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GLOBAL ANALYSIS IN DELAYED RATIO-DEPENDENT GAUSE-TYPE PREDATOR-PREY MODELS*

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Abstract A class of three-dimensional delayed Gause-type predator-prey model with ratio-dependent is considered. Firstly, we present some results, including the boundedness of solutions and the permanence of system. Secondly, we construct a Lyapunov function to give the global asymptotically stable of the positive equilibrium under some parameter conditions. Finally, we analyed the influence of the time delay on the system and showed that the occurrence of small range of periodic motion.

Keywords Gause-type model, ratio-dependent, permanence, global asymptotically stable, delay.

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1. Introduction

The interaction between the predator and the prey is the fundamental structure in population dynamics. Various predator-prey models have been studied and will continue to be studied in both ecology and mathematical ecology. Many biologists believe that if the unique positive equilibrium point of a predator-prey system is locally asymptotically stable, then it is globally asymptotically stable. The Lyapunov functional method and LaSalle's Invariance Principle have been used to prove the global stability of the positive equilibrium point [6, 13, 22, 26, 27, 30].

In predator-prey models, the response functions, which describe the number of preys consumed by per predator per unit of time, have different forms. The Michaelis-Menten or Holling type II response function of the form $p(x) = \frac{cx}{m+x}$ is the most general and useful one, and it is also called "prey-dependent". Freedman [8] presented the predator-prey model with prey-dependent functional response and logistic prey growth:

$$\begin{cases} x' = ax\Big(1 - \dfrac{x}{K}\Big) - \dfrac{cxy}{m+x}, \\ y' = y\Big(-d + \dfrac{fx}{m+x}\Big). \end{cases} \tag{1.1}$$

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Where x and y are the population densities of the prey, and the predator at time t, respectively. a, K, c, m, f, d are positive constants that stand for prey intrinsic growth rate, carrying capacity, capturing rate, half capturing saturation constant, conversion rate, predator death rate, respectively. This model has been well studied in many papers [2, 7, 10, 14]. However, in this model, the functional response depends only on the prey density, which has been questioned by several biologists through numerous laboratory experiments and observations [3, 4]. It has been recognized that predators might interfere, share or compete for each other's foraging and hence the functional response should depend on densities of both predators and preys [1, 5]. Therefore, Arditi and Ginzburg [3] proposed a model with ratio-dependent type functional response which is a function of the ratio of prey to predator abundance. This ratio-dependent predator-prey system displayed richer and more plausible dynamics than that of system (1.1). It allows the predator population or both populations to either become extinct or coexist, depending on the initial population values [9, 20, 21, 23].

On the other hand, one characteristic of predator-prey dynamics is the fluctuation of the population densities. A simple and natural approach of understanding this phenomenon is to incorporate a single discrete delay into the predator equation [16–19, 25, 29]. The delayed predator-prey model with the Michaelis-Menten type ratio-dependent functional response $p(\frac{y}{x})$ is in the form of

$$\begin{cases} x' = ax\Big(1 - \dfrac{x}{K}\Big) - \dfrac{cxy}{my + x}, \\ y' = y\Big(- d + \dfrac{fx(t-\tau)}{my(t-\tau) + x(t-\tau)}\Big). \end{cases} \tag{1.2}$$

The dynamics of the ratio-dependent predator-prey system (1.2) has been systematically studied by many authors. Beretta and Kuang [6] provided sufficient conditions for the positive equilibrium to be globally asymptotically stable by construct a Lyapunov function. Xiao and Li [28] studied the Hopf bifurcation and the stability of the periodic solutions by choosing the delay as the bifurcation parameter.

The dynamics properties of the population model becomes more complex along with the population species increasing [11, 12, 15, 24]. A delay ratio-dependent food chain model with Michaelis-Menten type functional response takes the form:

$$\begin{cases} x'(t) = ax\Big(1 - \dfrac{x}{K}\Big) - \dfrac{cxy}{my + x}, \\ y'(t) = y\Big[- d + \dfrac{fx(t-\tau)}{my(t-\tau) + x(t-\tau)} - \dfrac{hz}{nz + y}\Big], \\ z'(t) = z\Big[- s + \dfrac{py(t-\tau)}{nz(t-\tau) + y(t-\tau)}\Big]. \end{cases} \tag{1.3}$$

Where a, K, c, m, d, f are positive constants whose biological meaning are obvious. h, n, s, p are positive constants that stand for capturing rate, half capturing saturation constant, conversion rate, top-predator death rate, respectively. Hsu et al. [15] analyzed the corresponding ODE model, and gave the richness of the model in boundary dynamics while generating the extinction dynamics.

In this paper, we systematically study the boundary dynamics of system (1.3) and establish the global stability results by constructing suitable Lyapunov functions for the above delayed ratio-dependent Gause-type predator-prey systems.

The rest of our paper is organized as follows: In Section 2, we present results on the boundedness of solutions and permanence for system (1.3). In Section 3, we construct a Lyapunov function to provide sufficient conditions for the positive equilibrium (when they exist) of system (1.3) to be globally asymptotically stable. The paper ends with numerical simulations and a discussion of how the time delay affect the behavior of the system.

2. Permanence of the system

In this section, we shall present some preliminary results, including the boundedness of solutions and permanence. For the sake of convenience, letting $b = \dfrac{a}{K}$, then Eq.(1.3) takes the form:

$$\begin{cases} x'(t) = x(a - bx) - \dfrac{cxy}{my + x}, \\ y'(t) = y\Big[- d + \dfrac{fx(t-\tau)}{my(t-\tau) + x(t-\tau)} - \dfrac{hz}{nz + y}\Big], \\ z'(t) = z\Big[- s + \dfrac{py(t-\tau)}{nz(t-\tau) + y(t-\tau)}\Big]. \end{cases} \tag{2.1}$$

The initial conditions for the above delayed system take the form of

$$x_0(\theta) = \phi_1(\theta) \geqslant 0,\ y_0(\theta) = \phi_2(\theta) \geqslant 0,\ z_0(\theta) = \phi_3(\theta) \geqslant 0,$$

and

$$x(0) > 0, y(0) > 0, z(0) > 0,$$

where $\phi_t(\theta) = (\phi_1, \phi_2, \phi_3) \in C\big([-\tau, 0], \mathbb{R}^3_+\big)$, $\mathbb{R}^3_+ = \{(x, y, z) : x \geqslant 0, y \geqslant 0, z \geqslant 0\}$, and $\|\phi\| = \max\{|\phi_t(\theta)| : \theta \in [-\tau, 0]\}$ and $|\phi|$ is any norm in $\mathbb{R}^3$. As usual, we use the conventional notation $x_t(\theta) = x(t + \theta)$, for $\theta \in [-\tau, 0]$.

The system has a unique positive equilibrium if and only if one of the following is true:

(**a**) $f > d + \dfrac{h}{n}$ and $p > \max\big\{s,\ \dfrac{hs}{(f-d)n - h}\big\}$, when $ma \geqslant c$;

(**b**) $d + \dfrac{h}{n} - \dfrac{sh}{np} < f < \dfrac{hc(s-p) - npdc}{np(am - c)}$ and $p > s$, when $ma < c$.

In this paper, we always assume that the positive equilibrium exists and denote it by E^* (x^*, y^*, z^*). It's obvious that the first quadrant is positively invariant for system (2.1). Through the simple analysis, we have:

$$\limsup_{t\to+\infty} x(t) \leqslant K. \tag{2.2}$$

From system (2.1), we have

$$y(t) < y(0)\mathrm{e}^{(f-d)t}.$$

Clearly, if $d \geqslant f$, then $\limsup\limits_{t\to+\infty} y(t) = 0$. For the case $f > d$, we have the following results.

