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ORTHOGONAL ARRAYS CONSTRUCTED BY GENERALIZED KRONECKER PRODUCT*

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Abstract In this paper, we propose a new general approach to construct asymmetrical orthogonal arrays, namely generalized Kronecker product. The operation is not usual Kronecker product in the theory of matrices, but it is interesting since the interaction of two columns of asymmetrical orthogonal arrays can be often written out by the generalized Kronecker product. As an application of the method, some new mixed-level orthogonal arrays of run sizes 72 and 96 are constructed.

Keywords Mixed-level orthogonal arrays, generalized Kronecker product, difference matrices, projection matrices, permutation matrices.

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1. Introduction

An $n\times m$ matrix A, having k_i columns with p_i levels, $i=1,\dots,t$, t is an integer, $m=\sum_{i=1}^{t} k_i, p_i\neq p_j$, for $i\neq j$, is called an orthogonal array (OA) of strength d and size n if each $n\times d$ submatrix of A contains all possible $1\times d$ row vectors with the same frequency. Unless stated otherwise, we consider an OA of strength 2, using the notation $L_n(p_1^{k_1}\cdots p_t^{k_t})$ for such an array. An OA is said to have mixed level (or asymmetrical) if $t\geq 2$. The proceeding definition also includes the case $t=1$, and the array is usually called a symmetrical OA, denoted by $L_n(p^m)$. For simplicity, the symmetrical and asymmetrical will only be used when needed.

An essential concept for the construction of asymmetrical OAs is that of difference matrices. Using the notation for additive (or Abelian) groups, a difference matrix(or difference scheme) with level p is an $\lambda p\times m$ matrix with the entries from a finite additive group G of order p such that the vector differences of any two columns of the array, say d_i-d_j if $i\neq j$, contains every element of G exactly λ times. We will denote such an array by $D(\lambda p,m;p)$, although this notation suppresses the relevance of the group G. In most of our examples G will correspond to the additive group associated with Galois field $GF(p)$. The difference matrix $D(\lambda p,m;p)$ is

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called a generalized Hadamard matrix if $\lambda p = m$. In particular, $D(\lambda 2, \lambda 2; 2)$ is the usual Hadamard matrix.

If a $D(\lambda p, m; p)$ exists, it can always be constructed so that only one of its rows and one of its columns has the zero element of G. Deleting this column from $D(\lambda p, m; p)$, we obtain a difference matrix, denoted by $D^0(\lambda p, m-1; p)$, called an atom of difference matrix $D(\lambda p, m; p)$ or an atomic difference matrix. Without loss of generality, the matrix $D(\lambda p, m; p)$ can be written as

$$D(\lambda p, m; p) = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} = (0 \quad D^0(\lambda p, m-1; p)).$$

The property is important for the following discussions.

For two matrices $A = (a_{ij})_{n\times m}$ and $B = (b_{ij})_{s\times t}$ both with the entries from group G, define their Kronecker sum (Shrikhande [10]) to be

$$A \oplus B = (a_{ij} \oplus B)_{1\le i\le n, 1\le j\le m},$$

where each submatrix $a_{ij} \oplus B$ of $A \oplus B$ stands for the matrix obtained by adding a_{ij} to each entry of B. Shrikhande [10] showed that $A \oplus B$ is a difference matrix if both A and B are difference matrices. And, Zhang [14] showed that A is a difference matrix if both $A \oplus B$ and B are difference matrices.

A new theory or procedure of constructing asymmetrical OAs by using the orthogonal decompositions of projection matrices has been given by Zhang etc [15]. Suen [11], Suen etc [12] and Luo etc [5] have obtained some OAs by this procedure. Similarly, Leng etc [6] have constructed some fusion frames by investigating decompositions of positive matrices as weighted sums of orthogonal projections. The idea of the orthogonal decompositions of projection matrices for constructing designs comes from the theory of multilateral matrices in Zhang [14]–a mathematical technique to solve the problems of system with complexity. In general, the procedure of constructing asymmetrical OAs in our theory has been partitioned mainly into five parties: orthogonal-array addition, subtraction, multiplication, division and replacement. The technique, namely generalized Kronecker product (Definition 2.1) which belongs to the class of orthogonal-array multiplications, has also been proposed for the construction of asymmetrical OAs by Zhang [14] in the theory of multilateral matrices. Pang etc [8] have discussed the generalized concept of orthogonal-array multiplications, Zhang [16] has discussed the special technique of Kronecker sum from generalized Hadamard product and Zhang etc [17] have proposed a particular generalized Kronecker product about generalized difference matrices. Furthermore, Luo [5] has discussed the relationship between generalized difference matrices and mixed OAs. The relationship is similar to a general "expansive replacement method" for constructing mixed-level OAs of an arbitrary strength established by Jiang etc [3], a construction and decomposition of OAs with non-prime-power numbers of symbols on the complement of a Baer subplane demonstrated by Yamada etc [13], and the existence of mixed OAs with four and five factors of strength two investigated by Chen etc [1].

In this paper the generalized Kronecker product technique will be further explained and extended to construct some new asymmetrical (or mixed-level) orthogonal arrays by using the orthogonal decompositions of projection matrices.

Section 2 contains the basic concepts and main theorems while in Section 3 we describe the method of construction. Some new mixed level OAs with run sizes 72 and 96 are constructed in Section 4.

2. Basic Concepts and Main Theorems

In our procedure, an important idea is to find the relationship among difference matrices, projection matrices and permutation matrices. A matrix A is called a projection matrix if $A^T A = A$. The following notations are used.

Let 1_r be the $r \times 1$ vector of 1's, 0_r the $r \times 1$ vector of 0's, I_r the identity matrix of order r and $J_{r,s}$ the $r \times s$ matrix of 1's, also $J_r = J_{r,r}$. Of course, the two matrices $P_r = (1/r)1_r 1_r^T = (1/r)J_r$ and $\tau_r = I_r - P_r$ are projection matrices for any positive integer r.

Define

$$(r) = (0, \ldots, r-1)^T_{r\times 1}, e_i(r) = (0 \cdots 0 \overset{i}{1} 0 \cdots 0)^T_{r\times 1},$$

where $e_i(r)$ is the base vector of R^r (r-dim vector space) for any i. We can construct two permutation matrices as follows:

$$N_r = e_1(r)e_2^T(r) + \cdots + e_{r-1}(r)e_r^T(r) + e_r(r)e_1^T(r)$$

and

$$K(p,q) = \sum_{i=1}^{p}\sum_{j=1}^{q} e_i(p)e_j^T(q) \otimes e_j(q)e_i^T(p), \tag{2.1}$$

where $\otimes$ is the usual Kronecker product in the theory of matrices. The permutation matrices N_r and $K(p,q)$ have the following properties:

$$N_r \cdot (r) = 1 \oplus (r), \text{ mod } r, \text{ and } K(p,\lambda p)((\lambda p) \oplus (p)) = (p) \oplus (\lambda p).$$

Let $D = (d_{ij})_{\lambda p \times m}$ be a matrix over an additive group G of order p. Then for any given $d_{ij} \in G$ there exists an permutation matrix $\sigma(d_{ij})$ such that

$$\sigma(d_{ij})(p) = d_{ij} \oplus (p),$$

where the vector (p) with elements from G is the same as that of (r) if $p = r$. Define $H(\lambda p, m; p) = (\sigma(d_{ij}))_{\lambda p^2 \times mp}$, where each entry or submatrix $\sigma(d_{ij})$ of $H(\lambda p, m; p)$ is a $p \times p$ permutation matrix. And Zhang [14] has proved that the matrix $D = (d_{ij})_{\lambda p \times m}$ over some group G is a difference matrix $D(\lambda p, m; p)$ if and only if

$$H^T(\lambda p, m; p)H(\lambda p, m; p) = \lambda p(I_m \otimes \tau_p + J_m \otimes P_p),$$

where τ_p and P_p are the same as those of τ_r and P_r if $p = r$.

