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LIMIT CYCLE BIFURCATION FOR A NILPOTENT SYSTEM IN Z_3-EQUIVARIANT VECTOR FIELD*

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Abstract Our work is concerned with the problem on limit cycle bifurcation for a class of Z_3-equivariant Lyapunov system of five degrees with three third-order nilpotent critical points which lie in a Z_3-equivariant vector field. With the help of computer algebra system-MATHEMATICA, the first 5 quasi-Lyapunov constants are deduced. The fact of existing 12 small amplitude limit cycles created from the three third-order nilpotent critical points is also proved. Our proof is algebraic and symbolic, obtained result is new and interesting in terms of nilpotent critical points' Hilbert number in equivariant vector field.

Keywords Third-order nilpotent critical point, Z_3-equivariant, limit cycle bifurcation, Quasi-Lyapunov constant.

MSC(2010) 34C07.

1. Introduction

Consider an autonomous planar ordinary differential equation having three third-order nilpotent critical points $(1,0), (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ with the form

$$\begin{aligned}\frac{dx}{d\tau} &= P_1(x,y) + P_2(x,y) + P_3(x,y) + P_4(x,y) + P_5(x,y) \equiv P(x,y),\\ \frac{dy}{d\tau} &= Q_1(x,y) + Q_2(x,y) + Q_3(x,y) + Q_4(x,y) + Q_5(x,y) \equiv Q(x,y),\end{aligned} \tag{1.1}$$

where

$$\begin{aligned}P_1(x,y) =& -3B_{10}x - 3A_{10}y,\ P_2(x,y) = 12(1-2A_{10})xy - 3B_{02}(x^2-y^2),\\ P_3(x,y) =& 3(3B_{02}x + 6B_{10}x + \delta x + 6y - 6A_{10}y)(x^2+y^2),\\ P_4(x,y) =& -3(3B_{02} + 8B_{10} + 2\delta)x^4 + 6(3A_{13} - 12 + 8A_{10})x^3y\\ &+ 6(9B_{02} + 12B_{10} + 4\delta)x^2y^2 + 6(12 - 8A_{10} - A_{13})xy^3 - (9B_{02} + 2\delta)y^4,\\ P_5(x,y) =& 3(B_{02}+3B_{10}+\delta)x^5 + 3(15-A_{10}-6A_{13})x^4y + 6(7B_{02}+3B_{10}+3\delta)x^3y^2\end{aligned}$$

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$$+2(12A_{13}-20-3A_{10})x^2y^3-(9B_{02}-9B_{10}+\delta)xy^4+(19-3A_{10}-6A_{13})y^5,$$

$$Q_1(x,y)=3A_{10}x-3B_{10}y,\ Q_2(x,y)=6B_{02}xy+6(1-2A_{10})(x^2-y^2),$$

$$Q_3(x,y)=3(-6x+6A_{10}x+3B_{02}y+6B_{10}y+\delta y)(x^2+y^2),$$

$$Q_4(x,y)=6(3-2A_{10})x^4-12(3B_{02}+2B_{10}+\delta)x^3y+18(A_{13}-6+4A_{10})x^2y^2 + 4(9B_{02}+18B_{10}+5\delta)xy^3+6(3-2A_{10}-A_{13})y^4,$$

$$Q_5(x,y)=3(A_{10}-2)x^5+3(7B_{02}+3B_{10}+3\delta)x^4y+6(6A_{13}-15+A_{10})x^3y^2 + 2(9B_{10}-\delta-9B_{02})x^2y^3+(20+3A_{10}-12A_{13})xy^4+(9B_{02}+9B_{10}+5\delta)y^5,$$

in which A_{10}, B_{10}, A_{13}, $B_{02}\in R$, δ is a small parameter and $\delta\to 0$.

System (1.1) is invariable under the following transformations

$$x'=x\cos\frac{2}{3}\pi-y\sin\frac{2}{3}\pi,\ y'=x\sin\frac{2}{3}\pi+y\cos\frac{2}{3}\pi,$$

hence system (1.1) lies in a Z_3-equivariant vector field and it can be verified that three symmetric critical points $(1,0),(-\frac{1}{2},\frac{\sqrt{3}}{2})$ and $(-\frac{1}{2},-\frac{\sqrt{3}}{2})$ satisfy (1.1). This paper is devoted to investigating the limit cycle bifurcation problem from the three critical points of the above system.

Clearly, it can be obtained that the three singular points of system (1.1) are three nilpotent critical points. In the qualitative theory of ordinary differential equation, investigation on limit cycle bifurcation for a critical point P of a planar analytic vector field X is a hot topic, and many results occur in published references, for example: Ref. [14, 15] and references therein discussed this problem and reported the recent new improvements. Let $H(n)$ be the Hilbert number of n-degrees system in planar vector field, Shi [25] showed $H(2)\geq 4$, Yu and Han [28] gave $H(3)\geq 12$, Jibin Li and Yirong Liu [16] obtained $H(3)\geq 13$. [17–19] gave the relation between focal values and singular point values, and offered a kind of method to compute the focal values which is called singular values method. [29] introduced this kind of method and we obtained many results on center-focus problem and bifurcation of limit cycles by making use of this kind of method, for example References [2–10].

