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HOPF-ZERO BIFURCATION OF A DELAYED PREDATOR-PREY MODEL WITH DORMANCY OF PREDATORS*

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Abstract In this paper, We investigate Hopf-zero bifurcation with codimension 2 in a delayed predator-prey model with dormancy of predators. First we prove the specific existence condition of the coexistence equilibrium. Then we take the mortality rate and time delay as two bifurcation parameters to find the occurrence condition of Hopf-zero bifurcation in this model. Furthermore, using the Faria and Magalhases normal form method and the center manifold theory, we obtain the third order degenerate normal form with two original parameters. Finally, through theoretical analysis and numerical simulations, we give a bifurcation set and a phase diagram to show the specific relations between the normal form and the original system, and explain the coexistence phenomena of several locally stable states, such as the coexistence of multi-periodic orbits, as well as the coexistence of a locally stable equilibrium and a locally stable periodic orbit.

Keywords Predator-prey model with dormancy of predators, Hopf-zero bifurcation, time delay, stability, periodic orbit.

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1. Introduction

In 2009, Kuwamura etc [16] proposed a minimum mathematical model of predator-prey systems with dormancy of predators to explain the paradox of enrichment. In the same year, Kuwamura etc [17] used the theory of fast-slow systems to demonstrate the mixed-mode oscillations and chaos bifurcated from a coexistence equilibrium. In 2012, based on the model in Kuwamura etc [16, p460], taking account of the effects of feedback time delay of the prey growth, we proposed a delayed model (see Wang etc [19, p1542]) with a nondimensionalized change of similar variable transforms (see Wang etc [19, p1543]), and obtained the following dimensionless

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system:

$$\begin{aligned}\dot{x} =&(1-\frac{x(t-\tau)}{N})x-\frac{mxy}{1+x},\\ \dot{y} =&[1+\tanh(\sigma(x-\eta))]\frac{mxy}{1+x}+\vartheta\alpha z-d_1 y,\\ \dot{z} =&[1-\tanh(\sigma(x-\eta))]\frac{mxy}{1+x}-(\alpha+d_2)z.\end{aligned} \tag{1.1}$$

where τ is the nondimensionalized change of T, which denotes the feedback time delay of the prey to the growth of the species itself; and d_1 is the nondimensionalized change of the mortality rates of the active predator(see Kuwamura etc [16, p461] and Wang etc [19, p1542] for more details about the model). In Wang etc [19], we used time delay as a parameter to discuss the global dynamical behavior, such as the stability of the coexistence equilibrium, Hopf bifurcation, and the chaos of system (1.1). In addition, we found some other phenomena: the coexistence of three periodic orbits shown in Figs 1, 4, and 5 in Wang etc [19, p1546-1548]; as well as the coexistence of a locally stable coexistence equilibrium and a locally stable periodic orbit (When $\tau = 0.63 < \tau_2^{(0)} = 0.6790$, the coexistence equilibrium E^* from the initial value $(0.22, 0.96, 0.59)$ is locally asymptotically stable shown as Fig 2 of Wang etc [19, p1547]; at the same time the global Hopf bifurcation graph from $(E^*, \tau_2^{(0)})$ shows that there exists a local stable periodic orbit from the additional initial value shown in Fig 3 of Wang etc [19, p1547]).

Since the coexistence of three periodic orbits is an uncommon phenomenon and may lead to chaos of the system, the cause of the coexistence of three periodic solutions coexist needs to be further studied. In Wang etc [19], based on theoretical analysis, we know that among three periodic orbits, one is bifurcated from a coexistence equilibrium; while how the other two periodic orbits are generated remains unknown. In this paper, we introduce two bifurcation parameters to find some theoretical explanations for the coexistence of multi-periodic orbits as well as the coexistence of a stable equilibrium and a stable periodic orbit. From some research papers, we know that, through the analysis of Hopf-zero bifurcation of dynamic systems, some complex behavior(multi-periodic phenomenon, quasi-periodic phenomenon, even chaotic phenomenon) in predator-prey systems with delay [8,20], neural systems [4,9,10,12,21,23], oscillator systems [13,14,22], and financial system Ding etc [5], have been revealed.

In this paper, according to the existence condition and the corresponding characteristic equation of the coexistence equilibrium, we choose the mortality rate and time delay τ as two bifurcation parameters to discuss the occurrence condition of Hopf-zero bifurcation of system (1.1). In addition, by analyzing the structure of its normal form, we discuss whether Hopf-zero bifurcation of the system may lead to the coexistence of the three periodic solutions in Wang etc [19, p1547].

