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A KIND OF BIFURCATION OF LIMIT CYCLES FROM A NILPOTENT CRITICAL POINT*

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Abstract In this paper, an interesting and new bifurcation phenomenon that limit cycles could be bifurcated from nilpotent node (focus) by changing its stability is investigated. It is different from lowing its multiplicity in order to get limit cycles. We prove that $n^2 + n - 1$ limit cycles could be bifurcated by this way for $2n + 1$ degree systems. Moreover, this upper bound could be reached. At last, we give two examples to show that $N(3) = 1$ and $N(5) = 5$ respectively. Here, $N(n)$ denotes the number of small-amplitude limit cycles around a nilpotent node (focus) with n being the degree of polynomials in the vector field.

Keywords Nilpotent critical point, limit cycle, bifurcation.

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1. Introduction

One of the most intriguing aspects of the dynamics of real planar polynomial vector fields is the close relationship between the center conditions and bifurcation of limit cycles. Bifurcation of limit cycles from a high-order critical point in plane is becoming more and more important, there have been many results about this problem. The B-T bifurcation from a saddle-node point was discussed in [16,19,21,22]. Bifurcation of limit cycles from a degenerate critical point was investigated by Han and Yu, see [6]. Especially, there were many results about bifurcations of limit cycles from a nilpotent critical point, see [3,4,9,17,18,20] and [5,7,8,12–15].

The following planar real systems

$$\frac{dx}{dt} = y + \sum_{i+j=2}^{\infty} a_{ij}x^i y^j = \Phi(x, y), \quad \frac{dy}{dt} = \sum_{i+j=2}^{\infty} b_{ij}x^i y^j = \Psi(x, y), \tag{1.1}$$

whose functions of right hand are analytic in a neighborhood of the origin will be discussed in this paper. The linear parts of (1.1) has double zero eigenvalues but the matrix of the linearized system of (1.1) at the origin is not identically null. The origin $O(0, 0)$ of system (1.1) is called a nilpotent singular point.

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The paper will be divided into five sections. Some preliminary knowledge will be given in section two. In section three, stability and bifurcation of limit cycle from a nilpotent node (focus) will be discussed thoroughly. At last, we will give two examples to show our results in section four and five.

2. Preliminary Knowledge

In [1] the author gave a definition of the multiplicity. In [10], the authors also gave a definition of the multiple number of a critical point of system

$$\begin{aligned} \frac{dx}{dt} &= \sum_{k+j=0}^{n} a_{kj}x^k y^j = P(x,y), \\ \frac{dy}{dt} &= \sum_{k+j=0}^{m} b_{kj}x^k y^j = Q(x,y), \end{aligned} \tag{2.1}$$

by using the crossing number of two algebraic curves given by the right-hand sides of a polynomial system, where the crossing number of two algebraic curves is the number of intersection points of two algebraic curves. There is a natural equivalence between the algebraic viewpoint and the geometric viewpoint.

Definition 2.1. Suppose (x_0, y_0) is an isolate critical point (2.1)(real or complex), if the crossing number of $P(x,y)=0$ and $Q(x,y)=0$ at (x_0,y_0) is N, then the point (x_0,y_0) is called a N-multiple singular point of (2.1), N is called the multiplicity of the point (x_0,y_0).

Definition 2.1 yields that

Proposition 2.1. *Suppose that the function $y=y(x)$ satisfies $\Phi(x,y(x))=0$ and $y(0)=0$, if $\Psi(x,y(x)) = Ax^N + o(x^N)$, $A \neq 0$, then the the origin of (1.1) is a N-multiple singular point.*

Under a small perturbation of system, a multiple critical point can decompose into much lower multiple critical points. Now, we consider the perturbed system of (1.1) and (2.1)

$$\frac{dx}{dt} = \Phi(x,y) + h(x,y,\varepsilon), \quad \frac{dy}{dt} = \Psi(x,y) + g(x,y,\varepsilon), \tag{2.2}$$

and

$$\frac{dx}{dt} = P(x,y) + h(x,y,\varepsilon), \quad \frac{dy}{dt} = Q(x,y) + g(x,y,\varepsilon), \tag{2.3}$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_l)$ is a finite dimension small parameters, $h(x,y,\varepsilon)$ and $g(x,y,\varepsilon)$ are power series of (x,y,ε) with nonzero convergence radius, and $h(x,y,0) = 0, g(x,y,0)=0$. From Theorem 1 in [10] and Theorem 2.1 in [14], it is easy to get the following theorem.