Lemma 2.1. *For system (2.1), if $f > d$, then*

$$\limsup_{t\to+\infty} y(t) \leqslant \frac{(f-d)K\mathrm{e}^{(f-d)\tau}}{md} \triangleq \bar{y}. \tag{2.3}$$

Proof. There exists a $T > 0$, such that for $t > T$, $x(t) < K$. From system (2.1), we have

$$y'(t) \leqslant (f-d)y(t).$$

Therefore, when $t > \tau$, we have

$$y(t) \leqslant y(t-\tau)\mathrm{e}^{(f-d)\tau},$$

which is equivalent to $t > \tau$, $y(t-\tau) \geqslant y(t)\mathrm{e}^{-(f-d)\tau}$.

Therefore, for $t > T + \tau$, and $x(t-\tau) < K$, we have

$$\begin{aligned} y'(t) &\leqslant y(t)\Big(-d + \frac{fx(t-\tau)}{my(t-\tau)+x(t-\tau)}\Big) \\ &\leqslant y(t)\Big(-d + \frac{fK}{K+my(t-\tau)}\Big) \\ &\leqslant y(t)\Big(-d + \frac{fK}{K+m\mathrm{e}^{-(f-d)\tau}y(t)}\Big) \\ &= y(t)\Big(\frac{(f-d)K - md\mathrm{e}^{-(f-d)\tau}y(t)}{K+m\mathrm{e}^{-(f-d)\tau}y(t)}\Big). \end{aligned}$$

The comparison argument implies that

$$\limsup_{t\to+\infty} y(t) \leqslant \frac{(f-d)K\mathrm{e}^{(f-d)\tau}}{md}.$$

□

Lemma 2.2. *For system (2.1), if $f > d$, $p > s$, then*

$$\limsup_{t\to+\infty} z(t) \leqslant \frac{(p-s)(f-d)K\mathrm{e}^{(f-d+p-s)\tau}}{mnds} \triangleq \bar{z}. \tag{2.4}$$

Proof. From Lemma 2.1, there exists a $T > 0$, such that for $t > T$, $y(t) < \bar{y}$. And from the third equation of system (2.1), we have

$$z'(t) \leqslant (p-s)z(t).$$

Therefore, for $t > \tau$,

$$z(t) \leqslant z(t-\tau)\mathrm{e}^{(p-s)\tau}.$$

Thus, for $t > T + \tau$, and $y(t-\tau) < \bar{y}$, we have

$$\begin{aligned} z'(t) &\leqslant z(t)\Big(-s + \frac{p\bar{y}}{\bar{y}+nz(t-\tau)}\Big) \\ &\leqslant z(t)\Big(-s + \frac{p\bar{y}}{\bar{y}+n\mathrm{e}^{(s-p)\tau}z(t)}\Big) \\ &= z(t)\Big(\frac{(p-s)\bar{y} - ns\mathrm{e}^{(s-p)\tau}z(t)}{\bar{y}+n\mathrm{e}^{(s-p)\tau}z(t)}\Big). \end{aligned}$$

The comparison argument shows that

$$\limsup_{t\to+\infty} z(t) \leqslant \frac{(p-s)(f-d)K\mathrm{e}^{(f-d+p-s)\tau}}{mnds} \triangleq \bar{z}.$$

$\square$

According to Lemma 2.1 and 2.2, we conclude that system (2.1) is dissipative. The system (2.1) is said to be permanent if there exists the constants δ, Δ, $0 < \delta < \Delta$, such that

$$\min\Big\{\liminf_{t\to+\infty} x(t),\qquad \liminf_{t\to+\infty} y(t),\qquad \liminf_{t\to+\infty} z(t)\Big\} \geqslant \delta,$$
$$\max\Big\{\limsup_{t\to+\infty} x(t),\qquad \limsup_{t\to+\infty} y(t),\qquad \limsup_{t\to+\infty} z(t)\Big\} \leqslant \Delta$$

for all solutions of (2.1) with the initial conditions are ture. Next, we will prove the permanence of the system (2.1).

For system (2.1), if $ma > c$, then

$$x'(t) > x(t)\Big(a - \frac{c}{m} - bx(t)\Big). \tag{2.5}$$

We have

$$\liminf_{t\to+\infty} x(t) \geqslant \frac{ma-c}{bm} \triangleq \underline{x}.$$

There exists a T_1, such that when $t > T_1$, $x(t) > \frac{\underline{x}}{2}$. Therefore, for $t > T_1$,

$$y'(t) \geqslant y(t)\Big(-(d+\frac{h}{n}) + \frac{f\cdot \underline{x}/2}{my(t-\tau)+\underline{x}/2}\Big).$$

Thus, for $t > T_1 + \tau$, since

$$y(t-\tau) \leqslant y(t)\mathrm{e}^{(d+h/n)\tau},$$

we have

$$y'(t) \geqslant y(t)\Big[\frac{(f-d-h/n)\underline{x}/2 - (d+h/n)m\mathrm{e}^{(d+h/n)\tau}y(t)}{my(t-\tau)+\underline{x}/2}\Big]. \tag{2.6}$$

Hence, for large t, if $f > d + h/n$, then

$$\liminf_{t\to\infty} y(t) \geqslant (f-d-h/n)\underline{x}\mathrm{e}^{-(d+h/n)\tau}(2(d+h/n)m)^{-1} \triangleq \underline{y}.$$

Hence there exists a T_2, such that $y(t) > \dfrac{\underline{y}}{2}$ when $t > T_2$. From (2.1), we have

$$z'(t) \geqslant z(t)\Big(-s + \frac{p\frac{\underline{y}}{2}}{nz(t-\tau)+\frac{\underline{y}}{2}}\Big).$$

Using the fact that, for large t,

$$z(t-\tau) \leqslant z(t)\mathrm{e}^{s\tau},$$

we have

$$z'(t) \geqslant z(t)\frac{(p-s)\underline{y}/2 - nsz(t-\tau)}{nz(t-\tau)+\underline{y}/2}.$$

That is,

$$z'(t) \geqslant z(t)\frac{(p-s)\underline{y}/2 - nsz(t)\mathrm{e}^{s\tau}}{nz(t-\tau)+\underline{y}/2}, \tag{2.7}$$

which yields that for $p > s$,

$$\liminf_{t\to\infty} z(t) \geqslant (p-s)\underline{y}\mathrm{e}^{-s\tau}(2ns)^{-1} \triangleq \underline{z}.$$

The above arguments imply that for system (2.1), we set

$$\delta = \min\{\underline{x},\ \underline{y},\ \underline{z}\}, \qquad \Delta = \max\{K,\ \bar{y},\ \bar{z}\}.$$

So we have

$$\min\Big\{\liminf_{t\to+\infty} x(t), \qquad \liminf_{t\to+\infty} y(t), \qquad \liminf_{t\to+\infty} z(t)\Big\} \geqslant \delta,$$
$$\max\Big\{\limsup_{t\to+\infty} x(t), \qquad \limsup_{t\to+\infty} y(t), \qquad \limsup_{t\to+\infty} z(t)\Big\} \leqslant \Delta.$$

Therefore, we have the following theorem.

Theorem 2.1. *When $f > d + h/n$, $p > s$ and $ma > c$, system (2.1) is permanent.*

Next we examine conditions that render certain species extinct. Scenarios include the extinction of species x (and hence y and z), the extinction of y (and hence z) but not x, the extinction of top predator z (but not x and y). Firstly, we give stability conditions where all three species extinct.

