

GREEN'S FUNCTIONS OF PARTIAL DIFFERENTIAL EQUATIONS WITH INVOLUTIONS*

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Abstract In this paper we develop a way of obtaining Green's functions of partial differential equations with linear involutions by reducing the equation to a higher-order PDE without involutions. The developed theory is applied to a model of heat transfer in a conducting plate which is bent in half.

Keywords Green's functions, PDEs, linear involution, heat equation.

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1. Introduction

The study of differential equations with involutions dates back to the work of Silberstein [10] who, in 1940, obtained the solution of the equation $f(x) = f(1/x)$. In the field of differential equations there has been quite a number of publications (see for instance the monograph on the subject of reducible differential equations of Wiener [11]) but most of them relate to ordinary differential equations (ODEs). There has also been some work in partial differential equations (PDEs), for instance [11] or [2], where they study a PDE with reflection.

In what Green's functions for equations with involutions is concerned, we find in [3] the first Green's function for ODEs with reflection and in [4] we have a framework that allows the reduction of any differential equation with reflection and constant coefficients. This setting is established in a general way, so it can be used as well for other operators (the Hilbert transform, for instance) or in other yet unexplored problems, like PDEs [8]. In this work we take this last approach and find a way of reducing general linear PDEs with linear involutions to usual PDEs.

The paper is structured as follows. In Section 2 we develop an abstract framework, with definitions and adequate notation in order to treat linear PDEs as elements of a vector space consisting of symmetric tensors. This will allow us to systematize the algebraic transformations necessary in order to obtain the desired reduction of the problem. In Section 3 we start providing a simple example that

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shows how the general process works and then prove the main result of the paper, Theorem 3.1, that permits a general reduction in the case of order two involutions. We end the Section with a problem with an order 3 involution (Example 3.2), illustrating that the same principles could be applied to higher order involutions. Finally, in Section 4, we describe a way to obtain Green’s functions for PDEs with linear involutions and apply it to a model of the process of heat transfer in a conducting plate which is bent in half with the two halves separated by some insulating material. We study the problem for different kinds of boundary conditions and a general heat source.

2. Definitions and notation

2.1. Derivatives

Let \mathbb{F} be \mathbb{R} or \mathbb{C} , $n \in \mathbb{N}$ and $\Omega \subset V := \mathbb{F}^n$ a connected open subset. For $p \geq 2$, note by $V^{\odot p}$ the space of symmetric tensors of order p , that is, the space of tensors of order p modulus the permutations of their components. We note $V^{\odot 1} = V$ and $V^{\odot 0} = \mathbb{F}$. For the convenience of the reader, we summarize now the properties and operations of the symmetric tensors:

- $V^{\odot p} := \{v_1^1 \odot \dots \odot v_1^k + \dots + v_r^1 \odot \dots \odot v_r^k : v_j^s \in \mathbb{F}^n; j = 1, \dots, r; s = 1, \dots, k; r, k \in \mathbb{N}\}$.
- $(v_1^1 \odot \dots \odot v_1^k) \odot (v_1^{k+1} \odot \dots \odot v_1^p) = v_1^1 \odot \dots \odot v_1^p; v_1^s \in \mathbb{F}^n; s = 1, \dots, p; p \in \mathbb{N}$.
- $v_1 \odot v_2 = v_2 \odot v_1; v_1, v_2 \in \mathbb{F}^n$.
- $\lambda(v_1 \odot v_2) = (\lambda v_1) \odot v_2; v_1, v_2 \in \mathbb{F}^n$.
- $(v_1 + v_2) \odot v_3 = v_1 \odot v_3 + v_2 \odot v_3; v_1, v_2, v_3 \in \mathbb{F}^n$.
- $0 \odot v_1 = 0; v_1 \in \mathbb{F}^n$.

With these properties, $V^{\odot p}$ is an \mathbb{F} -vector space of dimension $\binom{n+p-1}{p}$.

For every $v = (v_1, \dots, v_n) \in V$, we define the directional derivative operator as

$$\begin{aligned} \mathcal{C}^1(\Omega, \mathbb{F}) &\xrightarrow{D_v} \mathcal{C}(\Omega, \mathbb{F}) \\ y &\longmapsto v_1 \frac{\partial y}{\partial x^1} + \dots + v_n \frac{\partial y}{\partial x^n} \end{aligned}$$

If ∇y denotes the gradient vector of y , then $D_v(y) = v^T \nabla y$. Observe that $D_{\lambda u + v} = \lambda D_u + D_v$ for every $u, v \in \mathbb{F}^n$ and $\lambda \in \mathbb{F}$, that is, D_v is linear in v . Also, for $u, v \in \mathbb{F}^n$, if $y \in \mathcal{C}^2(\Omega, \mathbb{F})$, then $D_u(D_v y) = D_v(D_u y)$. Furthermore, $D_u \circ D_v$ is bilinear –that is, linear in both u and v , so we can write the identification $D_v \circ D_u \equiv D_{v \odot u}^2$, where $v \odot u$ denotes de symmetric tensor product of u and v . In the same way, we define the composition of higher order derivatives by $D_{\omega_2}^p \circ D_{\omega_1}^q = D_{\omega_2 \odot \omega_1}^{p+q}$ where $\omega_1 \in V^{\odot q}$ and $\omega_2 \in V^{\odot p}$, $p, q \in \mathbb{N}$.