On the other hand, the permutation matrices $\sigma(d_{ij})$ are often obtained by the permutation matrices N_r and $K(p,q)$. Furthermore, by the permutation matrices $\sigma(d_{ij})$ and $K(\lambda p, p)$, the Kronecker sum (Shrikhande [10]) of difference matrices can be written as

$$\begin{aligned}(p) \oplus D(\lambda p, m; p) &= K(p, \lambda p)[D(\lambda p, m; p) \oplus (p)] \\ &= K(p,\lambda p)(\sigma(d_{ij})(p))_{\lambda p^2 \times m} \\ &= K(p,\lambda p)(S_1(0_{\lambda p} \oplus (p)), \ldots, S_m(0_{\lambda p} \oplus (p))) \\ &= (Q_1((p) \oplus 0_{\lambda p}), \ldots, Q_m((p) \oplus 0_{\lambda p})),\end{aligned}$$

where

$$Q_j = K(p, \lambda p) S_j K(p, \lambda p)^T, S_j = \text{ diag}(\sigma(d_{1j}), \ldots, \sigma(d_{rj})), (r = \lambda p), \qquad (2.2)$$

are permutation matrices for any $j = 1, \ldots, m$ and where $0_{\lambda p} \oplus (p) = 1_{\lambda p} \otimes (p)$ holds for the additive group associated with Galois Field GF(p). Therefore, both the projection matrices P_r and τ_r and the permutation matrices $N_r, K(p, q), Q_j$ and S_j(defined in Eqs. (2.1) and (2.2)) are often used to construct the asymmetrical OAs in our procedure.

Definition 2.1. Let $k(x, y)$ be a map from $\Omega_1 \times \Omega_2$ to V, where $\Omega_1 \times \Omega_2 = \{(x, y) : x \in \Omega_1, y \in \Omega_2\}$ and Ω_1, Ω_2, V are some sets. For two matrices $A = (a_{ij})_{n\times m}$ with entries from Ω_1 and $B = (b_{uv})_{s\times t}$ with entries from Ω_2, define their generalized Kronecker product, denoted by $\overset{k}{\otimes}$, as follows

$$A \overset{k}{\otimes} B = (k(a_{ij}, b_{uv}))_{ns\times mt} = (k(a_{ij}, B))_{1\leq i\leq n, 1\leq j\leq m},$$

where each submatrix $k(a_{ij}, B) = (k(a_{ij}, b_{uv}))_{s\times t}$ of $A \overset{k}{\otimes} B$ stands for the matrix obtained by operating a_{ij} to each entry of B under the map $k(x, y)$.

Unless stated otherwise, we consider that the sets Ω_1 and Ω_2 are finite, using the vector notations (p) and (q) for such two sets. When V is a row-vector space of m-dimensions, the map $k(i, j)$ can be represented by a $pq \times m$ matrix D, i.e.,

$$k : (p) \overset{k}{\otimes} (q) = D = (d_{(1)}, \ldots, d_{(pq)})^T,$$

with $k(i, j) = d^T_{(iq+j+1)}$(or $k(i, j)$ is the $(iq + j + 1)$th row of D where $\Omega_1 = (p) = (0, 1, \cdots, p-1)^T$ and $\Omega_2 = (q) = (0, 1, \cdots, q-1)^T$ hereinafter the same). For this case in the following discussions, the generalized Kronecker product $\overset{k}{\otimes}$ will only be defined as $(p) \overset{k}{\otimes} (q) = D$.

Note 1. Using the notation for a finite multiplicative group G, i.e., let $\Omega_1 = \Omega_2 = V = G$ (a finite multiplicative group) and $k(i, j) = ij$. Then the generalized Kronecker product $\overset{k}{\otimes}$ is really the usual Kronecker product in the theory of matrices, denoted by $\otimes$.

Note 2. Using the notation for a finite additive (or Abelian) group G, i.e., let $\Omega_1 = \Omega_2 = V = G$ (a finite additive group) and $k(i, j) = i + j$, then the generalized Kronecker product $\overset{k}{\otimes}$ will be the usual Kronecker sum (Shrikhande [10], denoted by $\oplus$.

Note 3. Furthermore, if the Ω_1, Ω_2 and V are additive (or abelian) groups $G_1 = (\lambda p)$ and $G_2 = (p)$ of order λp, p and a row-vector space of m-dimensions respectively, and if $k(i, j)$ is the $(ip+j+1)$th row of $D^0(\lambda p, m-1; p) \oplus (p)$ (i.e., the usual Kronecker sum $\oplus$ of $D^0(\lambda p, m-1; p)$ and (p) (Shrikhande [10]), the generalized Kronecker product $\overset{k}{\otimes}$ is really denoted by $(\lambda p) \overset{k}{\otimes} (p) = D^0(\lambda p, m-1; p) \oplus (p)$, namely normal Kronecker sum.

Note 4. In general, if the Ω_1, Ω_2 and V are multiplicative (or additive) groups $G_1 = (p)$ and $G_2 = (q)$ of order p, q and a row-vector space of m-dimensions respectively, and if $k(i, j)$ is the $(iq + j + 1)$th row of L (an orthogonal array) for any i, j, the generalized Kronecker product $\overset{k}{\otimes}$ can be only defined as $(p) \overset{k}{\otimes} (q) = L$, namely an orthogonal-array product.

The generalized Kronecker product $\overset{k}{\otimes}$ has many properties similar to the usual Kronecker product $\otimes$ and the Kronecker sum $\oplus$ (Shrihande [10]). Such as

$$\begin{aligned} K(p,q) \cdot (p) \overset{k}{\otimes} (q) &= (q) \overset{k}{\otimes} (p), (\text{ if } k(i,j) = k(j,i) \text{which is a row vector}), \\ (0_q \oplus a) \overset{k}{\otimes} (p) &= 0_q \oplus [a \overset{k}{\otimes} (p)], \\ (a,b) \overset{k}{\otimes} (p) &= [a \overset{k}{\otimes} (p), b \overset{k}{\otimes} (p)]. \end{aligned}$$

The generalized Kronecker operations $\otimes, \oplus, \overset{k}{\otimes}$ are very useful for the construction of asymmetrical OAs and many other designs.

For example, if define

$$(2) \overset{k}{\otimes} (2) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad (4) \overset{k}{\otimes} (2) = \begin{pmatrix} 0\,0\,0 \\ 0\,1\,1 \\ 1\,0\,1 \\ 1\,1\,0 \end{pmatrix} \oplus (2),$$

and

$$(3) \overset{k}{\otimes} (3) = \begin{pmatrix} 0\,0 \\ 1\,2 \\ 2\,1 \end{pmatrix} \oplus (3), \quad (6) \overset{k}{\otimes} (3) = \begin{pmatrix} 0\,0\,0\,0\,0 \\ 0\,1\,1\,2\,2 \\ 1\,2\,0\,1\,2 \\ 1\,0\,2\,2\,1 \\ 2\,1\,2\,1\,0 \\ 2\,2\,1\,0\,1 \end{pmatrix} \oplus (3),$$

then the following arrays

$$\begin{aligned} ((2) \oplus 0_4, 0_2 \oplus (4)) \overset{k}{\otimes} (2) &= (((2) \oplus 0_2) \overset{k}{\otimes} (2), (0_2 \oplus (4)) \overset{k}{\otimes} (2)), \\ ((3) \oplus 0_6, 0_3 \oplus (6)) \overset{k}{\otimes} (3) &= (((3) \oplus 0_6) \overset{k}{\otimes} (3), (0_3 \oplus (6)) \overset{k}{\otimes} (3)), \end{aligned}$$

are all OAs (Theorem 2.5). The array product is an essential operation of the generalized Kronecker product for constructing asymmetrical arrays.

Definition 2.2. Let A be an OA of strength 1, i.e.,

$$A = (a_1, \ldots, a_m) = (T_1(0_{r_1} \oplus (p_1)), \ldots, T_m(0_{r_m} \oplus (p_m))),$$

where $r_i p_i = n, T_i$ is a permutation matrix for any $i = 1, \ldots, m$. The following projection matrix,

$$A_j = T_j(P_{r_j} \otimes \tau_{p_j})T_j^T, \tag{2.3}$$

is called the matrix image (MI) of the jth column a_j of A, denoted by $m(a_j) = A_j$ for $j = 1, \ldots, m$. In general, the MI of a subarray of A is defined as the sum of the MI's of all its columns. In particular, we denote the MI of A by $m(A)$.