Theorem 2.1. *Suppose the origin of system (1.1) (or (2.1)) is a N-multiple singular point, then when $||\varepsilon|| << 1$, the sum of multiplicity of all complex singular point in a sufficiently small neighborhood of origin of (2.2) (or (2.3)) is exactly N.*

Example 2.1. From Proposition 2.1, the multiplicity of the origin of system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = Ax^N + yg(x, y) \tag{2.4}$$

is exactly N, $A \neq 0$, $g(x, y)$ is analytic in a neighborhood of origin. When $||\varepsilon|| \ll 1$, there are m critical points $(\varepsilon_k, 0)$ in a neighborhood of origin of system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = A\prod_{k=1}^{m}(x - \varepsilon_k)^{l_k} + yg(x, y), \tag{2.5}$$

and their multiplicity are l_k, $k = 1, 2, \cdots, m$, where $l_1 + l_2 + \ldots + l_m = N$.

Theorem 2.2. *Suppose the index of the origin of system* (1.1) (*or* (2.1)) *is* k*, then when* $||\varepsilon|| << 1$*, the sum of index of all real singular point in a sufficiently small neighborhood of origin of* (2.2) (*or* (2.3))*is exactly* k*.*

Liu et al. gave the following definition in order to compute Lyapunov constant in [11].

Definition 2.2. Let f_k, g_k be polynomials with respect to $a_{ij}{}'s, b_{ij}{}'s$, $k = 1, 2, \cdots$. If for an integer m, there exist polynomials with respect to $a_{ij}{}'s$, $b_{ij}{}'s$: $\xi_1^{(m)}$, $\xi_2^{(m)}$, $\cdots$, $\xi_{m-1}^{(m)}$, such that

$$f_m = g_m + \left(\xi_1^{(m)} f_1 + \xi_2^{(m)} f_2 + \cdots + \xi_{m-1}^{(m)} f_{m-1}\right). \tag{2.6}$$

Then, we say that f_m and g_m is algebraic equivalent, written by $f_m \sim g_m$. If for any integer m, we have $f_m \sim g_m$, we say that the sequences of functions $\{f_m\}$ and $\{g_m\}$ are algebraic equivalent, written by $\{f_m\} \sim \{g_m\}$.

The authors have proved that a nilpotent-node (nilpotent-focus) point with multiplicity $2m+1$ could be broken into a nilpotent-node (nilpotent-focus) with multiplicity $2m-1$ and two complex singular points by a small parameters perturbation in [14]. If the stability at the elementary focus and nilpotent singular point is different, limit cycle will be bifurcated out from sufficiently small neighborhood of the element focus. In this paper, bifurcation of limit cycles from a nilpotent-node (nilpotent-focus) point will be investigated by changing the stability of the nilpotent-node (nilpotent-focus) point when the multiplicity is not decreased. It is different from [14].

3. Stability and bifurcation of limit cycle at nilpotent node (focus)

Using theorem proved in [23], see also in [2], we have

Proposition 3.1. *Suppose that the function* $y = y(x)$ *satisfies* $\Phi(x, y(x)) = 0$, $y(0) = 0$*, and*

$$\begin{aligned} &\Psi(x, y(x)) = \alpha_{2m+1}x^{2m+1} + o(x^{2m+1}), \qquad \alpha_{2m+1} < 0, \\ &\left(\frac{\partial\Phi}{\partial x} + \frac{\partial\Psi}{\partial y}\right)_{y=y(x)} = \beta_{2n}x^{2n} + o(x^{2n}), \quad \beta_{2n} \neq 0, \end{aligned} \tag{3.1}$$

where n, m are positive integers, then the origin of (1.1) *is a node with multiplicity* $2m+1$, *and the origin is a node if and only if one of the following conditions is satisfied:*

$$\begin{aligned} &C_1: \ 2n < m, \quad \alpha_{2m+1} < 0; \\ &C_2: \ 2n = m, \quad \alpha_{2m+1} < 0, \quad \beta_{2n}^2 + 4(m+1)\alpha_{2m+1} \geqslant 0. \end{aligned} \tag{3.2}$$