Theorem 2.2. *Assume $cm^{-1} > a+d+\frac{h}{n}$, $hn^{-1} > f-d+s-\frac{fm}{c}\big(a+d+\frac{h}{n}\big)$ and $p < s(1+n\beta^{-1})$, where $\beta = \dfrac{hc}{(f-d+s)c - fm(a+d+\frac{h}{n})}$. Then there exist positive solutions $(x(t),\ y(t),\ z(t))$ of system (2.1) such that $\lim\limits_{t\to+\infty}(x(t),\ y(t),\ z(t)) = (0,\ 0,\ 0)$.*

Proof. Assume $cm^{-1} > a + d + \frac{h}{n}$ in system (2.1). Then there is an $\alpha > 0$, such that $c(m+\alpha)^{-1} = a + d + \frac{h}{n}$. Let $\delta = x(0)/y(0) < \alpha$. We claim that for all $t > 0$, $x(t)/y(t) < \alpha$ and $\lim\limits_{t\to+\infty} x(t) = 0$. Otherwise, there is a first time t_1, $x(t_1)/y(t_1) = \alpha$ and for $t \in [0,\ t_1)$, $x(t)/y(t) < \alpha$. Then for $t \in [0, t_1]$, we have

$$x'(t) \leqslant x(a - c/(m + x/y)) \leqslant x\Big(a - \frac{c}{m+\alpha}\Big) = -\Big(d + \frac{h}{n}\Big)x(t),$$

which implies that $x(t) \leqslant x(0)\mathrm{e}^{-(d+\frac{h}{n})t}$. However, for all $t \geqslant 0$, $y(t)/z(t) \geqslant 0$, and

$$y'(t) \geqslant \Big(-d - \frac{h}{n}\Big)y(t),$$

which implies that $y(t) \geqslant y(0)\mathrm{e}^{-(d+\frac{h}{n})t}$. This shows that for $t \in [0, t_1]$,

$$x(t)/y(t) \leqslant x(0)/y(0) = \delta < \alpha,$$

which is a contradiction to the existence of t_1. Thus, we have $\lim\limits_{t\to+\infty} x(t)=0$.

Assume $hn^{-1} > f-d+s-\frac{fm}{c}(a+d+\frac{h}{n})$, then there is an $\beta>0$, such that $h(n+\beta)^{-1} = f-d+s-\frac{fm}{c}(a+d+\frac{h}{n})$. Let $\zeta = y(0)/z(0) < \beta$. We claim that for all $t>0$, $y(t)/z(t)<\beta$ and $\lim\limits_{t\to+\infty} y(t)=0$. Otherwise, there is another time t_2, $y(t_2)/z(t_2)=\beta$ and for $t\in[0,t_2)$, $y(t)/z(t)<\beta$. For $t\geqslant 0$, $x(t)/y(t)\leqslant\alpha$. These imply that for $t\geqslant\tau$,

$$y'(t)\leqslant y(t)\Big(-d+\frac{f\alpha}{m+\alpha}-\frac{h}{n+\frac{y}{z}}\Big).$$

Then for $t\in[0,\ t_2]$, we have

$$\begin{aligned}
y'(t) &\leqslant y(t)\Big(-d+\frac{f\alpha}{m+\alpha}-\frac{h}{n+\beta}\Big)\\
&= y(t)\Big(-d+\frac{f(c-m(a+d+\frac{h}{n}))}{c}-\frac{h}{n+\beta}\Big)\\
&= y(t)\Big(-d+\frac{fc-fm(a+d+\frac{h}{n})}{c}-f+d-s+\frac{fm(a+d+\frac{h}{n})}{c}\Big)\\
&= -sy(t),
\end{aligned}$$

which implies that $y(t)\leqslant y(0)\mathrm{e}^{-st}$. However, for all $t\geqslant 0$, $y(t)/z(t)\geqslant 0$ and

$$z'(t)\geqslant -sz(t).$$

This means $z(t)\geqslant z(0)\mathrm{e}^{-st}$, for $t\in[0,t_2]$,

$$y(t)/z(t)\leqslant y(0)/z(0)=\zeta<\beta,$$

which is a contradiction to the existence of t_2. Thus, we have $\lim\limits_{t\to+\infty} y(t)=0$.

Therefore, we have for all $t\geqslant 0$, $y(t)/z(t)\leqslant y(0)/z(0)=\beta$, for $t\geqslant\tau$,

$$z'(t)\leqslant z(t)(-s+p\beta/(n+\beta))\triangleq -lz(t),$$

which implies that $\lim\limits_{t\to+\infty} z(t)=0$, which completes the proof. □

If the positive equilibrium $E^*(x^*,y^*,z^*)$ does not exist, then the following propositions are right after a simple calculation.

Proposition 2.1. *Let $f\leqslant d$, $p\leqslant s$, then $\lim\limits_{t\to+\infty} y(t)=0$ and $\lim\limits_{t\to+\infty} z(t)=0$; If $p\leqslant s$, then $\lim\limits_{t\to+\infty} z(t)=0$.*

Next, we suggest that if the middle predator has a low capacity consume causing its extinction, then the prey species will persist and the top predator will extinct at the same time.

Theorem 2.3. *If $ma>c, f\leqslant d$ and $p\leqslant s$, then there exist positive solutions $(x(t),y(t),z(t))$ of system (2.1) such that $\lim\limits_{t\to+\infty}(x(t),y(t),z(t))=(K,0,0)$.*

Proof. If $f\leqslant d$ and $p\leqslant s$, then it is obvious from the previous argument that $\lim\limits_{t\to+\infty} y(t)=0$, and $\lim\limits_{t\to+\infty} z(t)=0$. When $ma>c$, we have $\liminf\limits_{t\to+\infty} x(t)\geqslant \underline{x}$, $\underline{x}=\dfrac{ma-c}{bm}$. Hence, for any $\varepsilon\in(0,\ a)$, there exists $T=T(\varepsilon)$, such that for $t>T$,

$$x(t)(a-\varepsilon-bx(t))\leqslant x'(t)\leqslant x(t)(a-bx(t)),$$

which implies that

$$\lim_{t\to+\infty} x(t) = ab^{-1} = K.$$

This proves the theorem. □

3. Global stability of the coexistence equilibrium

Consider the system of differential equations

$$x' = f(x), \tag{3.1}$$

where $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is continuous. We call V a Lyapunov function on $G \subseteq \Omega$ for system (3.1) if

(**a**) V is continuous on G,

(**b**) If V is not continuous at $\bar{x} \in \bar{G}$ (the closure of G) then $\lim\limits_{x\to\bar{x}, x\in G} V(x) = +\infty$,

(**c**) $\dot{V}(x) = \mathrm{grad}V(x) \cdot f(x) \leqslant 0$ on G.

Furthermore, LaSalle's Invariant Principle [11] is stated as follows: Assume that V is a Lyapunov function of (3.1) on G. Define $S = \{x \in \bar{G} \cap \Omega : \dot{V}(x) = 0\}$. Let D be the maximal compact invariant set in S. Then every bounded trajectory (for $t \geqslant 0$) of (3.1) that remains in G approaches the set D as $t \to \infty$.

For a given mathematical model in population biology, in most cases the only locally asymptotically stable equilibrium is expected to be globally stable. However, it is quite difficult to construct a Lyapunov function for establishing the global stability of the equilibrium. If we are able to construct a Lyapunov function for the system then the global stability follows directly from the above Lasalle's Invariance Principle.

