BIFURCATION ANALYSIS OF AN SIRS EPIDEMIC MODEL WITH STANDARD INCIDENCE RATE AND SATURATED TREATMENT FUNCTION*

Yixian Gao[1], Weipeng Zhang[1,†], Dan Liu[2] and Yanju Xiao[1]

Abstract An epidemic model with standard incidence rate and saturated treatment function of infectious individuals is proposed to understand the effect of the capacity for treatment of infective individuals on the disease spread. The treatment function in this paper is a continuous and differential function which exhibits the effect of delayed treatment when the rate of treatment is lower and the number of infected individuals is getting larger. It is proved that the existence and stability of the disease-free and endemic equilibria for the model are not only related to the basic reproduction number but also to the capacity for treatment of infective individuals. And a backward bifurcation is found when the capacity is not enough. By computing the first Lyapunov coefficient, we can determine the type of Hopf bifurcation, i.e., subcritical Hopf bifurcation or supercritical Hopf bifurcation. We also show that under some conditions the model undergoes Bogdanov-Takens bifurcation. Finally, numerical simulations are given to support some of the theoretical results.

Keywords Epidemic model, saturated treatment, stability, bifurcation.

MSC(2010) 34C23, 34D20.

1. Introduction

Recently, a lot of research work in the literature has been devoted to detecting the dynamical behavior of all kinds of epidemic models to prevent and control the spread of the infectious disease. We found that the incidence rate is a key factor in conducting the rich dynamical behaviors in many related literatures [1–3, 6, 9, 11, 14, 16, 18, 20, 21, 24–27, 29, 32, 36–40, 42, 44, 45]. Let $S(t)$ represent the number of susceptible individuals, $I(t)$ represent the number of infected individuals and $R(t)$ be the number of the recovered individuals at time t, respectively. Bilinear incidence rate is common in most epidemic models, i.e., $\beta I(t)S(t)$, where β is the probability

of transmission per contact. Besides, many other types of incidence rate are also adopted recently. The general incidence rate

$$f(I)S = \frac{\beta I^p S}{1 + \alpha I^q} \tag{1.1}$$

was cited extensively in literature and was proposed by Liu et al. [26]. We can see that when $p = 1$ and $\alpha = 0$ or $q = 0$, the incidence rate changes into bilinear incidence rate. Moreover, when $p = q = 2$, Ruan and Wang in [38] studied the global dynamics of an SIRS model with the incidence rate function

$$f(I)S = \frac{\beta I^2 S}{1 + \alpha I^2}, \tag{1.2}$$

and they also exhibited that the epidemic model undergoes a Bogdanov-Takens bifurcation. Yorke and London adopted a special incidence rate in [43], that is,

$$f(I)S = \beta(1 - cI)IS. \tag{1.3}$$

The continuous-time Susceptible-Infected-Recovered-Susceptible (SIRS) epidemic model with standard incidence rate

$$f(I)S = \frac{\beta IS}{S + I + R} \tag{1.4}$$

was studied in [30], where the authors discussed the stability of both the disease-free equilibrium and the endemic equilibrium for the model. To have a better understanding of the dynamics of the system, Wei and Cui [40] explored an SIS epidemic model with standard incidence rate function

$$f(I)S = \frac{\beta IS}{I + S}, \tag{1.5}$$

and they found that the model undergoes rich dynamic behaviors and backward bifurcation.

It is common to see in recent research works [18, 23, 25, 33, 37, 38, 44–46, 48] that the researchers began to add treatment function into the epidemic models to prevent the spread of the infectious diseases. Generally speaking, the treatment function of the infective individuals is always supposed to be proportional to the number of the infective individuals. But the treatment of a disease should have a maximal capacity and the treatment resources should be quite large. So, we should adopt a suitable treatment function for the epidemic disease. Wang and Ruan [38] showed a constant treatment function of diseases in an SIR model as follows

$$T(I) = \begin{cases} r, \ I > 0, \\ 0, \ I = 0. \end{cases} \tag{1.6}$$

In this SIR model, they showed that the model undertakes saddle-node bifurcation, Hopf bifurcation and Bogdanov-Takens bifurcation. Moreover, [37] adopted a new type of treatment function, that is,

$$T(I) = \begin{cases} kI, \ 0 \le I \le I_0, \\ m, \ I > I_0. \end{cases} \tag{1.7}$$

This piecewise linear treatment function means that the treatment rate is proportional to the number of the infective individuals when the treatment capacity has not been reached. With this treatment function, Wang [37] found that a backward bifurcation takes place in an SIR epidemic model. In [13], J.C. Eckalbar and W.L. Eckalbar introduced an SIR epidemic model with a quadratic treatment function, that is,

$$T(I) = \max\{rI - gI^2, 0\}, \quad r, g > 0. \tag{1.8}$$

They found that the model has four equilibria at most, and the system undertakes backward bifurcation and limit cycles under certain conditions.

Now, as seen in [23, 40, 41, 44, 46–48], saturated treatment function is frequently adopted in different models. In [44], Zhang and Liu took a continuous and differentiable saturated treatment function

$$T(I) = \frac{rI}{1 + \alpha I}, \tag{1.9}$$

where $r > 0$, $\alpha \geq 0$. r represents the cure rate and α measures the effect extent of the infected being delayed for treatment. We can realize that the treatment function $T(I) \sim rI$ when I is small enough, whereas $T(I) \sim r/\alpha$ when I is large enough. In [44], the authors found that $R_0 = 1$ is a critical threshold. The disease will be eradicated when the delayed effect is weak. Otherwise, a backward bifurcation will take place. Recently, saturated-type treatment functions have been adopted in all kinds of epidemic models, such as for SIR [44, 46], for SIS [40, 41, 47] as well as for SEIR [23, 48] models and so on. It is well known that in many developing countries the number of patients that need to be treated may exceed the carry capacity of local hospitals because of the restrictions on medical conditions. Hence, saturated treatment function is a suitable choice for this case.

In the real world, some infectious diseases confer temporary immunity. After a period of time, such infections with loss of immunity become susceptible again after infection. This type of disease can be modeled by the SIRS type. To the best of our knowledge, the SIRS epidemic models with different types of incidence rates have been extensively investigated in the literatures about epidemic models [2, 19, 21, 26, 28, 32, 34, 36, 49], etc. But there is no much research about the saturated treatment function.

Motivated by these points, this paper considers the following SIRS epidemic model with standard incidence rate and saturated treatment function.

$$\begin{cases} \frac{dS}{dt} = B - dS - \frac{\lambda IS}{N} + \nu R, \\ \frac{dI}{dt} = \frac{\lambda IS}{N} - (d + r)I - \frac{\beta I}{1 + \alpha I}, \\ \frac{dR}{dt} = rI - (d + \nu)R + \frac{\beta I}{1 + \alpha I}, \end{cases} \tag{1.10}$$

where S and I denotes the number of susceptible and infective, respectively. R denotes the number of removed individuals and B is the rate of recruitment of individuals. $N \equiv S + I + R$ is the total population size, d is the natural death rate and r is the recovery rate, ν is the rate at which recovered individuals lose their immunity (acquired by infection) and return to susceptible class. The standard incidence rate is $\frac{\lambda IS}{N}$, where λ is the probability of infection per contact per unit time. The saturated treatment function $h(I) \triangleq \frac{\beta I}{1 + \alpha I}$, where β is positive and α is nonnegative.

Our study shows that the SIRS model may also exhibit multiple stable equilibria even when the basic reproductive number R_0 is less than unity. In most classical epidemic models, the disease will be eradicated if $R_0 < 1$ and persist if $R_0 > 1$. However, recent work has shown that there are cases for which the necessary condition $R_0 < 1$ is not sufficient to completely remove the disease from the population. Here, we prove that the disease will be eradicated if and only if $R_0 < R_0^* < 1$. This paper focuses on the detailed dynamics analysis of the model (1.10). The stability of the disease-free equilibria and endemic equilibria is investigated. We show that the system exhibits backward bifurcation, Hopf bifurcation and Bogdanov-Takens bifurcation under some conditions.

The organization of this paper is as follows. In Section 2, we discuss the existence of the equilibria and backward bifurcation by reducing the model to a two dimensional system. In Section 3, we investigate the stability analysis of the equilibria. In Section 4, we explore the Hopf bifurcation of system (1.10). In Section 5, we show that Bogdanov-Takens bifurcation happens in system (1.10). In Section 6, we make numerical simulations for the model. The paper ends with a brief discussion of the mathematical results and epidemiological implications in Section 7.

2. Model equilibria

From the model (1.10), we note that the equation for the total population is given by $\frac{dN}{dt} = B - dN$. Since $N \to B/d$ as $t \to \infty$, it follows that at any equilibrium $E^* = (S^*, I^*, R^*)$, $N^* = S^* + I^* + R^* = B/d$, and

$$\Omega = \{(S, I, R) : S, I, R \geq 0, S + I + R = B/d\}$$

is a positively invariant region for the model. Henceforth, we restrict our attention to the dynamics of the model in Ω.

Since Ω is a positively invariant region for the model (1.10), assuming that the size of the population has reached its limiting value, i.e., $N \equiv B/d = S + I + R$, and using $R = B/d - S - I$, we can reduce the model (1.10) to the following model

$$\begin{cases} \frac{dS}{dt} = B - dS - \frac{\lambda d SI}{B} + \nu(B/d - S - I), \\ \frac{dI}{dt} = \frac{\lambda d SI}{B} - (d + r)I - \frac{\beta I}{1 + \alpha I}. \end{cases} \tag{2.1}$$

In the absence of the disease ($I = 0$), the model (2.1) exhibits a unique disease-free equilibrium, given by $E_0 = (B/d, 0)$. The Jacobian matrix of (2.1) at E_0 is

$$M(E_0) = \begin{pmatrix} -d - \nu & -\lambda - \nu \\ 0 & \lambda - d - r - \beta \end{pmatrix}. \tag{2.2}$$

In the following, let us recall how to calculate the basic reproduction number R_0 by using the next generation matrix [10, 12], where the whole population is divided into n compartments in which there are $m < n$ infected compartments. Let $x_i, i = 1, 2, 3, \ldots, m$ be the numbers of infected individuals in the ith infected compartment at time t. Now, the epidemic model is

$$\frac{dx_i}{dt} = \mathcal{F}_i(x) - \mathcal{V}_i(x),$$

where $\mathcal{V}_i(x) = [\mathcal{V}_i^-(x) - \mathcal{V}_i^+(x)]$, $i = 1, 2, \ldots, n$. $\mathcal{F}_i = 0$, if $i > m$.