The rest of the article is organized as follows. In Sect. 2, we give and prove the existence condition of the coexistence equilibrium. Furthermore, in Sect. 3, we obtain the occurrence conditions of Hopf-zero bifurcation from the boundary equilibrium $(N, 0, 0)$. In Sect. 4, we obtain the normal forms of the Hopf-zero bifurcation with the original parameters of the system. In Sect. 5, we perform some bifurcation analysis and numerical simulation, which are shown to verify the theoretical predictions. In Sect. 6, biological implications of our results are given.

2. Existence conditions of coexistence equilibrium

Clearly, there is always a trivial equilibrium $E_0 = (0, 0, 0)$ and the boundary equilibrium $E_1 = (N, 0, 0)$ in system (1.1). If there exists an the coexistence equilibrium $E_* = (x^*, y^*, z^*)$ in system (1.1), then the coexistence equilibrium satisfies the following equations

$$\begin{aligned} d_1 =& \frac{mx^*}{1+x^*}[\tanh(\sigma(x^*-\eta)) + 1 + \frac{\vartheta\alpha}{\alpha+d_2}(1-\tanh(\sigma(x^*-\eta)))]; \\ y^* =& (1-\frac{x^*}{N})\frac{1+x^*}{m}; \\ z^* =& \frac{m}{\alpha+d_2}(1-\tanh(\sigma(x^*-\eta)))\frac{x^*y^*}{1+x^*}. \end{aligned} \tag{2.1}$$

Biologically, the coexistence equilibrium indicates a mode of persistent coexistence of the predator and prey. In this paper, since z denotes the density of predators with dormant state (resting eggs), the coexistence equilibrium of system (1.1) implies that $x^* > 0, y^* > 0$, $z^* \geq 0$. In addition, from the mathematical point of view, the premise of studying equilibrium stability and bifurcation lies in the proof of the equilibrium existence. Thus the proof of the existence of the coexistence equilibrium in Wang etc [19, p1543] needs to be added into our discussion.

Next, we discuss the existence condition of the coexistence equilibrium $E_* = (x^*, y^*, z^*)$. According to expression (2.1), let

$$g(x) = \frac{mx}{1+x}[((\alpha+d_2)\tanh(\sigma(x-\eta)) + 1) + \vartheta\alpha(1-\tanh(\sigma(x-\eta))]. \tag{2.2}$$

Through analysis, we obtain the following results:

Proposition 2.1. *If* $0 < d_1 < g(N)/(\alpha+d_2)$*, then there exists a unique coexistence equilibrium* $E_* = (x^*, y^*, z^*)$ *in system* (1.1)*, where* $0 < x^* < N$*,* x^**,* y^* *and* z^* *are defined in* (2.1).

Proof. Suppose that there exists a coexistence equilibrium $E_* = (x^*, y^*, z^*)$. Then E_* must satisfy (2.1) if $x^* > 0$. We derive that $g(x)$ satisfies

$$g'(x) > 2m\vartheta\alpha/(1+x)^2 > 0.$$

If $0 < d_1 < g(N)/(\alpha+d_2)$. Then

$$g(N) > (\alpha+d_2)d_1 > 0.$$

Since $g(0) = 0$, there must be a unique x^* satisfying $0 < x^* < N$ and

$$g(x^*) = (\alpha+d_2)d_1.$$

Based on the expression of y^* , z^* and the inequality $0 < x^* < N$, we know that $y^* > 0$ and $z^* \geq 0$. Therefore, there exists a unique coexistence equilibrium $E_* = (x^*, y^*, z^*)$ in system (1.1). The proof is complete. □

3. Occurrence condition of Hopf-zero bifurcation

If system (1.1) undergoes Hopf-zero bifurcation at an equilibrium, then its corresponding characteristic equation has a zero root except for a pair of imaginary roots. In Wang etc [19, p1543], the corresponding characteristic equation with the linearization of system (1.1) at E^* has the following form

$$(\lambda^3+\tilde{a}_2\lambda^2+\tilde{a}_1\lambda+\tilde{a}_0)+(\tilde{b}_2\lambda^2+\tilde{b}_1\lambda)e^{-\lambda\tau}=0, \tag{3.1}$$