Recall that $b = \frac{a}{K}$, Eq.(2.1) can be rewritten into

$$\begin{cases} x'(t) = x\Big\{ -b(x-x^*) + c\Big[\dfrac{y^*}{x^*}U\Big(\dfrac{x^*}{y^*}\Big) - \dfrac{y}{x}U\Big(\dfrac{x}{y}\Big)\Big]\Big\}, \\ y'(t) = fy\Big[U\Big(\dfrac{x(t-\tau)}{y(t-\tau)}\Big) - U\Big(\dfrac{x^*}{y^*}\Big)\Big] + hy\Big[\dfrac{z^*}{y^*}V\Big(\dfrac{y^*}{z^*}\Big) - \dfrac{z}{y}V\Big(\dfrac{y}{z}\Big)\Big], \\ z'(t) = pz\Big[V\Big(\dfrac{y(t-\tau)}{z(t-\tau)}\Big) - V\Big(\dfrac{y^*}{z^*}\Big)\Big]. \end{cases} \tag{3.2}$$

Where $x^*, y^*, z^* > 0$ are the components of the positive equilibrium $E^*(x^*, y^*, z^*)$ and $b = \frac{a}{K}$, $U(\xi) = \frac{\xi}{m+\xi}$, $V(\eta) = \frac{\eta}{n+\eta}$. Denote by $U^* = U(u^*) = U(\frac{x^*}{y^*})$, $V^* = V(v^*) = V(\frac{y^*}{z^*})$. Furthermore, we take the following variable change:

$$(x, y, z) \to \Big(x, u = \frac{x}{y}, v = \frac{y}{z}\Big).$$

If $(x, u, v) \to (x^*, u^*, v^*)$, then $(x, y, z) \to (x^*, y^*, z^*)$.

$$\begin{aligned} u' &= \Big(\frac{x}{y}\Big)' = \frac{x'y - xy'}{y^2} = \frac{x'}{y} - \frac{x}{y^2}y' \\ &= \frac{x}{y}\Big[-b(x-x^*) + c\Big(\frac{U^*}{u^*} - \frac{U}{u}\Big)\Big] - \frac{x}{y}[f(U(u(t-\tau)) - U^*)] - \frac{x}{y}\Big[h\Big(\frac{V^*}{v^*} - \frac{V}{v}\Big)\Big] \\ &= \frac{x}{y}\Big[-b(x-x^*) + c\Big(\frac{U^*}{u^*} - \frac{U}{u}\Big) - f(U(u(t-\tau)) - U^*) - h\Big(\frac{V^*}{v^*} - \frac{V}{v}\Big)\Big], \end{aligned}$$

and

$$\begin{aligned} v' &= \Big(\frac{y}{z}\Big)' = \frac{y'z - yz'}{z^2} = \frac{y'}{z} - \frac{y}{z^2}z' \\ &= \frac{y}{z}\Big[f(U(u(t-\tau)) - U^*) + h\Big(\frac{V^*}{v^*} - \frac{V}{v}\Big) - p(V(v(t-\tau)) - V^*)\Big]. \end{aligned}$$

Therefore, system (3.2) is transformed into the following form with the variables (x, u, v),

$$\begin{cases} x' = x\Big[-b(x-x^*) + c\Big(\frac{U^*}{u^*} - \frac{U}{u}\Big)\Big], \\ u' = u\Big[-b(x-x^*) + c\Big(\frac{U^*}{u^*} - \frac{U}{u}\Big) - f(U(u(t-\tau)) - U^*) - h\Big(\frac{V^*}{v^*} - \frac{V}{v}\Big)\Big], \\ v' = v\Big[f(U(u(t-\tau)) - U^*) + h\Big(\frac{V^*}{v^*} - \frac{V}{v}\Big) - e(V(v(t-\tau)) - V^*)\Big]. \end{cases} \tag{3.3}$$

Because

$$\frac{U^*}{u^*} - \frac{U}{u} = \frac{U}{uu^*}(u-u^*) - \frac{1}{u^*}(U - U^*), \tag{3.4}$$

and

$$\frac{V^*}{v^*} - \frac{V}{v} = \frac{V}{vv^*}(v-v^*) - \frac{1}{v^*}(V - V^*). \tag{3.5}$$

Define

$$\omega_1 = x - x^*, \omega_2 = u - u^*, \omega_3 = v - v^*, \tag{3.6}$$

such that $\omega_1 \geqslant -x^*, \omega_2 \geqslant -u^*, \omega_3 \geqslant -v^*$, and

$$\begin{cases} g_1(\omega_2) = U - U^* = \dfrac{m\omega_2}{(m+u)(m+u^*)}, \\ g_2(\omega_3) = V - V^* = \dfrac{n\omega_3}{(n+v)(n+v^*)}, \\ g_1(\omega_2(t-\tau)) = U(u(t-\tau)) - U^*, \quad g_2(\omega_3(t-\tau)) = V(v(t-\tau)) - V^*. \end{cases} \tag{3.7}$$

Observe that:

$$\begin{cases} g_1'(\omega_2) = \dfrac{m}{(m+u^*+\omega_2)^2}, \\ g_2'(\omega_3) = \dfrac{n}{(n+v^*+\omega_3)^2}. \end{cases} \tag{3.8}$$

According to (3.4)–(3.8), we have

$$\begin{cases} c\Big(\dfrac{U^*}{u^*} - \dfrac{U}{u}\Big) = \dfrac{c}{u^*}\Big[\dfrac{U(u)}{u}\omega_2 - g_1(\omega_2)\Big] = \dfrac{c}{m}g_1(\omega_2), \\ h\Big(\dfrac{V^*}{v^*} - \dfrac{V}{v}\Big) = \dfrac{h}{v^*}\Big[\dfrac{V(v)}{v}\omega_2 - g_2(\omega_3)\Big] = \dfrac{h}{n}g_2(\omega_3). \end{cases} \tag{3.9}$$

Finally, we get

$$\begin{cases} \omega_1' = (\omega_1 + x^*)\Big[-b\omega_1 + \dfrac{c}{m}g_1(\omega_2)\Big], \\ \omega_2' = (\omega_2 + u^*)\Big[-b\omega_1 + \dfrac{c}{m}g_1(\omega_2) - fg_1(\omega_2(t-\tau)) - \dfrac{h}{n}g_2(\omega_3)\Big], \\ \omega_3' = (\omega_3 + v^*)\Big[fg_1\omega_2(t-\tau) + \dfrac{h}{n}g_2(\omega_3) - eg_2(\omega_3(t-\tau))\Big]. \end{cases} \tag{3.10}$$

The positive equilibrium (x^*, u^*, v^*) in (3.3) corresponds to $\omega_1(t) = \omega_2(t) = \omega_3(t) = 0$ for all $t \in \mathbb{R}$. Therefore, the global asymptotical stability of the trivial equilibrium of (3.10) implies that of the positive equilibrium of (3.2).