For an elementary singular point, singular values method is a kind of valid method to study limit cycle bifurcation. But singular values method offered by [17–19] is invalid to investigate the limit cycle problem from a nilpotent critical point. Of course, in terms of limit cycle problem around a nilpotent critical point, some significant results have be published, for example: Han etc [11] studied the limit cycle bifurcation in near-hamiltonian systems by perturbing a nilpotent center, A. Algaba etc [1] gave local bifurcation of limit cycles and integrability of a class of nilpotent systems of differential equations, Han etc [12] investigated polynomial Hamiltonian systems with a nilpotent critical point, Jiang [13] considered the limit cycle bifurcation for a quartic near-Hamiltonian system by perturbing a nilpotent center. Liu and Li [20] offered a kind of method (namely integral factor method) to study the limit cycle bifurcation behavior for third-order nilpotent critical points of the following dynamical system:

$$\frac{dx}{dt}=y+\sum_{i+j=2}^{\infty}a_{ij}x^iy^j=X(x,y),\ \frac{dy}{dt}=\sum_{i+j=2}^{\infty}b_{ij}x^iy^j=Y(x,y), \tag{1.2}$$

in which the function $y=y(x)$ satisfies $X(x,y)=0,y(0)=0$.

[21–24, 26] made use of this kind of method offered by [20] and obtained some examples aout limit cycle bifurcation behavior from a nilpotent critical point. For the simultaneous Hopf bifurcation of several nilpotent critical points, it is hardly seen in published references. Our work in this paper will show this kind of result. We will employ the theory about equivariant vector field in [4] and the integral factor method introduced in [20] to carry out our investigation of system (1.1). For system (1.1), we will discuss several cases, and being based on these cases, we investigate the center-focus problem and prove the singular point (1,0) of system (1.1) can bifurcate 4 small limit cycles. From the equivariant symmetrical quality, the fact of existing 12 small amplitude limit cycles created from the three third-order nilpotent critical points is also proved. Our proof is algebraic and symbolic, obtained result is new and interesting in terms of nilpotent critical points' Hilbert number in equivariant vector field.

We will organize this paper as follows. In Section 2, we stated some preliminary knowledge given in [20] which is useful to carry out our work in this paper. In Section 3, we gave the linear recursive formulae to compute the first 5 quasi-Lyapunov constants of each one critical point of three symmetric critical points $(1,0)$, $(-\frac{1}{2},\frac{\sqrt{3}}{2})$ and $(-\frac{1}{2},-\frac{\sqrt{3}}{2})$ and obtained some related expressions. Moreover, we gave the condition that each one critical point of three symmetric critical points could become a 5th-order weak focus, and the example with 12 small limit cycles is also shown in the result of Theorem 3.4.

2. Our method to investigate the bifurcation of nilpotent critical point

In [20], the second author offered a kind of method to study the center-focus problem of third-order nilpotent critical point of the planar dynamical systems. The work focuses on the following system:

$$\begin{aligned}\frac{dx}{dt} &= y+\mu x^2+\sum_{i+2j=3}^{\infty} a_{ij}x^i y^j = X(x,y),\\ \frac{dy}{dt} &= -2x^3+2\mu xy+\sum_{i+2j=4}^{\infty} b_{ij}x^i y^j = Y(x,y),\end{aligned} \tag{2.1}$$

in which

$$X_k(x,y)=\sum_{i+j=k} a_{ij}x^i y^j,\ Y_k(x,y)=\sum_{i+j=k} b_{ij}x^i y^j.$$

Under the transformation of generalized polar coordinates

$$x=r\cos\theta,\quad y=r^2\sin\theta, \tag{2.2}$$

system (2.1) becomes

$$\begin{aligned}\frac{dr}{dt} &= \frac{\cos\theta[\sin\theta(1-2\cos^2\theta)+\mu(\cos^2\theta+2\sin^2\theta)]}{1+\sin^2\theta}r^2+o(r^2),\\ \frac{d\theta}{dt} &= \frac{-r}{2(1+\sin^2\theta)(\cos^4\theta+\sin^2\theta)}+o(r).\end{aligned} \tag{2.3}$$

Hence

$$\frac{dr}{d\theta} = \frac{-\cos\theta[\sin\theta(1-2\cos^2\theta)+\mu(\cos^2\theta+2\sin^2\theta)]}{2(\cos^4\theta+\sin^2\theta)}r + o(r). \tag{2.4}$$

Let

$$r = \tilde{r}(\theta, h) = \sum_{k=1}^{\infty} \nu_k(\theta)h^k \tag{2.5}$$

be a solution of (2.4) satisfying the initial condition $r|_{\theta=0} = h$, where h is a small enough real number and

$$\nu_1(\theta) = (\cos^4\theta + \sin^2\theta)^{\frac{-1}{4}} \exp\left(\frac{-\mu}{2}\arctan\frac{\sin\theta}{\cos^2\theta}\right),$$
$$\nu_1(k\pi) = 1, \quad k = 0, \pm 1, \pm 2, \cdots. \tag{2.6}$$

Because for all sufficiently small r $(r > 0)$, from the second expression of (2.3), we have $d\theta/dt < 0$. In a small neighborhood, the successor function of system (2.1) can be defined as follows:

$$\Delta(h) = \tilde{r}(-2\pi, h) - h = \sum_{k=2}^{\infty} \nu_k(-2\pi)h^k. \tag{2.7}$$

For system (2.1), [20] gave the following results.

Lemma 2.1 ([20]). *For any positive integer m, $\nu_{2m+1}(-2\pi)$ has the form*

$$\nu_{2m+1}(-2\pi) = \sum_{k=1}^{m} \zeta_k^{(m)} \nu_{2k}(-2\pi), \tag{2.8}$$

where $\zeta_k^{(m)}$ is a polynomial of $\nu_j(\pi), \nu_j(2\pi), \nu_j(-2\pi), (j = 2, 3, \cdots, 2m)$ with rational coefficients.

