_{m=1}^\infty$ which satisfies $f_1(u_{m_j}) \overset{W^{-1,q_0}(\Omega)}{\rightharpoonup} f_1(u_0)$.

Since we have one-to-one correspondence between the classes (Theorem 2.1)

$$\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega) \underset{\varphi^{-1}}{\overset{\varphi}{\longleftrightarrow}} W_0^{1,q_0}(\Omega)$$

with the homeomorphism

$$\varphi(\tau) \equiv |\tau|^{p_0-2}\tau, \ \varphi^{-1}(\tau) \equiv |\tau|^{-\frac{p_0-2}{p_0-1}}\tau$$

for $\forall m \geq 1$

$$|u_m|^{p_0-2} u_m \in W_0^{1,q_0}(\Omega)$$

and since $W_0^{1,q_0}(\Omega)$ is a reflexive space, there exists a subsequence $\{u_{m_j}\}_{m=1}^\infty \subset \{u_m\}_{m=1}^\infty$ such that

$$|u_{m_j}|^{p_0-2} u_{m_j} \overset{W_0^{1,q_0}(\Omega)}{\rightharpoonup} \xi.$$

Now we show that $\xi = |u_0|^{p_0-2} u_0$.

According to compact embedding, $W_0^{1,q_0}(\Omega) \hookrightarrow L^{q_0}(\Omega)$

$$\exists \left\{u_{m_{j_k}}\right\}_{m=1}^\infty \subset \{u_{m_j}\}_{m=1}^\infty, \ \left|u_{m_{j_k}}\right|^{p_0-2} u_{m_{j_k}} \overset{L^{q_0}(\Omega)}{\to} \xi.$$

Since $\varphi^{-1} : L^{q_0}(\Omega) \longrightarrow L^{p_0}(\Omega)$ continuous then

$$u_{m_{j_k}} \overset{L^{p_0}(\Omega)}{\to} \varphi^{-1}(\xi),$$

hence we have

$$u_{m_{j_k}} \underset{a.e}{\overset{\Omega}{\to}} \varphi^{-1}(\xi).$$

So we obtain $\varphi^{-1}(\xi) = u_0$, equivalently $\xi = |u_0|^{p_0-2} u_0$.

From this, we conclude that for $\forall v \in W_0^{1,p_0}(\Omega)$,

$$\begin{aligned}\langle f_1\left(u_{m_{j_k}}\right), v\rangle =& \sum_{i=1}^n \langle -D_i\left(\left|u_{m_{j_k}}\right|^{p_0-2} D_i u_{m_{j_k}}\right), v\rangle \\ &\underset{m_j \nearrow \infty}{\longrightarrow} \sum_{i=1}^n \langle -D_i\left(|u_0|^{p_0-2} D_i u_0\right), v\rangle \\ =& \langle f_1(u_0), v\rangle,\end{aligned}$$

hence, the result is obtained.

Now we shall show the weak compactness of f_2. Since

$$c : \mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega) \cap L^{\alpha(x)}(\Omega) \to L^{\alpha^*(x)}(\Omega)$$

is bounded by Lemma 3.2, then $\forall m \geq 1,\ f_2(u_m) \equiv \{c(x, u_m)\}_{m=1}^{\infty} \subset L^{\alpha^*(x)}(\Omega)$. Also $L^{\alpha^*(x)}(\Omega)$ $(1 < (\alpha^*)^- < \infty)$ is a reflexive space thus $\{u_m\}_{m=1}^{\infty}$ has a subsequence $\{u_{m_j}\}_{m=1}^{\infty}$ such that

$$c\left(x, u_{m_j}\right) \overset{L^{\alpha^*(x)}(\Omega)}{\rightharpoonup} \psi.$$

Since the compact embedding, $\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega) \hookrightarrow L^{p_0}(\Omega)$ exists

$$\exists \left\{u_{m_{j_k}}\right\}_{m=1}^{\infty} \subset \left\{u_{m_j}\right\}_{m=1}^{\infty},\ u_{m_{j_k}} \overset{L^{p_0}(\Omega)}{\rightarrow} u_0,$$

thus

$$u_{m_{j_k}} \underset{a.e}{\overset{\Omega}{\rightarrow}} u_0$$

and using the continuity of $c(x, .)$ for almost $x \in \Omega$, we get

$$c(x, u_{m_{j_k}}) \underset{a.e}{\overset{\Omega}{\rightarrow}} c(x, u_0),$$

so, we arrive at $\psi = c(x, u_0)$ i.e. $f_2(u_{m_{j_k}}) \overset{W^{-1,q_0}(\Omega)+L^{\alpha^*(x)}(\Omega)}{\rightharpoonup} f_2(u_0)$. □

Now we give the proof of main theorem of this section.

Proof of Theorem 3.1. Since $g \equiv Id$, so it is a linear bounded map and satisfies the conditions of (1). Also from Lemma 3.1-Lemma 3.3, it follows that the mappings f and g satisfy all the conditions of Theorem 2.3. If we apply Theorem 2.3 to problem (1.3), we obtain that $\forall h \in W^{-1,q_0}(\Omega) + L^{\alpha^*(x)}(\Omega)$ the equation

$$\sum_{i=1}^{n} - \int_{\Omega} \left[D_i\left(|u|^{p_0-2} D_i u\right) + c(x, u)\right] w dx$$
$$= \int_{\Omega} h(x) w dx,\ w \in W_0^{1,p_0}(\Omega) \cap L^{\alpha(x)}(\Omega)$$

has a solution in $\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega) \cap L^{\alpha(x)}(\Omega)$. □

It can be easily seen from the proof of Theorem 3.1, the results which are given below are valid for α satisfying special conditions.