In order to structure the Lyapunov functional, we firstly define the function

$$V_1(\omega_t) = A\Big[\omega_1 - x^* \ln\Big(\frac{x^* + \omega_1}{x^*}\Big)\Big] + \int_{u^*}^{u} \frac{U(\omega) - U(u^*)}{\omega} \mathrm{d}\omega + \frac{h}{nf} \int_{v^*}^{v} \frac{V(\omega) - V(v^*)}{\omega} \mathrm{d}\omega, \tag{3.11}$$

where $A \in \mathbb{R}_+$ is an arbitrary constant.

$$\begin{aligned}
\dot{V}_1(\omega_t)|_{(3.10)} &= A\omega_1\Big(-b\omega_1 + \frac{c}{m} g_1(\omega_2)\Big) + g_1(\omega_2)\Big[-b\omega_1 - \Big(f - \frac{c}{m}\Big) g_1(\omega_2) \\
&\quad + f\int_{t-\tau}^{t} g_1'(\omega_2)\omega_2'(v)\mathrm{d}v - \frac{h}{n} g_2(\omega_3)\Big] + \frac{h}{nf} g_2(\omega_3)\Big[f g_1(\omega_2) \\
&\quad - f\int_{t-\tau}^{t} g_1'(\omega_2)\omega_2'(v)\mathrm{d}v + \Big(\frac{h}{n} - p\Big) g_2(\omega_3) + p\int_{t-\tau}^{t} g_2'(\omega_3)\omega_3'(v)\mathrm{d}v\Big] \\
&= -Ab\omega_1^2 + \Big(\frac{Ac}{m} - b\Big) g_1(\omega_2)\omega_1 + \frac{c}{m} g_1^2(\omega_2) + \Big(\frac{h}{n} - p\Big)\frac{h}{nf} g_2^2(\omega_3) \\
&\quad + f g_1(\omega_2)\int_{t-\tau}^{t} g_1'(\omega_2) u(\omega)\Big[\frac{c}{m} g_1(\omega_2(\omega)) - f g_1(\omega_2(\omega - \tau)) \\
&\quad - b\omega_1(\omega) - \frac{h}{n} g_2(\omega_3(\omega))\Big]\mathrm{d}\omega - \frac{h}{n} g_2(\omega_3)\int_{t-\tau}^{t} g_1'(\omega_2) u(\omega)\Big[-b\omega_1(\omega) \\
&\quad + \frac{c}{m} g_1(\omega_2(\omega)) - f g_1(\omega_2(\omega - \tau)) - \frac{h}{n} g_2(\omega_3(\omega))\Big]\mathrm{d}\omega - f g_1^2(\omega_2) \\
&\quad + \frac{ph}{nf} g_2(\omega_3)\int_{t-\tau}^{t} g_2'(\omega_3) v(\omega)\Big[f g_1(\omega_2(\omega)) + \frac{h}{n} g_2(\omega_3(\omega)) \\
&\quad - p g_2(\omega_3(\omega - \tau))\Big]\mathrm{d}\omega.
\end{aligned}$$

Note that

$$\begin{aligned}
&g_i(\omega_{i+1}(t))u(\omega)\omega_1(\omega) \leqslant \frac{1}{2}(g_i^2(\omega_{i+1}(t)) + u^2(\omega)\omega_1^2(\omega)), \\
&g_i(\omega_{i+1}(t))u(\omega)g_1(\omega_2(\omega)) \leqslant \frac{1}{2}(g_i^2(\omega_{i+1}(t)) + u^2(\omega)g_1(\omega_2(\omega))), \\
&g_i(\omega_{i+1}(t))u(\omega)g_1\omega_2(\omega - \tau) \leqslant \frac{1}{2}(g_i^2(\omega_{i+1}(t)) + u^2(\omega)g_1^2(\omega_2(\omega - \tau))), \\
&g_i(\omega_{i+1}(t))u(\omega)g_2\omega_3(\omega) \leqslant \frac{1}{2}(g_i^2(\omega_{i+1}(t)) + u^2(\omega)g_2^2(\omega_3(\omega))),
\end{aligned}$$

where $i = 1, 2$, and

$$\begin{aligned}
&g_2(\omega_3(t))v(\omega)g_1(\omega_2(\omega)) \leqslant \frac{1}{2}(g_2^2(\omega_3(t)) + v^2(\omega)g_1^2(\omega_2(\omega))), \\
&g_2(\omega_3(t))v(\omega)g_2(\omega_3(\omega)) \leqslant \frac{1}{2}(g_2^2(\omega_3(t)) + v^2(\omega)g_2^2(\omega_3(\omega))), \\
&g_2(\omega_3(t))v(\omega)g_2(\omega_3(\omega - \tau)) \leqslant \frac{1}{2}(g_2^2(\omega_3(t)) + v^2(\omega)g_2^2(\omega_3(\omega - \tau))).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\dot{V}_1(\omega_t)|_{(3.10)} \leqslant & -Ab\omega_1^2 + \big(\frac{Ac}{m} - b\big)g_1(\omega_2)\omega_1 - \big(f - \frac{c}{m}\big)g_1^2(\omega_2) + \frac{h^2}{n^2 f}g_2^2(\omega_3)\\ & +\Big[\frac{f}{2}\big(b + \frac{c}{m} + f + \frac{h}{n}\big)g_1^2(\omega_2(t))\int_{t-\tau}^{t} g_1'(\omega_2(\omega))\mathrm{d}\omega\\ & +\frac{h}{2n}\Big(b + \frac{c}{m} + f + \frac{h}{n}\Big)g_2^2(\omega_3(t))\Big] + \frac{h}{2n}\int_{t-\tau}^{t} g_1(\omega_2(\omega))u^2\Big[\frac{c}{m}g_1^2(\omega_2(\omega))\\ & +fg_1^2(\omega_2(\omega-\tau)) + \frac{h}{n}g_2^2(\omega_3(\omega)) + b\omega_1^2(\omega)\Big]\mathrm{d}\omega\\ & +\frac{f}{2}\int_{t-\tau}^{t} g_1(\omega_2(\omega))u^2\Big[b\omega_1^2(\omega) + \frac{c}{m}g_1^2(\omega_2(\omega)) + fg_1^2(\omega_2(\omega-\tau))\\ & +\frac{h}{n}g_2^2(\omega_3(\omega))\Big]\mathrm{d}\omega + \frac{ph}{2nf}\big(f + \frac{h}{n} + p\big)g_2^2(\omega_3(t))\int_{t-\tau}^{t} g_2'(\omega_3(\omega))\mathrm{d}\omega\\ & -p\frac{h}{nf}g_2^2(\omega_3) + \frac{ph}{2nf}\int_{t-\tau}^{t} g_2'(\omega_3(\omega))v^2(\omega)\Big[fg_1^2(\omega_2(\omega))\\ & +\frac{h}{n}g_2^2(\omega_3(\omega)) + pg_2^2(\omega_3(\omega-\tau))\Big]\mathrm{d}\omega. \end{aligned} \tag{3.12}$$

From (3.8), we have

$$\begin{aligned} g_1'(\omega_2) &= \frac{m}{(m+u^*+\omega_2)^2} < \frac{1}{m}, & g_2'(\omega_3) &= \frac{n}{(n+v^*+\omega_3)^2} < \frac{1}{n},\\ g_1'(\omega_2)u^2 &= m\frac{u^2}{(m+u)^2} < m, & g_2'(\omega_3)v^2 &= n\frac{v^2}{(n+v)^2} < n. \end{aligned}$$