In Definition 2.2, for a given column $a_j = T_j(0_{r_j} \oplus (p_j))$, the matrices A_j defined in equation (2.3) are unique though the permutation matrix T_j introduced here is not unique.

If a design is an OA, then the MI's of its columns has some interesting properties which can be used to construct OAs. For example, by the definition, we have

$$m(0_r) = P_r \text{ and } m((r)) = \tau_r.$$

Theorem 2.1. *For any permutation matrix T and any orthogonal array L with strength at least one, we have*

$$m(T(L \oplus 0_r)) = T(m(L) \otimes P_r)T^T \text{ and } m(T(0_r \oplus L)) = T(P_r \otimes m(L))T^T.$$

Theorem 2.2. *Let the array A be an OA of strength 1, i.e.,*

$$A = (a_1, \ldots, a_m) = (T_1(0_{r_1} \oplus (p_1)), \ldots, T_m(0_{r_m} \oplus (p_m))),$$

where $r_i p_i = n, T_i$ is a permutation matrix, for $i = 1, \ldots, m$.

The following statements are equivalent.

(1) A is an OA of strength 2.

(2) The MI of A is a projection matrix.

(3) The MI's of any two columns of A are orthogonal, i.e., $m(a_i)m(a_j) = 0 (i \neq j)$.

(4) The projection matrix τ_n can be decomposed as

$$\tau_n = m(a_1) + \ldots + m(a_m) + \triangle,$$

where $rk(\triangle) = n - 1 - \sum_{j=1}^{m}(p_j - 1)$ is the rank of the matrix $\triangle$.

Definition 2.3. An OA A is said to be saturated if $\sum_{j=1}^{m}(p_j - 1) = n - 1$ (or, equivalently, $m(A) = \tau_n$).

Corollary 2.1. *Let (L, H) and K be OAs of run size n. Then (K, H) is an orthogonal array if $m(K) \leq m(L)$, where $m(K) \leq m(L)$ means that the difference $m(L) - m(K)$ is nonnegative definite.*

Corollary 2.2. *Suppose L and H are OAs. Then $K = (L, H)$ is also an OA if $m(L)$ and $m(H)$ are orthogonal, i.e., $m(L)m(H) = 0$. In this case $m(K) = m(L) + m(H)$.*

By Corollaries 2.1 and 2.2, in order to construct an OA L_n of run size n, we should decompose the projection matrix τ_n into $C_1 + \cdots + C_k$ such that $C_i C_j = 0$ for $i \neq j$ and find OAs H_j such that $m(H_j) \leq C_j$ for $j = 1, 2, \cdots, k$, because the array $L_n = (H_1, \cdots, H_k)$ is an OA of run size n. The method of constructing OAs by using the orthogonal decompositions of projection matrices is also called orthogonal-array addition (Zhang etc [15]).

Definition 2.4. An OA L_n is called satisfactory if there doesn't exist any OA K such that (L_n, K) is an OA.

Theorem 2.3. *(Optimality) Let p, q and r be integers satisfying $p, q \geq 2, n = pqr$ and $(p, q) = 1$ where $(p, q) = 1$ means the maximal common divisor of p and q is 1. Then there is not any OA K of run size n such that $m(K) \leq \tau_p \otimes I_r \otimes \tau_q$.*

Note 5. Satisfactory OAs and maximal OAs are different concepts, because OA $L_n(p_1^{k_1} \cdots p_t^{k_t})$ is called maximal if $(k_1, \cdots, k_t)$ is maximal for fixed $(n, p_1, \cdots, p_t)$. A maximal OA must be a satisfactory OA, but a satisfactory OA is not always a maximal OA.

Theorem 2.4. *Let $D^0(\lambda p, m-1; p)$ be an atom of difference matrix $D(\lambda p, m; p)$. Then $D^0(\lambda p, m-1; p) \oplus (p)$ is an OA whose MI (defined in (2.3)) is less than or equal to $\tau_{\lambda p} \otimes \tau_p$.*

These theorems and corollaries can be found in Zhang [14].

The following definition is a main idea in our procedure of generalized Kronecker product for constructing the asymmetrical OAs.

Definition 2.5. Let $L_{n_1} = [L_{n_1}(p_1^{x_1}), \ldots, L_{n_1}(p_s^{x_s})]$ and $L_{n_2} = [L_{n_2}(q_1^{y_1}), \ldots, L_{n_2}(q_t^{y_t})]$ be two OAs. If for given i, j the map $k_{ij}(s,t)$ of generalized Kronecker product $\overset{k_{ij}}{\otimes}$ is $k_{ij} : (p_i) \overset{k_{ij}}{\otimes} (q_j) = H_{ij}$ (an OA) such that $m(H_{ij}) \leq \tau_{p_i} \otimes \tau_{q_j}$, then we define the orthogonal-array product of L_{n_1} and L_{n_2} as

$$L_{n_1} \overset{K}{\otimes} L_{n_2} = [\ldots, L_{n_1}(p_i^{x_i}) \overset{k_{ij}}{\otimes} L_{n_2}(q_j^{y_j}), \ldots],$$

where $K = \{k_{ij}; i = 1, 2, \ldots, s, j = 1, 2, \ldots, t\}$.

The following theorem is a main result in our procedure of generalized Kronecker product for constructing the asymmetrical OAs.

Theorem 2.5. *Suppose that*

$$L_{n_1} = [L_{n_1}(p_1^{x_1}), \ldots, L_{n_1}(p_s^{x_s})]$$

and

$$L_{n_2} = [L_{n_2}(q_1^{y_1}), \ldots, L_{n_2}(q_t^{y_t})]$$

are two orthogonal arrays. Then the array product of L_{n_1} and L_{n_2} , i.e., $L_{n_1} \overset{K}{\otimes} L_{n_2}$, is also OA whose MI is less than or equal to $m(L_{n_1}) \otimes m(L_{n_2})$.

Proof. Without loss of generality, the OAs L_{n_1} and L_{n_2} can be written as

$$L_{n_1} = [S_1(0_{r_1} \oplus (p_1)), \ldots, S_{m_1}(0_{r_{m_1}} \oplus (p_{m_1}))]$$

and

$$L_{n_2} = [Q_1((q_1) \oplus 0_{t_1}), \ldots, Q_{m_2}((q_{m_2}) \oplus 0_{t_{m_2}})]$$

where $r_i p_i = n_1, t_j q_j = n_2$, and S_j, Q_j are permutation matrices for any i, j. By Theorems 2.1 and 2.2, we have

$$\begin{aligned} m(L_{n_1}) \otimes m(L_{n_2}) &= [\sum_{i=1}^{m_1} S_i(P_{r_i} \otimes \tau_{p_i}) S_i^T] \otimes [\sum_{j=1}^{m_2} Q_j(\tau_{q_j} \otimes P_{t_j}) Q_j^T] \\ &= \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} (S_i \otimes Q_j)(P_{r_i} \otimes \tau_{p_i} \otimes \tau_{q_j} \otimes P_{t_j})(S_i \otimes Q_j)^T \end{aligned}$$

is an orthogonal decomposition of projection matrix $m(L_{n_1}) \otimes m(L_{n_2})$. If there exists an OA H_{ij} such that $m(H_{ij}) \leq \tau_{p_i} \otimes \tau_{q_j}$, i.e., $k_{ij} : (p_i) \overset{k_{ij}}{\otimes} (q_j) = H_{ij}$ for any i, j, then we have

$$(\ldots, (S_i \otimes Q_j)(0_{r_i} \oplus (p_i) \overset{k_{ij}}{\otimes} (q_j) \oplus 0_{t_j}), \ldots)$$

is an OA by Theorems 2.1 and 2.2 , Corollary 2.1 and Definition 2.5. The proof is completed. □

By Theorem 2.4, the OAs H_{ij} in Definition 2.5 can be taken into

$$D^0(p_i, u_i; q_j) \oplus (q_j) \text{ or } (p_i) \oplus D^0(q_j, v_j; p_i),$$

for any i, j.