In this way, a linear partial differential equation is given by

$$Ly := \sum_{k=0}^m D_{\omega_k}^k y = 0, \tag{2.1}$$

where $\omega_k \in V^{\odot k}$ for $k = 1, \dots, m$ and $D_{\omega_0}^0 u \equiv \omega_0 u$ where $\omega_0 \in \mathbb{F}$ (that is, $V^{\odot 0} := \mathbb{F}$). Now, the operator L can be identified with $\omega_0 + \omega_1 + \dots + \omega_n$, which is an element of the symmetric tensor algebra

$$S^*V := \bigoplus_{k=0}^{\infty} V^{\odot k} = \mathbb{F} \oplus V \oplus (V \odot V) \oplus (V \odot V \odot V) \oplus \dots$$

It is interesting to point out the the Hilbert space completion of S^*V , that is, $F_+(V) := \overline{S^*V}$, is called the *symmetric* or *bosonic Fock space*, which is widely used in quantum mechanics [5].

2.2. Involutions

Definition 2.1. Let Ω be a set and $A : \Omega \rightarrow \Omega$, $p \in \mathbb{N}$, $p \geq 2$. We say that A is an *order p involution* if

1. $A^p \equiv A \circ \overset{p}{\dots} \circ A = \text{Id}$,
2. $A^j \neq \text{Id}$, $j = 1, \dots, p - 1$.

We will consider linear involutions in \mathbb{F}^n . They are characterized by the following theorem.

Theorem 2.1 ([1]). *A necessary and sufficient condition for a linear transformation A on a finite dimensional complex vector space V to be an involution of order p is that $A = \alpha_1 P_1 + \dots + \alpha_k P_k$ where α_j is a p -th root of the unity, and P_1, \dots, P_k are projections such that $P_j P_l = 0$, $i \neq j$ and $P_1 + \dots + P_k = \text{Id}$.*

Remark 2.1. As an straightforward consequence of this result we have that there are only order two linear involutions in \mathbb{R}^n . This is because the only real p -th roots of the unity are contained in $\{\pm 1\}$.

The characterization provided in Theorem 2.1 can be rewritten in the following way.

Corollary 2.1. *A necessary and sufficient condition for a linear transformation A on V to be an involution of order p is that $A = U^{-1} \Lambda U$ where $\Lambda, U \in \mathcal{M}_n(\mathbb{F})$, U is invertible and Λ is a diagonal matrix where the elements of the diagonal are p -th roots of the unity.*

Proof. Consider the characterization of involutions given by Theorem 2.1. Take the vector subspaces $H_j := P_j V$, $j = 1, \dots, k$. Then, $V = H_1 \oplus \dots \oplus H_k$. Take U^{-1} to be the matrix of which its columns are, consecutively, a basis of H_k . Hence, $A = U^{-1} \Lambda U$ where Λ is a diagonal matrix of diagonal

$$(\alpha_1, \dots, \alpha_1, \alpha_2, \dots, \alpha_2, \dots, \alpha_k, \dots, \alpha_k),$$

where every α_j is repeated according to the dimension of H_k . □

2.3. Pullbacks and equations

Let $\mathcal{F}(\Omega, \mathbb{F})$ be the set of functions from $\Omega \subset \mathbb{F}^n$ to \mathbb{F} . We define the pullback operator by a function $\varphi \in \mathcal{F}(\Omega, \Omega)$ as

$$\begin{aligned} \mathcal{F}^1(\Omega, \mathbb{F}) &\xrightarrow{\varphi^*} \mathcal{F}(\Omega, \mathbb{F}) \\ y &\longmapsto y \circ \varphi \end{aligned}$$