It is different from the center-focus problem for the elementary critical points, we know from Lemma 2.1 that when $k > 1$ for the first non-zero $\nu_k(-2\pi), k$ is an even integer.

Definition 2.1. (1) For any positive integer m, $\nu_{2m}(-2\pi)$ is called the mth-order focal value of system (2.1) at the origin;

(2) If $\nu_2(-2\pi) \neq 0$, the origin of system (2.1) is called 1th-order weak focus. If there is an integer $m > 1$, such that $\nu_2(-2\pi) = \nu_4(-2\pi) = \cdots = \nu_{2m-2}(-2\pi) = 0$, $\nu_{2m}(-2\pi) \neq 0$, then, the origin of system (2.1) is called mth-order weak focus;

(3) If for all positive integer m, we have $\nu_{2m}(-2\pi) = 0$, then the origin of system (2.1) is called a center.

Moreover the computation of focal value of system (2.1) is also given in [20], which is shown in the following several lemmas.

Lemma 2.2 ([20]). *If the origin of system* (2.1) *is s-class or* ∞*-class, one can construct successively the terms of the formal power series* $M(x,y) = x^4+y^2+o(r^4)$, *such that*

$$\frac{\partial}{\partial x}\left(\frac{X}{M^{s+1}}\right)+\frac{\partial}{\partial y}\left(\frac{Y}{M^{s+1}}\right) = \frac{1}{M^{s+2}}\sum_{m=1}^{\infty}(2m-4s-1)\lambda_m[x^{2m+4}+o(r^{2m+4})], \tag{2.9}$$

i.e.,

$$\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)M-(s+1)\left(\frac{\partial M}{\partial x}X+\frac{\partial M}{\partial y}Y\right) = \sum_{m=1}^{\infty}\lambda_m[(2m-4s-1)x^{2m+4}+o(r^{2m+4})]. \tag{2.10}$$

Lemma 2.3 ([20]). *For system* (2.1), *if there exists a natural number s and formal series* $M(x,y) = x^4+y^2+o(r^4)$, *such that* (2.9) *holds, then*

$$\{\nu_{2m}(-2\pi)\} \sim \{\sigma_m(s,\mu)\lambda_m\}, \tag{2.11}$$

where

$$\sigma_m(s,\mu) = \frac{1}{2}\int_0^{2\pi}\frac{(1+\sin^2\theta)\cos^{2m+4}\theta}{(\cos^4\theta+\sin^2\theta)^{s+2}}\nu_1^{2m-4s-1}(\theta)d\theta. \tag{2.12}$$

Definition 2.2. If there exists a natural number s and a formal series $M(x,y) = x^4+y^2+o(r^4)$, such that (2.9) holds, then λ_m is called the m-th quasi-Lyapunov constants of the origin of system (2.1).

Lemma 2.4 ([20]). *For any positive integer s and a given number sequence*

$$\{c_{0\beta}\}, \quad \beta \geq 3, \tag{2.13}$$

one can construct successively the terms with the coefficients $c_{\alpha\beta}$ *satisfying* $\alpha \neq 0$ *of the formal series*

$$M(x,y) = y^2 + \sum_{\alpha+\beta=3}^{\infty}c_{\alpha\beta}x^\alpha y^\beta = \sum_{k=2}^{\infty}M_k(x,y), \tag{2.14}$$

such that

$$\frac{\partial}{\partial x}\left(\frac{X}{M^{s+1}}\right)+\frac{\partial}{\partial y}\left(\frac{Y}{M^{s+1}}\right) = \frac{1}{M^{s+2}}\sum_{m=5}^{\infty}\omega_m(s,\mu)x^m, \tag{2.15}$$

where for all k, $M_k(x,y)$ *is a k-homogeneous polynomial of* x, y *and* $s\mu = 0$.

Obviously, (2.15) can also be written as

$$\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)M-(s+1)\left(\frac{\partial M}{\partial x}X+\frac{\partial M}{\partial y}Y\right) = \sum_{m=3}^{\infty}\omega_m(s,\mu)x^m. \tag{2.16}$$

It is clear that (2.16) is linear with respect to the function M, so that we can easily find the following recursive formulae for the calculation of $c_{\alpha\beta}$ and $\omega_m(s,\mu)$.

Lemma 2.5 ([20]). *For $\alpha \geq 1, \alpha+\beta \geq 3$ in* (2.14) *and* (2.15), $c_{\alpha\beta}$ *can be uniquely determined by the recursive formula*

$$c_{\alpha\beta} = \frac{1}{(s+1)\alpha}(A_{\alpha-1,\beta+1} + B_{\alpha-1,\beta+1}). \tag{2.17}$$

For $m \geq 1$, $\omega_m(s,\mu)$ can be uniquely determined by the recursive formula

$$\omega_m(s,\mu) = A_{m,0} + B_{m,0}, \tag{2.18}$$

where

$$A_{\alpha\beta} = \sum_{k+j=2}^{\alpha+\beta-1} [k-(s+1)(\alpha-k+1)]a_{kj}c_{\alpha-k+1,\beta-j},$$
$$B_{\alpha\beta} = \sum_{k+j=2}^{\alpha+\beta-1} [j-(s+1)(\beta-j+1)]b_{kj}c_{\alpha-k,\beta-j+1}. \tag{2.19}$$