For example, in Definition 2.1, we can define

$$(2) \overset{k}{\otimes} (2) = D^0(2, 1; 2) \oplus (2), (4) \overset{k}{\otimes} (2) = D^0(4, 3; 2) \oplus (2),$$
$$(3) \overset{k}{\otimes} (3) = D^0(3, 2; 3) \oplus (3), (6) \overset{k}{\otimes} (3) = D^0(6, 5; 3) \oplus (3), \ldots,$$

where each of the above maps $k(i, j)$'s can be defined by the corresponding formula.

By Theorem 2.5, finding all OAs H such that $m(H) \leq \tau_p \otimes \tau_q$ is also an essential operation of the generalized Kronecker product for constructing asymmetrical OAs. If there exists an OA H such that $m(H) = \tau_p \otimes \tau_q$, then the OA H is called the interaction of two columns $(p \oplus 0_q)$ and $0_p \oplus (q)$. Thus the operation of finding the generalized Kronecker products is similar to that of finding the interactions in experiment designs.

In fact, Theorems 2.4 and 2.5 in our this article are same as Theorem 3.6 and Corollaries 2.4-2.5 in Zhang [16], respectively. But Theorem 3.6 and Corollaries 2.4-2.5 in Zhang [16] are no proof and have stated in cited.

3. General Methods for Constructing OAs by Generalized Kronecker Product

Our procedure of constructing mixed-level OAs by using the generalized Kronecker product based on the orthogonal decomposition of the projection matrix τ_n consists of the following three steps:

Step 1. Orthogonally decompose the projection matrix τ_n :

$$\tau_n = T_1(A_1 \otimes B_1)T_1^T + \cdots + T_{k_1}(A_{k_1} \otimes B_{k_1})T_{k_1}^T + C_1 + \cdots + C_{k_2} + \triangle,$$

where all $A_i, B_j, C_s, \triangle$ are projection matrices and all T_t are permutation matrices.

Step 2. Find OAs H_i^1, H_j^2 and H_s from some known OAs such that

$$m(H_i^1) \leq A_i, m(H_j^2) \leq B_j \text{ and } m(H_s) \leq C_s.$$

Step 3. Lay out the new OA L by Theorem 2.5, Corollaries 2.1 and 2.2:

$$L = (T_1(H_1^1 \overset{K_1}{\otimes} H_1^2), \ldots, T_{k_1}(H_{k_1}^1 \overset{K_{k_1}}{\otimes} H_{k_1}^2), H_1, \ldots, H_{k_2}),$$

where all $\overset{K_1}{\otimes}, \ldots, \overset{K_{k_1}}{\otimes}$ are orthogonal-array products.

In applying Step 1, the following orthogonal decomposition of τ_n is very useful,

$$\begin{aligned}\tau_{pq} &= I_p \otimes \tau_q + \tau_p \otimes P_q = \tau_p \otimes P_q + P_p \otimes \tau_q + \tau_p \otimes \tau_q = \tau_p \otimes I_q + P_p \otimes \tau_q,\\ \tau_{prq} &= \tau_p \otimes I_r \otimes P_q + P_p \otimes \tau_{rq} + \tau_p \otimes I_r \otimes \tau_q = \tau_{pr} \otimes P_q + P_p \otimes I_r \otimes \tau_q + \tau_p \otimes I_r \otimes \tau_q.\end{aligned} \tag{3.1}$$

These equations are easy to verify from $\tau_p = I_p - P_p, P_{pq} = P_p \otimes P_q$ and $I_{pq} = I_p \otimes I_q$.

The following properties play very a useful role in the procedure:

Corollary 3.1 (Two-factor method)**.** *Let L_p^1, L_p^2, L_q^1 and L_q^2 be OAs. Then*

$$(L_p^1 \oplus 0_q, 0_p \oplus L_q^1, L_p^2 \overset{K}{\otimes} L_q^2)$$

is an OA.

Proof. The proof follows from Theorem 2.5 and the orthogonal decomposition of τ_{pq} (in Eq.(3.1)):

$$\tau_{pq} = \tau_p \otimes P_q + P_p \otimes \tau_q + \tau_p \otimes \tau_q.$$

□

Corollary 3.2 (Three-factor method)**.** *Let $n = prq$ and L_{pr}, L_{rq}, L_q be OAs of run sizes pr, rq, q, respectively. If there exist OAs $L_{pr}^{(-)}$, $L_{pr}^{(=)}$ and $L_{rq}^{(-)}$ such that $m(L_{pr}^{(-)}), m(L_{pr}^{(=)}) \le \tau_p \otimes I_r$ and $m(L_{rq}^{(-)}) \le I_r \otimes \tau_q$, then*

$$[L_{pr} \oplus 0_q, 0_p \oplus L_{rq}^{(-)}, L_{pr}^{(=)} \overset{K}{\otimes} L_q]$$

and

$$[L_{pr}^{(-)} \oplus 0_q, 0_p \oplus L_{rq}, L_{pr}^{(=)} \overset{K}{\otimes} L_q]$$

are OAs.

Proof. The proof follows from Theorem 2.5 and the orthogonal decompositions of τ_{prq} (in Eq. (3.1)):

$$\tau_{prq} = \tau_{pr} \otimes P_q + P_p \otimes [I_r \otimes \tau_q] + [\tau_p \otimes I_r] \otimes \tau_q$$

and

$$\tau_{prq} = [\tau_p \otimes I_r] \otimes P_q + P_p \otimes \tau_{rq} + [\tau_p \otimes I_r] \otimes \tau_q.$$

On Corollary 3.2 (Three factor method), it is useful to mention that the two OAs $L_{pr}^{(-)}$ and $L_{pr}^{(=)}$ in the second constructed array are not necessarily the same.

□

4. Constructions of OAs with Run Sizes 72 and 96

4.1. Construction of OA $L_{72}(2^{61}3^14^1)$

Since $72 = 18 \times 2 \times 2$, by Corollary 3.2 (Three-factor method), we have

$$[L_{36}^{(-)} \oplus 0_2, 0_{18} \oplus (4), L_{36}^{(=)}(2^{34}) \overset{k}{\otimes} (2)]$$

is an OA for any OAs $L_{36}^{(-)}$ and $L_{36}^{(=)}(2^{34})$ such that $m(L_{36}^{(-)}) \leq \tau_{18} \otimes I_2$ and $m(L_{36}^{(=)}(2^{34})) = \tau_{18} \otimes I_2$.

Now we want to find an OA $L_{36}^{(=)}(2^{34})$ whose MI is equal to $\tau_{18} \otimes I_2$. Many forms of OA $L_{36}(2^{35})$ can be constructed such as Plackett etc [9]. Without loss of generality, the first column can always be changed to be $0_{18} \oplus (2)$ by some row permutation. Deleting the column $0_{18} \oplus (2)$ from $L_{36}(2^{35})$, we obtain an OA in Table 1, denoted by $L_{36}^{(=)}(2^{34})$, whose MI is equal to $\tau_{18} \otimes I_2$ since $\tau_{18} \otimes I_2 = \tau_{36} - P_{18} \otimes \tau_2$.

By Theorem 2.4, there exists a generalized Kronecker product $(2) \overset{k}{\otimes} (2) = (0\ 1\ 1\ 0)^T = (2) \oplus (2)$, mod 2, i.e., the Kronecker sum (Shrikhande [10]). Therefore by Theorem 2.5 the Kronecker sum $L_{36}^{(=)}(2^{34}) \oplus (2)$ is an OA whose MI is $\tau_{18} \otimes I_2 \otimes \tau_2$.

On the other hand, a satisfactory OA $L_{36}(2^{28}3^1)$ which has a 2-level column $0_{18} \oplus (2)$ (in Table 1) can be obtained by an approach similar to that by Zhang etc [15] through complicated computing by using Theorem 2.3. Deleting the column $0_{18} \oplus (2)$ from $L_{36}(2^{28}3^1)$, we obtain an OA, denoted by $L_{36}^{(-)}(2^{27}3^1)$, whose MI is less than $\tau_{18} \otimes I_2$ since $\tau_{18} \otimes I_2 = \tau_{36} - P_{18} \otimes \tau_2$.