Corollary 3.1. *Let **(i)** holds. If* $1 < \alpha^- \leq \alpha(x) \leq \alpha^+ < p_0$ *i.e.* $\Omega \equiv \Omega_1$ *and* $c_0 \in L^{\beta_2(x)}(\Omega)$, $c_1 \in L^{\alpha^*(x)}(\Omega)$ *where* $\beta_2(x) \equiv \frac{p_0\alpha^*(x)}{p_0-\alpha(x)}$ *then* $\forall h \in W^{-1,q_0}(\Omega)$ *problem* (1.3) *has a generalized solution in the space* $\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)$.

Corollary 3.2. *Let **(i)**, **(ii)** hold. If* $1 < \alpha^- \leq \alpha(x) \leq \alpha^+ < \tilde{p}$ *i.e.* $\Omega \equiv \Omega_1 \cup \Omega_2$ *and* $c_2 \in L^{\frac{p_0}{p_0-\alpha_1(x)}}(\Omega_2)$, $c_3 \in L^1(\Omega_2)$, $c_0 \in L^{\beta_3(x)}(\Omega)$, $c_1 \in L^{\beta_4(x)}(\Omega)$ *where* $\beta_3(x) \equiv \begin{cases} \frac{p_0\alpha^*(x)}{p_0-\alpha(x)}, & \text{if } x \in \Omega_1, \\ \frac{\tilde{p}\alpha^*(x)}{\tilde{p}-\alpha(x)}, & \text{if } x \in \Omega_2, \end{cases}$ *and* $\beta_4(x) \equiv \begin{cases} \alpha^*(x), & \text{if } x \in \Omega_1, \\ q_0, & \text{if } x \in \Omega_2, \end{cases}$ *then* $\forall h \in W^{-1,q_0}(\Omega)$ *problem* (1.3) *has a generalized solution in the space* $\mathring{S}_{1,(p_0-2)q_0,q_0}(\Omega)$.

4. Existence Results for Main Problem (1.1)

4.1. Preliminary results

In this subsection, we prove some necessary results. Throughout this section, we take $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with Lipschitz boundary.

Lemma 4.1. *Assume that* $\zeta : \Omega \longrightarrow [1,\infty)$ *is measurable function that satisfy* $1 \leq \zeta^{-} \leq \zeta(x) \leq \zeta^{+} < \infty$ *also* $\beta > 1$, $\epsilon > 0$. *Then for every* $u \in L^{\zeta(x)+\epsilon}(\Omega)$

$$\int_{\Omega} |u|^{\zeta(x)} \left|\ln|u|\right|^{\beta} dx \leq M_1 \int_{\Omega} |u|^{\zeta(x)+\epsilon} dx + M_2$$

is satisfied. Here $M_1 \equiv M_1(\epsilon,\beta) > 0$ *and* $M_2 \equiv M_2(\epsilon,\beta,|\Omega|) > 0$ *are constants.*

Proof. For given $\epsilon > 0$ one can easily see that by calculus there exist $M_0 = M_0(\epsilon) > 0$ such that

$$\ln|t| \leq M_0(\epsilon)|t|^{\epsilon}, \ t \in \mathbb{R} - \{0\}$$

holds. Hence on the set $\{x \in \Omega : |u(x)| \geq 1\}$ the inequality

$$|u|^{\zeta(x)} \left|\ln|u|\right|^{\beta} \leq M_0(\epsilon,\beta)|u|^{\zeta(x)+\epsilon}$$

is satisfied. On the other hand, since $\lim\limits_{t\to 0^+} t^{\epsilon}|\ln t|^{\beta} = 0$ and for every fixed $x_0 \in \Omega$, $\lim\limits_{t\to 0^+} \frac{|t|^{\zeta(x_0)}|\ln|t||^{\beta}}{t^{\zeta(x_0)+\epsilon}+1} = 0$, we have the inequality $|u|^{\zeta(x)-1}|u||\ln|u||^{\beta} \leq \tilde{M}_0\left(|u|^{\zeta(x)+\epsilon}+1\right)$ on the set $\{x \in \Omega : |u(x)| < 1\}$ for some $\tilde{M}_0 = \tilde{M}_0(\epsilon,\beta) > 0$. So the proof is completed by the combination of these inequalities. $\square$

Let $\rho : \Omega \longrightarrow \mathbb{R}$, $2 \leq \rho^{-} \leq \rho(x) \leq \rho^{+} < \infty$, $\rho \in C^1(\bar{\Omega})$ and m be a number which satisfies $\rho(x) \geq m \geq 2$ a.e. $x \in \Omega$, $m_1 \equiv \frac{m}{m-1}$, $\psi(x) \equiv \frac{\rho(x)-m}{m-1}$, $x \in \Omega$ and $\varphi : \Omega \longrightarrow \mathbb{R}$ is measurable function satisfy $1 \leq \varphi^{-} \leq \varphi(x) \leq \varphi^{+} < \infty$.