Furthermore, we easily get

Theorem 3.1. *Suppose that the function $y = y(x)$ satisfies $\Phi(x, y(x)) = 0$, $y(0) = 0$, and* (3.1) *holds, then multiplicity of the origin of system* (1.1) *is* $2m+1$, *Lyapunov constants are*

$$V_n = \beta_{2n}, \tag{3.3}$$

namely it is stable when $\beta_{2n} < 0$ and unstable when $\beta_{2n} > 0$.

Proof. From the discussions in [20] and [5], under conditions in Theorem 3.1, system (1.1) could be transformed into Liénard system

$$\frac{du}{d\tau} = v, \quad \frac{dv}{d\tau} = \alpha_{2m+1}u^{2m+1} + \beta_{2n} v u^{2n} g(u) \tag{3.4}$$

by the following analytic changes

$$\begin{aligned} u &= x + \sum_{k+j=2}^{\infty} a'_{kj} x^k y^j, \\ v &= y + \sum_{k+j=2}^{\infty} b'_{kj} x^k y^j, \\ \frac{dt}{d\tau} &= 1 + \sum_{k+j=1}^{\infty} c'_{kj} x^k y^j, \end{aligned} \tag{3.5}$$

where $g(u)$ is analytic at $u = 0$, and $g(0) = 1$. Let $V = v^2 - \frac{1}{m+1}\alpha_{2m+1}u^{2m+2}$,

$$\left.\frac{dV}{d\tau}\right|_{(3.4)} = 2\beta_{2n} v^2 u^{2n} g(u). \tag{3.6}$$

So the conclusion in Theorem 3.1 holds. □

The Theorem 3.1 leads to the following theorem

Theorem 3.2. *Suppose that the function $y = y(x)$ satisfies $\Phi(x, y(x)) = 0$, $y(0) = 0$, and*

$$\begin{aligned} &\Psi(x, y(x)) = \alpha_{2m+1}x^{2m+1} + o(x^{2m+1}), \qquad \alpha_{2m+1} < 0, \\ &\left(\frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y}\right)_{y=y(x)} = \sum_{k=1}^{n} \beta_{2k}\left(x^{2k} + o(x^{2k})\right), \quad \beta_{2n} \neq 0, \end{aligned} \tag{3.7}$$

where n, m are positive integers, then there exist $n-1$ limit cycles in a neighborhood of origin of system (1.1) *when*

$$0 < |\beta_2| \ll |\beta_4| \ll \cdots |\beta_{2n}|, \quad \beta_{2k}\beta_{2k+2} < 0, \ k = 1, 2, \cdots, n-1. \tag{3.8}$$

Example 3.1. From Theorem 3.2, when (3.8) holds, there exist $n-1$ limit cycles in a neighborhood of origin of system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x^{2m+1} + y\sum_{k=1}^{n} \beta_{2k} x^{2k}. \tag{3.9}$$

Suppose O is a nilpotent-node of system

$$\frac{dx}{dt} = y + \sum_{k+j=2}^{2n+1} a_{kj} x^k y^j, \quad \frac{dy}{dt} = \sum_{k+j=2}^{2n+1} b_{kj} x^k y^j, \tag{3.10}$$

we denote the number of limit cycles which could be bifurcated from origin of (3.10) by changing the stability of the nilpotent-node point when the multiplicity is not decreased by $N(2n+1)$. It is easy to know that multiplicity of the nilpotent-node point O is not more than $(2n+1)^2$ from Bezout theorem and definition 2.1. Combining with 3.1, we could get

Theorem 3.3.

$$N(2n+1) \leq n^2 + n - 1. \tag{3.11}$$

We will give two examples in Section 4 and Section 5 to show that the upper bound is achieved when $n = 1$ and $n = 2$ in (3.11) respectively, namely $N(3) = 1,\ N(5) = 5$.