By Corollary 3.2 (Three-factor method), we obtain an OA $L_{72}(2^{61}3^1 4^1)$ as follows:

$$L_{72}(2^{61}3^1 4^1) = [L_{36}^{(-)}(2^{27}3^1) \oplus 0_2, 0_{18} \oplus (4), L_{32}^{(=)}(2^{34}) \oplus (2)],$$

which is satisfactory since $\triangle = \tau_{72} - m(L_{72}(2^{61}3^1)) \leq (\tau_{36} - m(L_{36}(2^{28}3^1))) \otimes P_2$. The OA is new, which is not included in Hedayat etc [2] and Kuhfeld [4] yet.

Furthermore, replacing the OA $L_{36}(2^{28}3^1)$ by any one of OAs $L_{36}(2^x \cdots)$ which has at least a 2-level column, we will able to construct an OA for this family which are included in Table 2.

4.2. Construction of OA $L_{72}(2^{28}3^{11}6^1 12^1)$

Since $72 = 12 \times 3 \times 2$, by Corollary 3.2 (Three - factor method), we have

$$[L_{36}^{(-)} \oplus 0_2, 0_{12} \oplus (6), L_{36}^{(=)}(2^{28}) \overset{k}{\otimes} (2)]$$

is an OA for any OAs $L_{36}^{(-)}$ and $L_{36}^{(=)}(2^{28})$ such that $m(L_{36}^{(-)}) \leq \tau_{12} \otimes I_3$ and $m(L_{36}^{(=)}(2^{28})) \leq \tau_{12} \otimes I_3$. Similarly to Section 4.1, we can find an OA $L_{36}^{(=)}(2^{28})$ from $L_{36}(3^1 2^{28})$ (in Table 1) such that $m(L_{36}^{(=)}(2^{28})) \leq \tau_{12} \otimes I_3$. Thus the Kronecker sum $L_{36}^{(=)}(2^{28}) \oplus (2)$ is an OA whose MI is $\tau_{12} \otimes I_3 \otimes \tau_2$.

On the other hand, there is a saturated OA $L_{36}(3^{12}12^1)$ which has a 3-level column $0_{12} \oplus (3)$. Deleting the column $0_{12} \oplus (3)$ from $L_{36}(3^{12}12^1)$, we obtain an OA, denoted by $L_{36}^{(-)}(3^{11}12^1)$, whose MI is equal to $\tau_{12} \otimes I_3$ since $\tau_{12} \otimes I_3 = \tau_{36} - P_{12} \otimes \tau_3$.

By Corollary 3.2 (Three-factor method), we obtain an OA $L_{72}(2^{28}3^{11}6^1 12^1)$ as follows:

$$L_{72}(2^{28}3^{11}6^1 12^1) = [L_{36}^{(-)}(3^{11}12^1) \oplus 0_2, 0_{12} \oplus (6), L_{36}^{(=)}(2^{28}) \oplus (2)],$$

which is satisfactory since $\triangle = \tau_{72} - m(L_{72}(3^{11}6^1 12^1)) \leq (\tau_{36} - m(L_{36}(2^{28}3^1))) \otimes \tau_2$. This OA is new, which is not included in Hedayat etc [2] and Kuhfeld(2006) yet.

Furthermore, replacing the OA $L_{36}(3^{12}12^1)$ by any one of OAs $L_{36}(3^x \cdots)$ which has at least a 3-level column, we will able to construct an OA for this family. There are at least 11 new OAs of run size 72 for this family which are included in Table 2.

Table 1. Orthogonal arrays $L_{36}(\cdots)$ used in Sections 4.1 and 4.2.

No.	$B_1 - B_8$	$B_9 - B_{17}$	$B_{18} - B_{26}$	$B_{27} - B_{35}$	CF
1	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0	0 0
2	1 0 0 1 1 1 1 1	0 0 0 0 0 0 0 0 0	1 0 1 1 0 0 1 1 1	0 1 0 0 1 1 1 1 1	1 1
3	0 1 1 1 0 0 1 1	1 0 0 0 1 0 1 0 1	1 1 1 0 1 0 0 1 0	1 0 0 0 0 1 1 1 0	2 2
4	1 1 0 0 1 0 0 1	0 1 1 1 1 0 0 0 1	1 1 1 0 0 1 1 0 0	0 0 1 0 0 1 0 1 1	0 3
5	0 0 0 1 1 0 1 0	1 1 1 1 1 1 1 1 1	1 0 1 1 0 0 0 0 0	0 1 0 0 1 1 0 0 0	1 4
6	1 1 1 0 1 0 1 0	1 0 0 0 0 1 1 1 0	1 0 0 0 0 1 1 1 0	0 1 1 1 0 1 0 1 0	2 5
7	0 0 1 0 1 1 0 1	0 0 0 1 0 1 1 0 1	0 1 1 1 0 1 0 1 0	1 1 1 0 1 0 0 1 0	0 0
8	1 1 0 1 0 1 0 0	0 0 0 1 0 1 1 0 1	1 0 1 0 1 1 0 0 1	1 1 0 1 0 1 0 0 1	1 1
9	0 1 1 1 0 0 1 1	0 0 1 0 0 1 0 1 1	1 1 0 1 0 1 0 0 1	0 1 1 0 0 0 1 0 1	2 2
10	1 1 0 0 1 0 0 1	1 0 0 1 1 1 0 1 0	0 1 1 1 1 0 0 0 1	0 1 0 1 0 0 1 1 0	0 3
11	0 1 1 1 1 1 0 0	1 1 1 0 1 0 0 1 0	0 0 1 0 0 1 0 1 1	1 1 0 0 0 0 0 1 1	1 4
12	1 1 1 0 1 0 1 0	0 1 1 0 0 0 1 0 1	0 1 1 0 0 0 1 0 1	1 1 0 1 1 0 1 0 0	2 5
13	0 0 0 0 0 0 0 0	0 1 0 0 1 1 1 1 1	0 1 0 0 1 1 1 1 1	0 1 0 0 1 1 1 1 1	0 0
14	1 0 0 1 1 1 1 1	0 1 0 0 1 1 1 1 1	0 0 0 0 0 0 0 0 0	1 0 1 1 0 0 1 1 1	1 1
15	0 1 1 1 0 0 1 1	0 1 0 1 0 0 1 1 0	0 0 1 1 1 1 1 0 0	0 0 0 1 1 0 0 1 1	2 2
16	1 1 0 0 1 0 0 1	1 1 1 0 0 1 1 0 0	1 0 0 1 1 1 0 1 0	1 0 0 0 1 0 1 0 1	0 3
17	0 0 0 1 1 0 1 0	1 0 1 1 0 0 0 0 0	0 1 0 0 1 1 0 0 0	1 1 1 1 1 1 1 1 1	1 4
18	1 1 1 0 1 0 1 0	0 0 0 1 1 0 0 1 1	0 0 0 1 1 0 0 1 1	1 0 1 0 1 1 0 0 1	2 5
19	0 0 1 0 1 1 0 1	0 0 1 0 1 0 1 1 0	1 1 0 1 1 0 1 0 0	1 1 0 1 0 1 0 0 1	0 0
20	1 1 0 1 0 1 0 0	0 0 1 0 1 0 1 1 0	0 1 1 1 0 1 0 1 0	0 0 1 1 1 1 1 0 0	1 1
21	0 1 0 0 0 1 1 1	0 1 1 1 0 1 0 1 0	1 1 0 0 0 0 0 1 1	1 0 0 1 1 1 0 1 0	2 2
22	1 0 1 1 0 0 0 1	0 1 1 1 0 1 0 1 0	0 0 1 0 1 0 1 1 0	1 1 1 0 0 1 1 0 0	0 3
23	0 1 1 1 1 1 0 0	1 1 0 1 0 1 0 0 1	0 1 0 1 0 0 1 1 0	0 0 0 1 0 1 1 0 1	1 4
24	1 0 1 0 0 1 1 0	0 1 1 1 1 0 0 0 1	1 0 0 1 1 1 0 1 0	0 1 0 1 0 0 1 1 0	2 5
25	0 0 0 0 0 0 0 0	1 0 1 1 0 0 1 1 1	1 0 1 1 0 0 1 1 1	1 0 1 1 0 0 1 1 1	0 0
26	1 0 0 1 1 1 1 1	1 0 1 1 0 0 1 1 1	0 1 0 0 1 1 1 1 1	0 0 0 0 0 0 0 0 0	1 1
27	0 1 0 0 0 1 1 1	1 1 0 1 1 0 1 0 0	0 0 0 1 0 1 1 0 1	1 1 1 0 0 1 1 0 0	2 2
28	1 0 1 1 0 0 0 1	1 1 0 1 1 0 1 0 0	1 1 0 0 0 0 0 1 1	0 1 1 1 1 0 0 0 1	0 3
29	0 0 0 1 1 0 1 0	0 1 0 0 1 1 0 0 0	1 1 1 1 1 1 1 1 1	1 0 1 1 0 0 0 0 0	1 4
30	1 0 1 0 0 1 1 0	1 0 0 1 1 1 0 1 0	1 1 1 0 0 1 1 0 0	1 0 0 0 1 0 1 0 1	2 5
31	0 0 1 0 1 1 0 1	1 1 0 0 0 0 0 1 1	1 0 1 0 1 1 0 0 1	0 0 1 1 1 1 1 0 0	0 0
32	1 1 0 1 0 1 0 0	1 1 0 0 0 0 0 1 1	1 1 0 1 1 0 1 0 0	1 1 1 0 1 0 0 1 0	1 1
33	0 1 0 0 0 1 1 1	1 0 1 0 1 1 0 0 1	0 0 1 0 1 0 1 1 0	0 1 1 1 1 0 0 0 1	2 2
34	1 0 1 1 0 0 0 1	1 0 1 0 1 1 0 0 1	0 0 0 1 0 1 1 0 1	1 0 0 1 1 1 0 1 0	0 3
35	0 1 1 1 1 1 0 0	0 0 1 1 1 1 1 0 0	1 0 0 0 1 0 1 0 1	0 0 1 0 1 0 1 1 0	1 4
36	1 0 1 0 0 1 1 0	1 1 1 0 0 1 1 0 0	0 1 1 1 1 0 0 0 1	0 0 1 0 0 1 0 1 1	2 5