Now we introduce the following class of functions[¶] for $u : \Omega \longrightarrow \mathbb{R}$,

$$\tilde{T}_0 \equiv \left\{ u \in L^1(\Omega) \mid \sum_{i=1}^{n} \left\| |u|^{\psi(x)} D_i u \right\|_{L^m(\Omega)} + \|u\|_{L^{\frac{nm_1(\rho(x)-1)}{n-m_1}}(\Omega)} < \infty \right\} \cap L^{\varphi(x)+\psi(x)}(\Omega) \cap \{u\mid_{\partial\Omega} \equiv 0\}$$

and

$$T_0 \equiv \left\{ u \in L^1(\Omega) \mid \sum_{i=1}^{n} \left\| |u|^{\rho(x)-2} D_i u \right\|_{L^{m_1}(\Omega)} + \|u\|_{L^{\frac{nm_1(\rho(x)-1)}{n-m_1}}(\Omega)} < \infty \right\} \cap L^{\varphi(x)+\psi(x)}(\Omega) \cap \{u\mid_{\partial\Omega} \equiv 0\}.$$

Following lemma indicates the relation of these classes with Sobolev and generalized Lebesgue spaces.

Lemma 4.2. *Let the functions* ρ, φ, ψ *and number* m *be defined as above, then the following statements hold:*

$$\text{(a) } \phi : \Omega \times \mathbb{R} \longrightarrow \mathbb{R},\ \phi(x,\tau) \equiv |\tau|^{\rho(x)-2}\tau$$

is a bijection between the spaces T_0 *and* $W_0^{1,m_1}(\Omega) \cap L^{\frac{\varphi(x)+\psi(x)}{(m-1)(\psi(x)+1)}}(\Omega)$.

$$\text{(b) } \tilde{\phi} : \Omega \times \mathbb{R} \longrightarrow \mathbb{R},\ \tilde{\phi}(x,\tau) \equiv |\tau|^{\psi(x)}\tau$$

is a bijection between $\tilde{T}_0$ *and* $W_0^{1,m}(\Omega) \cap L^{\frac{\varphi(x)+\psi(x)}{\psi(x)+1}}(\Omega)$.

[¶]In general, there might not be an embedding between $L^{\varphi(x)+\psi(x)}(\Omega)$ and $L^{\frac{nm_1(\rho(x)-1)}{n-m_1}}(\Omega)$.

Proof. Since the proofs of (a) and (b) are similar, we only prove (a).

First let us show that for $u \in T_0$, $v \equiv |u|^{\rho(x)-2} u \equiv \phi(x,u) \in W_0^{1,m_1}(\Omega) \cap L^{\frac{\varphi(x)+\psi(x)}{(m-1)(\psi(x)+1)}}(\Omega)$. Since by direct calculations

$$\sigma_{\frac{\varphi+\psi}{(m-1)(\psi+1)}}(v) = \sigma_{\varphi+\psi}(u),$$

so from this equality, we obtain $v \in L^{\frac{\varphi(x)+\psi(x)}{(m-1)(\psi(x)+1)}}(\Omega)$.

On the other hand for $\forall i = \overline{1..n}$

$$\begin{aligned}
\|D_i v\|_{m_1}^{m_1} &= \int_\Omega \left| D_i \left(|u|^{\rho(x)-2} u \right) \right|^{m_1} dx \\
&= \int_\Omega \left| (\rho(x)-1) |u|^{\rho(x)-2} D_i u + (D_i \rho) |u|^{\rho(x)-2} u \ln |u| \right|^{m_1} dx \\
&\le C_0 \int_\Omega |u|^{m_1(\rho(x)-2)} |D_i u|^{m_1} dx + C_1 \int_\Omega |u|^{m_1(\rho(x)-1)} |\ln |u||^{m_1} dx,
\end{aligned}$$

here $C_0 = C_0 \left(m_1, \|\rho\|_{C(\Omega)} \right)$, $C_1 = C_1 \left(m_1, \|\rho\|_{C^1(\bar\Omega)} \right) > 0$ are constants.

As for sufficiently small $\varepsilon > 0$, $m_1(\rho(x)-1) + \varepsilon \le \frac{nm_1(\rho(x)-1)}{n-m_1}$ holds, applying Lemma 4.1 to second integral we obtain

$$\|D_i v\|_{m_1}^{m_1} \le C_0 \left\| |u|^{\rho(x)-2} D_i u \right\|_{m_1}^{m_1} + C_2 \int_\Omega |u|^{m_1(\rho(x)-1)+\varepsilon} dx + C_3.$$

Applying Young's inequality to second integral, we get

$$\|D_i v\|_{m_1}^{m_1} \le C_0 \left\| |u|^{\rho(x)-2} D_i u \right\|_{m_1}^{m_1} + C_2 \sigma_{\frac{nm_1(\rho(x)-1)}{n-m_1}}(u) + \tilde{C}_3,$$

where $C_2 = C_2 \left(m_1, \|\rho\|_{C^1(\bar\Omega)}, \varepsilon \right) > 0$ and $\tilde{C}_3 = \tilde{C}_3 \left(m_1, \|\rho\|_{C^1(\bar\Omega)}, |\Omega| \right) > 0$. Hence from last inequality, we get $v \in W_0^{1,m_1}(\Omega)$.