4. N(3)=1

In this section, we will prove that the following cubic system

$$\begin{aligned} \frac{dx}{dt} &= y + x^2 + \varepsilon^2 y^2 + \varepsilon^2 x^2 y - xy^2 + \varepsilon y^3 = X(x,y), \\ \frac{dy}{dt} &= -2xy - 2\varepsilon y^2 - 2x^3 - 2\varepsilon x^2 y - 2y^3 = Y(x,y). \end{aligned} \tag{4.1}$$

has only one limit cycle in a neighborhood of the origin, namely $N(3) = 1$. For system (4.1), a solution for $X(x, y(x)) = 0$ and $y(0) = 0$ is

$$y = y(x) = -x^2 + x^5 + \varepsilon x^6 + \varepsilon^2 x^7 + (-2 + \varepsilon^3)x^8 + o(x^8), \tag{4.2}$$

and

$$\begin{aligned} &Y(x, y(x)) = -2x^9 + o(x^9), \\ &\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)\bigg|_{y=y(x)} = 2\varepsilon x^2(1 - \varepsilon x) - 7x^4 + o(x^4). \end{aligned} \tag{4.3}$$

From (4.3), $\beta_2 = 2\varepsilon, \quad \beta_4 = -7, \quad \alpha_9 = -2 < 0$, then $\Delta = \beta_4^2 + 20\alpha_9 = 9 > 0$ when $\varepsilon = 0$, Theorem 3.2 shows that

Theorem 4.1. *The origin of system* (4.1) *is a nilpotent node of multiplicity 9, and there is a limit cycle in a neighborhood of the origin of system* (4.1) *when* $0 < \varepsilon \ll 1$.

5. $N(2)=5$

In this section, we will prove that the upper bound could be reached when $n=2$. A class of Z_2-equivariant quintic with 25-multiple nilpotent node $O(0,0)$

$$\begin{aligned}\frac{dx}{dt}&=y+\sum_{k+j=3}a_{kj}x^ky^j+\sum_{k+j=5}a_{kj}x^ky^j=X(x,y),\\ \frac{dy}{dt}&=\sum_{k+j=3}b_{kj}x^ky^j+\sum_{k+j=5}b_{kj}x^ky^j=Y(x,y),\end{aligned}\tag{5.1}$$