Assume A is a linear order p involution on Ω (Ω has to be such that $\Omega = A(\Omega)$). From now on, we will omit the composition signs. Observe that, for $v \in V$, $x \in \Omega$ and $y \in \mathcal{C}^1(\Omega, \mathbb{F})$,

$$\begin{aligned} ((D_v A^*)y)(x) &= D_v(y(Ax)) = v^T \nabla(y(Ax)) = v^T A^T \nabla y(Ax) \\ &= (Av)^T \nabla y(Ax) = D_{Av} \nabla y(Ax) = (A^* D_{Av})y(x), \end{aligned}$$

or, written briefly, $D_v A^* = A^* D_{Av}$. All the same, for $v_1, \dots, v_j \in V$,

$$D_{v_1 \odot \dots \odot v_j}^j A^* = A^* D_{Av_1 \odot \dots \odot Av_j}^j.$$

If $\omega_k = v_1 \odot \dots \odot v_k \in V^{\odot k}$, we denote $A\omega_k \equiv Av_1 \odot \dots \odot Av_j$. This way, $D_{\omega_k}^j A^* = A^* D_{A\omega_k}^j$.

We can consider now linear partial differential equations with linear involutions of the form

$$Ly := \sum_{j=0}^{p-1} \sum_{k=0}^m (A^*)^j D_{\omega_k^j}^k y = 0,$$

where $\omega_k^j \in V^{\odot k}$ for $k = 0, \dots, m$; $j = 0, \dots, p - 1$. This time we can identify L with

$$\left(\omega_1^0 + \dots + \omega_m^0, \omega_1^1 + \dots + \omega_m^1, \dots, \omega_1^{p-1} + \dots + \omega_m^{p-1} \right) \in (S^*V)^p.$$

The interest in these equations appears when they can be reduced to usual partial differential equations.

Definition 2.2 ([4]). If $\mathbb{F}[D]$ is the ring of polynomials on the usual differential operator D and \mathcal{A} is any operator algebra containing $\mathbb{F}[D]$, then an equation $Lx = 0$, where $L \in \mathcal{A}$, is said to be a *reducible differential equation* if there exists $R \in \mathcal{A}$ such that $RL \in \mathbb{F}[D]$.

In our present case, the first projection of the algebra $(S^*V)^p$ is precisely the algebra of partial differential operators on n variables $\text{PD}_n[\mathbb{F}]$, so we want to find elements $R \in (S^*V)^p$ such that they nullify the last $p - 1$ components of L .

3. Reducing the operators

We start with an illustrative example.

Example 3.1. Let $V = \mathbb{R}^2$, $v = (v_1, v_2) \in V$ and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A is an order 2 involution. Consider the equation

$$v_1 \frac{\partial y}{\partial x_1}(x) + v_2 \frac{\partial y}{\partial x_2}(x) + y(Ax) = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2. \tag{3.1}$$

Here we work with the operator $L = D_v + A^*$. Take then $R = D_{-Av} + A^*$ and consider the identity operator Id . We have that

$$\begin{aligned} RL &= (D_{-Av} + A^*)(D_v + A^*) = D_{-Av}D_v + A^*D_v + D_{-Av}A^* + (A^*)^2 \\ &= D_{-Av \odot v} + A^*D_v + A^*D_{-AAv} + \text{Id} = D_{-Av \odot v} + A^*D_v + A^*D_{-v} + \text{Id} \\ &= D_{-Av \odot v} + A^*D_v - A^*D_v + \text{Id} = D_{-Av \odot v} + \text{Id}. \end{aligned}$$

Hence, every two-times differentiable solution of equation (3.1) has to be a solution of the partial differential equation

$$-v_1^2 \frac{\partial^2 y}{\partial x_1^2}(x) + v_2^2 \frac{\partial^2 y}{\partial x_2^2}(x) + y = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

Remark 3.1. With the notation we have introduced, it is extremely important the use of parentheses. Observe that every $\omega \in (\mathbb{F}^n)^{\odot k}$ can be expressed as $\omega = v_1^1 \odot \dots \odot v_1^k + \dots + v_r^1 \odot \dots \odot v_r^k$ for some $v_j^s \in \mathbb{F}^n$, $j = 1, \dots, r$, $s = 1, \dots, k$; $r, k \in \mathbb{N}$. Hence, for $c \in \mathbb{F}$,

$$\begin{aligned} (cA)\omega &= cAv_1^1 \odot \dots \odot cAv_1^k + \dots + cAv_r^1 \odot \dots \odot cAv_r^k \\ &= c^k (Av_1^1 \odot \dots \odot Av_1^k) + \dots + c^k (Av_r^1 \odot \dots \odot Av_r^k) \\ &= c^k (A\omega) \equiv c^k A\omega. \end{aligned}$$

Theorem 3.1. Let A be an order 2 linear involution on \mathbb{F}^n . Let $L \in (S^*V)^p$ be defined as in (2.1). Then there exists $R \in (S^*V)^p$ defined as

$$Ry := \sum_{j=0}^{p-1} \sum_{k=0}^m (A^*)^j D_{\xi_k^j}^k y = 0,$$

where $\xi_k^0 = -A\omega_k^0$, $\xi_k^1 = \omega_k^1$, for $k = 0, 1, \dots$, such that $RL \in \text{PD}_n[\mathbb{F}]$. Furthermore, L and R commute.