where
$\tilde{a}_2=\alpha+d_1+d_2-\frac{mx^*}{1+x^*}[\tanh(\sigma(x^*-\eta))+1]-\frac{mx^*y^*}{(1+x^*)^2}$,
$\tilde{a}_1=\frac{mx^*y^*}{(1+x^*)^2}\{\sigma mx^*[1-\tanh^2(\sigma(x^*-\eta))]-(\alpha+d_1+d_2)+m[\tanh(\sigma(x^*-\eta))+1]\}$,
$\tilde{a}_0=\frac{my^*}{(1+x^*)^2}\{x^*\sigma mx^*[1-\tanh^2(\sigma(x^*-\eta))](\alpha+d_2-\theta\alpha)+d_1(\alpha+d_2)\}$,
$\tilde{b}_2=\frac{x^*}{N}$,
$\tilde{b}_1=\frac{x^*}{N}\{\alpha+d_1+d_2-\frac{mx^*}{1+x^*}[\tanh(\sigma(x^*-\eta))+1]\}$.
If equation has a zero root $\lambda=0$, then $\tilde{a}_0=0$. Since

$$x^*\sigma mx^*[1-\tanh^2(\sigma(x^*-\eta))](\alpha+d_2-\theta\alpha)+d_1(\alpha+d_2)>0$$

with the parameter values $N=2.025$, $m=1.2$, $\sigma=20$, $\eta=0.5$, $d_1=0.4$, $\theta=0.2$, $\alpha=0.8$ and $d_2=0.05$ in Wang etc [19, p1546], we know if $\tilde{a}_0=0$, then $y^*=0$, which is contradictory to $y^*=0.9412$. That is to say, if system (1.1) undergoes Hopf-zero bifurcation at an equilibrium, then the second coordinate of the equilibrium is zero.

Therefore, we study the following characteristic equation with the linearizations of system (1.1) at E_0 and E_1, respectively.

$$(\lambda-1)(\lambda+d_1)(\lambda-\alpha-d_2)=0 \tag{3.2}$$

and

$$(\lambda+e^{-\lambda\tau})(\lambda^2+(\alpha+d_1+d_2-\frac{mN}{1+N}[\tanh(\sigma(N-\eta))+1])\lambda+d_1(\alpha+d_2)-g(N))=0. \tag{3.3}$$

Since $\alpha>0,\ d_2>0$, equation (3.2) has a negative root and two positive roots for an arbitrary $\tau\geq 0$, thus E_0 is unstable when $\tau\geq 0$. If $d_1=g(N)/(\alpha+d_2)$ holds, then equation (3.3) has a zero root; and if $\tau=\frac{\pi}{2}+2k\pi$ $(k=0,\pm1,\pm2,\cdots)$ holds, then equation (3.3) has a pair of pure imaginary roots. By direct calculation, we obtain:

Proposition 3.1. *If $0\leq\tau<\pi/2$ and $d_1>g(N)/(\alpha+d_2)$ hold, then the equilibrium E_1 of system (1.1) is locally stable; if $d_1=g(N)/(\alpha+d_2)$ holds, then system (1.1) undergoes a Hopf-zero bifurcation at the equilibrium E_1 when $\tau=\pi/2+2k\pi$, where $k=0,\pm1,\pm2,\cdots$, and $g(N)$ is defined in (2.2) when $x=N$.*

Proof. If $0\leq\tau<\pi/2$ and $d_1>g(N)/(\alpha+d_2)$ hold, then all the solutions of equation (3.3) have negative real parts, so the equilibrium E_1 of system (1.1) is locally stable. By Corollary 2.3 in Ruan etc [18, p867] and (3.3), we know that if $d_1=g(N)/(\alpha+d_2)$ and $\tau=\pi/2$ hold, then all the roots of equation (3.3) have negative real parts except a zero and a pair of pure imaginary eigenvalues. Thus we obtain the desired results. The proof is complete. □

From Proposition 3.1, we want to know the dynamic behavior of the boundary equilibrium $(N,0,0)$. Then we need to further investigate the properties of Hopf-zero bifurcation of singularity $(N,0,0)$.

4. Normal form of Hopf-zero bifurcation

In order to study the properties of Hopf-zero bifurcation of $(N,0,0)$, we perform the center manifold reduction and normal form computation for system (1.1).