Conversely for all $v \in W_0^{1,m_1}(\Omega) \cap L^{\frac{\varphi(x)+\psi(x)}{(m-1)(\psi(x)+1)}}(\Omega)$ let us show that $w \equiv |v|^{-\frac{\rho(x)-2}{\rho(x)-1}} v \equiv \phi^{-1}(x,v) \in T_0$. As

$$\sigma_{\varphi+\psi}(w) = \sigma_{\frac{\varphi+\psi}{(m-1)(\psi+1)}}(v)$$

according to this equality, we have $w \in L^{\varphi(x)+\psi(x)}(\Omega)$. Furthermore from definition of T_0 and the Luxemburg norm, we have

$$\begin{aligned}
&\left\| |w|^{\rho(x)-2} D_i w \right\|_{m_1}^{m_1} + \sigma_{\frac{nm_1(\rho(x)-1)}{n-m_1}}(w) \\
&= \int_\Omega |v|^{\frac{m(\rho(x)-2)}{\rho(x)-1}} \left| D_i \left(|v|^{-\frac{\rho(x)-2}{\rho(x)-1}} v \right) \right|^{m_1} dx + \int_\Omega |v|^{\frac{nm_1}{n-m_1}} dx \\
&= \int_\Omega |v|^{\frac{m(\rho(x)-2)}{\rho(x)-1}} \left| \left(\tfrac{1}{\rho(x)-1} \right) |v|^{-\frac{\rho(x)-2}{\rho(x)-1}} D_i v + \left(\tfrac{-D_i(\rho)}{(\rho(x)-1)^2} \right) |v|^{-\frac{\rho(x)-2}{\rho(x)-1}} v \ln |v| \right|^{m_1} dx \\
&\quad + \int_\Omega |v|^{\frac{nm_1}{n-m_1}} dx \le C_4 \int_\Omega |D_i v|^{m_1} dx + C_5 \int_\Omega |v|^{m_1} |\ln |v||^{m_1} dx + \int_\Omega |v|^{\frac{nm_1}{n-m_1}} dx,
\end{aligned}$$

here $C_4 = C_4 \left(m_1, \|\rho\|_{C(\bar\Omega)} \right) > 0$ and $C_5 = C_5 \left(m_1, \|\rho\|_{C^1(\bar\Omega)} \right) > 0$.

Estimating the second integral on the right hand side of last inequality with the help of Lemma 4.1, we obtain

$$\leq C_4 \int_{\Omega} |D_i v|^{m_1} \, dx + C_6 \int_{\Omega} |v|^{\frac{nm_1}{n-m_1}} \, dx + C_7.$$

Considering the embedding $W_0^{1,m_1}(\Omega) \subset L^{\frac{nm_1}{n-m_1}}(\Omega)$ [3] in the last inequality, we obtain $w \in T_0$.

To end the proof, observe that for every fixed $x_0 \in \Omega$, $\phi(x_0,\tau) \equiv \phi(\tau) = |\tau|^{\rho(x_0)-2}\tau$ and $\phi^{-1}(x_0,t) \equiv \phi^{-1}(t) = |t|^{-\frac{\rho(x_0)-2}{\rho(x_0)-1}} t$ are strictly monotone functions thus we verify that ϕ is a bijection between T_0 and $W_0^{1,m_1}(\Omega) \cap L^{\frac{\varphi(x)+\psi(x)}{(m-1)(\psi(x)+1)}}(\Omega)$. □

Remark 4.1. Under the conditions of Lemma 4.2, T_0 and $\tilde{T}_0$ are metric spaces. On the class T_0, metric is defined as given below: $\forall u, v \in T_0$

$$d_{T_0}(u,v) \equiv \|\phi(u) - \phi(v)\|_{L^{\frac{\varphi(x)+\psi(x)}{(m-1)(\psi(x)+1)}}(\Omega)} + \|\phi(u) - \phi(v)\|_{W_0^{1,m_1}(\Omega)}$$

it easy to see that $d_{T_0}(.,.) : T_0 \longrightarrow \mathbb{R}$ satisfies the metric axioms and moreover ϕ and ϕ^{-1} are continuous in the sense of topology defined on T_0 with this metric. Hence we get that ϕ is a homeomorphism between T_0 and $W_0^{1,m_1}(\Omega) \cap L^{\frac{\varphi(x)+\psi(x)}{(m-1)(\psi(x)+1)}}(\Omega)$. By the same way we can show that $\tilde{\phi}$ is a homeomorphism between $\tilde{T}_0$ and $W_0^{1,m}(\Omega) \cap L^{\frac{\varphi(x)+\psi(x)}{\psi(x)+1}}(\Omega)$.