Proof. For convenience, define ξ_k^j and ω_k^j outside the index range $j = 0, \dots, p-1$, $k = 0, \dots, m$ to be zero. In general,

$$\begin{aligned} RL &= \sum_{l=0}^{p-1} \sum_{r=0}^m (A^*)^l D_{\xi_r^l}^r \left(\sum_{j=0}^{p-1} \sum_{k=0}^m (A^*)^j D_{\omega_k^j}^k \right) = \sum_{l,j=0}^{p-1} \sum_{r,k=0}^m (A^*)^l D_{\xi_r^l}^r (A^*)^j D_{\omega_k^j}^k \\ &= \sum_{l,j=0}^{p-1} \sum_{r,k=0}^m (A^*)^{l+j} D_{A^j \xi_r^l}^r D_{\omega_k^j}^k = \sum_{l,j=0}^{p-1} \sum_{r,k=0}^m (A^*)^{l+j} D_{A^j \xi_r^l \odot \omega_k^j}^{r+k} \\ &= \sum_{l,j=0}^{p-1} (A^*)^{l+j} \left(\sum_{s=0}^{2m} \sum_{k=0}^s D_{A^j \xi_{s-k}^l \odot \omega_k^j}^s \right) = \sum_{l,j=0}^{p-1} (A^*)^{l+j} \left(\sum_{s=0}^{2m} D_{\sum_{k=0}^s A^j \xi_{s-k}^l \odot \omega_k^j}^s \right). \end{aligned}$$

In the particular case $p = 2$, we have that

$$\begin{aligned} RL &= \sum_{s=0}^{2m} D_{\sum_{k=0}^s \xi_{s-k}^0 \odot \omega_k^0}^s + \sum_{s=0}^{2m} D_{\sum_{k=0}^s A \xi_{s-k}^1 \odot \omega_k^1}^s \\ &\quad + A^* \left(\sum_{s=0}^{2m} D_{\sum_{k=0}^s \xi_{s-k}^1 \odot \omega_k^0}^s + \sum_{s=0}^{2m} D_{\sum_{k=0}^s A \xi_{s-k}^0 \odot \omega_k^1}^s \right) \end{aligned}$$

$$= \sum_{s=0}^{2m} D_{\sum_{k=0}^s}^s (\xi_{s-k}^0 \odot \omega_k^0 + A \xi_{s-k}^1 \odot \omega_k^1) + A^* \left(\sum_{s=0}^{2m} D_{\sum_{k=0}^s}^s (\xi_{s-k}^1 \odot \omega_k^0 + A \xi_{s-k}^0 \odot \omega_k^1) \right).$$

So it is enough to check that, for $s = 0, \dots, 2m$,

$$\sum_{k=0}^s (\xi_{s-k}^1 \odot \omega_k^0 + A \xi_{s-k}^0 \odot \omega_k^1) = 0.$$

Substituting the ξ_k^j by their given values,

$$\begin{aligned} \sum_{k=0}^s (\xi_{s-k}^1 \odot \omega_k^0 + A \xi_{s-k}^0 \odot \omega_k^1) &= \sum_{k=0}^s (\omega_{s-k}^1 \odot \omega_k^0 - A^2 \omega_{s-k}^0 \odot \omega_k^1) \\ &= \sum_{k=0}^s (\omega_{s-k}^1 \odot \omega_k^0 - \omega_{s-k}^0 \odot \omega_k^1) = \sum_{k=0}^s \omega_{s-k}^1 \odot \omega_k^0 - \sum_{k=0}^s \omega_{s-k}^0 \odot \omega_k^1 \\ &= \sum_{k=0}^s \omega_{s-k}^1 \odot \omega_k^0 - \sum_{k=0}^s \omega_k^0 \odot \omega_{s-k}^1 = 0. \end{aligned}$$

Let us see that L and R commute.

$$LR = \sum_{s=0}^{2m} D_{\sum_{k=0}^s}^s (\omega_{s-k}^0 \odot \xi_k^0 + A \omega_{s-k}^1 \odot \xi_k^1) + A^* \left(\sum_{s=0}^{2m} D_{\sum_{k=0}^s}^s (\omega_{s-k}^1 \odot \xi_k^0 + A \omega_{s-k}^0 \odot \xi_k^1) \right).$$