In the above equations, $\mathcal{F}_i(x)$ represents the rate of appearance of new infections in compartment i. $\mathcal{V}_i^+$ represents the rate of transfer of individuals into compartment i by all other means, and $\mathcal{V}_i^-(x)$ represents the rate of transfer of individuals out of compartment i. The above model can also be written as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathcal{F}(x) - \mathcal{V}(x),$$

where

$$x = \left(x_1, x_2, \ldots, x_n \right)^T,$$

$$\mathcal{F}(x) = \left(\mathcal{F}_1(x), \mathcal{F}_2(x), \ldots, \mathcal{F}_n(x) \right)^T,$$

and

$$\mathcal{V}(x) = \left(\mathcal{V}_1(x), \mathcal{V}_2(x), \ldots, \mathcal{V}_n(x) \right)^T.$$

Let x_0 be the disease-free equilibrium. The values of the Jacobian matrices $F(x)$ and $V(x)$ are

$$D\mathcal{F}(x_0) = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$D\mathcal{V}(x_0) = \begin{pmatrix} V & 0 \\ J_3 & J_4 \end{pmatrix},$$

respectively.

Here, F and V are $m \times m$ matrices, defined as $F = \frac{\partial \mathcal{F}_i}{\partial x_j}(x_0)$ and $V = \frac{\partial \mathcal{V}_i}{\partial x_j}(x_0)$. Now, the matrix FV^{-1} is known as the next-generation matrix. The largest eigenvalue or spectral radius of FV^{-1} is the basic reproduction number of the model, i.e., $R_0 = \rho(FV^{-1})$.

Then, we calculate the basic reproduction number of model (2.1). Firstly, system (2.1) is written as the following model

$$\begin{cases} \frac{dI}{dt} = \frac{\lambda d S I}{B} - (d+r)I - \frac{\beta I}{1+\alpha I}, \\ \frac{dS}{dt} = B - dS - \frac{\lambda d S I}{B} + \nu(B/d - S - I). \end{cases} \quad (2.3)$$

We get

$$\mathcal{F} = \begin{pmatrix} \frac{\lambda d S I}{B} \\ 0 \end{pmatrix}$$

and

$$\mathcal{V} = \begin{pmatrix} (d+r)I + \frac{\beta I}{1+\alpha I} \\ -B + dS + \frac{\lambda d S I}{B} - \nu(B/d - S - I) \end{pmatrix}.$$

The infected compartment are I, giving $m = 1$. In system (2.3), a disease-free equilibrium is $x_0 = (0, B/d)$. Then

$$F = \lambda, \quad V = d + r + \beta,$$

giving

$$V^{-1} = \frac{1}{d+r+\beta}$$

and the basic reproduction number

$$R_0 = \rho(FV^{-1}) = \frac{\lambda}{d+r+\beta}.$$

For convenience, we define

$$R_0^* \triangleq \frac{4B\lambda^2 d\alpha(d+\nu)(d+r+\nu)}{[(d+\nu)B\alpha(d+r-\lambda)+d\lambda(d+r+\nu+\beta)]^2+4\lambda^2 d\alpha(d+\nu)B(d+r+\nu)}. \quad (2.4)$$

Next, we consider all endemic equilibria in system (2.1) and get the following theorem.

Theorem 2.1. *The following results hold.*

(A) *Let $\alpha = 0$. Then system (2.1) has a unique endemic equilibrium when $R_0 > 1$, and has no endemic equilibrium when $R_0 \leq 1$.*

(B) *Let $\alpha > 0$. If $b > 0$, then system (2.1) has a unique endemic equilibrium when $R_0 > 1$, and no endemic equilibrium when $R_0 \leq 1$.*

(C) *Let $\alpha > 0$. If $b < 0$, then system (2.1) has a unique endemic equilibrium when $R_0 \geq 1$, and no endemic equilibrium when $R_0 < R_0^*$, and two endemic equilibria E_1 and E_2 when $R_0^* \leq R_0 < 1$, and $E_1 = E_2$ When $R_0 = R_0^*$.*

Proof. In order to find the endemic equilibrium in the presence of the disease ($I \neq 0$), we consider the model (2.1). An endemic equilibrium always satisfies

$$\begin{cases} B - dS - \frac{\lambda dSI}{B} + \nu(\frac{B}{d} - S - I) = 0, \\ \frac{\lambda dSI}{B} - (d+r)I - \frac{\beta I}{1+\alpha I} = 0. \end{cases} \quad (2.5)$$

Since $I \neq 0$, we can solve $S = \frac{B}{d\lambda}(d+r+\frac{\beta}{1+\alpha I})$ by the second equation of (2.5), then substitute it into the first equation of (2.5), and get

$$B - \frac{(d+\nu)B}{d\lambda}(d+r+\frac{\beta}{1+\alpha I}) - (d+r+\frac{\beta}{1+\alpha I})I + \frac{B\nu}{d} - \nu I = 0.$$

Then we obtain the following equation form

$$aI^2 + bI + c = 0, \quad (2.6)$$

where

$$a = d\lambda\alpha(d+r+\nu),$$
$$b = (d+\nu)B\alpha(d+r-\lambda) + d\lambda(d+r+\nu+\beta),$$
$$c = B(d+\nu)(d+r+\beta-\lambda).$$

This equation may admit positive solution

$$I_1 = \frac{-b-\sqrt{b^2-4ac}}{2a}, \qquad I_2 = \frac{-b+\sqrt{b^2-4ac}}{2a}.$$

Obviously, if $R_0 = 1$ then $c = 0$, if $R_0 > 1$ then $c < 0$, and if $R_0 < 1$ then $c > 0$. From (2.6), It follows that the results (A), (B) and (C) hold.   □

As well known, many different approaches can be used to prove the occurrence of a backward bifurcation for a system. For example, the normal form theory [14] or the Castillo-Chavez and Song method [8] or the qualitative approach proposed by Brauer [7] which is based on the analysis of the equilibria curve in the neighborhood of the critical threshold $R_0 = 1$. In this paper, we use this last method to prove our results. Then, we have the following theorem.

Theorem 2.2. *If $\alpha > 0$, $b < 0$, then system* (2.1) *has a backward bifurcation at* $R_0 = 1$ *(see Figure 1).*

Proof. In order to prove the bifurcation curve (the graph of I as a function of R_0) in Figure 1, we think of β as a variable with the other parameters as constants. Through implicit differentiation of the equation (2.6) with respect to β, we get

$$(2aI + b)\frac{dI}{d\beta} = -Id\lambda - B(d + \nu) < 0. \tag{2.7}$$

From equation (2.7), we know the sign of $\frac{dI}{d\beta}$ is opposite with that of $2aI + b$. And from the definition of R_0 we know that R_0 decreases when β increases. It implies that the bifurcation curve has positive slope at equilibrium values with $2aI + b > 0$, and negative slope at equilibrium values with $2aI + b < 0$. If there is no backward bifurcation at $R_0 = 1$, then the unique endemic equilibrium for $R_0 > 1$ satisfies

$$2aI + b = \sqrt{b^2 - 4ac} > 0,$$

and the bifurcation curve has positive slope at all points where $I > 0$. When $\alpha > 0$, $b < 0$, if there is a backward bifurcation at $R_0 = 1$, then there is an interval on which there are two endemic equilibria given by

$$2aI + b = \pm\sqrt{b^2 - 4ac}.$$

The bifurcation curve has negative slope at the smaller one and positive slope at the larger one. This shows that if $\alpha > 0$, $b < 0$, there is a backward bifurcation as well as a positive equilibrium at $R_0 = 1$.   □

From Figure 1, we can see that there is a critical value R_0^* as a new threshold if a backward bifurcation takes place, that is to say, the disease will die out when $R_0 < R_0^*$. Now, we give a corollary of a backward bifurcation at $R_0 = 1$ as follows. Set

$$\alpha_0 := \frac{d\lambda(d + r + \nu + \beta)}{B\beta(d + \nu)}.$$

Corollary 2.1. *When $\alpha > \alpha_0$, then system* (2.1) *has a backward bifurcation at* $R_0 = 1$.

Proof. When $R_0 = 1$, which is equivalent to

$$\lambda = d + r + \beta. \tag{2.8}$$

From $\alpha > \alpha_0 = \frac{d\lambda(d+r+\nu+\beta)}{B\beta(d+\nu)}$, it follows that $\alpha B\beta(d + \nu) > d\lambda(d + r + \nu + \beta)$. Furthermore, we get

$$\alpha B\beta(d + \nu) + \alpha B(d + \nu)(d + r) > d\lambda(d + r + \nu + \beta) + \alpha B(d + \nu)(d + r).$$

From (2.8), we have

$$(d+\nu)B\alpha(d+r) + d\lambda(d+r+\nu+\beta) < (d+\nu)B\lambda\alpha, \qquad (2.9)$$

i.e., $b < 0$. From Theorem 2.2, it follows that system (2.1) has a backward bifurcation at $R_0 = 1$. Therefore, the proof is complete. $\qquad \square$

3. Stability analysis

By (2.2) and the basic reproduction number $R_0 = \frac{\lambda}{d+r+\beta}$, it is obvious that $M(E_0)$ has negative eigenvalues if $\lambda - d - r - \beta < 0$, i.e., $R_0 < 1$. Then we have the following result.