$$L_{36}(2^{35}) = (B_1 - B_{35}) = [0_{18} \oplus (2), L_{36}^{(=)}(2^{34})],$$
$$L_{36}(3^1 2^{28}) = (CB_1 B_9 - B_{35}) = [0_{12} \oplus (3), L_{36}^{(=)}(2^{28})],$$
$$L_{36}(6^1 2^{18}) = (FB_{18} - B_{35}) = [0_6 \oplus (6), L_{36}^{(=)}(2^{18})].$$

Similarly, by using OAs $L_{36}(2^{18}6^1) = [0_6 \oplus (6), L_{36}^{(=)}(2^{18})]$ (in Table 1) and $L_{36}(6^x \cdots) = [0_6 \oplus (6), L_{36}^{(-)}(\cdots)]$, we can construct the following OAs

$$[L_{36}^{(-)}(\cdots) \oplus 0_2, 0_{12} \oplus (12), L_{36}^{(=)}(2^{18}) \overset{k}{\otimes} (2)].$$

There are at least 7 new OAs of run size 72 for this family which are included in Table 2.

There are lots of asymmetrical OAs with moderate run sizes (of course run size 72) which can be obtained by only using the simple procedures of both the two factor method and the three factor method. The generalized Kronecker product (or orthogonal-array product) is more useful for constructing larger arrays from lesser

Table 2. Orthogonal arrays $L_{72}(\cdots)$ Obtained in Sections 4.1 and 4.2.

No.	Krown OAs $L_{36}(\cdots)$	Obtained OAs $L_{72}(\cdots 4^1)$	Obtained OAs $L_{72}(\cdots 6^x)$	Obtained OAs $L_{72}(\cdots 12^1)$
1	$L_{36}(2^{35})$	$L_{72}(2^{68}4^1)$	–	–
2	$L_{36}(2^{28}3^1)$(new)	$L_{72}(2^{61}3^14^1)$(new)	$L_{72}(2^{56}6^1)$ (new)	–
3	$L_{36}(2^{20}3^2)$	$L_{72}(2^{53}3^24^1)$	$L_{72}(2^{48}3^16^1)$	–
4	$L_{36}(2^{18}3^16^1)$	$L_{72}(2^{51}3^16^14^1)$	$L_{72}(2^{46}6^2)$	$L_{72}(2^{36}3^112^1)$
5	$L_{36}(2^{16}3^4)$	$L_{72}(2^{49}3^44^1)$	$L_{72}(2^{44}3^36^1)$	–
6	$L_{36}(2^{16}9^1)$	$L_{72}(2^{49}9^14^1)$	–	–
7	$L_{36}(2^{13}6^2)$	$L_{72}(2^{46}6^24^1)$	–	$L_{72}(2^{31}6^112^1)$ (new)
8	$L_{36}(2^{11}3^{12})$	$L_{72}(2^{44}3^{12}4^1)$	$L_{72}(2^{39}3^{11}6^1)$ (new)	–
9	$L_{36}(2^{11}3^26^1)$	$L_{72}(2^{44}3^26^14^1)$	$L_{72}(2^{39}3^16^2)$	$L_{72}(2^{29}3^212^1)$
10	$L_{36}(2^{10}3^86^1)$	$L_{72}(2^{43}3^86^14^1)$	$L_{72}(2^{38}3^76^2)$ (new)	$L_{72}(2^{28}3^812^1)$
11	$L_{36}(2^{10}3^16^2)$	$L_{72}(2^{43}3^16^24^1)$	$L_{72}(2^{38}6^3)$	$L_{72}(2^{28}3^16^112^1)$ (new)
12	$L_{36}(2^93^46^2)$	$L_{72}(2^{42}3^46^24^1)$	$L_{72}(2^{37}3^36^3)$	$L_{72}(2^{27}3^46^112^1)$
13	$L_{36}(2^86^3)$	$L_{72}(2^{41}6^34^1)$	–	$L_{72}(2^{26}6^212^1)$ (new)
14	$L_{36}(2^43^{13})$	$L_{72}(2^{37}3^{13}4^1)$	$L_{72}(2^{32}3^{12}6^1)$	–
15	$L_{36}(2^43^16^3)$	$L_{72}(2^{37}3^16^34^1)$	$L_{72}(2^{32}6^4)$ (new)	$L_{72}(2^{22}3^16^212^1)$ (new)
16	$L_{36}(2^33^96^1)$	$L_{72}(2^{36}3^96^14^1)$	$L_{72}(2^{31}3^86^2)$	$L_{72}(2^{21}3^912^1)$
17	$L_{36}(2^33^26^3)$	$L_{72}(2^{36}3^26^34^1)$	$L_{72}(2^{31}3^16^4)$ (new)	$L_{72}(2^{21}3^26^212^1)$ (new)
18	$L_{36}(2^23^{12}6^1)$	$L_{72}(2^{35}3^{12}6^14^1)$	$L_{72}(2^{30}3^{11}6^2)$(new)	$L_{72}(2^{20}3^{12}12^1)$
19	$L_{36}(2^23^56^2)$	$L_{72}(2^{35}3^56^24^1)$	$L_{72}(2^{30}3^46^3)$(new)	$L_{72}(2^{20}3^56^112^1)$
20	$L_{36}(2^218^1)$	$L_{72}(2^{35}18^14^1)$	–	–
21	$L_{36}(2^13^86^2)$	$L_{72}(2^{34}3^86^24^1)$	$L_{72}(2^{29}3^76^3)$(new)	$L_{72}(2^{19}3^86^112^1)$
22	$L_{36}(2^13^36^3)$	$L_{72}(2^{34}3^36^34^1)$	$L_{72}(2^{29}3^26^4)$(new)	$L_{72}(2^{19}3^36^212^1)$ (new)
23	$L_{36}(3^{13}4^1)$	–	$L_{72}(2^{28}3^{12}6^14^1)$	–
24	$L_{36}(3^{12}12^1)$	–	$L_{72}(2^{28}3^{11}6^112^1)$(new)	–
25	$L_{36}(3^76^3)$	–	$L_{72}(2^{28}3^66^4)$(new)	$L_{72}(2^{18}3^76^212^1)$ (new)
26	$L_{36}(4^19^1)$	–	–	–
	...	...	...	...