Now,

$$\begin{aligned} \sum_{k=0}^s (\omega_{s-k}^0 \odot \xi_k^0 + A \omega_{s-k}^1 \odot \xi_k^1) &= \sum_{k=0}^s \omega_k^0 \odot \xi_{s-k}^0 + \sum_{k=0}^s A \omega_{s-k}^1 \odot \omega_k^1 \\ &= \sum_{k=0}^s \xi_{s-k}^0 \odot \omega_k^0 + \sum_{k=0}^s A \xi_{s-k}^1 \odot \omega_k^1. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{k=0}^s (\omega_{s-k}^1 \odot \xi_k^0 + A \omega_{s-k}^0 \odot \xi_k^1) &= \sum_{k=0}^s (\omega_{s-k}^1 \odot (-A \omega_k^0) + A \omega_{s-k}^0 \odot \omega_k^1) \\ &= \sum_{k=0}^s (-\omega_{s-k}^1 \odot A \omega_k^0 + A \omega_{s-k}^0 \odot \omega_k^1) \\ &= \sum_{k=0}^s (-\omega_k^1 \odot A \omega_{s-k}^0 + A \omega_{s-k}^0 \odot \omega_k^1) = 0. \end{aligned}$$

Hence, the result is proven. □

Similar reductions can be found for higher order involutions, although the coefficients may have a much more complex expression.

Example 3.2. Let A be and order 3 linear involution in \mathbb{C}^n , $v \in \mathbb{C}^n \setminus \{0\}$ and consider the operator $L = D_v + A^*$. Define now

$$R := D_{v \odot A^2 v} - A^* D_{A^2 v} + (A^*)^2.$$

Observe that second derivatives occur in R but not in L . We have that

$$\begin{aligned} RL &= D_{v \circ A^2 v} D_v - A^* D_{A^2 v} D_v + (A^*)^2 D_v + D_{v \circ A^2 v} A^* - A^* D_{A^2 v} A^* + (A^*)^2 A^* \\ &= D_{v \circ v \circ A^2 v} - A^* D_{v \circ A^2 v} + (A^*)^2 D_v + A^* D_{v \circ A^2 v} - (A^*)^2 D_v + \text{Id} \\ &= D_{v \circ v \circ A^2 v} + \text{Id}. \end{aligned}$$

Unfortunately, we do not have commutativity in general:

$$\begin{aligned} LR &= D_v D_{v \circ A^2 v} - D_v A^* D_{A^2 v} + D_v (A^*)^2 + A^* D_{v \circ A^2 v} - (A^*)^2 D_{A^2 v} + \text{Id} \\ &= D_{v \circ v \circ A^2 v} - A^* D_{A^* v \circ A^2 v} + (A^*)^2 D_{A^2 v} + A^* D_{v \circ A^2 v} - (A^*)^2 D_{A^2 v} + \text{Id} \\ &= D_{v \circ v \circ A^2 v} + A^* D_{(v - A^* v) \circ A^2 v} + \text{Id}. \end{aligned}$$

In the particular case v is a fixed point of A , $RL = LR$.

The obtaining of a general expression for associated operators in the case of order 3 involutions and the conditions under which such operators commute is an interesting open problem.

4. Green's functions

Consider now the following problem

$$Lu = h; B_\lambda u = 0, \lambda \in \Lambda, \tag{4.1}$$

where $L \in (S^*V)^p$, $h \in L^1(\mathbb{F}^n, \mathbb{F})$, the $B_\lambda : \mathcal{C}(\mathbb{F}^n, \mathbb{F}) \rightarrow \mathbb{F}$ are linear functionals, $\lambda \in \Lambda$ and Λ is an arbitrary set.

Let $R \in (S^*V)^p$, $f \in L^1(\mathbb{F}^n, \mathbb{F})$ and consider the problem

$$RLv = f; B_\lambda v = 0, B_\lambda Rv = 0, \lambda \in \Lambda. \tag{4.2}$$

Given a function $G : \mathbb{F}^n \times \mathbb{F}^n \rightarrow \mathbb{F}$, we define the operator H_G such that $H_G(h)|_x := \int_{\mathbb{F}^n} G(x, s)h(s) ds$ for every $h \in L^1(\mathbb{F}^n, \mathbb{F})$, assuming such an integral is well defined. Also, given an operator R for functions of one variable, define the operator R_{\vdash} as $R_{\vdash}G(t, s) := R(G(\cdot, s))|_t$ for every s , that is, the operator acts on G as a function of its first variable.

We have now the following theorem relating problems (4.1) and (4.2). The proof for the case of ordinary differential equations can be found in [4]. The case of PDEs is analogous.