Theorem 3.1. *The disease-free equilibrium E_0 is locally asymptotically stable when $R_0 < 1$ (see Figure 2), and E_0 is unstable when $R_0 > 1$ (see Figure 3).*

By Figure 1, we know that system (2.1) have a unique endemic equilibrium when $R_0 > 1$, then we consider the local stability of the unique endemic equilibrium when $R_0 > 1$.

Theorem 3.2. *When $R_0 > 1$ and $0 \leq \alpha < \frac{d\lambda}{B\beta}$, the unique endemic equilibrium E^* is locally asymptotically stable.*

Proof. From Theorem 2.1, we know system (2.1) has a unique endemic equilibrium E^* when $R_0 > 1$. Moreover, the Jacobian matrix of system (2.1) is

$$M = \begin{pmatrix} -d - \frac{d\lambda I}{B} - \nu & -\frac{d\lambda S}{B} - \nu \\ \frac{d\lambda I}{B} & \frac{d\lambda S}{B} - (d+r) - \frac{\beta}{(1+\alpha I)^2} \end{pmatrix}. \qquad (3.1)$$

From the second equation of (2.5), we have

$$\frac{d\lambda S}{B} = d + r + \frac{\beta}{1 + \alpha I}. \qquad (3.2)$$

From (3.2), the Jacobian matrix M reduces to

$$M = \begin{pmatrix} -d - \frac{d\lambda I}{B} - \nu & -(d+r+\nu) - \frac{\beta}{1+\alpha I} \\ \frac{d\lambda I}{B} & \frac{\beta}{1+\alpha I} - \frac{\beta}{(1+\alpha I)^2} \end{pmatrix}.$$

We obtain

$$\det(M) = \frac{I}{(1+\alpha I)^2}[\frac{d\lambda}{B}(d+r+\nu)(1+\alpha I)^2 + \frac{d\lambda\beta}{B} - (d+\nu)\alpha\beta],$$

which is positive if and only if

$$\frac{d\lambda}{B}(d+r+\nu)(1+\alpha I)^2 + \frac{d\lambda\beta}{B} > (d+\nu)\alpha\beta. \qquad (3.3)$$

In fact, it holds that $\frac{d\lambda}{B}(d+r+\nu)(1+\alpha I)^2 + \frac{d\lambda\beta}{B} > \frac{d\lambda}{B}(d+r+\nu) + \frac{d\lambda\beta}{B} = \frac{d\lambda}{B}(d+r+\nu+\beta)$. So $\det(M) > 0$ if

$$\frac{d\lambda}{B}(d+r+\nu+\beta) > (d+\nu)\alpha\beta,$$

which is equivalent to

$$\alpha < \frac{d\lambda(d + r + \nu + \beta)}{B\beta(d + \nu)}.$$

The trace of M is given by

$$tr(M) = \frac{1}{(1 + \alpha I)^2}[\alpha\beta I - (d + \nu + \frac{d\lambda I}{B})(1 + \alpha I)^2], \qquad (3.4)$$

which is negative if

$$(d + \nu + \frac{d\lambda I}{B})(1 + \alpha I)^2 > \alpha\beta I.$$

Again, we have

$$(d + \nu + \frac{d\lambda I}{B})(1 + \alpha I)^2 > d + \nu + \frac{d\lambda I}{B} > \frac{d\lambda I}{B}.$$

So only if

$$\frac{d\lambda I}{B} > \alpha\beta I,$$

we can obtain

$$\alpha < \frac{d\lambda}{B\beta}.$$

And also

$$\alpha < \frac{d\lambda}{B\beta} < \frac{d\lambda(d + r + \nu + \beta)}{B\beta(d + \nu)}.$$

The proof is complete.   □

Now we consider the case that there are two endemic equilibria E_1 and E_2. Let M_i be the Jacobian matrix at E_i, $i = 1, 2$.

Theorem 3.3. *When $B\alpha(d + \nu)(d + r - \lambda) + d\lambda(d + r + \nu) > 0$, the endemic equilibrium E_1 is a saddle.*

Proof. Since $I_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $\triangle = b^2 - 4ac$, we have $I_1 = \frac{-b - \sqrt{\triangle}}{2a}$. Thus

$$\det(M_1) = \frac{I_1}{B(1 + \alpha I_1)^2}[d\lambda(d + r + \nu)(1 + \alpha I_1)^2 + d\lambda\beta - B\alpha\beta(d + \nu)]$$

$$\triangleq \frac{I_1}{B(1 + \alpha I_1)^2} \times \psi(I_1).$$

From the existence of E_1, we know that $b < 0$ and $R_0 < 1$, then we obtain

$$\psi(0) < 0.$$

Again, $\psi'(I_1) = 2\alpha d\lambda(d + r + \nu)(1 + \alpha I_1) > 0$, so $\psi(I_1)$ is a monotone increasing function. It follows that there is a unique $I^* > 0$ such that

$$\psi(I_1) = 0, \quad \text{when} \quad I_1 = I^*,$$
$$\psi(I_1) < 0, \quad \text{when} \quad 0 < I_1 < I^*,$$
$$\psi(I_1) > 0, \quad \text{when} \quad I_1 > I^*,$$

where

$$I^* = \frac{1}{\alpha}\sqrt{\frac{B\alpha\beta(d+\nu) - d\lambda\beta}{d\lambda(d+r+\nu)}} - \frac{1}{\alpha}.$$

Besides,

$$I_1 = I^* + \frac{P - \sqrt{\triangle}}{2\alpha d\lambda(d+r+\nu)},$$

where

$$P = d\lambda(d+r+\nu-\beta) - 2\sqrt{[B\alpha\beta(d+\nu) - \lambda\beta]d\lambda(d+r+\nu)}$$
$$+ B\alpha(d+\nu)(\lambda - d - r),$$

and

$$\triangle = [(d+\nu)B\alpha(d+r-\lambda) + d\lambda(d+r+\nu+\beta)]^2$$
$$- 4d\lambda\alpha(d+r+\nu)B(d+\nu)(d+r+\beta-\lambda),$$

After tedious calculations (see Appendix A), we show that $I_1 < I^*$. So we get $\det(M_1) < 0$. Hence the endemic equilibria E_1 is a saddle. The proof is complete.

□

In order to explore the stability of the endemic equilibrium E_2, define

$$m_1 := a^2 C - a\Lambda C - abE + \Lambda b^2, \quad m_2 := a^2 D - acE + bc\Lambda,$$

where Λ, E, C and D are defined in (3.7).

Theorem 3.4. *If $\eta > 0$, then endemic equilibrium E_2 is locally asymptotically stable; if $\eta < 0$, then endemic equilibrium E_2 is unstable, where $\eta := 2am_2 + m_1(\sqrt{b^2 - 4ac} - b)$.*

Proof. Since

$$\begin{aligned}
\det(M_2) &= \frac{I_2}{B(1+\alpha I_2)^2}[d\lambda(d+r+\nu)(1+\alpha I_1)^2 + d\lambda\beta - B\alpha\beta(d+\nu)] \\
&= \frac{I_2}{B(1+\alpha I_2)^2} \times \psi(I_2).
\end{aligned} \tag{3.5}$$

Similar to the arguments of Theorem 3.3, we have $I_2 > I^*$. Therefore, $\det(M_2) > 0$. Now, we only consider the trace of the endemic equilibrium E_2,

$$\begin{aligned}
tr(M_2) &= - d - \nu - \frac{d\lambda I_2}{B} + \frac{\beta}{1+\alpha I_2} - \frac{\beta}{(1+\alpha I_2)^2} \\
&= - \frac{1}{B(1+\alpha I_2)^2}[d\lambda\alpha^2 I_2^3 + (2d\lambda\alpha + B(d+\nu)\alpha^2)I_2^2 \\
&\quad + (2B(d+\nu)\alpha + d\lambda - B\alpha\beta)I_2 - B(d+\nu)] \\
&= - \frac{\Lambda I_2^3 + EI_2^2 + CI_2 + D}{B(1+\alpha I_2)^2},
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
\Lambda &= d\lambda\alpha^2, & E &= 2d\lambda\alpha + B(d+\nu)\alpha^2, \\
C &= 2B(d+\nu)\alpha + d\lambda - B\alpha\beta, & D &= B(d+\nu),
\end{aligned} \tag{3.7}$$

then $sgn(tr(M_2)) = -sgn(G(I_2))$, where

$$G(x) = \Lambda x^3 + Ex^2 + Cx + D.$$

Using the expression of $m_1 = a^2C - a\Lambda C - abE + \Lambda b^2$ and $m_2 = a^2D - acE + bc\Lambda$, one has

$$G(I_2) = (aI_2^2 + bI_2 + c)\varphi_0 + \frac{m_1 I_2 + m_2}{a^2},$$

where $\varphi_0 = \frac{\Lambda I_2}{a} + \frac{m_1 I_2 + m_2}{a^2}$. From $aI_2^2 + bI_2 + c = 0$, it follows that $sgn(tr(M_2)) = -sgn(G(I_2)) = -sgn(m_1 I_2 + m_2)$. From the expression of I_2, we conclude

$$sgn(m_1 I_2 + m_2) = sgn(2am_2 + m_1(\sqrt{b^2 - 4ac} - b)) \triangleq sgn(\eta).$$

Thus, E_2 is locally asymptotically stable if $\eta > 0$ and E_2 is unstable if $\eta < 0$. The proof is complete. $\qquad\square$

Now, we consider the global stability of the disease-free equilibrium E_0. Let $N = S + I + R$ be the total population size. We note that the equation for total population is given by $\frac{dN}{dt} = B - dN$. It follows that $\lim_{t \to +\infty} N(t) = \frac{B}{d}$. Let

$$\Omega = \{(S, I, R) : S, I, R \geq 0, S + I + R = B/d\},$$

which is positively invariant with respect to system (2.1).