4.2. Solvability of Problem (1.1)

In this section, we consider the main problem (1.1) and investigate the existence of weak solution of that problem by the help of the results that established in Subsection 4.1 and Theorem 3.1. So, we study

$$\begin{cases} -\Delta\left(|u|^{p(x)-2} u\right) + a(x,u) = h(x), \\ u \mid_{\partial\Omega} = 0, \end{cases}$$

under the following conditions:

(I) *$p : \Omega \longrightarrow \mathbb{R}$, $2 \leq p^- \leq p(x) \leq p^+ < \infty$ and $p \in C^1(\bar{\Omega})$, $a : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Caratheodory function and for the measurable function $\xi : \Omega \longrightarrow \mathbb{R}$ satisfies $1 < \xi^- \leq \xi(x) \leq \xi^+ < \infty$, the inequality*

$$|a(x,\tau)| \leq a_0(x) |\tau|^{\xi(x)-1} + a_1(x), \ \ (x,\tau) \in \Omega \times \mathbb{R}, \tag{4.1}$$

holds. Here a_0, a_1 are nonnegative, measurable functions defined on Ω.

Let $\eta_0 \in (0,1)$ is sufficiently small. We separate Ω to disjoint sets because of the same reason for which is required problem (1.3).

$$\begin{aligned} \Omega_1 &\equiv \{x \in \Omega \mid 1 \leq \xi^- \leq \xi(x) \leq p(x) - \eta_0\}, \\ \Omega_2 &\equiv \{x \in \Omega \mid p(x) - \eta_0 < \xi(x) \leq \tilde{p}(x)\}, \\ \Omega_3 &\equiv \{x \in \Omega \mid \tilde{p}(x) \leq \xi(x) \leq \xi^+ < \infty\}, \end{aligned} \tag{*}$$

here $\tilde{p} : \Omega \longrightarrow \mathbb{R}$ *is a measurable function which satisfies* $2 \leq p(x) < \tilde{p}(x)$ *a.e.* $x \in \Omega$ *and will be defined later.*

(II) *There exists a measurable function* $\xi_1 : \Omega_2 \longrightarrow \mathbb{R}$ *which satisfy* $2 \leq \xi_1^- \leq \xi_1(x) \leq \xi_1^+ < p(x)$, *such that on* $\Omega_2 \times \mathbb{R}$, $a(x,\tau)$ *fulfills the inequality*

$$a(x,\tau)\tau \geq -a_2(x)|\tau|^{\xi_1(x)} - a_3(x), \quad (x,\tau) \in \Omega_2 \times \mathbb{R}, \tag{4.2}$$

here a_2, a_3 *are nonnegative, measurable functions defined on* Ω_2.

(III) *On* $\Omega_3 \times \mathbb{R}$, $a(x,\tau)$ *satisfies the inequality*

$$a(x,\tau)\tau \geq a_4(x)|\tau|^{\xi(x)} - a_5(x), \quad (x,\tau) \in \Omega_3 \times \mathbb{R}, \tag{4.3}$$

here ξ *is the same function as in* (4.1) *and* $a_4(x) \geq \bar{A}_0 > 0$ *a.e.* $x \in \Omega_3$ *and* a_5 *is nonnegative, measurable function defined on* Ω_3.

Let p_1 be a number holds $p(x) \geq p_1 \geq 2$ a.e. $x \in \Omega$, $q_1 \equiv \frac{p_1}{p_1-1}$, $\gamma(x) \equiv \frac{p(x)-p_1}{p_1-1}$ and $\theta(x) \equiv \frac{\xi(x)+\gamma(x)}{\gamma(x)+1}$, $x \in \Omega$.

We introduce the following class of functions:

$$\tilde{P}_0 \equiv \left\{ u \in L^1(\Omega) \mid \sum_{i=1}^n \left\| |u|^{\gamma(x)} D_i u \right\|_{L^{p_1}(\Omega)} + \|u\|_{L^{\frac{nq_1(p(x)-1)}{n-q_1}}(\Omega)} < \infty \right\} \cap L^{\xi(x)+\gamma(x)}(\Omega) \cap \{u\mid_{\partial\Omega} \equiv 0\}$$

and

$$P_0 \equiv \left\{ u \in L^1(\Omega) \mid \sum_{i=1}^n \left\| |u|^{p(x)-2} D_i u \right\|_{L^{q_1}(\Omega)} + \|u\|_{L^{\frac{nq_1(p(x)-1)}{n-q_1}}(\Omega)} < \infty \right\} \cap L^{\xi(x)+\gamma(x)}(\Omega) \cap \{u\mid_{\partial\Omega} \equiv 0\}.$$

Remark 4.2. From Lemma 4.2 and Remark 4.1, it follows that $\tilde{P}_0$ and P_0 are metric spaces (also pn-spaces). Furthermore $\phi_0(x,\tau) \equiv |\tau|^{p(x)-2}\tau$ is a homeomorphism between P_0 and $W_0^{1,q_1}(\Omega) \cap L^{\frac{\theta(x)}{p_1-1}}(\Omega)$ and $\phi_1(x,\tau) \equiv |\tau|^{\gamma(x)}\tau$ is a homeomorphism between $\tilde{P}_0$ and $W_0^{1,p_1}(\Omega) \cap L^{\theta(x)}(\Omega)$.

We investigate (1.1) for $h \in W^{-1,q_1}(\Omega) + L^{\theta^*(x)}(\Omega)$ (θ^* is conjugate function of θ). Solution of problem (1.1) is understood in the following sense.