Notice that in (2.18), *we set*

$$c_{00} = c_{10} = c_{01} = 0, c_{20} = c_{11} = 0,\ c_{02} = 1, c_{\alpha\beta} = 0,\ \text{if}\ \alpha < 0\ \text{or}\ \beta < 0. \tag{2.20}$$

By choosing $\{c_{\alpha\beta}\}$, such that

$$\omega_{2k+1}(s,\mu) = 0, \quad k = 1, 2, \cdots, \tag{2.21}$$

we can obtain a solution group of $\{c_{\alpha\beta}\}$ of (2.21), thus, we have

$$\lambda_m = \frac{\omega_{2m+4}(s,\mu)}{2m-4s-1}. \tag{2.22}$$

Obviously, the recursive formulae in Lemma 2.5 is linear with respect to all $c_{\alpha\beta}$ which offered a good way for us to compute quasi-Lyapunov constants with help of computer algebraic system like MATHEMATICA.

3. Quasi-Lyapunov constants of system (1.1) and bifurcation of limit cycles

In order to obtain the expressions of quasi-Lyapunov constants of the three symmetric critical points $(1,0)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ of system (1.1) and study the limit cycle bifurcation. May as well make the following transformations:

$$x = x_1 + 1,\ y = y_1,\ dt = 3d\tau, \tag{3.1}$$

system (1.1) is changed into

$$\frac{dx_1}{dt} = y_1 + \delta x_1^2 + \sum_{i+2j=3}^{5} a_{ij}x_1^i y_1^j,$$
$$\frac{dy_1}{dt} = -2x_1^3 + 2\delta x_1 y_1 + \sum_{i+2j=4}^{5} b_{ij}x_1^i y_1^j. \tag{3.2}$$

Comparing system (2.1) and system (3.2), it is clear that the origin of system (3.2) is a nilpotent critical point. According to the quality of equivariant vector field and translation's invariable property, system (3.2) has three symmetric nilpotent critical points $(0,0)$, $(-\frac{3}{2},\frac{\sqrt{3}}{2})$ and $(-\frac{3}{2},-\frac{\sqrt{3}}{2})$ which have the same topological property and bifurcation behavior. Hence the study on the origin of system (3.2) will can derive the similar property for the three symmetric nilpotent critical points $(1,0)$, $(-\frac{1}{2},\frac{\sqrt{3}}{2})$ and $(-\frac{1}{2},-\frac{\sqrt{3}}{2})$ of system (1.1). Next we will investigate the bifurcation behavior of the origin of system (3.2).

According to Lemma 2.4, for system (3.2), we can find a positive integer s and a formal series $M(x,y)=x^4+y^2+o(r^4)$, such that (2.14) holds. Applying the recursive formulae presented in Lemma 2.5 to carry out calculations in MATHEMATICA, we have

$$\begin{aligned}
&\omega_3=\omega_4=\omega_5=0,\quad \omega_6=-4(B_{02}+B_{10})(-1+4s),\\
&\omega_7\sim 220B_{10}-144A_{10}B_{10}-45A_{13}B_{10}-440B_{10}s+288A_{10}B_{10}s\\
&\qquad\qquad +90A_{13}B_{10}s+3c_{03}+3sc_{03},\\
&\omega_8\sim \frac{1}{5}(33434-27676A_{10}+3600A_{10}^2-17382A_{13}+8037A_{10}A_{13}+2160A_{13}^2)B_{10}(-3+4s),\\
&\omega_9\sim \frac{1}{120}B_{10}(-850764+1444946A_{10}-801820A_{10}^2+159120A_{10}^3+228852A_{13}-265212\\
&\qquad \times A_{10}A_{13}+49725A_{10}^2A_{13}-475200B_{10}^2+311040A_{10}B_{10}^2+97200A_{13}B_{10}^2)(s-1).
\end{aligned}\tag{3.3}$$

From (2.22) and (3.3), we obtain the first two quasi-Lyapunov constants of system (3.3):

$$\begin{aligned}
&\lambda_1\sim 4(B_{02}+B_{10}),\\
&\lambda_2\sim -\frac{1}{5}(33434-27676A_{10}+3600A_{10}^2-17382A_{13}+8037A_{10}A_{13}+2160A_{13}^2)B_{10}.
\end{aligned}\tag{3.4}$$

In order to let $\omega_9\sim 0$, may as well let $s=1$. Then we can obtain the first quasi-Lyapunov constants of the origin for system (3.2) as follows:

Theorem 3.1. *The first 5 quasi-Lyapunov constants of the origin for system* (3.2) *are as follows:*

$$\begin{aligned}
&\lambda_1\sim 4(B_{02}+B_{10});\\
&\lambda_2\sim -\frac{1}{5}B_{10}(33434-27676A_{10}+3600A_{10}^2-17382A_{13}+8037A_{10}A_{13}+2160A_{13}^2);\\
&\lambda_3\sim -\frac{1}{8400}B_{10}(-289275320+579302848A_{10}-167333682A_{10}^2-124367700A_{10}^3\\
&\qquad +34847280A_{10}^4+97161960A_{13}-194957964A_{10}A_{13}+74661534A_{10}^2A_{13}\\
&\qquad +10889775A_{10}^3A_{13}+73407600B_{10}^2-146370240A_{10}B_{10}^2+68117760A_{10}^2B_{10}^2\\
&\qquad -14968800A_{13}B_{10}^2+21286800A_{10}A_{13}B_{10}^2);\\
&\lambda_4\sim \frac{2}{24310125}B_{10}m_1m_2/(n_1^3n_2^2);
\end{aligned}$$