Note. These OAs(new) in above table are new, which are not included in Hedayat etc [2] and Kuhfeld [4] yet.

ones.

4.3. Construction of OA $L_{96}(2^{12}4^{20}24^1)$

Consider the three-step procedure of generalized Kronecker product in Section 3. The following is a recipe for constructing the OA $L_{96}(2^{12}4^{20}24^1)$ (Zhang [16]) by using the three-step procedure of generalized Kronecker product for case $k_1 = 3$, $k_2 = 1$ and $\triangle = 0$.

The specific result of OA $L_{96}(2^{12}4^{20}24^1)$ has been given in Zhang [16], but gave no detail construction process. To illustrate the three-step procedure of the generalized Kronecker product to construct OAs in this paper, this section will give the special structure of this OA.

Step 1. Orthogonally decompose the projection matrix τ_{96}. From Eq.(3.1), we have

$$\tau_{96} = I_{24} \otimes \tau_4 + \tau_{24} \otimes P_4. \tag{4.1}$$

Based on the Abelian group $G = \{0, 1, 2, 3\}$ of order 4 with the addition table:

$$(4) \oplus (4)^T = \begin{pmatrix} 0\,1\,2\,3 \\ 1\,0\,3\,2 \\ 2\,3\,0\,1 \\ 3\,2\,1\,0 \end{pmatrix}, \tag{4.2}$$

consider the particular form of difference matrix $D(12, 12; 4)$ (Zhang [14]) as follows

$$D(12,12;4) = \begin{pmatrix} 0\,0\,0 & 1\,1\,1 & 2\,2\,2 & 3\,3\,3 \\ 0\,0\,0 & 2\,2\,2 & 3\,3\,3 & 1\,1\,1 \\ 0\,0\,0 & 3\,3\,3 & 1\,1\,1 & 2\,2\,2 \\ 1\,2\,3 & 1\,2\,3 & 1\,2\,3 & 1\,2\,3 \\ 1\,2\,3 & 2\,3\,1 & 2\,3\,1 & 2\,3\,1 \\ 1\,2\,3 & 3\,1\,2 & 3\,1\,2 & 3\,1\,2 \\ 2\,3\,1 & 1\,2\,3 & 2\,3\,1 & 3\,1\,2 \\ 2\,3\,1 & 2\,3\,1 & 3\,1\,2 & 1\,2\,3 \\ 2\,3\,1 & 3\,1\,2 & 1\,2\,3 & 2\,3\,1 \\ 3\,1\,2 & 1\,2\,3 & 3\,1\,2 & 2\,3\,1 \\ 3\,1\,2 & 2\,3\,1 & 1\,2\,3 & 3\,1\,2 \\ 3\,1\,2 & 3\,1\,2 & 2\,3\,1 & 1\,2\,3 \end{pmatrix},$$

By Definition 2.5, we obtain

$$\begin{aligned} D(12,12;4) \overset{k}{\otimes} (4) = [&((4) \oplus 0_3) \overset{k}{\otimes} (4), T_1(((4) \oplus 0_3) \overset{k}{\otimes} (4)), \\ &T_2(((4) \oplus 0_3) \overset{k}{\otimes} (4)), T_3(((4) \oplus 0_3) \overset{k}{\otimes} (4))\ ], \end{aligned}$$

where the map $k(i, j)$ of generalized Kronecker product $(4) \overset{k}{\otimes} (4)$ over above Abelian group G of order 4 satisfies

$$k : (4) \overset{k}{\otimes} (4) = \begin{pmatrix} 0\,0\,0 \\ 1\,2\,3 \\ 2\,3\,1 \\ 3\,1\,2 \end{pmatrix} \oplus (4) = \begin{pmatrix} 0\,1\,2\,3 & 1\,0\,3\,2 & 2\,3\,0\,1 & 3\,2\,1\,0 \\ 0\,1\,2\,3 & 2\,3\,0\,1 & 3\,2\,1\,0 & 1\,0\,3\,2 \\ 0\,1\,2\,3 & 3\,2\,1\,0 & 1\,0\,3\,2 & 2\,3\,0\,1 \end{pmatrix}^T,$$

and the permutation matrices T_1, T_2, T_3 are defined as

$$\begin{aligned} T_1 &= \mathrm{diag}(\sigma(1), \sigma(2), \sigma(3), K(3,3) \otimes I_4), \\ T_2 &= \mathrm{diag}(\sigma(2), \sigma(3), \sigma(1), [\mathrm{diag}(I_3, N_3, N_3^2)K(3,3)] \otimes I_4), \\ T_3 &= \mathrm{diag}(\sigma(3), \sigma(1), \sigma(2), [\mathrm{diag}(I_3, N_3^2, N_3)K(3,3)] \otimes I_4), \end{aligned}$$

in which the permutation matrices N_3 and $K(3,3)$ are defined in (1) and $\sigma(j)(4) = j \oplus (4)$ over above Abelian group G of order 4 for $j = 0,1,2,3$. For example, by the notations of the permutation matrices I_2 and N_2 in (1), we can take

$$\sigma(0) = I_4, \quad \sigma(1) = I_2 \otimes N_2, \quad \sigma(2) = N_2 \otimes I_2, \quad \sigma(3) = N_2 \otimes N_2.$$

By Theorem 2.5 and Eq. (2.2), we obtain

$$\begin{aligned} I_{12} \otimes \tau_4 &= m(D(12,12;4) \overset{k}{\otimes} (4)) \\ &= \sum_{i=0}^{3} m(T_i((4) \overset{k}{\otimes} 1_3 \otimes (4))) \\ &= \sum_{i=0}^{3} T_i m((4) \overset{k}{\otimes} 1_3 \otimes (4)) T_i^T \\ &= \sum_{i=0}^{3} T_i(\tau_4 \otimes P_3 \otimes \tau_4) T_i^T, \end{aligned}$$

where $T_0 = I_{48}$. By Theorem 2.5 and Eq.(4.3), an orthogonal decomposition of projection matrix $I_{24} \otimes \tau_4$ can be obtained as follows:

$$\begin{aligned} I_{24} \otimes \tau_4 &= I_2 \otimes [I_{12} \otimes \tau_4] \\ &= I_2 \otimes \left(\sum_{i=0}^{3} T_i(\tau_4 \otimes P_3 \otimes \tau_4) T_i^T \right) \\ &= \sum_{i=0}^{3} (I_2 \otimes T_i)(I_2 \otimes \tau_4 \otimes P_3 \otimes \tau_4)(I_2 \otimes T_i)^T, \\ &= \sum_{i=0}^{3} S_i (I_2 \otimes \tau_4 \otimes P_3 \otimes \tau_4) S_i^T, \end{aligned}$$

where $S_0 = I_{96}$, $S_i = I_2 \otimes T_i$, $i = 1,2,3$.

Denoted that $M_i = S_i K(8,12)$ for $i = 0,1,2,3$. Since $K(8,12)(P_3 \otimes \tau_4 \otimes I_2 \otimes \tau_4)K(8,12)^T = I_2 \otimes \tau_4 \otimes P_3 \otimes \tau_4$, from Eq.(4.1) and above equation we obtain an orthogonal decomposition of projection matrix τ_{96} as follows:

$$\tau_{96} = I_{24} \otimes \tau_4 + \tau_{24} \otimes P_8 = \sum_{i=0}^{3} M_i (P_3 \otimes \tau_4 \otimes I_2 \otimes \tau_4) M_i^T + \tau_{24} \otimes P_4. \tag{4.3}$$

The above decompositions are orthogonal because of the orthogonality in each step.