Let us choose A in (3.12) such that $\frac{Ac}{m} = b$, that's $A = \frac{mb}{c}$, then (3.12) becomes:

$$\begin{aligned}\dot{V}_1(\omega_t)|_{(3.10)} \leqslant & -\frac{mb^2}{c}\omega_1^2 - \Big(f - \frac{c}{m}\Big)g_1^2(\omega_2) - \Big(p - \frac{h}{n}\Big)\frac{h}{nf}g_2^2(\omega_3) + \frac{f}{2m}\Big(b + \frac{c}{m}\\ & +f + \frac{h}{n}\Big)\tau g_1^2(\omega_2(t)) + \Big[\frac{h}{2mn}\Big(b + \frac{c}{m} + f + \frac{h}{n}\Big) + \frac{ph}{2n^2 f}\Big(f + \frac{h}{n}\\ & +p\Big)\Big]\tau g_2^2(\omega_3(t)) + m\Big(\frac{h}{2n} + \frac{f}{2}\Big)\int_{t-\tau}^{t}\Big[b\omega_1^2(\omega) + \frac{c}{m}g_1^2(\omega_2(\omega))\\ & +fg_1^2(\omega_2(\omega-\tau)) + \frac{h}{n}g_2^2(\omega_3(\omega))\Big]\mathrm{d}\omega + \frac{ph}{2f}\int_{t-\tau}^{t}\Big[fg_1^2(\omega_2(\omega))\\ & +\frac{h}{n}g_2^2(\omega_3(\omega)) + pg_2^2(\omega_3(\omega-\tau))\Big]\mathrm{d}\omega. \end{aligned}$$

We introduce another function:

$$\begin{aligned} V_2(\omega_t) = & V_1(\omega_t) + m\Big(\frac{h}{2n} + \frac{f}{2}\Big)\int_{t-\tau}^{t}\mathrm{d}s\int_{s}^{t}\Big[b\omega_1^2(\omega) + \frac{c}{m}g_1^2(\omega_2(\omega)) + fg_1^2(\omega_2(\omega-\tau))\\ & +\frac{h}{n}g_2^2(\omega_3(\omega))\Big]\mathrm{d}\tau + \frac{ph}{2f}\int_{t-\tau}^{t}\mathrm{d}s\int_{s}^{t}\Big[fg_1^2(\omega_2(\omega)) + \frac{h}{n}g_2^2(\omega_3(\omega))\\ & +pg_2^2(\omega_3(\omega-\tau))\Big]\mathrm{d}\tau. \end{aligned} \tag{3.13}$$

Differentiating V_2 along (3.10) gives:

$$\begin{aligned}
\dot{V}_2(\omega_t)|_{(3.10)} \leqslant & -\frac{mb^2}{c}\omega_1^2 - \Big(f-\frac{c}{m}\Big)g_1^2(\omega_2) - \Big(p-\frac{h}{n}\Big)\frac{h}{nf}g_2^2(\omega_3) + \frac{f}{2m}\Big(b+\frac{c}{m}+f \\
& +\frac{h}{n}\Big)\tau g_1^2(\omega_2(t)) + \Big[\frac{h}{2mn}\Big(b+\frac{c}{m}+f+\frac{h}{n}\Big)+\frac{ph}{2n^2f}\Big(f+\frac{h}{n} \\
& +p\Big)\Big]\tau g_2^2(\omega_3(t)) + \Big(\frac{h}{2n}+\frac{f}{2}\Big)\tau c g_1^2(\omega_2) + mf\Big(\frac{h}{2n}+\frac{f}{2}\Big)\tau g_1^2(\omega_2(t-\tau)) \\
& +\tau m\Big(\frac{h}{2n}+\frac{f}{2}\Big)b\omega_1^2 + \frac{ph}{2}\tau g_1^2(\omega_2) + \frac{ph^2}{2fn}\tau g_2^2(\omega_3) + \frac{p^2h}{2f}\tau g_2^2(\omega_3(t-\tau)) \\
= & -\Big[\frac{mb}{c} - \tau m(\frac{h}{2n}+\frac{f}{2})\Big]b\omega_1^2 - \Big(f-\frac{c}{m}\Big)g_1^2(\omega_2) - \Big(p-\frac{h}{n}\Big)\frac{h}{nf}g_2^2(\omega_3) \\
& +\Big[\frac{f}{2m}\Big(b+\frac{c}{m}+f+\frac{h}{n}\Big)+\Big(\frac{h}{2n}+\frac{f}{2}\Big)+\frac{ph}{2}\Big]\tau g_1^2(\omega_2) \\
& +\Big[\frac{h}{2mn}\Big(b+\frac{c}{m}+f+\frac{h}{n}\Big)+\frac{ph}{2n^2f}\Big(f+\frac{h}{n}+p\Big)+\frac{mh}{n}\Big(\frac{h}{2n}+\frac{f}{2}\Big) \\
& +\frac{ph^2}{2fn}\Big]\tau g_2^2(\omega_3) + mf\Big(\frac{h}{2n}+\frac{f}{2}\Big)\tau g_1^2(\omega_2(t-\tau)) + \frac{p^2h}{2f}\tau g_2^2(\omega_3(t-\tau)).
\end{aligned}$$

Therefore, we introduce another function again:

$$V_3(\omega_t) = V_2(\omega_t) + mf\Big(\frac{h}{2n}+\frac{f}{2}\Big)\tau\int_{t-\tau}^{t} g_1^2(\omega_2(s))\mathrm{d}s + \frac{p^2h}{2f}\tau\int_{t-\tau}^{t} g_2^2(\omega_3(s))\mathrm{d}s, \quad (3.14)$$

its time derivative along (3.10) gives:

$$\begin{aligned}
\dot{V}_3(\omega_t)|_{(3.10)} \leqslant & -\Big[\frac{mb}{c}-\tau m\Big(\frac{h}{2n}+\frac{f}{2}\Big)\Big]b\omega_1^2 - \Big[\Big(f-\frac{c}{m}\Big)-\frac{f\tau}{2m}\Big(b+\frac{c}{m}+f+\frac{h}{n}\Big) \\
& -\Big(\frac{h}{2n}+\frac{f}{2}\Big)\tau - \frac{ph}{2}\tau - mf\tau\Big(\frac{h}{2n}+\frac{f}{2}\Big)\Big]g_1^2(\omega_2) \\
& -\Big[\Big(p-\frac{h}{n}\Big)\frac{h}{nf} - \frac{h\tau}{2mn}\Big(b+\frac{c}{m}+f+\frac{h}{n}\Big) - \frac{ph\tau}{2n^2f}\Big(f+\frac{h}{n}+p\Big) \\
& -\frac{mh\tau}{n}\Big(\frac{h}{2n}+\frac{f}{2}\Big) - \frac{ph^2\tau}{2nf} - \frac{p^2h\tau}{2f}\Big]g_2^2(\omega_3).
\end{aligned}$$

If $fm-c>0$, $np-h>0$ and $\tau<\tau^*$, where

$$\begin{aligned}
\tau^* = \min\Big\{ & \frac{2nb}{c(h+nf)}, \frac{2n(mf-c)}{nf(b+\frac{c}{m}+f+\frac{h}{n})+m(h+nf)+mnph+m^2(h+nf)}, \\
& \frac{2mnh(np-h)}{nfh(b+\frac{c}{m}+f+\frac{h}{n})+mph(f+\frac{h}{n}+p)+m^2hf(h+f)+mnph(h+np)}\Big\},
\end{aligned}$$

then according to (3.11), (3.13) and (3.14), we construct a Lyapunov function utilizing LaSalle's Invariant Principle [11]. So we have the following theorem.

Theorem 3.1. *If $fm-c>0$, $np-h>0$, and $\tau<\tau^*$, then the positive equilibrium of (3.3) is global asymptotically stable in* $\mathbb{R}^3_+$.