in which

$$
\begin{aligned}
n_1 =&24310125(2356900 + 11443764A_{10} - 43655643A_{10}^2 + 40303008A_{10}^3,\\
n_2 =&32387320 - 64985988A_{10} + 24887178A_{10}^2 + 3629925A_{10}^3\\
&- 4989600B_{10}^2 + 7095600A_{10}B_{10}^2,\\
m_1 =&432457081000 - 2114784149580A_{10} + 3981317097582A_{10}^2\\
&- 3350088088065A_{10}^3 + 1027908919008A_{10}^4,\\
m_2 =&- 6142640287366513311570570724713552000000\\
&+ 11895851535416392363529935837437747488000A_{10}\\
&- 10776636289683898324329445595518827807360 0A_{10}^2\\
&+ 605142289372630676169104721115880813657600A_{10}^3\\
&- 2352438812212562063077837428522473268313536A_{10}^4\\
&+ 6694951588284207648221497734807879888755424A_{10}^5\\
&- 14378941971565333990605851497597007406345024A_{10}^6\\
&+ 23660892635875446177771110792017371263606544A_{10}^7\\
&- 29945373654876502119865249451818830180871122A_{10}^8\\
&+ 28949796635772205843294709064058792967619477A_{10}^9\\
&- 20951746648069853909581155290035572225488775A_{10}^{10}\\
&+ 10880769933820920499526730545255068428443463A_{10}^{11}\\
&- 3688349239049582685636429795170020500377607A_{10}^{12}\\
&+ 593860206291618558791899091595216005083944A_{10}^{13}\\
&+ 70928422593915180614456799184414384006272A_{10}^{14}\\
&- 49546081528776009218616097368818315132928A_{10}^{15}\\
&+ 6539589788630853482250264834573907132416A_{10}^{16}\\
&- 802227631135331722996195085425644000000B_{10}^2\\
&+ 13766121628592628487321707952731557760000A_{10}B_{10}^2\\
&- 109870672550053815434477721222916449408000A_{10}^2B_{10}^2\\
&+ 539367774670193512722507735588740085350400A_{10}^3B_{10}^2\\
&- 1814365150916952602269207140999990597919584A_{10}^4B_{10}^2\\
&+ 4407819179526408462154136365477519243954368A_{10}^5B_{10}^2\\
&- 7934611459360495609646786611605935000531808A_{10}^6B_{10}^2\\
&+ 10670525482074046319346659296170888441554784A_{10}^7B_{10}^2\\
&- 10642811245583301737458089555883855600145295A_{10}^8B_{10}^2\\
&+ 7666723483493406466055510944502940641714064A_{10}^9B_{10}^2\\
&- 3750683150060150457609564508262855072917008A_{10}^{10}B_{10}^2\\
&+ 1059469698742662320655624194646063359207808A_{10}^{11}B_{10}^2\\
&- 60931471241140097715899127790127908227072A_{10}^{12}B_{10}^2
\end{aligned}
$$

$$
\begin{aligned}
&- 563813373280217323862464418586474121789 44A_{10}^{13}B_{10}^{2}\\
&+ 1278327053705216608295074393002682299187 2A_{10}^{14}B_{10}^{2},
\end{aligned}
$$

and

(1) If $m_1 = 0$*, then* $\lambda_5 \sim 0$.

(2) If $m_2 = 0$*, then* $\lambda_5 \sim B_{10}m_3m_4^3m_5/(n_3^3n_4^4)$,

in which

$$
\begin{aligned}
m_3 =& 282 - 857A_{10} + 648A_{10}^2,\\
m_4 =& - 891364034594813025551327872695160000 00\\
&+ 152956906984362538748018977252572864 0000A_{10}\\
&- 122078525055615350482753023581018277 12000A_{10}^2\\
&+ 599297527411326125247230817320822317 05600A_{10}^3\\
&- 201596127879661400252134126777776733 102176A_{10}^4\\
&+ 489757686614045384683792929497502138 217152A_{10}^5\\
&- 881623495484499512182976290178437222 281312A_{10}^6\\
&+ 118561394245267181326073992179676538 2394976A_{10}^7\\
&- 118253458284258908193978772843153951 1127255A_{10}^8\\
&+ 851858164832600718450612327166993404 634896A_{10}^9\\
&- 416742572228905606401062723140317230 324112A_{10}^{10}\\
&+ 117718855415851368961736021627340373 245312A_{10}^{11}\\
&- 677016347123778863509990308779198980 3008A_{10}^{12}\\
&- 626459303644685915402738242873860135 3216A_{10}^{13}\\
&+ 142036339300579623143897154778075811 0208A_{10}^{14},\\
m_5 =& - 429153322621986110300383359835263714786469083016100768000000\\
&+ 12135294214967177193691107989616593544872181416571940858240000A_{10}\\
&- 163897155533505425205714209384503303011448079638983707806476800A_{10}^2\\
&+ 1405458899632513797814865718369576308680157087239724151212279040A_{10}^3\\
&- 8580379214231524333047555721080053068636193062186819630778777600A_{10}^4\\
&+ 39632258996948626961851032850273632985183691001346805309372557952A_{10}^5\\
&- 143658107082931901022664968059807755306616478851422792652055543104A_{10}^6\\
&+ 418266618559351085176026372150260324745269063380489253399889289632A_{10}^7\\
&- 992826567841577804692961345333382602120563046223675339755111354664A_{10}^8\\
&+ 1938437857458792187913663363694430297505906459660447044028337679596A_{10}^9\\
&- 3125827727412598988220287444403917881423245359370468029748128387298A_{10}^{10}\\
&+ 4161688243397171335465311073143463419921007938617617260337924679601A_{10}^{11}\\
&- 4551575255893649975534205220888035223309835894495444049193477357875A_{10}^{12}\\
&+ 4044112846771249001086874969333447246445935246911269088347024135609A_{10}^{13}\\
&- 2860605704668885309659262607792026218816474670013658211635701481840A_{10}^{14}\\
&+ 1551690152172988433027301122675729000514736248520512286370856969030A_{10}^{15}\\
&- 595605313963884008298209700833328386298368547714197176072036536978A_{10}^{16}
\end{aligned}
$$