First, let $\hat{x} = x - N$, $\hat{y} = y$, $\hat{z} = z$, and still denote $\hat{x}$,$\hat{y}$,$\hat{z}$ by x, y, z. By rescaling the time by $t \to t/\tau$ to normalize the delay, and by expanding the functions $\tanh(\sigma(x-\eta))$ and $xy/(1+x)$ in system (1.1), we obtain

$$\begin{aligned}\dot{x} =& - a_1\tau y - \tau x(t-1) - \tau x(t)x(t-1)/N - a_2\tau xy + a_3\tau x^2 y + h.o.t.,\\ \dot{y} =& \tau(a_1 b_1 - d_1)y + \tau\vartheta\alpha z + \tau a_2(b_1 + Nb_2)xy - \tau a_3(b_1 - b_2 - b_3)x^2 y + h.o.t.,\\ \dot{z} =& \tau(a_1(2-b_1)y - (\alpha + d_2)z + a_2(2 - b_1 - Nb_2)xy\\ &- a_3(2 - b_1 + b_2 + b_3)x^2 y) + h.o.t.,\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}a_1 =& \frac{mN}{1+N}; a_2 = \frac{m}{(1+N)^2}; a_3 = \frac{m}{(1+N)^3},\\ b_1 =& 1 + \tanh(\sigma(N-\eta)); b_2 = \sigma(1+N)(1 - \tanh^2(\sigma(N-\eta))),\\ b_3 =& \sigma^2 N(1+N)^2 \tanh(\sigma(N-\eta))(\tanh^2(\sigma(N-\eta)) - 1).\end{aligned}$$

Next, we introduce two bifurcation parameters by $d_1 = a_1[b_1 + \frac{(2-b_1)\vartheta\alpha}{\alpha+d_2}] + \mu_1$ and $\tau = \frac{\pi}{2} + \mu_2$ in Eq. (4.1), and denote $\mu = (\mu_1, \mu_2)$. Select $C = C([-1,0], \mathbb{R}^3)$ to denote the space of all the continuous functions from $[-1,0]$ to $\mathbb{R}^3$. For each $X_t \in C$, define $X_t(\theta) = X(t+\theta)$ and $\|X_t\| = \sup\limits_{-1<\theta<0} |X_t(\theta)|$. Then Eq. (4.1) can be written as

$$\dot{X}(t) = L(\mu)X_t + F(X_t, \mu), \tag{4.2}$$

where

$$L(\mu)X_t = \begin{pmatrix} -a_1(\frac{\pi}{2}+\mu_2)y - (\frac{\pi}{2}+\mu_2)x(t-1) \\ (\frac{\pi}{2}+\mu_2)(\frac{(b_1-2)a_1\vartheta\alpha}{\alpha+d_2} - \mu_1)y + (\frac{\pi}{2}+\mu_2)\vartheta\alpha z \\ (\frac{\pi}{2}+\mu_2)a_1(2-b_1)y - (\frac{\pi}{2}+\mu_2)(\alpha+d_2)z \end{pmatrix}$$

and

$$F(X_t,\mu) = \begin{pmatrix} -\frac{1}{N}(\frac{\pi}{2}+\mu_2)x(t)x(t-1) - a_2(\frac{\pi}{2}+\mu_2)xy + a_3(\frac{\pi}{2}+\mu_2)x^2 y \\ (\frac{\pi}{2}+\mu_2)a_2(b_1+Nb_2)xy - (\frac{\pi}{2}+\mu_2)a_3(b_1-b_2-b_3)x^2 y \\ (\frac{\pi}{2}+\mu_2)a_2(2-b_1-Nb_2)xy - (\frac{\pi}{2}+\mu_2)a_3(2-b_1+b_2+b_3)x^2 y \end{pmatrix} + h.o.t.$$

Let $(\mu_1, \mu_2) = (0,0)$ for $\varphi \in C([-1,0], \mathbb{R}^3)$, $L_0\varphi = \int_{-1}^{0} d\eta(\theta)\varphi(\xi)d\xi$, and select

$$\eta(\theta) = \begin{cases} -\frac{\pi}{2}(A_1 + A_2), & \theta = -1, \\ -\frac{\pi}{2}A_1, & \theta \in (-1,0), \\ 0, & \theta = 0, \end{cases}$$

with

$$A_1 = \begin{pmatrix} 0 & -a_1 & 0 \\ 0 & a_1 b_1 - d_1 & \vartheta\alpha \\ 0 & a_1(2-b_1) & -(\alpha+d_2) \end{pmatrix}; \quad A_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the linearization equation of (4.1) at the trivial equilibrium is