Proof. It's obvious that $V_3(\omega_t)\in C^1(\mathbb{R}^3_+,\mathbb{R})$, $V_3(0,0,0)=0$, and $V_3(\omega_1,\ \omega_2,\ \omega_3)>0$ for $(\omega_1,\ \omega_2,\ \omega_3)\in\mathbb{R}^3_+/(0,0,0)$. When $fm-c>0, np-h>0$, and $\tau<\tau^*$

hold, then $\dot{V}_3(\omega_t) \leqslant 0$. Let $S = \{(\omega_1(t),\ \omega_2(t),\ \omega_3(t)) \in \mathbb{R}^3_+\colon\ V_3(\omega_t) = 0\}$, M be the maximal compact invariant set in S. Through simple calculation, we have $\dot{V}_3(\omega_t)|_{(3.10)} = 0 \Longleftrightarrow \omega_1 = 0,\ \omega_2 = 0,\ \omega_3 = 0$. Therefore, $M = \{(0,0,0)\}$. By LaSalle's Invariant Principle [11], we prove the theorem. $\square$

Discussion

In this paper, we analyzed a delayed ratio-dependent Gause-type predator-prey food chain model. The existence of positive equilibrium E^* is discussed along with numerical simulation. With the parameter values: a = 0.612, b = 0.612, c = 0.362, m = 0.705, f = 0.709, d = 0.1182, h = 0.212, n = 0.366, s = 0.2145, p = 0.6209. We show that E^* is asymptotically stable with $\tau = 1.9667 < \tau^* = 2.0486$ (See Fig.1), which is consistent with Theorem 3.1. Further we find that starting from several different initial conditions, the solutions still tent to E^* when 2.0486 $< \tau <$ 2.4347 (See Fig.2); However, a periodic solution appears when τ= 2.4347 (See Fig.3). With the increase of the time delay, the periodic solution has a tendency to burst (See Fig.4). The numerical results show that: when τ is small enough, the positive equilibrium is global asymptotical stable which is in line with the theoretical result. When τ increases to a critical value, the positive equilibrium loses the global asymptotical stability, and a periodic orbit appears. When τ is large enough, the periodic orbit will rupture. It reveals that the delay has an important impact on dynamical system.

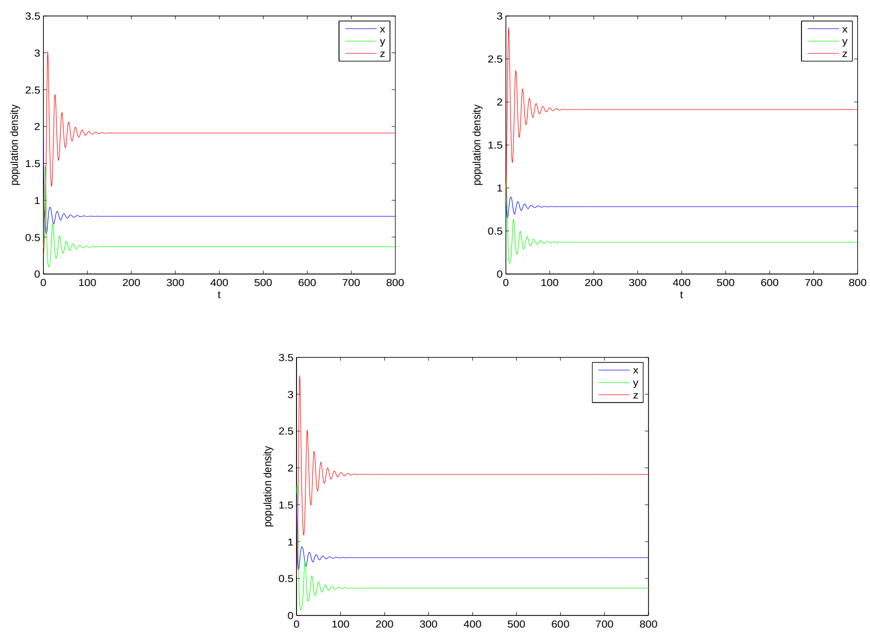

Figure 1. $E^*(0.7081,\ 0.3901,\ 1.960)$ is asymptotically stable when $\tau = 1.9667 < \tau^* = 2.0486$ with different initial values (green:(1.85, 0.351, 0.243); blue:(0.975, 1, 0.804); red:(1.705, 1.651, 0.524)).

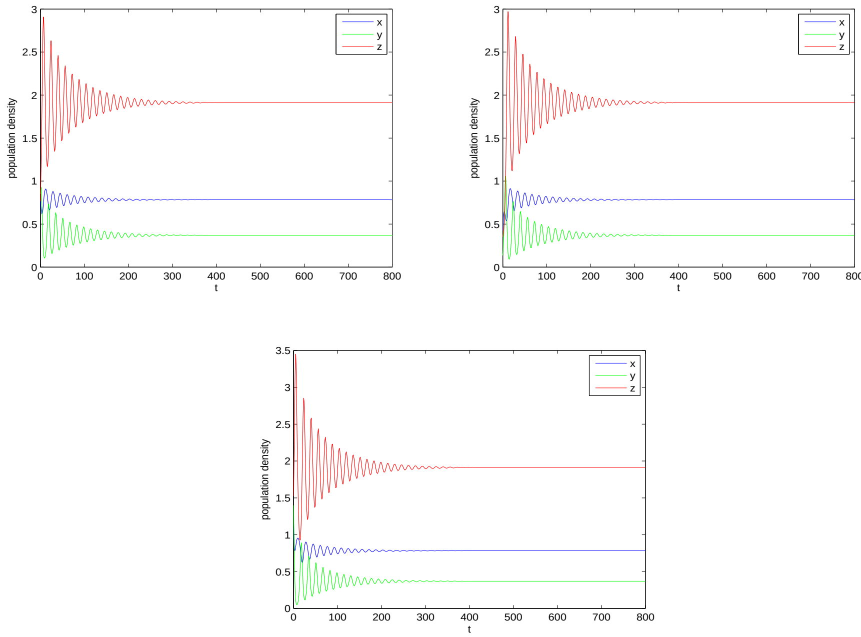

Figure 2. $E^*(0.7081,\ 0.3901,\ 1.960)$ is asymptotically stable when $\tau = 2.2677 < \tau^* = 2.0486$ with different initial values (green:(0.85, 0.135, 1.400); blue:(0.775, 0.425, 1.224); red:(0.765, 0.351, 1.524)).

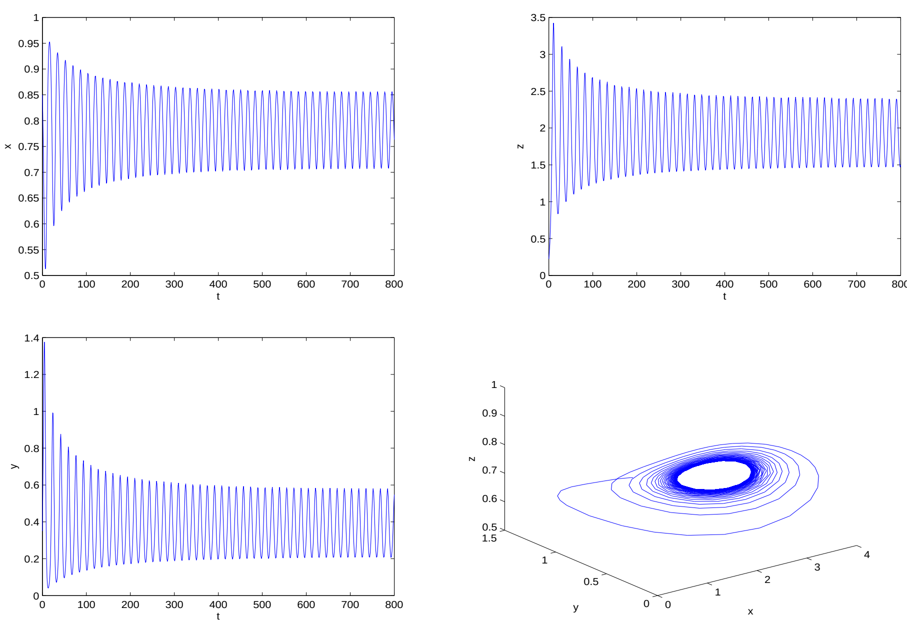

Figure 3. The cycle fluctuation diagram when $\tau = 2.4347$.