Definition 4.1. A function $u \in P_0$ is called the weak solution of problem (1.1) if it satisfies the equality

$$-\int_\Omega \Delta\left(|u|^{p(x)-2}u\right) w dx + \int_\Omega a(x,u) w dx = \int_\Omega h(x) w dx$$

for all $w \in W_0^{1,p_1}(\Omega) \cap L^{\theta(x)}(\Omega)$.

We define the following functions:

$\mu_1(x) \equiv \frac{p(x)+\gamma(x)}{p(x)-\xi_1(x)}$, $x \in \Omega_2$, $\mu_2(x) \equiv \frac{\xi_1(x)+\gamma(x)}{\xi_1(x)}$, $x \in \Omega_2$, $\mu_3(x) \equiv \frac{\xi(x)+\gamma(x)}{\xi(x)}$, $x \in \Omega_3$,

$$\mu_4(x) \equiv \begin{cases} \theta^*(x) & \text{if } x \in \Omega_1 \\ q_1 & \text{if } x \in \Omega_2 \cup \Omega_3 \end{cases}, \ \mu(x) \equiv \begin{cases} \frac{p_1\theta^*(x)}{p_1-\theta(x)} & \text{if } x \in \Omega_1 \\ \frac{\tilde{p}_1\theta^*(x)}{\tilde{p}_1-\theta(x)} & \text{if } x \in \Omega_2 \\ \infty & \text{if } x \in \Omega_3 \end{cases}, \ \tilde{p}_1 \equiv \frac{np_1}{n-q_1},$$

and $\tilde{p}(x) \equiv \tilde{p}_1(\gamma(x)+1) - \gamma(x)$. Here $p_1 \equiv \frac{p(x)+\gamma(x)}{\gamma(x)+1}$, $q_1 \equiv \frac{p(x)+\gamma(x)}{p(x)-1}$, $x \in \Omega$.

Now we state the main theorem of this article that is the solvability theorem for problem (1.1).

Theorem 4.1. *Assume that **(I)-(III)** hold. If* $a_2 \in L^{\mu_1(x)}(\Omega_2)$, $a_3 \in L^{\mu_2(x)}(\Omega_2)$, $a_5 \in L^{\mu_3(x)}(\Omega_3)$, $a_4 \in L^{\infty}(\Omega_3)$, $a_1 \in L^{\mu_4(x)}(\Omega)$ *and* $a_0 \in L^{\mu(x)}(\Omega)$, *then* $\forall h \in W^{-1,q_1}(\Omega) + L^{\theta^*(x)}(\Omega)$ *problem* (1.1) *has a generalized solution in* P_0.

As mentioned in introduction part, we investigate problem (1.1) such a different method. In order to study that problem, firstly we transform it to equivalent problem by the transformation $\phi_1 : u \to |u|^{\gamma(x)} u$. Following equality can be obtained easily,

$$-\Delta\left(|u|^{p(x)-2} u\right) = -\Delta\left(\left||u|^{\gamma(x)} u\right|^{p_1-2} |u|^{\gamma(x)} u\right)$$
$$= (p_1 - 1)\sum_{i=1}^{n} -D_i\left(\left||u|^{\gamma(x)} u\right|^{p_1-2} D_i\left(|u|^{\gamma(x)} u\right)\right).$$

Once applying the transformation $u \to |u|^{\gamma(x)} u$ in a, we denote the established function with b i.e.

$$b\left(x, |u|^{\gamma(x)} u\right) \equiv a(x, u)$$

it is clear that $b : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is also Caratheodory function.

Also, it is obvious that

$$|u|^{\gamma(x)} u \mid_{\partial\Omega} = 0 \Leftrightarrow u \mid_{\partial\Omega} = 0.$$

Consequently (1.1) can be written in the form;

$$\begin{cases} (p_1 - 1)\sum\limits_{i=1}^{n} -D_i\left(\left||u|^{\gamma(x)} u\right|^{p_1-2} D_i\left(|u|^{\gamma(x)} u\right)\right) + b\left(x, |u|^{\gamma(x)} u\right) = h(x), \\ |u|^{\gamma(x)} u \mid_{\partial\Omega} = 0. \end{cases} \tag{4.4}$$

Denote $v \equiv |u|^{\gamma(x)} u$ then by (4.4) we establish the following

$$\begin{cases} (p_1 - 1)\sum\limits_{i=1}^{n} -D_i\left(|v|^{p_1-2} D_i v\right) + b(x, v) = h(x), \\ v \mid_{\partial\Omega} = 0, \end{cases} \tag{4.5}$$

which is equivalent to (1.1) as a immediate consequence of Lemma 4.2. Obviously, problem (4.5) is same as the problem (1.3) which we have studied in Section 3.

Lemma 4.3. *Under the conditions of Theorem 4.1 for* $\forall h \in W^{-1,q_1}(\Omega) + L^{\theta^*(x)}(\Omega)$, *problem* (4.5) *has generalized solution, in the sense of Definition 3.1, in the space* $\mathring{S}_{1,(p_1-2)q_1,q_1}(\Omega) \cap L^{\theta(x)}(\Omega)$.

Proof. For the proof we only need to prove that $b(x, v)$ in problem (4.5) satisfies all the conditions of Theorem 3.1.