$$
\begin{aligned}
&+ 124634018826691773309014936985512382371781410580598531586799407856A_{10}^{17}\\
&+ 12707155555768148718289695054317509770186922978936586377214756688A_{10}^{18}\\
&- 19207760170838338003056363394420583033032604503432355199178235136A_{10}^{19}\\
&+ 6398296296419108836162869487210778344228427466591266160451092480A_{10}^{20}\\
&- 989103619616268000037970335256517352476646245400735468934135808A_{10}^{21}\\
&+ 55204271873258557861072808480380095062766407616829302125887488A_{10}^{22};
\end{aligned}
$$

$$
n_3 = 40111706250(2356900 + 11443764A_{10} - 43655643A_{10}^2 + 40303008A_{10}^3,
$$

$$
\begin{aligned}
n_4 =& - 7463023134133691009821563460000000\\
&+ 107820926146853624122090307940480000A_{10}\\
&- 713116430844566663885617359077836800A_{10}^2\\
&+ 2844541270346291948127963306964276224A_{10}^3\\
&- 7583853608339658577135198612660468512A_{10}^4\\
&+ 14136219152432887673712877866097657920A_{10}^5\\
&- 18681974327081692670059096122237915744A_{10}^6\\
&+ 17304088268303913256376502538827153504A_{10}^7\\
&- 10730206834827871187516152380596328573A_{10}^8\\
&+ 3932181821913156193706272128137142816A_{10}^9\\
&- 490885141160583277010515864933899264A_{10}^{10}\\
&- 171179345278968761993762160825237504A_{10}^{11}\\
&+ 53795148521586502820501181828169728A_{10}^{12}.
\end{aligned}
$$

In the above expression of λ_k, *we have already let* $\lambda_1 = \cdots = \lambda_{k-1} = 0, k = 2, 3, 4, 5.$

Remark 3.1. During the fourth quasi-Lyapunov constants, we let $c_{04=0}$ and

$$
\begin{aligned}
c_{05} =& -\frac{1}{1016064000l_1^2l_2}B_{10}(-570812271801995156168696998992000000\\
&+ 8486796181480990243075546513775680000A_{10}\\
&- 58649810277852428866597040485128960000A_{10}^2\\
&+ 250857331467417569846231396148840755200A_{10}^3\\
&- 745791876509154022563466443064046198400A_{10}^4\\
&+ 1638406950848060402124582375482477375040A_{10}^5\\
&- 2751140861270943125890650846000072145920A_{10}^6\\
&+ 3582818108502159657904282984896401624640A_{10}^7\\
&- 3609785433079235161842709070563584143940A_{10}^8\\
&+ 2758849232877219908075206476622273441650A_{10}^9\\
&- 1537592606521372091373180323034517206480A_{10}^10\\
&+ 583893081981748485144727583977582974720A_{10}^{11}\\
&- 133385426777069984381167969593129861120A_{10}^{12}\\
&+ 13606633559233472552807161160383856640A_{10}^{13}
\end{aligned}
$$

$$
\begin{aligned}
&- 7258551026345105317985375600400000\,0B_{10}^2\\
&+ 94893829791907227788157785134416000\,0A_{10}B_{10}^2\\
&- 56545636530970999984723234112504160\,00A_{10}^2B_{10}^2\\
&+ 20201882163845266443842682280516895040A_{10}^3B_{10}^2\\
&- 47870670883372691899984862186960801280A_{10}^4B_{10}^2\\
&+ 78536965778783114818611442092433332336A_{10}^5B_{10}^2\\
&- 90266968523403962542479801722394096936A_{10}^6B_{10}^2\\
&+ 71740865548384843804060446514909710396A_{10}^7B_{10}^2\\
&- 37779078560075046313420861378141900623A_{10}^8B_{10}^2\\
&+ 11999285863572021085818690522673108320A_{10}^9B_{10}^2\\
&- 1834454493657903415001297676340294656A_{10}^{10}B_{10}^2\\
&+ 507107210825229040690472080389242\,88A_{10}^{11}B_{10}^2),
\end{aligned}
$$

in which

$$
\begin{aligned}
l_1 =& 2356900 + 11443764A_{10} - 43655643A_{10}^2 + 40303008A_{10}^3,\\
l_2 =& 32387320 - 64985988A_{10} + 24887178A_{10}^2 + 3629925A_{10}^3\\
&- 4989600B_{10}^2 + 7095600A_{10}B_{10}^2.
\end{aligned}
$$

By analyzing the construction of quasi-Lyapunov constants of Theorem 3.1, we can obtain the following result.