Theorem 3.5. *If $R_0 < R_0^*$, the disease-free equilibrium $E_0(\frac{B}{d}, 0)$ is globally asymptotically stable, i.e., the disease dies out.*

Proof. Suppose $R_0 < R_0^*$. From the (H_3) of Theorem 2.2, we know the model has no endemic equilibrium. From the corollary of Poincaré-Bendixson theorem [17], we know there is no periodic orbits in Ω as there is a disease-free equilibrium in Ω. Since Ω is a bounded positively invariant region and E_0 is the only equilibrium in Ω, the local stability of E_0 implies that every solution initiating in Ω approaches E_0. Thus, the disease free equilibrium E_0 is globally asymptotically stable. The proof is complete. $\qquad\square$

Now we analyze the global dynamics of the unique endemic equilibrium when $R_0 > 1$.

Theorem 3.6. *If $R_0 > 1$ and $0 \leq \alpha < \frac{d\lambda}{B\beta}$, then system (2.1) has no limit cycle.*

Proof. We use Dulac theorem to exclude the existence of any limit cycle. Let

$$P(S, I) = B - dS - \frac{d\lambda SI}{B} + \nu(\frac{B}{d} - S - I),$$
$$Q(S, I) = \frac{d\lambda SI}{B} - (d + r)I - \frac{\beta I}{1 + \alpha I},$$

and take the Dulac function

$$D = \frac{1}{I}.$$

From $0 \leq \alpha < \frac{d\lambda}{B\beta}$, it follows that $\alpha\beta < \frac{d\lambda}{B}$. In addition,

$$
\begin{aligned}
\frac{\partial(PD)}{\partial S} + \frac{\partial(QD)}{\partial I} &= -\frac{d}{I} - \frac{d\lambda}{B} - \frac{\nu}{I} + \frac{\alpha\beta}{(1+\alpha I)^2} \\
&= \frac{1}{(1+\alpha I)^2}[-(d+\nu)(1+\alpha I)^2 - \frac{d\lambda}{B}I(1+\alpha I)^2 + \alpha\beta I] \\
&< \frac{1}{(1+\alpha I)^2}[-(d+\nu)(1+\alpha I)^2 - \frac{d\lambda}{B}I(1+\alpha I)^2 + \frac{d\lambda I}{B}] \\
&= \frac{1}{(1+\alpha I)^2}[-(d+\nu)(1+\alpha I)^2 - \frac{d\lambda I}{B}((1+\alpha I)^2 - 1)] \\
&< 0.
\end{aligned}
$$

Hence, the system (2.1) has no limit cycle. The proof is complete.   □

   Therefore, we obtain the global result of the unique endemic equilibrium.

Theorem 3.7. *If $R_0 > 1$ and $0 \leq \alpha < \frac{d\lambda}{B\beta}$, the unique endemic equilibrium E^* is globally asymptotically stable (see Figure 5).*

4. Hopf bifurcation

In this section, we study the Hopf bifurcation of system (2.1). From the above discussion, we know that there is no closed orbit surrounding E_0 or E_1 because the S-axis is invariant with respect to system (2.1) and E_1 is always a saddle. Therefore, Hopf bifurcation can only occur at E_2.

   By the proof of Theorem 3.4, it shows that $tr(M_2) = 0$ if and only if $\eta = 0$, and $det(M_2) > 0$ when E_2 exists. Therefore, the eigenvalues of M_2 are a pair of pure imaginary roots if and only if $\eta = 0$. The direct calculations show that

$$
\frac{d(tr(M_2))}{d\eta}\Big|_{\eta=0} = -\frac{1}{2a^3(1+\alpha I_2)^2 B} < 0.
$$

By [14, Theorem 3.4.2], $\eta = 0$ is the Hopf bifurcation point for system (2.1).

   Now, we consider the equivalent system of (2.1)

$$
\begin{cases}
\frac{dS}{dt} = B(1+\alpha I) - dS(1+\alpha I) - \frac{d\lambda IS(1+\alpha I)}{B} + \nu(\frac{B}{d} - S - I)(1+\alpha I), \\
\frac{dI}{dt} = \frac{d\lambda IS(1+\alpha I)}{B} - (d+r)I(1+\alpha I) - \beta I.
\end{cases}
\tag{4.1}
$$

Let $S = x + S_2$ and $I = y + I_2$, then (4.1) becomes

$$
\begin{cases}
\frac{dx}{dt} = a_{11}x + a_{12}y + c_1 y^2 + c_2 xy - \frac{d\lambda\alpha}{B}xy^2 = a_{11}x + a_{12}y + f_1(x,y), \\
\frac{dy}{dt} = a_{21}x + a_{22}y + c_3 y^2 + c_4 xy + \frac{d\lambda\alpha}{B}xy^2 = a_{21}x + a_{22}y + f_2(x,y),
\end{cases}
\tag{4.2}
$$

where

$$
\begin{aligned}
a_{11} =& -[d + \nu + (d\alpha + \frac{d\lambda}{B} + d\nu)I_2 + \frac{d\lambda\alpha}{B}I_2^2], \\
a_{12} =& B\alpha + \frac{\nu B\alpha}{d} - \nu - (d\alpha + \frac{d\lambda}{B} + \alpha\nu)S_2 - 2\frac{d\lambda\alpha}{B}I_2 S_2 - 2\nu\alpha I_2, \\
c_1 =& -\frac{d\lambda\alpha}{B}S_2 - \nu\alpha, \\
c_2 =& -(d\alpha + \frac{d\lambda}{B} + \nu\alpha + 2\frac{d\lambda\alpha}{B}I_2), \\
a_{21} =& \frac{d\lambda}{B}I_2 + \frac{d\lambda\alpha}{B}I_2{}^2, \\
a_{22} =& \frac{d\lambda}{B}S_2 + 2\frac{d\lambda\alpha}{B}I_2 S_2 - (d + r) - 2\alpha(d + r)I_2 - \beta, \\
c_3 =& \frac{d\lambda\alpha}{B}S_2 - \alpha(d + r), \\
c_4 =& \frac{d\lambda}{B} + \frac{2d\lambda\alpha}{B}I_2.
\end{aligned}
\tag{4.3}
$$

Let $\hat{E}^*$ denote the origin of x-y plane. Since $E_2 = (S_2, I_2)$ satisfies Eq. (2.5), we obtain

$$
\begin{aligned}
\det(M(\hat{E}^*)) =& a_{11}a_{22} - a_{12}a_{21} \\
=& \frac{I_2}{B(1 + \alpha I_2)^2}(1 + \alpha I_2)^2 \times \psi(I_2) \\
=& \frac{I_2}{B} \times \psi(I_2).
\end{aligned}
$$

From the proof of Theorem 3.4, it follows that $\psi(I_2)$ is always positive. It is easy to verify that $a_{11} + a_{22} = 0$ if and only if $\eta = 0$. Set

$$
D = \sqrt{\det(M(\hat{E}^*))},
$$

then

$$
D^2 = a_{11}a_{22} - a_{12}a_{21}.
$$

Let $X = x$ and $Y = a_{11}x + a_{12}y$, then system (2.1) becomes

$$
\begin{cases}
\frac{dX}{dt} = Y + f_1(X, \frac{Y - a_{11}X}{a_{12}}), \\
\frac{dv}{dt} = -D^2 X + a_{11}f_1(X, \frac{Y - a_{11}X}{a_{12}}) + a_{12}f_2(X, \frac{Y - a_{11}X}{a_{12}}).
\end{cases}
\tag{4.4}
$$

Again, set $u = -X$ and $v = Y/D$, then system (4.4) becomes

$$
\begin{cases}
\frac{du}{dt} = -Dv + F_1(u, v), \\
\frac{dv}{dt} = Du + F_2(u, v),
\end{cases}
\tag{4.5}
$$

where

$$F_1(u,v) = -f_1(-u, \frac{Dv+a_{11}u}{a_{12}}),$$

$$= \frac{c_2 u}{a_{12}}(Dv+a_{11}u) - (Dv+a_{11}u)^2(\frac{c_1}{a_{12}^2} + \frac{ud\lambda\alpha}{Ba_{12}^2}),$$

$$F_2(u,v) = \frac{a_{11}}{D}f_1(-u, \frac{Dv+a_{11}u}{a_{12}}) + \frac{a_{12}}{D}f_2(-u, \frac{Dv+a_{11}u}{a_{12}})$$

$$= (Dv+a_{11}u)^2(\frac{a_{11}c_1}{Da_{12}^2} + \frac{a_{11}ud\lambda\alpha}{BDa_{12}^2}) + \frac{c_3}{Da_{12}} - \frac{d\lambda\alpha u}{BDa_{12}}$$

$$- (Dv+a_{11}u)(\frac{a_{11}c_2 u}{Da_{12}} + \frac{c_4 u}{D}).$$

Set

$$\sigma = \frac{1}{16}[F_1uuu+F_1uvv+F_2uuv+F_2vvv]+\frac{1}{16D}[F_1uv(F_1uu+F_1vv)-F_2uv(F_2uu$$
$$+ F_2vv) - F_1uuF_2uu + F_1vvF_2vv],$$

where F_1uv denotes $(\partial^2 F_1/\partial u\partial v)(0,0)$, F_2uv denotes $(\partial^2 F_2/\partial u\partial v)(0,0)$, etc. Then

$$\sigma = \frac{1}{8B^2D^3a_{12}^4}[BD^3a_{12}^2d\lambda\alpha(-a_{11}^2-D^2-2a_{11}a_{12})+D^2(BDc_2a_{12}-2a_{11}BDc_1$$
$$- 4Da_{11}ud\lambda\alpha)(Ba_{11}a_{12}c_2-Ba_{11}^2c_1-3a_{11}^2ud\lambda\alpha-2a_{11}Dvd\lambda\alpha - BD^2c_1$$
$$- D^2du\lambda\alpha)-(2a_{11}D(Ba_{11}c_1+a_{11}ud\lambda\alpha+Ba_{12}c_3-a_{12}du\lambda\alpha)$$
$$+ 2D(Dv+a_{11}u)(a_{11}-a_{12})d\lambda\alpha-BDa_{12}(a_{11}c_2+a_{12}c_4))(a_{11}^2(Ba_{11}c_1$$
$$+ a_{11}ud\lambda\alpha+Bc_3a_{12}-a_{12}ud\lambda\alpha)+2a_{11}(Dv+a_{11}u)(a_{11} - a_{12})d\lambda\alpha$$
$$+ D^2(Ba_{11}c_1+a_{11}ud\lambda\alpha+Ba_{12}c_3-a_{12}du\lambda\alpha))-D(2Bc_2a_{11}a_{12}-2Ba_{11}^2c_1$$
$$- 6a_{11}^2ud\lambda\alpha-4a_{11}Dvd\lambda\alpha)(a_{11}^2(Ba_{11}c_1+a_{11}ud\lambda\alpha+Bc_3a_{12}-a_{12}ud\lambda\alpha)$$
$$+ 2a_{11}(Dv+a_{11}u)(a_{11}-a_{12})d\lambda\alpha)-2D^5(Bc_1+ud\lambda\alpha)(Ba_{11}c_1+a_{11}ud\lambda\alpha$$
$$+ Ba_{12}c_3-a_{12}du\lambda\alpha)].$$

By [14, Theorem 3.4.2 and (3.4.11)], we have the following Theorem.