If we rewrite the inequality (4.1) in terms of v, we have

$$|b(x, v)| \leq a_0(x)|v|^{\theta(x)-1} + a_1(x) \text{ on } \Omega \times \mathbb{R}. \tag{4.6}$$

Also in terms of θ the sets Ω_i, $i = 1, 2, 3$ can be written equivalently, by simple calculations, in the form which is given below i.e. the inequalities which define the sets Ω_i in (*) are equivalent to the ones given below.

$$\Omega_1 = \left\{ x \in \Omega \mid 1 \leq \theta^- \leq \theta(x) \leq p_1 - \tilde{\eta}_0 \right\},$$
$$\Omega_2 = \left\{ x \in \Omega \mid p_1 - \tilde{\eta}_0 < \theta(x) \leq \tilde{p}_1 \right\},$$
$$\Omega_3 = \left\{ x \in \Omega \mid \tilde{p}_1 \leq \theta(x) \leq \theta^+ < \infty \right\},$$

where $\tilde{\eta}_0 = \tilde{\eta}_0(\eta_0) > 0$ is sufficiently small.

Now we prove that under the conditions of Theorem 4.1, conditions **(ii)** and **(iii)** of Theorem 3.1 hold. From (4.2), (4.3), we have the following inequalities for b:

$$b(x, v) v \geq -a_2(x) |v|^{\frac{\xi_1(x)+\gamma(x)}{\gamma(x)+1}} - a_3(x) |v|^{\frac{\gamma(x)}{\gamma(x)+1}} \text{ on } \Omega_2 \times \mathbb{R} \tag{4.7}$$

and

$$b(x, v) v \geq a_4(x) |v|^{\theta(x)} - a_5(x) |v|^{\frac{\gamma(x)}{\gamma(x)+1}} \text{ on } \Omega_3 \times \mathbb{R}. \tag{4.8}$$

Coefficients and exponents in (4.7) and (4.8) satisfy the followings: Since $\xi_1(x) < p(x)$ a.e. $x \in \Omega_2$ so we have

$$\tfrac{\xi_1(x)+\gamma(x)}{\gamma(x)+1} < p_1 \text{ a.e. } x \in \Omega_2. \tag{4.9}$$

Applying Young's inequality to the second term in (4.7), we arrive at

$$a_3(x) |v|^{\frac{\gamma(x)}{\gamma(x)+1}} \leq (a_3(x))^{\frac{\xi_1(x)+\gamma(x)}{\xi_1(x)}} + |v|^{\frac{\xi_1(x)+\gamma(x)}{\gamma(x)+1}}. \tag{4.10}$$

Using ϵ-Young's inequality, we estimate the second term in (4.8)

$$a_5(x) |v|^{\frac{\gamma(x)}{\gamma(x)+1}} \leq C(\epsilon) (a_5(x))^{\frac{\xi(x)+\gamma(x)}{\xi(x)}} + \epsilon |v|^{\theta(x)}, \tag{4.11}$$

for sufficiently small $\epsilon > 0$ such that $a_4(x) - \epsilon \geq \tilde{A}_0 > 0$. From (4.6)–(4.11), we get that under the conditions of Theorem 4.1 the transformed problem (4.5) satisfies all conditions of Theorem 3.1. Finally by Theorem 3.1, (4.5) has a weak solution in $\mathring{S}_{1,(p_1-2)q_1,q_1}(\Omega) \cap L^{\theta(x)}(\Omega)$ for $\forall h \in W^{-1,q_1}(\Omega) + L^{\theta^*(x)}(\Omega)$. $\square$

Remark 4.3. We prove that by Lemma 4.3, $|u|^{\gamma(x)} u$ is solution of problem (4.4) and $|u|^{\gamma(x)} u \in \mathring{S}_{1,(p_1-2)q_1,q_1}(\Omega) \cap L^{\theta(x)}(\Omega)$.

Lemma 4.4. *Let the conditions of Theorem 4.1 hold. If* $|u|^{\gamma(x)} u \in \mathring{S}_{1,(p_1-2)q_1,q_1}(\Omega) \cap L^{\theta(x)}(\Omega)$ *then* $u \in P_0$.

Proof. As $v = |u|^{\gamma(x)} u \in L^{\theta(x)}(\Omega)$ and $\sigma_\theta(v) = \sigma_{\xi+\gamma}(u)$ thus we have $u \in L^{\xi(x)+\gamma(x)}(\Omega)$.

Since we have the embedding (Theorem 2.2)

$$\mathring{S}_{1,(p_1-2)q_1,q_1}(\Omega) \subset L^{\tilde{p}_1}(\Omega),$$

thus $|u|^{\gamma(x)} u \in L^{\tilde{p}_1}(\Omega)$ which implies that $u \in L^{\frac{nq_1(p(x)-1)}{n-q_1}}(\Omega)$.

Moreover using this fact and Lemma 4.1, one can see that

$$|u|^{p(x)-2} u \ln |u| = |u|^{\gamma(x)+(\gamma(x)+1)(p_1-2)} u \ln |u| \in L^{q_1}(\Omega).$$

Also by the definition of $\mathring{S}_{1,(p_1-2)q_1,q_1}(\Omega)$, it follows that $v \in \mathring{S}_{1,(p_1-2)q_1,q_1}(\Omega) \Leftrightarrow |v|^{p_1-2} D_i v \in L^{q_1}(\Omega)$.