Theorem 3.2. *The origin for system* (3.2) *can become a 5th-order nilpotent weak focus if and only if the following condition holds.*

$$
\begin{aligned}
&B_{10} \neq 0,\ B_{02} = -B_{10}, m_1 \neq 0,\ m_2 = 0,\\
&A_{13} = \frac{2}{3(32387320 - 64985988A_{10} + 24887178A_{10}^2 + 3629925A_{10}^3 - 4989600B_{10}^2 + 7095600A_{10}B_{10}^2)}\\
&\quad \times (144637660 - 289651424A_{10} + 83666841A_{10}^2 - +62183850A_{10}^3 - 17423640A_{10}^4\\
&\quad - 36703800B_{10}^2 + 73185120A_{10}B_{10}^2 - 34058880A_{10}^2B_{10}^2).
\end{aligned}
$$

Proof. From the expressions of $\lambda_k, k = 1, 2, 3, 4, 5$, clearly the necessity holds. Next we prove the sufficiency. In order to prove the origin for system (3.2) is a 5th-order weak focus, we only need to prove that there exists a group of solutions about $B_{10}, A_{10}, A_{13}, B_{02}$ such that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0, \lambda_5 \neq 0$.

From $\lambda_1 = 0$, it is clear that $B_{02} = -B_{10}$. According to $\lambda_2 = 0$, we can obtain $A_{13}^2 = (-33434 + 27676A_{10} - 3600A_{10}^2 + 17382A_{13} - 8037A_{10}A_{13})/2160$ and the expression of λ_3. Next $\lambda_3 = 0$ will deduce

$$
\begin{aligned}
&A_{13} = \frac{2}{3(32387320 - 64985988A_{10} + 24887178A_{10}^2 + 3629925A_{10}^3 - 4989600B_{10}^2 + 7095600A_{10}B_{10}^2)}\\
&\quad \times (144637660 - 289651424A_{10} + 83666841A_{10}^2 - +62183850A_{10}^3 - 17423640A_{10}^4\\
&\quad - 36703800B_{10}^2 + 73185120A_{10}B_{10}^2 - 34058880A_{10}^2B_{10}^2).
\end{aligned}
$$

Next let $\lambda_4 = 0$, then $m_1 = 0$ or $m_2 = 0$. At this time, $m_1 = 0$ will deduce $\lambda_5 = 0$, and $m_2 = 0$ will deduce $\lambda_5 \sim B_{10}m_3m_4^3m_5/(n_3^3n_4^4)$. And under the condition of Theorem 3.2, the all 6 groups of real number solutions of $\lambda_1 = \lambda_2 = \lambda_3 = m_2 = 0$

about $B_{10}, A_{10}, A_{13}, B_{02}$ can be obtained as follows:

$(S_1)\, A_{10} \approx -4.7269849,\ B_{10} \approx 52.3894621,\ B_{02} \approx -52.3894621,\ A_{13} \approx 19.9598497,$
$(S_2)\, A_{10} \approx -4.7269849,\ B_{10} \approx -52.3894621,\ B_{02} \approx 52.3894621,\ A_{13} \approx 19.9598497,$
$(S_3)\, A_{10} \approx 1.5929351,\ B_{10} \approx -3.1514449,\ B_{02} \approx 3.1514449,\ A_{13} \approx -0.2912932,$
$(S_4)\, A_{10} \approx 1.5929351,\ B_{10} \approx 3.1514449,\ B_{02} \approx -3.1514449,\ A_{13} \approx -0.2912932,$
$(S_5)\, A_{10} \approx 1.3374828,\ B_{10} \approx -0.5427501,\ B_{02} \approx 0.5427501,\ A_{13} \approx 3.7000000,$
$(S_6)\, A_{10} \approx 1.3374828,\ B_{10} \approx 0.5427501,\ B_{02} \approx -0.5427501,\ A_{13} \approx 3.7000000.$

The above solutions will all deduce $\lambda_5 \neq 0$, namely the following values about λ_5:

$$\begin{aligned}
\lambda_5|_{S_1} &\approx -4.0068953148810686711729517293348436213028251491220564 1 \times 10^{16},\\
\lambda_5|_{S_2} &\approx 4.00689531488106867117295172933484362130282514912205641 \times 10^{16},\\
\lambda_5|_{S_3} &\approx 2.30773698915477796189833886114626487065494961430698472 7 \times 10^{7},\\
\lambda_5|_{S_4} &\approx -2.307736989154777961898338861146264870654949614306984727 \times 10^{7},\\
\lambda_5|_{S_5} &\approx -3.595047779564351267699853740246172798278066519338043 83 \times 10^{8},\\
\lambda_5|_{S_6} &\approx 3.59504777956435126769985374024617279827806651933804383 \times 10^{8}.
\end{aligned}$$

Hence the conclusion of Theorem 3.2 holds. □

According the equivariant symmetric quality, one can obtain the following theorem.

Theorem 3.3. *The three symmetric critical points* $(1, 0), (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ *and* $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ *of system* (1.1) *become three nilpotent weak focuses of 5th-order if and only if condition of Theorem 3.2 holds.*

After discussing the weak focus problem of system (1.1), we will consider the limit cycle bifurcation of system (1.1). From Theorem 3.2 and Theorem 3.3, we can obtain the following theorem.