Theorem 4.1. *Let $L, R \in (S^*V)^p$, $h \in L^1(\mathbb{F}^n, \mathbb{F})$. Assume L commutes with R and that there exists G such that H_G is well defined satisfying*

- (I) $(RL)_{\vdash}G = 0$,
- (II) $B_{\lambda \vdash}G = 0, \lambda \in \Lambda$,
- (III) $(B_\lambda R)_{\vdash}G = 0, \lambda \in \Lambda$,
- (IV) $RLH_G h = H_{(RL)_{\vdash}G} h + h$,
- (V) $LH_{R_{\vdash}G} h = H_{L_{\vdash}R_{\vdash}G} h + h$,
- (VI) $B_\lambda H_G = H_{B_{\lambda \vdash}G}, \lambda \in \Lambda$,
- (VII) $B_\lambda RH_G = B_\lambda H_{R_{\vdash}G} = H_{(B_\lambda R)_{\vdash}G}, \lambda \in \Lambda$.

Then, $v := H_G f$ is a solution of problem (4.2) and $u := H_{R_{\vdash}G} h$ is a solution of problem (4.1).

4.1. A model of stationary heat transfer in a bent plate

We now consider a circular plate which is bent in half, with each of the two distinct halves separated by a very small distance which may be filled with some kind of (imperfect) heat insulating material (see Figure 1).

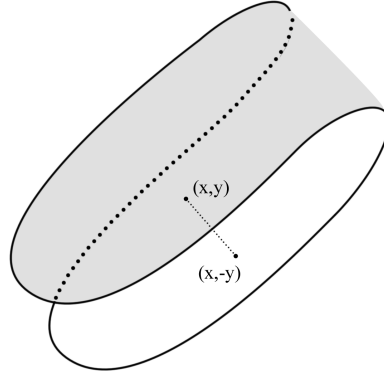


Figure 1. A section of the plate bent in half.

The heat equation which determines the temperature u on the plate for this situation is given by

$$\frac{\partial u}{\partial t}(t, x, y) = \alpha \left[\frac{\partial^2 u}{\partial x^2}(t, x, y) + \frac{\partial^2 u}{\partial y^2}(t, x, y) \right] + \beta[u(t, x, -y) - u(t, x, y)],$$

where

$$\frac{\partial u}{\partial t}(t, x, y) = \alpha \left[\frac{\partial^2 u}{\partial x^2}(t, x, y) + \frac{\partial^2 u}{\partial y^2}(t, x, y) \right],$$

is the usual heat equation with heat transfer coefficient $\alpha > 0$ and the term that goes with $\beta > 0$ relates to the heat transfer from the corollaryresponding point in the other half of the plate due to Newton's law of cooling.

If we consider the associated stationary problem

$$\alpha \left[\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) \right] + \beta[u(x, -y) - u(x, y)] = 0,$$

it can be rewritten in a convenient way as

$$Lu := \alpha \Delta u + \beta(A^* - \text{Id})u = 0,$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{and} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If we think of a circular plate in which the boundary is constantly cooled and the surface has a constant heat source given by a function h , we are imposing Dirichlet boundary conditions in the ball B of radius $\rho \in \mathbb{R}^+$ and considering the problem

$$Lu = h, \quad u|_{\partial B} = 0. \tag{4.3}$$

Observe that, Δ , expressed in tensor notation, is $\Delta = D_{\omega_2^0}$ where

$$\omega_2^0 = \frac{1}{2} [(1, 1) \odot (1, 1) + (1, -1) \odot (1, -1)].$$

Besides, $A\omega_2^0 = \omega_2^0$ and, thus, $\Delta A^* = A^* \Delta$. Hence, using Theorem 3.1, we have to take $R = -\alpha \Delta + \beta A^* + \beta \text{Id}$ and thus

$$\begin{aligned} RL &= -\alpha^2 \Delta^2 - \alpha \beta A^* \Delta + \alpha \beta \Delta + \alpha \beta A^* \Delta + \beta^2 \text{Id} - \beta^2 A^* + \alpha \beta \Delta + \beta^2 A^* - \beta^2 \text{Id} \\ &= -\alpha^2 \Delta^2 + 2\alpha \beta \Delta = (-\alpha^2 \Delta + 2\alpha \beta \text{Id}) \Delta. \end{aligned}$$

Now, the boundary conditions transformed by R are

$$0 = Ru = -\alpha \Delta u + \beta A^* u + \beta u = -\alpha \Delta u,$$

that is, the reduced problem becomes

$$RLu = Rh =: f, \quad u|_{\partial B} = 0, \quad \Delta u|_{\partial B} = 0, \quad (4.4)$$

which is equivalent to the sequence of problems

$$\Delta u = v, \quad u|_{\partial B} = 0, \quad (4.5)$$

$$(-\alpha^2 \Delta + 2\alpha \beta \text{Id})v = f, \quad v|_{\partial B} = 0. \quad (4.6)$$