where

$$\begin{aligned}
a_{30}=&1,\quad a_{21}=7\lambda_1,\quad a_{12}=\lambda_1\lambda_3,\\
a_{03}=&\frac{1}{8}(-1029\lambda_1^3+140\lambda_1^4+343\lambda_1^2\lambda_2-12\lambda_1^3\lambda_2-35\lambda_1\lambda_2^2+\lambda_2^3\\
&-28\lambda_1^2\lambda_3+4\lambda_1\lambda_2\lambda_3+16\lambda_4-56\lambda_1^2\lambda_5-8\lambda_1\lambda_2\lambda_5),\\
a_{50}=&0,\quad a_{41}=\lambda_1\lambda_5,\quad a_{32}=\lambda_4,\\
a_{23}=&\frac{1}{4}\lambda_1(-343\lambda_1^4+4\lambda_1^5+70\lambda_1^3\lambda_2-3\lambda_1^2\lambda_2^2-4\lambda_1^3\lambda_3\\
&+28\lambda_4-196\lambda_1^2\lambda_5+8\lambda_1^3\lambda_5+4\lambda_1\lambda_3\lambda_5-4\lambda_1\lambda_5^2),\\
a_{05}=&\frac{1}{16}(-50421\lambda_1^7+1960\lambda_1^8-16\lambda_1^9+19894\lambda_1^6\lambda_2-392\lambda_1^7\lambda_2-2744\lambda_1^5\lambda_2^2\\
&+16\lambda_1^6\lambda_2^2+154\lambda_1^4\lambda_2^3+686\lambda_1^2\lambda_2\lambda_4-24\lambda_1^3\lambda_2\lambda_4-70\lambda_1\lambda_2^2\lambda_4\\
&+2\lambda_2^3\lambda_4-56\lambda_1^2\lambda_3\lambda_4+8\lambda_1\lambda_2\lambda_3\lambda_4+16\lambda_4^2+14406\lambda_1^5\lambda_5\\
&+1960\lambda_1^6\lambda_5-64\lambda_1^7\lambda_5-4802\lambda_1^4\lambda_2\lambda_5-616\lambda_1^5\lambda_2\lambda_5+490\lambda_1^3\lambda_2^2\lambda_5\\
&+32\lambda_1^4\lambda_2^2\lambda_5-14\lambda_1^2\lambda_2^3\lambda_5+392\lambda_1^4\lambda_3\lambda_5+64\lambda_1^5\lambda_3\lambda_5\\
&-56\lambda_1^3\lambda_2\lambda_3\lambda_5-112\lambda_1^2\lambda_4\lambda_5-16\lambda_1\lambda_2\lambda_4\lambda_5-64\lambda_1^5\lambda_5^2+112\lambda_1^3\lambda_2\lambda_5^2),\\
a_{14}=&\frac{1}{8}\lambda_1(-7203\lambda_1^5+84\lambda_1^6+1813\lambda_1^4\lambda_2-4\lambda_1^5\lambda_2-133\lambda_1^3\lambda_2^2+3\lambda_1^2\lambda_2^3-84\lambda_1^4\lambda_3\\
&+4\lambda_1^3\lambda_2\lambda_3+8\lambda_3\lambda_4-1029\lambda_1^3\lambda_5+308\lambda_1^4\lambda_5+343\lambda_1^2\lambda_2\lambda_5-20\lambda_1^3\lambda_2\lambda_5\\
&-35\lambda_1\lambda_2^2\lambda_5+\lambda_2^3\lambda_5-84\lambda_1^2\lambda_3\lambda_5+4\lambda_1\lambda_2\lambda_3\lambda_5+56\lambda_1^2\lambda_5^2-8\lambda_1\lambda_2\lambda_5^2),\\
b_{30}=&0,\quad b_{21}=\lambda_1,\quad b_{12}=-\lambda_1(7\lambda_1-\lambda_2),\\
b_{03}=&\frac{1}{4}\lambda_1(49\lambda_1^2+4\lambda_1^3-14\lambda_1\lambda_2+\lambda_2^2),\\
b_{50}=&\lambda_1,\quad b_{41}=\lambda_1\lambda_2,\\
b_{32}=&\frac{1}{4}\lambda_1(-147\lambda_1^2+4\lambda_1^3+14\lambda_1\lambda_2+\lambda_2^2+4\lambda_1\lambda_3-4\lambda_1\lambda_5),\\
b_{23}=&\frac{1}{8}\lambda_1(-343\lambda_1^3+196\lambda_1^4+147\lambda_1^2\lambda_2-12\lambda_1^3\lambda_2-21\lambda_1\lambda_2^2+\lambda_2^3\\
&-84\lambda_1^2\lambda_3+12\lambda_1\lambda_2\lambda_3+8\lambda_4+56\lambda_1^2\lambda_5-16\lambda_1\lambda_2\lambda_5),\\
b_{14}=&-\frac{1}{8}\lambda_1(-7203\lambda_1^4+294\lambda_1^5+8\lambda_1^6+3430\lambda_1^3\lambda_2-84\lambda_1^4\lambda_2-588\lambda_1^2\lambda_2^2+6\lambda_1^3\lambda_2^2\\
&+42\lambda_1\lambda_2^3-\lambda_2^4-294\lambda_1^3\lambda_3-16\lambda_1^4\lambda_3+84\lambda_1^2\lambda_2\lambda_3-6\lambda_1\lambda_2^2\lambda_3\\
&+56\lambda_1\lambda_4-8\lambda_2\lambda_4+98\lambda_1^3\lambda_5+24\lambda_1^4\lambda_5-84\lambda_1^2\lambda_2\lambda_5+10\lambda_1\lambda_2^2\lambda_5),
\end{aligned}\tag{5.2}$$