Using these and the definition of v, we obtain the following equality for $\forall w \in L^{p_1}(\Omega)$:

$$\begin{aligned}&\langle |v|^{p_1-2} D_i v, w\rangle - \langle (D_i\gamma) |u|^{\gamma(x)+(\gamma(x)+1)(p_1-2)} u \ln |u|, w\rangle \\ =&\langle (1+\gamma(x)) |u|^{\gamma(x)+(\gamma(x)+1)(p_1-2)} D_i u, w\rangle.\end{aligned}$$

Applying Hölder inequality to the left hand side of above equality, we get

$$\begin{aligned}&C\left(\left\| |v|^{p_1-2} D_i v\right\|_{q_1} \|w\|_{p_1} + \left\| |u|^{p(x)-1} \ln |u|\right\|_{q_1} \|w\|_{p_1}\right) \\ \geq &\left| \langle (1+\gamma(x)) |u|^{\gamma(x)+(\gamma(x)+1)(p_1-2)} D_i u, w\rangle \right|,\end{aligned}$$

where $C = C\left(\|\gamma\|_{C^1(\Omega)}\right) > 0$.

From the last inequality, we obtain

$$\begin{aligned}&\left\| |v|^{p_1-2} D_i v\right\|_{q_1} + \left\| |u|^{p(x)-1} \ln |u|\right\|_{q_1} \\ \geq&\tilde{C}\left\| |u|^{\gamma(x)+(\gamma(x)+1)(p_1-2)} D_i u\right\|_{q_1} \\ =&\tilde{C}\left\| |u|^{p(x)-2} D_i u\right\|_{q_1}.\end{aligned}$$

Consequently we get that $u \in P_0$. $\square$

Now we give the proof of Theorem 4.1.

Proof of Theorem 4.1. For the proof we use Theorem 2.3 in general form. We introduce the following spaces and mappings in order to apply this theorem to prove Theorem 4.1.

$$S_0 \equiv P_0,\ Y \equiv W^{-1,q_1}(\Omega) + L^{\theta^*(x)}(\Omega),\ X_0 \equiv \tilde{P}_0$$

and

$$\begin{aligned}&Y_0^* \equiv Y^* \equiv W_0^{1,p_1}(\Omega) \cap L^{\theta(x)}(\Omega), \\ &f : S_0 \longrightarrow Y, \\ &f(u) \equiv -\Delta\left(|u|^{p(x)-2} u\right) + a(x,u), \\ &g : X_0 \subset S_0 \longrightarrow Y^*, \\ &g(u) \equiv |u|^{\gamma(x)} u.\end{aligned}$$

To apply this theorem we have to show the conditions of Theorem 2.3 is hold. Weak compactness and boundness of $f : P_0 \longrightarrow W^{-1,q_1}(\Omega) + L^{\theta^*(x)}(\Omega)$ follows from Remark 4.2, Lemma 4.3 and Lemma 4.4 by virtue of Lemma 3.2 and 3.3.

For $u \in \tilde{P}_0$ we have the equality

$$\begin{aligned}\langle f(u), g(u)\rangle =& \langle -\Delta\left(|u|^{p(x)-2} u\right) + a(x,u), |u|^{\gamma(x)} u\rangle \\ =& \langle -\Delta\left(\left||u|^{\gamma(x)} u\right|^{p_1-2} |u|^{\gamma(x)} u\right) + b\left(x, |u|^{\gamma(x)} u\right), |u|^{\gamma(x)} u\rangle \\ =& \langle (p_1-1)\sum_{i=1}^{n} -D_i\left(\left||u|^{\gamma(x)} u\right|^{p_1-2} D_i\left(|u|^{\gamma(x)} u\right)\right), |u|^{\gamma(x)} u\rangle \\ &+ \langle b\left(x, |u|^{\gamma(x)} u\right), |u|^{\gamma(x)} u\rangle \\ =& \langle (p_1-1)\sum_{i=1}^{n}\left(\left||u|^{\gamma(x)} u\right|^{p_1-2} D_i\left(|u|^{\gamma(x)} u\right)\right), D_i\left(|u|^{\gamma(x)} u\right)\rangle \\ &+ \langle b\left(x, |u|^{\gamma(x)} u\right), |u|^{\gamma(x)} u\rangle.\end{aligned}$$

Generating a "coercive pair" of the mappings f and g on $\tilde{P}_0$ follows from the above equality and Lemma 4.3 by virtue of Lemma 3.1.

Also as $g : \tilde{P}_0 \subset P_0 \longrightarrow W_0^{1,p_1}(\Omega) \cap L^{\theta(x)}(\Omega)$ is bounded and satisfies the conditions of (2) in Theorem 2.3. Thus we show that mappings f and g satisfy all the conditions of Theorem 2.3. Consequently applying that to problem (1.1), we obtain that $\forall h \in W^{-1,q_1}(\Omega) + L^{\theta^*(x)}(\Omega)$ the equation

$$-\int_\Omega \left[\Delta\left(|u|^{p(x)-2} u\right) + a(x,u)\right] w dx = \int_\Omega h(x) w dx,\ w \in W_0^{1,p_1}(\Omega) \cap L^{\theta(x)}(\Omega)$$

is solvable in P_0. □

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