Problem (4.5) is the well-known Poisson equation with Dirichlet conditions on the circle of radius ρ . The Green's function can be written in polar coordinates as

$$G_1(r, \varphi, \tilde{r}, \tilde{\varphi}) = \frac{-1}{4\pi} \ln \left[\frac{r^2 \tilde{r}^2 - 2\rho^2 r \tilde{r} \cos(\varphi - \tilde{\varphi}) + \rho^4}{\rho^2 r^2 - 2\rho^2 r \tilde{r} \cos(\varphi - \tilde{\varphi}) + \rho^2 \tilde{r}^2} \right].$$

See [9, Section 7.2.3]. On the other hand, problem (4.6) is a Helmholtz equation, and the Green's function can be described in terms of the eigenfunctions of the associated homogeneous problem (see [9, Section 7.3.3]). More concretely, the associated Green's function in polar coordinates is written as

$$\begin{aligned} &G_2(r, \varphi, \tilde{r}, \tilde{\varphi}) \\ &= \frac{1}{\alpha^2} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\left(\frac{\mu_{nm}^2}{\rho^2} + \frac{2\beta}{\alpha}\right) \|w_{nm}^{(1)}\|^2} \left[w_{nm}^{(1)}(r, \varphi) w_{nm}^{(1)}(\tilde{r}, \tilde{\varphi}) + w_{nm}^{(2)}(r, \varphi) w_{nm}^{(2)}(\tilde{r}, \tilde{\varphi}) \right], \end{aligned}$$

where μ_{nm} are the positive zeroes of the Bessel functions J_n , the eigenfunctions are given by

$$w_{nm}^{(1)} = J_n \left(\frac{\mu_{nm}}{\rho} r \right) \cos n\varphi, \quad w_{nm}^{(2)} = J_n \left(\frac{\mu_{nm}}{\rho} r \right) \sin n\varphi,$$

and

$$\|w_{nm}^{(1)}\|^2 = \frac{1}{2} \pi \rho^2 (1 + \delta_{n,0}) [J'_n(\mu_{nm})]^2,$$

where $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$.

Now, the Green's function associated to problem (4.4) is given by

$$G_3(r, \varphi, \tilde{r}, \tilde{\varphi}) = \int_0^\rho \int_0^{2\pi} G_2(r, \varphi, \hat{r}, \hat{\varphi}) G_1(\hat{r}, \hat{\varphi}, \tilde{r}, \tilde{\varphi}) d\hat{\varphi} d\hat{r}.$$

In conclusion, the Green’s function related to problem (4.3) is

$$G_4(\eta, \xi) = R_{\downarrow} G_3(\eta, \xi) = \int_0^\rho \int_0^{2\pi} R_{\downarrow} G_2(r, \varphi, \hat{r}, \hat{\varphi}) G_1(\hat{r}, \hat{\varphi}, \tilde{r}, \tilde{\varphi}) \, d\hat{\varphi} \, d\hat{r},$$

where R_{\downarrow} has to be expressed in polar coordinates in order to act in the first two variables of G_3 :

$$R = -\alpha \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right] + \beta A^* + \beta \text{Id}.$$

Also, it is known that $J'_n(z) = (n/z)J_n(z) - J_{n+1}(z)$, so

$$\begin{aligned} & R_{\downarrow} G_2(r, \varphi, \hat{r}, \hat{\varphi}) \\ &= \frac{1}{\alpha^2} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\left(\frac{\mu_{nm}^2}{\rho^2} + \frac{2\beta}{\alpha}\right) \|w_{nm}^{(1)}\|^2} \left[\tilde{w}_{nm}^{(1)}(r, \varphi) w_{nm}^{(1)}(\tilde{r}, \tilde{\varphi}) + \tilde{w}_{nm}^{(2)}(r, \varphi) w_{nm}^{(2)}(\tilde{r}, \tilde{\varphi}) \right], \end{aligned}$$