Step 2. First, we now want to find an OA $L_{32}(2^3 4^5)$ such that its MI is $\tau_4 \otimes I_2 \otimes \tau_4$. From Eq.(3.1) and some operations of matrices , we have the following orthogonal decomposition of projection matrix $\tau_4 \otimes I_2 \otimes \tau_4$:

$$\begin{aligned} &\tau_4 \otimes I_2 \otimes \tau_4 \\ =&(\tau_2 \otimes P_4 \otimes \tau_2 \otimes P_2 + P_2 \otimes \tau_2 \otimes P_2 \otimes P_2 \otimes \tau_2 + \tau_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 \otimes \tau_2) \\ &+ (\tau_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 \otimes P_2 + P_2 \otimes \tau_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 + \tau_2 \otimes P_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2) \\ &+ (\tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes P_2 + \tau_2 \otimes P_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 + P_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 \otimes \tau_2) \\ &+ (\tau_2 \otimes P_2 \otimes \tau_2 \otimes \tau_2 \otimes P_2 + \tau_2 \otimes \tau_2 \otimes P_2 \otimes P_2 \otimes \tau_2 + P_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2) \\ &+ (P_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes P_2 + \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 + \tau_2 \otimes P_4 \otimes \tau_2 \otimes \tau_2) \\ &+ P_2 \otimes \tau_2 \otimes P_2 \otimes \tau_2 \otimes P_2 + \tau_2 \otimes P_8 \otimes \tau_2 + \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2. \end{aligned} \tag{4.4}$$

By Theorem 2.4, we have $m((2)\oplus(2)) = \tau_2\otimes\tau_2$. If define the generalized Kronecker product $(2) \overset{k}{\otimes} (2) = (2)\oplus(2)$, then we can construct an OA $L_{32}(2^{18})$ whose MI is equal to $\tau_4\otimes I_2\otimes\tau_4$ as follows

$$\begin{aligned}
&L_{32}(2^{18})\\
=&[(4)\oplus 0_2, (4)\overset{k}{\otimes}(2)]\overset{k}{\otimes}(4)\\
=&[((2)\oplus 0_4\oplus(2)\oplus 0_2, 0_2\oplus(2)\oplus 0_2\oplus 0_2\oplus(2), (2)\oplus(2)\oplus 0_2\oplus(2)\oplus(2)),\\
&((2)\oplus(2)\oplus 0_2\oplus(2)\oplus 0_2, 0_2\oplus(2)\oplus(2)\oplus 0_2\oplus(2), (2)\oplus 0_2\oplus(2)\oplus(2)\oplus(2)),\\
&((2)\oplus(2)\oplus(2)\oplus(2)\oplus 0_2, (2)\oplus 0_2\oplus(2)\oplus 0_2\oplus(2), 0_2\oplus(2)\oplus 0_2\oplus(2)\oplus(2)),\\
&((2)\oplus 0_2\oplus(2)\oplus(2)\oplus 0_2, (2)\oplus(2)\oplus 0_2\oplus 0_2\oplus(2), 0_2\oplus(2)\oplus(2)\oplus(2)\oplus(2)),\\
&(0_2\oplus(2)\oplus(2)\oplus(2)\oplus 0_2, (2)\oplus(2)\oplus(2)\oplus 0_2\oplus(2), (2)\oplus 0_4\oplus(2)\oplus(2)),\\
&0_2\oplus(2)\oplus 0_2\oplus(2)\oplus 0_2, (2)\oplus 0_8\oplus(2), (2)\oplus(2)\oplus(2)\oplus(2)\oplus(2)],
\end{aligned} \tag{4.5}$$

where $0_2 = (0,0)^T, (2) = (0,1)^T, \oplus, \ldots$ are corresponding to $P_2, \tau_2, \otimes, \ldots$, respectively.

By the usual Hadamard product $\circ$ in matrix theory, we find the each of items in Eq.(4.4) corresponding to the each of items in Eq.(4.5) having the forms

$$(A + B + 32A\circ B) \text{ and } (a, b, a+b), \text{ respectively,}$$

where $A = m(a), B = m(b)$ and the addition $'+'$ of $a+b$ is the usual modulo 2. From the method of generalized Hadamard product $\overset{h}{\circ} = \diamond$ where $h(i,j) = 2i + j$, each of the items $(a, b, a+b)$ can be replaced by a 4-level column whose form is $a\diamond b$ where $[(2)\oplus 0_2]\diamond[0_2\oplus(2)] = (4)$. Thus we obtain an orthogonal array $L_{32}(2^3 4^5)$ whose MI is equal to $\tau_4\otimes I_2\otimes\tau_4$ and whose form is

$$\begin{aligned}
L_{32}(2^3 4^5) =& [0_2\oplus(2)\oplus 0_2\oplus(2)\oplus 0_2, (2)\oplus 0_8\oplus(2),\\
&(2)\oplus(2)\oplus(2)\oplus(2)\oplus(2), D(8,5;4)\oplus(4)],
\end{aligned}$$

in which the structure of difference matrix $D(8,5;4)$ can be obtained by using the definition of the generalized Hadamard product above $\diamond$ as follows:

$$D(8,5;4) = \begin{pmatrix} 0&0&0&0&0\\ 0&1&3&3&2\\ 1&3&3&2&1\\ 1&2&0&1&3\\ 2&2&1&3&3\\ 2&3&2&0&1\\ 3&1&2&1&2\\ 3&0&1&2&0 \end{pmatrix},$$

over the additive group $G = \{0,1,2,3\}$ with the addition table (4.2).

Step 3. By Corollaries 1, 2 and Eq.(4.3), we lay out the new OA

$$\begin{aligned}
L_{96}(2^{12}4^{20}24^1) =& [M_0(0_3\oplus L_{32}(2^3 4^5)), M_1(0_3\oplus L_{32}(2^3 4^5)),\\
&M_2(0_3\oplus L_{32}(2^3 4^5)), M_3(0_3\oplus L_{32}(2^3 4^5)), (24)\oplus 0_4].
\end{aligned}$$

By the definition of permutation matrices M_0, M_1, M_2, M_3 and the form of orthogonal array $L_{32}(2^3 4^5)$, we can change the OA $L_{96}(2^{12}4^{20}24^1)$ into the form

$$\begin{aligned} L_{96}(2^{12}4^{20}24^1) =& [D^1(12,4;2) \oplus 0_2 \oplus (2) \oplus 0_2, D^2(12,4;2) \oplus 0_4 \oplus (2), \\ & D^3(12,4;2) \oplus (2) \oplus (2) \oplus (2), D(24,20;4) \oplus (4), (24) \oplus 0_4], \end{aligned}$$

or the form

$$\begin{aligned} L_{96}(2^{12}4^{20}24^1) =& [(2) \oplus 0_2 \oplus D^1(12,4;2) \oplus 0_2, 0_2 \oplus (2) \oplus D^2(12,4;2) \oplus 0_2, \\ & (2) \oplus (2) \oplus D^3(12,4;2) \oplus (2), (4) \oplus D(24,20;4), 0_4 \oplus (24)]. \end{aligned}$$

Thus a new difference matrix $D(24, 20; 4)$ and a repeating-column difference matrix(See Zhang [16])

$$[D(24,20;4), D^1(12,4;2) \oplus 0_2, D^2(12,4;2) \oplus 0_2, D^3(12,4;2) \oplus (2)]$$

also can be obtained from the OA over the additive group $G = \{0, 1, 2, 3\}$ with the addition table (4.2). From the repeating-column difference matrix also can obtained equivalently a normal mixed difference matrix in Zhang [16].

Furthermore, replacing the column (24) by 24-run OAs:

$$L_{24}(2^{23}), L_{24}(2^{20}4^1), L_{24}(2^{13}4^1 3^1), L_{24}(2^{12}12^1), L_{24}(2^{11}4^1 6^1), L_{24}(3^1 8^1),$$

we can construct new mixed-level OAs as follows:

$$\begin{aligned} & L_{96}(2^{35}4^{20}), L_{96}(2^{32}4^{21}), L_{96}(2^{24}4^{20}12^1), L_{96}(2^{25}4^{21}3^1), \\ & L_{96}(2^{23}4^{21}6^1), L_{96}(2^{12}4^{20}3^1 8^1), \end{aligned}$$

respectively. Based on these OAs and by generalized Hadamard product $\diamond$, Zhang [16] had obtained and exhibited the following arrays:

$$L_{96}(2^{12}4^{20}24^1), L_{96}(2^{18}4^{22}12^1), L_{96}(2^{17}4^{23}6^1), L_{96}(2^{19}4^{23}3^1), L_{96}(2^{26}4^{23}).$$

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