$$\dot{X}(t) = L_0 X_t$$

and the bilinear form on $C^* \times C$ is

$$\begin{aligned}(\psi(s), \varphi(\theta)) =& \psi(0)\varphi(0) - \int_{-1}^{0}\int_{0}^{\theta} \psi(\xi-\theta)d\eta(\theta)\varphi(\xi)d\xi \\ =& \psi(0)\varphi(0) + \frac{\pi}{2}\int_{0}^{-1} \psi_1(\xi+1)\varphi_1(\xi)d\xi,\end{aligned}$$

where $\varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta), \varphi_3(\theta)) \in C$, $\Psi(s) = \begin{pmatrix} \psi_1(s) \\ \psi_2(s) \\ \psi_3(s) \end{pmatrix} \in C^*$. Then the space C is decomposed by $\Lambda = \{0, \pm\frac{\pi}{2}\mathrm{i}\}$ into $C = P + Q$, where $Q = \{\varphi \in C : (\psi, \varphi) = 0$ for all $\psi \in P^*\}$ and the bases for P and its adjoint P^* are

$$\Phi(\theta) = \begin{pmatrix} 1 & e^{\frac{\pi}{2}\mathrm{i}\theta} & e^{-\frac{\pi}{2}\mathrm{i}\theta} \\ -\frac{1}{a_1} & 0 & 0 \\ \frac{b_1-2}{\alpha+d_2} & 0 & 0 \end{pmatrix}, \quad -1 \le \theta \le 0$$

and

$$\Psi(s) = \begin{pmatrix} 0 & D_1(\alpha+d_2) & D_1\vartheta\alpha \\ D_2\left(1 - \left(\alpha + d_2 - \frac{a_1(b_1-2)\vartheta\alpha}{\alpha+d_2}\right)\mathrm{i}\right)\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}s} & D_2 a_1(\alpha+d_2+\mathrm{i})\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}s} & D_2 a_1\vartheta\alpha\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}s} \\ D_3\left(1 + \left(\alpha + d_2 - \frac{a_1(b_1-2)\vartheta\alpha}{\alpha+d_2}\right)\mathrm{i}\right)\mathrm{e}^{\frac{\pi}{2}\mathrm{i}s} & D_3 a_1(\alpha+d_2-\mathrm{i})\mathrm{e}^{\frac{\pi}{2}\mathrm{i}s} & D_3 a_1\vartheta\alpha e^{\frac{\pi}{2}\mathrm{i}s} \end{pmatrix},$$

$0 \le s \le 1$, where

$$\begin{aligned} D_1 =& \frac{-a_1(\alpha+d_2)}{(\alpha+d_2)^2 + \vartheta\alpha a_1(2-b_1)}, \\ D_2 =& \left\{1 + \frac{\pi}{2}\left[\alpha + d_2 - \frac{a_1(b_1-2)\vartheta\alpha}{\alpha+d_2}\right] + \left[\frac{\pi}{2} - \left(\alpha + d_2 - \frac{a_1(b_1-2)\vartheta\alpha}{\alpha+d_2}\right)\right]\mathrm{i}\right\}^{-1}, \end{aligned}$$

$D_3 = \bar{D}_2$. Thus the dual bases satisfy $\dot{\Phi} = \Phi B$ and $-\dot{\Psi} = B\Psi$ with

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\pi}{2}\mathrm{i} & 0 \\ 0 & 0 & -\frac{\pi}{2}\mathrm{i} \end{pmatrix},$$

$$\begin{aligned}\Psi(0) =& \begin{pmatrix} 0 & D_1(\alpha+d_2) & D_1\vartheta\alpha \\ D_2\left(1 - \left(\alpha + d_2 - \frac{a_1(b_1-2)\vartheta\alpha}{\alpha+d_2}\right)\mathrm{i}\right) & D_2 a_1(\alpha+d_2+\mathrm{i}) & D_2 a_1\vartheta\alpha \\ D_3\left(1 + \left(\alpha + d_2 - \frac{a_1(b_1-2)\vartheta\alpha}{\alpha+d_2}\right)\mathrm{i}\right) & D_3 a_1(\alpha+d_2-\mathrm{i}) & D_3 a_1\vartheta\alpha \end{pmatrix} \\ =& \begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{31} & \psi_{32} & \psi_{33} \end{pmatrix}.\end{aligned}$$

As in Faria etc [6, p205], we enlarge the phase space C into

$$BC := \left\{ \varphi : [-1, 0] \to \mathbb{R}^3 \text{is a continuous function on } [-1, 0) \text{and } \lim_{\theta \to 0^-} \varphi(\theta) \text{ exists} \right\}.$$