where

$$\begin{aligned} \tilde{w}_{nm}^{(1)} &= \left(\left(\frac{\mu_{nm}}{\rho} \right)^2 \left[\left(\frac{n\rho}{\mu_{nm}r} \right)^2 J_n - \left(1 + \frac{(n+1)\rho}{\mu_{nm}r} \right) J_{n+1} + J_{n+2} \right] \right. \\ &\quad \left. + \frac{n}{r} \left[\frac{\rho}{\mu_{nm}r} J_n - J_{n+1} \right] - n^2 \left(\frac{\mu_{nm}r}{\rho} \right)^{-2} J_n \right) \Big|_{\left(\frac{\mu_{nm}}{\rho}r\right)} \cos n\varphi, \\ \tilde{w}_{nm}^{(2)} &= \left(\left(\frac{\mu_{nm}}{\rho} \right)^2 \left[\left(\frac{n\rho}{\mu_{nm}r} \right)^2 J_n - \left(1 + \frac{(n+1)\rho}{\mu_{nm}r} \right) J_{n+1} + J_{n+2} \right] \right. \\ &\quad \left. + \frac{n}{r} \left[\frac{\rho}{\mu_{nm}r} J_n - J_{n+1} \right] - n^2 \left(\frac{\mu_{nm}r}{\rho} \right)^{-2} J_n \right) \Big|_{\left(\frac{\mu_{nm}}{\rho}r\right)} \sin n\varphi. \end{aligned}$$

Example 4.1. Inspired by the previous problem, we now change the term due to Newton’s law of cooling by a diffusion term in the following way.

$$\frac{\partial K}{\partial t}(t, x, y) = \alpha \left[\frac{\partial^2 K}{\partial x^2}(t, x, y) + \frac{\partial^2 K}{\partial y^2}(t, x, y) \right] + \beta \left[\frac{\partial^2 K}{\partial x^2}(t, x, -y) + \frac{\partial^2 K}{\partial y^2}(t, x, -y) \right],$$

where $\alpha, \beta > 0, \beta \neq \alpha$.

If we consider the associated stationary problem

$$\alpha \left[\frac{\partial^2 K}{\partial x^2}(x, y) + \frac{\partial^2 K}{\partial y^2}(x, y) \right] + \beta \left[\frac{\partial^2 K}{\partial x^2}(x, -y) + \frac{\partial^2 K}{\partial y^2}(x, -y) \right] = 0,$$

it can be rewritten as

$$LK := \alpha \Delta K + \beta A^* \Delta K = 0.$$

Using Theorem 3.1, we take $R = -\alpha \Delta + \beta A^* \Delta$ and then

$$RL = -\alpha^2 \Delta^2 - \alpha \beta \Delta A^* \Delta + \beta \alpha A^* \Delta^2 + \beta^2 (A^* \Delta)^2 = \beta^2 \Delta^2 - \alpha^2 \Delta^2 = (\beta^2 - \alpha^2) \Delta^2.$$

Now, if we consider the fundamental solution of the bi-Laplacian Δ^2 [6, equation (2.61)] we obtain a Green’s function given by

$$G_1(\eta, \xi) = \frac{1}{8\pi} \|\eta - \xi\|^2 \ln \|\eta - \xi\|, \quad \eta, \xi \in \mathbb{R}^2.$$

Hence, in that case, the Green's function associated to L is given by

$$G_2(\eta, \xi) = R_{\leftarrow} G_1(\eta, \xi) = (\beta - \alpha) \frac{\ln \|\eta - \xi\| + 1}{2\pi}, \quad \eta, \xi \in \mathbb{R}^2.$$

If we consider the problem

$$LK = h, \quad u|_{\partial B} = 0,$$

the reduced problem becomes

$$(\beta^2 - \alpha^2) \Delta^2 K = h, \quad u|_{\partial B} = 0, \quad Ru|_{\partial B} = 0. \quad (4.7)$$

Now, the condition $Ru = -\alpha \Delta u + \beta A^* \Delta u = 0$ is satisfied if we can guarantee that $\Delta u = 0$, so we can consider the problem

$$(\beta^2 - \alpha^2) \Delta^2 K = h, \quad u|_{\partial B} = 0, \quad \Delta u|_{\partial B} = 0. \quad (4.8)$$

For problem (4.8) we have that the Green's function is given by

$$G_3(\eta, \xi) = \frac{1}{8\pi} \|\eta - \xi\|^2 (\ln \rho - 1 + \ln \|\eta - \xi\|) + \frac{\rho^2}{8\pi}, \quad \eta, \xi \in \mathbb{R}^2.$$

Hence, the Green's function related to problem (4.7) is

$$G_4(\eta, \xi) = \frac{\ln \rho + \ln \|\eta - \xi\|}{2\pi}.$$

In general, the functions

$$G_5(\eta, \xi) = \frac{1}{8\pi} \|\eta - \xi\|^2 (\mu + \ln \|\eta - \xi\|) + \frac{\nu}{8\pi}, \quad \eta, \xi \in \mathbb{R}^2,$$

with $\mu, \nu \in \mathbb{R}$, are Green's functions related to the operator Δ^2 with different boundary conditions. The associated function for the operator L is given by

$$G_6(\eta, \xi) = \frac{1 + \mu + \log \|\eta - \xi\|}{2\pi}.$$

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