Theorem 4.1. *System* (2.1) *undergoes a Hopf bifurcation if $\eta = 0$. Moreover, if $\sigma \neq 0$, then a curve of periodic solutions bifurcates from the endemic equilibrium E_2 such that*

(a) *for $\sigma < 0$, then the model undergoes a supercritical Hopf bifurcation.*

(b) *for $\sigma > 0$, then the model undergoes a subcritical Hopf bifurcation.*

5. Bogdanov-Takens bifurcations

The Bogdanov-Takens bifurcation is a bifurcation of an equilibrium point in a two-parameter family of autonomous ODEs at which the critical equilibrium has a zero eigenvalue of (algebraic) mulitplicity two. For nearby parameter values, the system has two equilibria (a saddle and a nonsaddle) which collide and disappear via a saddle-node bifurcation. The nonsaddle equilibrium undergoes an Andronov-Hopf

bifurcation generating a limit cycle. This cycle degenerates into an orbit homoclinic to the saddle and disappears via a saddle homoclinic bifurcation [15].

The Bogdanov-Takens bifurcation (for short, BT bifurcation) is a type of codimension-2 bifurcation that emerges when (2.1) admits a unique degenerate equilibrium. Assume the following two assumptions hold.

(H_1) $b < 0$ and $b^2 - 4ac = 0$.

Then (2.5) admits a unique positive equilibrium $\bar{E}^* = (S^*, I^*)$, where

$$I^* = -\frac{b}{2a}, \quad S^* = \frac{B}{d\lambda}(d + r + \frac{\beta}{1 + \alpha I^*}). \tag{5.1}$$

The Jacobian matrix of system (2.1) at $\bar{E}^*$ is

$$M^* = \begin{pmatrix} -d - \frac{d\lambda I^*}{B} - \nu & -\frac{\beta}{(1+\alpha I^*)} - (d+r+\nu) \\ \frac{d\lambda I^*}{B} & \frac{\beta}{1+\alpha I^*} - \frac{\beta}{(1+\alpha I^*)^2} \end{pmatrix}. \tag{5.2}$$

Since we are interested in codimension 2 bifurcations, we assume further

(H_2) $(Bd\lambda\alpha r + d^2\lambda^2)b = 2aB(d + \nu)[d\lambda - B\alpha(d + \nu)]$.

By (5.2), we have

$$\begin{aligned} \det(M^*) =& \frac{I^*}{B(1 + \alpha I^*)^2}[d\lambda(d + r + \nu)(1 + \alpha I^*)^2 + d\lambda\beta - B\alpha\beta(d + \nu)] \\ =& 0, \end{aligned} \tag{5.3}$$

where the last equality follows from

$$\begin{aligned} (1 + \alpha I^*)^2 =& \frac{4a^2 - 4a\alpha b + \alpha^2 b^2}{4a^2} \\ =& \frac{4a^2 - 4a\alpha b + \alpha^2 4ac}{4a^2} \\ =& \frac{a - \alpha b + \alpha^2 c}{a} \\ =& \frac{B\alpha^2(d + \nu)\beta - d\lambda\alpha\beta}{a}. \end{aligned} \tag{5.4}$$

Furthermore, (H_2) implies that

$$tr(M^*) = 0. \tag{5.5}$$

Thus, (H_1) and (H_2) imply that the Jacobian matrix has a zero eigenvalue with multiplicity 2. This suggests that (2.1) may admit a Bogdanov-Takens singularity.

Theorem 5.1. *Suppose that (H_1), (H_2), $2b_1 + b_4 \neq 0$ and $b_3 \neq 0$ hold. Then the endemic equilibrium $\bar{E}^* = (S^*, I^*)$ of (2.1) is a cusp of codimension 2, i.e., it is a Bogdanov-Takens singularity.*

Proof. Using the transformation of $x = I - I^*$ and $y = S - S^*$, system (2.1) becomes

$$\begin{cases} \frac{dx}{dt} = a_1 x + a_2 y + \hat{a}_{21}xy + \hat{a}_{11}x^2 + P_1(x), \\ \frac{dy}{dt} = -\frac{a_1^2}{a_2}x - a_1 y - \hat{a}_{21}xy, \end{cases} \tag{5.6}$$

where $P_1(x)$ are smooth functions in x at least of the third order and

$$
\begin{aligned}
a_1 &= \frac{d\lambda}{B}S^* - d - r - \frac{\beta}{(1+\alpha I^*)^2} > 0, \\
a_2 &= \frac{d\lambda}{B}I^* > 0, \\
\hat{a}_{21} &= \frac{d\lambda}{B} > 0, \\
\hat{a}_{11} &= \frac{\alpha\beta}{(1+\alpha I^*)^3} > 0.
\end{aligned}
\tag{5.7}
$$

Set $X = x, Y = a_1 x + a_2 y$. Then (5.6) is transformed into

$$
\begin{cases}
\frac{dX}{dt} = Y + b_1 X^2 + b_2 XY + P_1(X), \\
\frac{dY}{dt} = b_3 X^2 + b_4 XY + P_2(X),
\end{cases}
\tag{5.8}
$$

where $P_i(X)$ are smooth functions in X at least of the third order and

$$
\begin{aligned}
b_1 &= \hat{a}_{11} - \frac{\hat{a}_{21}a_1}{a_2}, \\
b_2 &= \frac{\hat{a}_{21}}{a_2}, \\
b_3 &= a_1 \hat{a}_{11} - \frac{\hat{a}_{21}a_1^2}{a_2} + a_1 \hat{a}_{21}, \\
b_4 &= \frac{\hat{a}_{21}a_1}{a_2} - \hat{a}_{21}.
\end{aligned}
\tag{5.9}
$$

In order to obtain the canonical normal form, we use the following transformation of variables

$$
u = X - \frac{b_2}{2}X^2, v = Y + b_1 X^2.
\tag{5.10}
$$

Then, we obtain

$$
\begin{cases}
\frac{du}{dt} = v + R_1(u), \\
\frac{dv}{dt} = b_3 u^2 + (2b_1 + b_4)uv + R_2(u),
\end{cases}
\tag{5.11}
$$

where $R_i(u)$ are smooth functions of u at least of the third order.

Note that $b_3 \neq 0$ and $2b_1 + b_4 \neq 0$. It follows from [4, 5, 35] that (2.1) admits a Bogdanov-Takens bifurcation.   □

In the following, we will find the versal unfolding in terms of the original parameters in (2.1). In this way, we will know the approximate saddle-node, Hopf and homoclinic bifurcation curves. We choose r and ν as bifurcation parameters. Fix $B = B_0$, $d = d_0$, $\lambda = \lambda_0$, $\beta = \beta_0$ and $\alpha = \alpha_0$. Let $r = r_0 + \theta_1$ and $\nu = \nu_0 + \theta_2$, where θ_1 and θ_2 are parameters which vary in a small neighborhood of the origin.

Suppose that $B = B_0$, $d = d_0$, $\lambda = \lambda_0$, $\nu = \nu_0$, $r = r_0$, $\alpha = \alpha_0$ and $\beta = \beta_0$ satisfy (H_1) and (H_2). Consider the following system

$$
\begin{cases}
\frac{dS}{dt} = B_0 - d_0 S - \frac{\lambda_0 d_0 SI}{B} + (\nu_0 + \theta_2)(B_0/d_0 - S - I), \\
\frac{dI}{dt} = \frac{\lambda_0 d_0 SI}{B_0} - (d_0 + r_0 + \theta_1)I - \frac{\beta_0 I}{1 + \alpha_0 I}.
\end{cases}
\tag{5.12}
$$

By the transformations of $x = I - I^*$ and $y = S - S^*$, system (5.12) becomes

$$\begin{cases} \frac{dx}{dt} = -\theta_1 I^* + \hat{c}_1 x + \hat{c}_2 y + c_{11} x^2 + c_{12} xy + w_1(x), \\ \frac{dy}{dt} = (\frac{B_0}{d_0}\theta_2 - \theta_2 S^* - \theta_2 I^*) + \hat{c}_3 x + \hat{c}_4 y - c_{12} xy, \end{cases} \tag{5.13}$$

where $\theta = (\theta_1, \theta_2)$, $w_1(x)$ is a smooth function of x at least of the third order and

$$\hat{c}_1 = \frac{d_0}{B_0}\lambda_0 S^* - d_0 - r_0 - \theta_1 - \frac{\beta_0}{(1 + \alpha_0 I^*)^2}, \quad \hat{c}_2 = \frac{d_0}{B_0}\lambda_0 I^*,$$

$$\hat{c}_3 = -\frac{d_0}{B_0}\lambda_0 S^* - (\nu_0 + \theta_2), \quad \hat{c}_4 = -d_0 - \frac{d_0}{B_0}\lambda_0 I^* - (\nu_0 + \theta_2), \tag{5.14}$$

$$c_{11} = \frac{\alpha_0 \beta_0}{(1 + \alpha_0 I^*)^3}, \quad c_{12} = \frac{d_0}{B_0}\lambda_0.$$