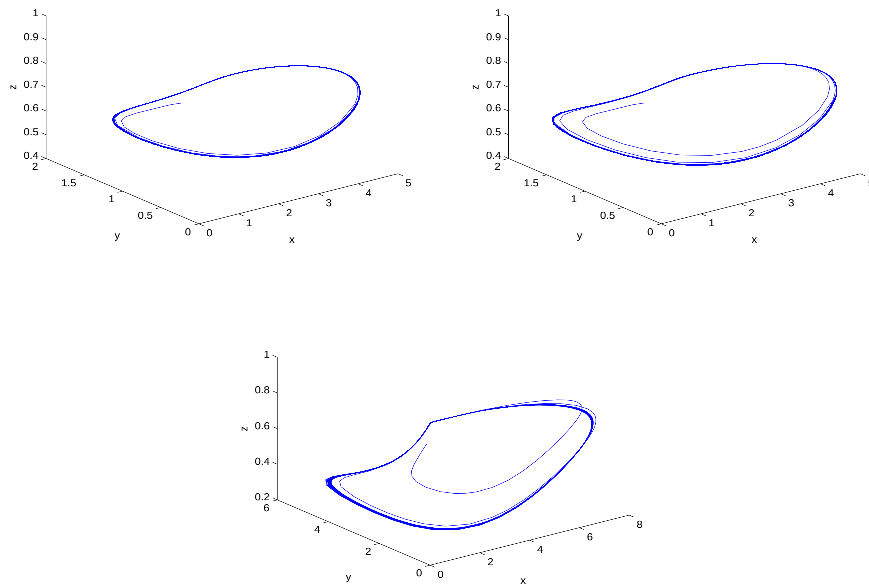

Figure 4. The cycle fluctuation diagrams when $\tau = 2.9337, \tau = 3.277$ and $\tau = 4.037$.

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References

[1] R. Arditi and H. R. Akcakaya, *Underestimation of mutual interference of predators*, Oecologia, 1990, 83(3), 358–361.

[2] R. Arditi and A. A. Berryman, *The biological control paradox*, Tree, 1991, 6(7), 32–32.

[3] R. Arditi and L. R. Ginzburg, *Coupling in predator-prey dynamics: ratio-dependence*, J. Theor. Biol., 1989, 139(3), 311–326.

[4] R. Arditi, L. R. Ginzburg and H. R. Akcakaya, *Variation in plankton densities among lakes: a case for ratio-dependent predation models*, Am. Nat., 1991, 138(5), 1287–1296.

[5] J. R. Beddington, *Mutual interference between parasites or predators and its effect on searching efficiency*, J. Anim. Ecol., 1975, 44(1), 331–340.

[6] E. Beretta and Y. Kuang, *Global analyses in some delayed ratio-dependent predator-prey systems*, Nonlinear Anal-theor., 1998, 32(3), 381–408.

[7] A. A. Berryman, *The origins and evolution of predator-prey theory*, Ecology, 1992, 73(5), 1530–1535.

[8] H. I. Freedman, *Deterministic Mathematical Models in Population Ecology*, Marcel Dekker, New York, 1980.

[9] M. Freeze, Y. Chang, and W. Feng, *Analysis of dynamics in a complex food chain with ratio-dependent functional response*, J. Appl. Anal. Comput., 2014, 4(1), 69–87.

[10] W. M. Getz, *Population dynamics: a per capita resource approach*, J. Theor. Biol., 1984, 108(4), 623–643.

[11] J. M. Ginoux, B. Rossetto and J. L. Jamet, *Chaos in a three-dimensional Volterra-Gause model of predator-prey type*, Int. J. Bifurcat. Chaos., 2005, 15(5), 1689–1708.

[12] A. A. Gomes, E. Manica and M. C. Varriale, *Applications of chaos control techniques to a three-species food chain*, Chaos, Soliton. Fract., 2008, 35(4), 432–441.

[13] H. B. Guo and M. Y. Li, *Global dynamics of a staged progression model for infectious diseases*, Math. Biosci. Eng., 2006, 3(3), 513–525.

[14] I. Hanski, *The functional response of predators worries about scale*, TREE, 1991, 6(5), 141–142.

[15] S. B. Hsu, T. W. Hwang and Y. Kuang, *A ratio-dependent food chain and its applications to biological control*, Math. Biosci., 2003, 181(1), 55–83.

[16] G. P. Hu and X. L. Li, *Stability and Hopf bifurcation for a delayed predator-prey model disease in the prey*. Chaos, Soliton. Fract., 2012, 45(3), 229–237.

[17] J. H. Huang and X. F. Zou, *Traveling wavefronts in diffusive and cooperative Lotka-Volterra system with delays*. J. Math. Anal. Appl., 2002, 271(2), 455–466.

[18] W. H. Jiang and J. J. Wei, *Bifurcation analysis in a limit cycle oscillator with delayed feedback*, Chaos, Soliton. Fract., 2005, 23(3), 817–831.

[19] W. H. Jiang and J. J. Wei, *Bifurcation analysis in van der Pol's oscillator with delayed feedback*, J. Comput. Appl. Math., 2008, 213(2), 604–615.

[20] C. Jost, O. Arino and R. Arditi, *About deterministic extinction in ratio-dependent predator-prey models*, B. Math. Biol., 1999, 61(1), 19–32.

[21] Y. Kuang and E. Beretta, *Global qualitative analysis of a ratio-dependent predator-prey system*, J. Math. Biol., 1998, 36(4), 389–406.

[22] M. Y. Li and H. Y. Shu, *Global dynamics of an in-host viral model with intracellular delay*, B. Math. Biol., 2010, 72 (6), 1492–1505.

[23] H. S. Mahato and M. Bohm, *Global existence and uniqueness of solution for a system of semilinear diffusion-reaction equations*, J. Appl. Anal. Comp., 2013, 3(4), 357–376.

[24] M. C. Varriale and A. A. Gomes, *A study of a three species food chain*, Ecol. Model., 1998, 110(2), 119–133.

[25] H. B. Wang and W. H. Jiang, *Hopf-pitchfork bifurcation in van der Pol's oscillator with nonlinear delayed feedback*, J. Math. Anal. Appl., 2010, 368(1), 9–18.

[26] J. N. Wang and W. H. Jiang. *Bogdanov-Takens singularity in the comprehensive national power model with time delays*, J. Appl. Anal. Comp., 2013, 3(1), 81–94.

[27] J. J. Wei and M. Y. Li, *Global existence of periodic solutions in a tri-neuron network model with delays*, Physica D., 2004, 198(1–2), 106–119.

[28] D. M. Xiao and W. X. Li, *Stability and bifurcation in a delayed ratio-dependent predator-prey system*, P. Edinburgh Math. Soc., 2003, 46(1), 205–220.

[29] R. Xu and M. A. J. Chaplain, *Persistence and global stability in a delayed Gause-type predator-prey system without dominating instantaneous negative feedbacks*, J. Math. Anal. Appl., 2002, 265(1), 148–162.

[30] T. S. Yi and X. F. Zou, *Global dynamics of a delay differential equation with spatial non-locality in an unbounded domain*, J. Diff. Eqs., 2011, 251(9), 2598–2611.