Theorem 3.4. *Each one of the three symmetric critical points* $(1, 0), (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ *and* $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ *of system* (1.1) *can bifurcate 4 small limit cycles under appropriate perturbations about parameter group* $(B_{02}, B_{10}, A_{10}, A_{13})$ *if one of the three given critical points of system* (1.1) *is a nilpotent weak focus of 5th-order, in sum 12 limit cycles can yield by simultaneous Hopf bifurcation.*

Proof. If one of the three symmetric critical points $(1, 0), (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ of system (1.1) is a nilpotent weak focus of 5th-order, then other two critical points are also two nilpotent weak focuses of 5th-order because system (1.1) lies in a Z_3-equivariant vector field. At this time, let values of parameter group $(B_{02}, B_{10}, A_{10}, A_{13})$ are (a, b, c, d), namely $\lambda_5 \neq 0$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ if $(B_{02}, B_{10}, A_{10}, A_{13}) = (a, b, c, d)$. Of course, during proving Theorem 3.2, we have show that this kind of solutions have 6 groups, i.e., $S_1 \sim S_6$, here we only express them as (a, b, c, d).

Give a suitable perturbation about these parameters, we may as well let

$$\lambda_1 = \varepsilon_1,\ \lambda_2 = \varepsilon_2,\ \lambda_3 = \varepsilon_3,\ \lambda_4 = \varepsilon_4, \tag{3.5}$$

in which $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ can be a group of arbitrary real numbers. When the three symmetric critical points of undisturbed system $(1.1)|_{\delta=0}$ are three nilpotent weak focuses of $5th$ order, then the jacobin of the functions group $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ with respect to the variables group $(B_{02}, B_{10}, A_{13}, A_{10})$

$$J = \frac{D(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}{D(B_{02},B_{10},A_{13},A_{10})} = \begin{vmatrix} \frac{\partial\lambda_1}{\partial B_{02}} & \frac{\partial\lambda_1}{\partial B_{10}} & \frac{\partial\lambda_1}{\partial A_{13}} & \frac{\partial\lambda_1}{\partial A_{10}} \\ \frac{\partial\lambda_2}{\partial B_{02}} & \frac{\partial\lambda_2}{\partial B_{10}} & \frac{\partial\lambda_2}{\partial A_{13}} & \frac{\partial\lambda_2}{\partial A_{10}} \\ \frac{\partial\lambda_3}{\partial B_{02}} & \frac{\partial\lambda_3}{\partial B_{10}} & \frac{\partial\lambda_3}{\partial A_{13}} & \frac{\partial\lambda_3}{\partial A_{10}} \\ \frac{\partial\lambda_4}{\partial B_{02}} & \frac{\partial\lambda_4}{\partial B_{10}} & \frac{\partial\lambda_4}{\partial A_{13}} & \frac{\partial\lambda_4}{\partial A_{10}} \end{vmatrix} = f(B_{02},B_{10},A_{13},A_{10}),$$

in which $f(B_{02},B_{10},A_{13},A_{10})$ is a function about $B_{02},B_{10},A_{13},A_{10}$. We can obtain that $f(B_{10},A_{13},A_{10}) \neq 0$ if the three symmetric critical points of undisturbed system $(1.1)|_{\delta=0}$ are three nilpotent weak focuses of $5th$ order. In fact, if they become nilpotent weak focuses of $5th$ order, then

$$\begin{aligned} &f(B_{02},B_{10},A_{13},A_{10})|_{S_1} \approx 5.374760669957396 \times 10^{24} \neq 0, \\ &f(B_{02},B_{10},A_{13},A_{10})|_{S_2} \approx 5.374760669957396 \times 10^{24} \neq 0, \\ &f(B_{02},B_{10},A_{13},A_{10})|_{S_3} \approx 6.18776801205598707424489 \times 10^{15} \neq 0, \\ &f(B_{02},B_{10},A_{13},A_{10})|_{S_4} \approx 6.18776801205598707424489 \times 10^{15} \neq 0, \\ &f(B_{02},B_{10},A_{13},A_{10})|_{S_5} \approx 1.59504777956435126769985 \times 10^{18} \neq 0, \\ &f(B_{02},B_{10},A_{13},A_{10})|_{S_6} \approx 1.59504777956435126769985 \times 10^{18} \neq 0. \end{aligned}$$

Hence, according to existence theorem of implicit function, equations groups (3.5) have a group of solutions as follows:

$$\begin{aligned} &B_{02} = a + f_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4),\ B_{10} = b + f_2(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4), \\ &A_{13} = c + f_3(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4),\ A_{10} = d + f_4(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4). \end{aligned} \tag{3.6}$$

Obviously, given perturbations by (3.6) will let (3.5) hold. Because $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ can be a group of arbitrary real numbers, one can give many kinds of perturbations's methods by (3.6). Hence, there exists a vector on small parameters $\varepsilon_k = \left(\varepsilon_1^{(k)}, \varepsilon_2^{(k)}, \varepsilon_3^{(k)}, \varepsilon_4^{(k)}\right)$ such that the first 5 quasi-Lyapunov constants of nilpotent singular point (1,0) of system (1.1) satisfy

$$\begin{aligned} &\lambda_1\lambda_2 < 0, \quad \lambda_2\lambda_3 < 0, \quad \lambda_3\lambda_4 < 0, \quad \lambda_4\lambda_5 < 0, \\ &|\lambda_1| \ll |\lambda_2| \ll |\lambda_3| \ll |\lambda_4| \ll |\lambda_5|. \end{aligned} \tag{3.7}$$

According to (3.5), we only need to let $|\varepsilon_1| \ll |\varepsilon_2| \ll |\varepsilon_3| \ll |\varepsilon_4|$, then (3.7) holds. According to Theorem 4 in [27], 4 small limit cycles can occur near singular point (1,0) of system (1.1). From the quality of equivariant vector field, each one of singular points $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ of system (1.1) also can bifurcate 4 limit cycles. Hence, in sum 12 small limit cycles can bifurcate from disturbed system (1.1) by simultaneous Hopf bifurcation. □

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