Using the change of variables $X = x$, $Y = -\theta_1 I^* + \hat{c}_1 x + \hat{c}_2 y + c_{11} x^2 + c_{12} xy + w_1(x)$ and rewriting X, Y as x and y, respectively, we have

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = e_0 + e_1 x + e_2 y + e_{11} x^2 + e_{12} xy + e_{22} y^2 + w_2(x, y, \theta), \end{cases} \tag{5.15}$$

where $\theta = (\theta_1, \theta_2)$, $w_2(x, y, \theta)$ is a smooth function of x, y, θ_1 and θ_2 at least of the third order and

$$e_0 = \hat{c}_2(\frac{B_0}{d_0}\theta_2 - \theta_2 S^* - \theta_2 I^*) + \theta_1 \hat{c}_4 I^*,$$

$$e_1 = c_{12}(\frac{B_0}{d_0}\theta_2 - \theta_2 S^* - \theta_2 I^*) + \hat{c}_2 \hat{c}_3 - \hat{c}_1 \hat{c}_4 - c_{12}\theta_1 I^*,$$

$$e_2 = \hat{c}_1 + \hat{c}_4 + \frac{c_{12}\theta_1}{\hat{c}_2}I^*,$$

$$e_{11} = c_{12}\hat{c}_3 - \hat{c}_4 c_{11} + \hat{c}_1 c_{12}, \tag{5.16}$$

$$e_{12} = -c_{12} + 2c_{11} - \frac{c_{12}\hat{c}_1}{\hat{c}_2} - \frac{c_{12}^2 \theta_1 I^*}{\hat{c}_2^2},$$

$$e_{22} = \frac{c_{12}}{\hat{c}_2}.$$

Next, we introduce a new time variable τ by $dt = (1 - \frac{c_{12}}{\hat{c}_2}x)d\tau$. Rewriting τ as t, we obtain

$$\begin{cases} \frac{dx}{dt} = y(1 - \frac{c_{12}}{\hat{c}_2}x), \\ \frac{dy}{dt} = (1 - \frac{c_{12}}{\hat{c}_2}x)(e_0 + e_1 x + e_2 y + e_{11} x^2 + e_{12} xy + e_{22} y^2 + w_2(x, y, \theta)). \end{cases} \tag{5.17}$$

Let $X = x$, $Y = y(1 - \frac{c_{12}}{\hat{c}_2}x)$ and rename X and Y as x and y, we have

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = e_0 + f_1 x + e_2 y + f_{11} x^2 + f_{12} xy + w_3(x, y, \theta), \end{cases} \tag{5.18}$$

where $\theta = (\theta_1, \theta_2)$, $w_3(x, y, \theta)$ is a smooth function of x, y, θ_1 and θ_2 at least of the third order and

$$f_1 = -2e_0 \frac{c_{12}}{\hat{c}_2} + e_1,$$

$$f_{11} = e_{11} - 2 \frac{e_1 c_{12}}{\hat{c}_2} + \frac{e_0 c_{12}^2}{\hat{c}_2^2}, \tag{5.19}$$

$$f_{12} = e_{12} - \frac{e_2 c_{12}}{\hat{c}_2}.$$

Now, we assume that $f_{11} \neq 0$ and $f_{12} \neq 0$ when θ_i are small. Set $x = X - \frac{e_2}{f_{12}}$ and rewrite X as x, we can get

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = g_0 + g_1 x + f_{11} x^2 + f_{12} xy + w_4(x, y, \theta), \end{cases} \tag{5.20}$$

where $\theta = (\theta_1, \theta_2)$, $w_4(x, y, \theta)$ is a smooth function of x, y, θ_1 and θ_2 at least of the third order and

$$g_0 = e_0 - \frac{f_1 e_2}{f_{12}} + \frac{f_{11} e_2^2}{f_{12}^2},$$

$$g_1 = f_1 - \frac{2 f_{11} e_2}{f_{12}}, \tag{5.21}$$

Note that $f_{11} \neq 0$ and $f_{12} \neq 0$ when θ_i are small. Making the final change of variables by $X = \frac{f_{12}^2 x}{f_{11}}$, $Y = \frac{f_{12}^3 y}{f_{11}^2}$ and $\tau = \frac{f_{11} t}{f_{12}}$, then denoting them again by x, y and t, respectively, we obtain

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = \tau_1(\theta_1, \theta_2) + \tau_2(\theta_1, \theta_2) x + x^2 + xy + w_5(x, y, \theta), \end{cases} \tag{5.22}$$

where $\tau_1 = \frac{g_0 f_{12}^4}{f_{11}^3}$, $\tau_2 = \frac{g_1 f_{12}^2}{f_{11}^2}$, $\theta = (\theta_1, \theta_2)$, $w_5(x, y, \theta)$ is a smooth function of x, y, θ_1 and θ_2 at least of the third order.

By the theorems in Bogdanov [4,5] and Takens [35] or Kuznetsov [22], we obtain the following local representations of the bifurcation curves in a small neighborhood of the origin.

Theorem 5.2. *Suppose that* $(B, d, \lambda, \beta, r, \nu, \alpha) = (B_0, d_0, \lambda_0, \beta_0, r_0, \nu_0, \alpha_0)$ *satisfy* (H_1), (H_2), $f_{11} \neq 0$ *and* $f_{12} \neq 0$ *when* θ_i *are small. Then* (2.1) *admits the following bifurcation behavior.*

(1) *There is a saddle-node bifurcation curve* $SN = \{(\theta_1, \theta_2) : 4g_0 f_{11} = g_1^2 + o(|(\theta_1, \theta_2)|^2), g_1 \neq 0\}$.

(2) *There is a Hopf bifurcation curve* $H = \{(\theta_1, \theta_2) : g_0 + o(|(\theta_1, \theta_2)|^2) = 0, g_1 < 0\}$.

(3) *There is a homoclinic bifurcation curve* $HL = \{(\theta_1, \theta_2) : 25 f_{11} g_0 + 6g_1^2 = o(|(\theta_1, \theta_2)|^2), g_1 < 0\}$.

6. Numerical Simulations

In this section, using the software package MATLAB [50], the relationship between the proportion of infectious individuals and the basic reproduction number R_0 is illustrated in Figure 1. This figure shows that the model has a unique endemic equilibrium for $R_0 = R_0^*$, then the model has two endemic equilibria for $R_0^* < R_0 < 1$ and a unique endemic equilibrium for $R_0 \geq 1$. Furthermore, using PPLANE8 [31], some numerical simulations of system (2.1) are depicted in Figure 2–Figure 5. A disease free equilibrium exists for all parameters and is locally asymptotically stable when $R_0 < 1$ (see Figure 2) and unstable when $R_0 > 1$ (see Figure 3). From Theorem 2.1 (C), it follows that when $R_0^* \leq R_0 < 1$, the two endemic equilibria E_1 and E_2 will exist, and the stable manifolds of the saddle E_1 split R_+^2 into two regions. The disease is persistent in the upper region and dies out in the lower region (see Figure 4). By Theorem 3.7, when $R_0 > 1$ and the effect of the infected being delayed for treatment (i.e., $0 \leq \alpha < \frac{d\lambda}{B\beta}$) is controlled to some degree, system (2.1) has no periodic solutions, and all orbits approach to the unique endemic equilibrium E^* as time goes to infinity. Then the disease persists (see Figure 5).
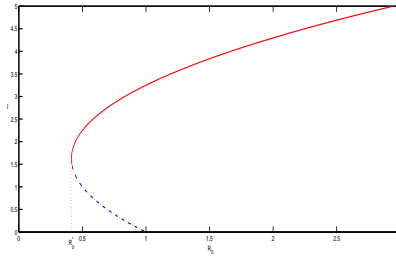

Figure 1. The bifurcation from the disease free equilibrium at $R_0 = 1$ is backward when $d = 0.1, r = 0.1, \lambda = 0.5, \alpha = 1, \nu = 0.2, B = 1, \beta = 0.1$, where $b = -0.065$.
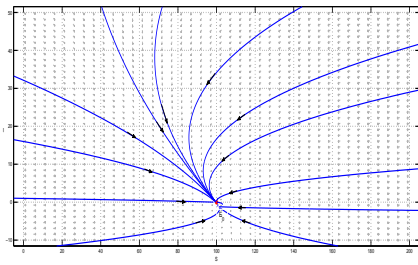

Figure 2. The disease-free equilibrium E_0 is locally asymptotically stable when $R_0 < 1$, with the parameter values $B = 2$, $d = 0.02$, $\lambda = 0.01$, $r = 0.04$, $\nu = 0.1$, $\beta = 0.08$, $\alpha = 2$.
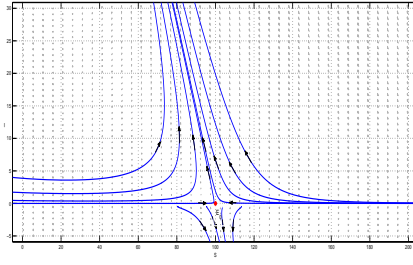
Figure 3. The disease-free equilibrium is unstable when $R_0 > 1$, with the parameter values $B = 2$, $d = 0.02$, $\lambda = 0.3$, $r = 0.04$, $\nu = 0.1$, $\beta = 0.08$, $\alpha = 2$.