$$b_{05} = \frac{1}{32}\lambda_1(-50421\lambda_1^5 - 6860\lambda_1^6 + 672\lambda_1^7 + 31213\lambda_1^4\lambda_2 + 2156\lambda_1^5\lambda_2 - 64\lambda_1^6\lambda_2 - 7546\lambda_1^3\lambda_2^2 - 196\lambda_1^4\lambda_2^2 + 882\lambda_1^2\lambda_2^3 + 4\lambda_1^3\lambda_2^3 - 49\lambda_1\lambda_2^4 + \lambda_2^5 - 1372\lambda_1^4\lambda_3 - 224\lambda_1^5\lambda_3 + 588\lambda_1^3\lambda_2\lambda_3 + 32\lambda_1^4\lambda_2\lambda_3 - 84\lambda_1^2\lambda_2^2\lambda_3 + 4\lambda_1\lambda_2^3\lambda_3 + 392\lambda_1^2\lambda_4 + 32\lambda_1^3\lambda_4 - 112\lambda_1\lambda_2\lambda_4 + 8\lambda_2^2\lambda_4 + 224\lambda_1^5\lambda_5 - 392\lambda_1^3\lambda_2\lambda_5 - 64\lambda_1^4\lambda_2\lambda_5 + 112\lambda_1^2\lambda_2^2\lambda_5 - 8\lambda_1\lambda_2^3\lambda_5) \tag{5.3}$$

will be investigated in this section.

Suppose that $y = y(x)$ is the only solution of $X(x, y(x)) = 0$ and $y(0) = 0$, $y(x)$ and $Y(x, y(x))$ are odd functions of x because (5.1) is Z_2-equivariant, and $\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)\Big|_{y=y(x)}$ is an even function of x. We have

$$\begin{aligned} &Y(x, y(x)) = \alpha_{25}x^{25} + o(x^{25}), \\ &\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)\Big|_{y=y(x)} = \sum_{k=1}^{6} \beta_{2k}x^{2k} + o(x^{12}), \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} &\alpha_{25} = -\frac{1}{16}\lambda_1^{10}(-343\lambda_1^2 + 4\lambda_1^3 + 70\lambda_1\lambda_2 - 3\lambda_2^2 - 4\lambda_1\lambda_3 + 8\lambda_1\lambda_5)^2, \\ &\beta_2 = 3 + \lambda_1, \quad \beta_4 \sim 3(56 + \lambda_2), \\ &\beta_6 \sim -\frac{3}{4}(-59 + 28\lambda_3 - 40\lambda_5), \\ &\beta_8 \sim -6(675 + \lambda_4 - 93\lambda_5), \\ &\beta_{10} \sim -\frac{27}{49}(477 + 4\lambda_5)(93 + \lambda_5), \\ &\beta_{12} \sim 972(477 + 4\lambda_5). \end{aligned} \tag{5.5}$$

Theorem 5.1. *If*

$$\begin{aligned} &\lambda_1 = -3 - \varepsilon_1, \quad \lambda_2 = -56 + \varepsilon_2, \\ &\lambda_3 = \frac{1}{4}(-523 + 4\varepsilon_3 + 40\varepsilon_5), \\ &\lambda_4 = -9324 - \varepsilon_4 + 651\varepsilon_5, \quad \lambda_5 = -93 + 7\varepsilon_5, \end{aligned} \tag{5.6}$$

then the origin of system (5.1) *is a nilpotent node with multiplicity 25, when*

$$0 < \varepsilon_1 \ll \varepsilon_2 \ll \varepsilon_3 \ll \varepsilon_4 \ll \varepsilon_5 \ll 1, \tag{5.7}$$

there exist 5 limit cycles in a neighborhood of system (5.1).

Proof. From (5.5), $\beta_{25} < 0$ when (5.6) and (5.6) hold, and

$$\begin{aligned} &\beta_2 = -\varepsilon_1, \quad \beta_4 \sim 3\varepsilon_2, \quad \beta_6 \sim -21\varepsilon_3, \quad \beta_8 \sim 6\varepsilon_4, \\ &\beta_{10} \sim -405\varepsilon_5 + o(\varepsilon_5), \quad \beta_{12} \sim 102060, \end{aligned} \tag{5.8}$$

and when $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 0$, we have

$$\Delta = \beta_{12}^2 + 52\alpha_{25} = 4198383900 > 0. \tag{5.9}$$

So the conclusion in Theorem 5.1 hold from Theorem 3.2. □

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