7. Discussion

In this paper, by combining qualitative and bifurcation analysis we have studied the global behavior of an SIRS epidemic model with standard incidence rate and saturated treatment function. Previous studies of analogous models with the treatment
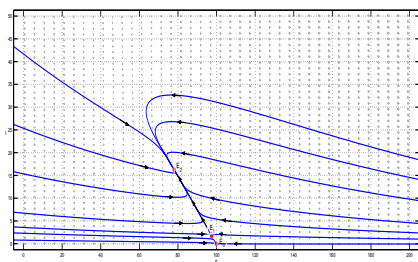
Figure 4. One region of disease persistence and the other region of disease extinction when $B = 2$, $d = 0.1$, $\lambda = 0.1$, $\beta = 0.08$, $\nu = 0.1$, $\alpha = 2$, $r = 0.04$.
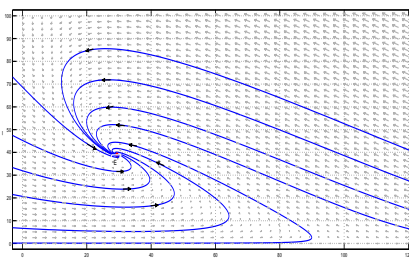

Figure 5. The unique endemic equilibrium E^* is globally asymptotically stable when $B = 2$, $d = 0.02$, $\lambda = 0.4$, $\beta = 0.08$, $\nu = 0.1$, $\alpha = 0.01$, $r = 0.04$.

function in [18, 23, 25, 33, 37, 38, 44–46, 48] mainly focused on simulations and only obtained the existence and stability of equilibria, backward bifurcation and Hopf bifurcation. Bogdanov-Takens bifurcation was rarely considered in the literatures except for [41]. Especially, in this paper the standard incidence rate $\frac{\lambda IS}{N}$ when $N = 1$ reduces to the bilinear incidence rate in [41]. In this paper, in order to analyze dynamics of the model (1.10), we need to reduce the three dimensional system to a two dimensional system and use more delicate computations and analysis.

In addition, in terms of the basic reproduction number $R_0 = \frac{\lambda}{d+r+\beta}$, our main results indicate that when $R_0 < R_0^*$, the disease-free equilibrium is globally asymptotically stable (see Theorem 3.5). Biologically, this indicates that the probability of infection (λ) is small enough and removal rate (death rate (d) and recovery rate (r) plus the cure rate (β)) is large enough such that $R_0 < R_0^*$, then the disease dies out. The aggressive control measures and policies, such as isolation, mask screening and improving medical level, etc., helped in reducing the infection rate and increasing the removal rate. Moreover, our results also suggest that the effect of the infected being delayed for treatment (α) to some degree can lead to a backward bifurcation (see Corollary 2.1). Therefore, in order to remove the backward bifurcation and control the disease, we should reduce the parameter α in a low range, that is to say, we should give the patients timely treatment such as improving medical facilities and quality of medical care and adding more medical professionals, etc.

We also study the Hopf bifurcation and obtain the criteria to judge its stability. Under some suitable conditions, when a stable limit cycle surrounds the endemic equilibrium, it means that the number of the infective tends to a periodic function and the disease will exhibit frequently regular oscillation. Hence, the disease become periodic outbreak as time evolves.

Finally, by Theorem 5.2, we present approximate expressions for saddle-node, Hopf and homoclinic bifurcation sets near the Bogdanov-Takens bifurcation points.

Acknowledgments

Appendix A

For the reader's convenience, in this section, we give the detailed calculations in the proof of Theorem 3.3.

Proof.   Since $I_1 = \frac{-b-\sqrt{b^2-4ac}}{2a}$ and $\triangle = b^2 - 4ac$, we have $I_1 = \frac{-b-\sqrt{\triangle}}{2a}$. Thus

$$
\begin{aligned}
\det(M_1) &= \frac{I_1}{B(1+\alpha I_1)^2}[d\lambda(d+r+\nu)(1+\alpha I_1)^2 + d\lambda\beta - B\alpha\beta(d+\nu)] \\
&\triangleq \frac{I_1}{B(1+\alpha I_1)^2} \times \psi(I_1).
\end{aligned}
$$

From the existence of E_1, we conclude that $b < 0$ and $R_0 < 1$, and obtain

$$
\begin{aligned}
\psi(0) =&d\lambda(d+r+\nu) + d\lambda\beta - B\alpha\beta(d+\nu) \\
=&d\lambda(d+r+\nu+\beta) - B\alpha\beta(d+\nu) \\
<&B\alpha(d{+}\nu)(\lambda - d - r){-}B\alpha\beta(d+\nu) \\
=&B\alpha(d+\nu)(\lambda - d - r - \beta) \\
=&B\alpha(d+\nu)(d+r+\beta)(R_0 - 1) \\
<&0.
\end{aligned} \tag{7.1}
$$

Again, $\psi'(I_1) = 2\alpha d\lambda(d+r+\nu)(1+\alpha I_1) > 0$, so $\psi(I_1)$ is a monotone increasing function. It follows that there is a unique $I^* > 0$ such that

$$
\begin{aligned}
\psi(I_1) &= 0, \quad \text{when} \quad I_1 = I^*, \\
\psi(I_1) &< 0, \quad \text{when} \quad 0 < I_1 < I^*, \\
\psi(I_1) &> 0, \quad \text{when} \quad I_1 > I^*,
\end{aligned}
$$

where

$$
I^* = \frac{1}{\alpha}\sqrt{\frac{B\alpha\beta(d+\nu) - d\lambda\beta}{d\lambda(d+r+\nu)}} - \frac{1}{\alpha}. \tag{7.2}
$$

Moreover,

$$
\begin{aligned}
I_1 =& -\frac{b}{2a} - \frac{\sqrt{\triangle}}{2a} \\
=&\frac{(d+\nu)B\alpha(\lambda - d - r) - d\lambda(d+r+\nu+\beta)}{2d\lambda\alpha(d+r+\nu)} - \frac{\sqrt{\triangle}}{2d\lambda\alpha(d+r+\nu)} \\
=&\frac{1}{\alpha}\sqrt{\frac{B\alpha\beta(d+\nu) - d\lambda\beta}{d\lambda(d+r+\nu)}} - \frac{1}{\alpha} + \frac{1}{\alpha} - \frac{1}{\alpha}\sqrt{\frac{B\alpha\beta(d+\nu) - d\lambda\beta}{d\lambda(d+r+\nu)}} \\
&+ \frac{(d+\nu)B\alpha(\lambda - d - r) - d\lambda(d+r+\nu+\beta)}{2d\lambda\alpha(d+r+\nu)} - \frac{\sqrt{\triangle}}{2d\lambda\alpha(d+r+\nu)} \\
=&I^* + \frac{P - \sqrt{\triangle}}{2\alpha d\lambda(d+r+\nu)},
\end{aligned} \tag{7.3}
$$

where

$$
\begin{aligned}
P =& d\lambda(d+r+\nu-\beta) - 2\sqrt{[B\alpha\beta(d+\nu) - \lambda\beta]d\lambda(d+r+\nu)} \\
&+ B\alpha(d+\nu)(\lambda - d - r),
\end{aligned} \tag{7.4}
$$

$$\triangle = [(d+\nu)B\alpha(d+r-\lambda) + d\lambda(d+r+\nu+\beta)]^2 \\ - 4d\lambda\alpha(d+r+\nu)B(d+\nu)(d+r+\beta-\lambda), \tag{7.5}$$

and

$$P^2 = d^2\lambda^2(d+r+\nu-\beta)^2 + [B\alpha(d+\nu)(\lambda-d-r)]^2 \\ + 4[B\alpha\beta(d+\nu)-d\lambda\beta]d\lambda(d+r+\nu)+2B\alpha d\lambda(d+\nu)(d+r+\nu-\beta)(\lambda-d-r) \\ - 4d\lambda(d+r+\nu-\beta)\sqrt{[B\alpha\beta(d+\nu)-d\lambda\beta]d\lambda(d+r+\nu)} \\ - 4B\alpha(d+\nu)(\lambda-d-r)\sqrt{[B\alpha\beta(d+\nu)-d\lambda\beta]d\lambda(d+r+\nu)}. \tag{7.6}$$

So

$$\triangle - P^2 = 4\sqrt{[B\alpha\beta(d+\nu)-d\lambda\beta]d\lambda(d+r+\nu)}[d\lambda(d+r+\nu-\beta)+B\alpha(d+\nu)(\lambda-d-r)] \\ - 8[B\alpha\beta(d+\nu)-d\lambda\beta]d\lambda(d+r+\nu). \tag{7.7}$$

In the following, we will show that

$$4\sqrt{[B\alpha\beta(d+\nu)-d\lambda\beta]d\lambda(d+r+\nu)}[d\lambda(d+r+\nu-\beta)+B\alpha(d+\nu)(\lambda-d-r)] \\ > 8[B\alpha\beta(d+\nu)-d\lambda\beta]d\lambda(d+r+\nu), \tag{7.8}$$

i.e.,

$$[d\lambda(d+r+\nu-\beta)+B\alpha(d+\nu)(\lambda-d-r)]^2 \\ > 4[B\alpha\beta(d+\nu)-d\lambda\beta]d\lambda(d+r+\nu). \tag{7.9}$$

Since $R_0^* < R_0$, it yields that $b^2 > 4ac$, i.e., $[B\alpha(d+\nu)(d+r-\lambda)+d\lambda(d+r+\nu-\beta)]^2 > 4d\lambda B\alpha(d+\nu)(d+r+\beta-\lambda)(d+r+\nu)$, so it follows that

$$[d\lambda(d+r+\nu-\beta)+B\alpha(d+\nu)(\lambda-d-r)]^2 \\ = [B\alpha(d+\nu)(d+r-\lambda)+d\lambda(d+r+\nu-\beta)]^2+4B\alpha(d+\nu)d\lambda(d+r+\nu-\beta)(\lambda-d-r) \\ > 4B\alpha d\lambda(d+\nu)(d+r+\nu)(d+r+\beta-\lambda)+4B\alpha d\lambda(d+\nu)(d+r+\nu-\beta)(\lambda-d-r) \\ = 4B\alpha d\lambda(d+\nu)\beta(2d+2r+\nu-\lambda) \\ = 4B\alpha d\lambda\beta(d+\nu)(d+r+\nu) + 4B\alpha d\lambda\beta(d+r-\lambda). \tag{7.10}$$

Then

$$4B\alpha d\lambda\beta(d+\nu)(d+r+\nu)+4B\alpha d\lambda\beta(d+r-\lambda)-4[B\alpha\beta(d+\nu)-d\lambda\beta]d\lambda(d+r+\nu) \\ = 4d\lambda\beta[B\alpha(d+\nu)(d+r-\lambda)+d\lambda(d+r+\nu)] > 0. \tag{7.11}$$

From (7.3), we know that $I_1 < I^*$. So we get $det(M_1) < 0$. Hence the endemic equilibria E_1 is a saddle. The proof is complete. $\square$

References

[1] M. E. Alexander and S. M. Moghadas, *Periodicity in an epidemic model with a generalized nonlinear incidence*, Math. Biosic., 2004, 189, 75–96.

[2] M. E. Alexander and S. M. Moghadas, *Bifurcation analysis of an SIRS epidemic model with generalized incidence*, SIAM J. Appl. Math., 2005, 65, 1794–1816.

[3] J. Arino, C. C. McCluskey and P. van den Driessche, *Global results for an epidemic model with vaccination that exhibits backward bifurcation*, SIAM J. Appl. Math., 2003, 64, 260–276.

[4] R. Bogdanov, *Bifurcations of a limit cycle for a family of vector fields on the plan*, Selecta Math. Sov., 1981, 1, 373–388.

[5] R. Bogdanov, *Versal deformations of a singular point on the plan in the case of zero eigen-values*, Selecta Math. Sov., 1981, 1, 389–421.

[6] F. Brauer, *Backward bifurcation in simple vaccination models*, J. Math. Anal. Appl., 2004, 289, 418–431.

[7] F. Brauer, *Backward bifurcations in simple vaccination/treatment models*, J. Biol. Dyn., 2011, 5(5), 410–418.

[8] C. Castillo-Chavez and B. J. Song, *Dynamical models of tuberculosis and their applications*, Math. Biosci. Eng., 2004, 1(2), 361–404.

[9] W. R. Derrick and P. van den Driessche, *Homoclinic orbits in a disease transmission model with nonlinear incidence and nonconsant population*, Discrete Contin. Dyn. Syst. Ser. B., 2003, 3, 299–309.

[10] O. Diekmann, J. A. P. Heesterbeek and J.A.J. Metz, *On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations*, J. Math. Biol., 1990, 28, 365–382.

[11] P. van den Driessche and J. Watmough, *A simple SIS epidemic model with a backward bifurcation*, J. Math. Biol., 2000, 40, 525–540.

[12] P. van den Driessche and J. Watmough, *Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease*, Math. Biosci., 2002, 180(1-2), 29–48.

[13] J. C. Eckalbar and W. L. Eckalbar, *Dynamics of an epidemic model with quadratic treatment*, Nonlinear Anal.: Real World Appl., 2011, 12, 320–332.

[14] J. Guckenheimer and P. J. Holmes, *Nonlinear oscillations, Dynamical Systems, and Bifurcation of Vector Fields*, Springer-Verlag, New York, 1996.

[15] J. Guckenheimer and Y. A. Kuznetsov, *Bogdanov-Takens bifurcation*, Scholarpedia, 2007, 2(1), 1854.

[16] K. P. Hadeler and P. van den Driessche, *Backward bifurcation of in epidemic control*, Math. Biosci., 1997, 146, 15–35.

[17] M. W. Hirsch, S. Smale and R. L. Devaney, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, Springer-Verlag, New York, 2013.

[18] Z. X. Hu, S. Liu and H. Wang, *Backward bifurcation of an epidemic model with standard incidence rate and treatment rate*, Nonlinear Anal.: Real World Appl., 2008, 9, 2302–2312.

[19] Z. X. Hu, P. Bi, W. B. Ma and S. G. Ruan, *Bifurcations of an SIRS epidemic model with nonlinear incidence rate*, Discrete Contin. Dyn. Syst. Ser. B, 2011, 15(1), 93–112.

[20] J. Hui and D.M. Zhu, *Global stability and periodicity on SIS epidemic models with backward bifurcation*, Comp. Math. Appl., 2005, 50, 1271–1290.

[21] Y. Jin, W. D. Wang and S. W. Xiao, *An SIRS model with a nonlinear incidence rate*, Chaos, Solitons and Fractals, 2007, 34, 1482–1497.

[22] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory*, Springer, New York, 1998.

[23] D. Lacitignola, *Saturated treatments and measles resurgence episodes in South Africa: a possible linkage*, Math. Biosci. Eng., 2013, 10, 1135–1157.

[24] G. H. Li and W. D. Wang, *Bifurcation analysis of an epidemic model with nonlinear incidence*, Appl. Math. Comput., 2009, 214, 411–423.

[25] X. Z. Li, W. S. Li and M. Ghosh, *Stability and bifurcation of an SIS epidemic model with treatment*, Chaos, Solitons and Fractals, 2009, 42, 2822–2832.

[26] W. M. Liu, S. A. Levin and Y. Iwasa, *Influence of nonlinear incidence rates upon the behavior of SIRS epidemiological models*, J. Math. Biol., 1986, 23, 187–204.

[27] X. B. Liu and L. J. Yang, *Stability analysis of an SEIQV epidemic model with saturated incidence rate*, Nonlinear Anal.: Real World Appl., 2012, 13(6), 2671–2679.

[28] M. Lizana and J. Rivero, *Multiparametric bifurcations for a model in epidemiology*, J. Math. Biol., 1996, 35, 21-36.

[29] M. Martcheva and H. R. Thieme, *Progression age enhance backward bifurcation in an epidemic model with superinfection*, J. Math. Biol., 2003, 46, 385–410.

[30] J. Mena-Lorca and H. W. Hethcote, *Dynamic models of infectious diseases as regulators of population sizes*, J. Math. Biol., 1992, 30, 693–716.

[31] J. C. Polking, *PPLANE8*, Rice University, 2009. http://math.rice.edu/ dfield/index.html.

[32] S. G. Ruan and W. D. Wang, *Dynamical behavior of an epidemic model with a nonlinear incidence rate*, J. Differential Equations, 2003, 188, 135–163.

[33] M. A. Safi, A. B. Gumel and E. H. Elbasha, *Qualitative analysis of an age-structured SEIR epidemic model with treatment*, Appl. Math. Comput., 2013, 219(22), 10627–10642.

[34] Z. G. Song, J. Xu and Q. H. Li, *Local and global bifurcations in an SIRS epidemic model*, Appl. Math. Comput., 2009, 214(2), 534–547.

[35] F. Takens, *Forced oscillations and bifurcation, in: Applications of Global Analysis I*, Comm. Math. Inst. Rijksuniversitat Utrecht., 1974, 3, 1–59.

[36] Y. L. Tang, D.Q. Huang, S.G. Ruan and W.N. Zhang, *Coexistence of limit cycles and homoclinic loops in a SIRS model with a nonlinear incidence rate*, SIAM J. Appl. Math., 2008, 2, 621–639.

[37] W. D. Wang, *Backward bifurcation of an epidemic model with treatment*, Math. Biosci., 2006, 201, 58–71.

[38] W. D. Wang and S. G. Ruan, *Bifurcation in an epidemic model with constant removal rate of the infectives*, J. Math. Anal. Appl., 2004, 291, 775–793.

[39] Z. W. Wang, *Backward bifurcation in simple SIS model*, Acta Math. Appl. Sin. Engl. Ser., 2009, 25, 127–136.

[40] J. J. Wei and J. A. Cui, *Dynamic of SIS epidemic model with the standard incidence rate and saturated treatment function*, Int. J. Biomath., 2012, 3, 1–18.

[41] Y. J. Xiao, W. P. Zhang, G. F. Deng and Z. H. Liu, *Stability and Bogdanov-Takens Bifurcation of an SIS Epidemic Model with Saturated Treatment Function*, Math. Probl. Eng., 2015, DOI: 10.1155/2015/745732.

[42] L. Xue and S. Caterina, *The network level reproduction number for infectious diseases with both vertical and horizontal transmission*, Math. Biosci., 2013, 243, 67–80.

[43] J. A. Yorke and W. P. London, *Recurrent outbreaks of measles, chickenpox and mumps II*, Amer. J. Epidemiol., 1973, 98, 469–482.

[44] X. Zhang and X. N. Liu, *Backward bifurcation of an epidemic model with saturated treatment function*, J. Math. Anal. Appl., 2008, 348, 433–443.

[45] X. Zhang and X. N. Liu, *Backward bifurcation and global dynamics of an SIS epidemic model with general incidence rate and treatment*, Nonlinear Anal.: Real World Appl., 2009, 10, 565–575.

[46] L. H. Zhou and M. Fan, *Dynamics of an SIR epidemic model with limited medical resources revisited*, Nonlinear Anal.: Real World Appl., 2012, 13, 312–324.

[47] T. T. Zhou, W. P. Zhang and Q. Y. Lu, *Bifurcation analysis of an SIS epidemic model with saturated incidence rate and saturated treatment function*, Appl. Math. Comput., 2014, 226, 288–305.

[48] X. Y. Zhou and J. A. Cui, *Analysis of stability and bifurcation for an SEIR epidemic model with saturated recovery rate*, Commun. Nonlinear Sci. Numer. Simul., 2011, 16(11), 4438–4450.

[49] Y. G. Zhou, D. M. Xiao and Y. L. Li, *Bifurcations of an epidemic model with non-monotonic incidence rate of saturated mass action*, Chaos Solitons Fractals, 2007, 32(5), 1903–1915.

[50] *MATLAB, Version 8.5.0 (R2015a)*, The MathWorks Inc., Natick, MA, 2015.