In BC, Eq. (4.2) becomes an abstract ODE,

$$\frac{\mathrm{d}}{\mathrm{d}t} u = Au + Y_0 \tilde{F}(u, \mu), \tag{4.3}$$

where $u \in C$, and A is defined by

$$A : C^1 \to BC, \quad Au = \dot{u} + Y_0[L_0 u - \dot{u}(0)]$$

and

$$\tilde{F}(u, \mu) = [L(\mu) - L_0]u + F(u, \mu),$$

where Y_0 is the 3×3 matrix-valued function defined by

$$Y_0(\theta) = \begin{cases} I, & \theta = 0, \\ 0, & -1 \le \theta < 0. \end{cases}$$

By the continuous projection $G : BC \mapsto P, \quad G(\varphi + Y_0 c) = \Phi[(\Psi, \varphi) + \Psi(0)c]$, we can decompose the enlarged phase space by $\Lambda = \{0, \pm\frac{\pi}{2}\mathrm{i}\}$ into $BC = P \oplus \mathrm{Ker}G$. Let $u_t = \Phi\tilde{x}(t) + \tilde{y}(\theta)$ where $\tilde{x}(t) = (x_1, x_2, x_3)^T$. Therefore Eq.(4.3) is decomposed into the system

$$\begin{aligned} \dot{\tilde{x}} &= B\tilde{x} + \Psi(0)\tilde{F}(\Phi\tilde{x} + \tilde{y}(\theta), \mu), \\ \dot{\tilde{y}} &= A_{Q_1}\tilde{y} + (I - G)Y_0\tilde{F}(\Phi\tilde{x} + \tilde{y}(\theta), \mu), \end{aligned} \tag{4.4}$$

where $\tilde{y}(\theta) \in Q^1 := Q \cap C^1 \subset \mathrm{Ker}G$, A_{Q_1} is the restriction of A as an operator from Q_1 to the Banach space $\mathrm{Ker}G$. By neglecting higher order terms with respect to parameters μ_1 and μ_2, Eq. (4.4) can be written as

$$\begin{aligned} \dot{\tilde{x}} &= B\tilde{x} + \tfrac{1}{2!}f_2^1(\tilde{x}, \tilde{y}, \mu) + \tfrac{1}{3!}f_3^1(\tilde{x}, \tilde{y}, \mu) + h.o.t, \\ \dot{\tilde{y}} &= A_{Q_1}\tilde{y} + \tfrac{1}{2!}f_2^2(\tilde{x}, \tilde{y}, \mu) + \tfrac{1}{3!}f_3^2(\tilde{x}, \tilde{y}, \mu) + h.o.t, \end{aligned} \tag{4.5}$$

where

$$\frac{1}{2!}f_2^1(\tilde{x}, \tilde{y}, \mu) = \begin{pmatrix} \psi_{11}F_2^1 + \psi_{12}F_2^2 + \psi_{13}F_2^3 \\ \psi_{21}F_2^1 + \psi_{22}F_2^2 + \psi_{23}F_2^3 \\ \psi_{31}F_2^1 + \psi_{32}F_2^2 + \psi_{33}F_2^3 \end{pmatrix}, \tag{4.6}$$

$$\frac{1}{3!}f_3^1(\tilde{x}, \tilde{y}, \mu) = \begin{pmatrix} \psi_{11}F_3^1 + \psi_{12}F_3^2 + \psi_{13}F_3^3 \\ \psi_{21}F_3^1 + \psi_{22}F_3^2 + \psi_{23}F_3^3 \\ \psi_{31}F_3^1 + \psi_{32}F_3^2 + \psi_{33}F_3^3 \end{pmatrix}, \tag{4.7}$$

$$\frac{1}{2!}f_2^2(\tilde{x}, \tilde{y}, \mu) = (I - G)Y_0 \begin{pmatrix} F_2^1 \\ F_2^2 \\ F_2^3 \end{pmatrix}; \quad \frac{1}{3!}f_3^2(\tilde{x}, \tilde{y}, \mu) = (I - G)Y_0 \begin{pmatrix} F_3^1 \\ F_3^2 \\ F_3^3 \end{pmatrix} \tag{4.8}$$

with

$$
\begin{aligned}
F_2^1 =& -a_1\mu_2\left(-\frac{x_1}{a_1}+y_2(0)\right)-b_{15}(x_1+x_2+x_3+y_1(0))(x_1-\mathrm{i}x_2+\mathrm{i}x_3+y_1(-1))\\
&-\mu_2(x_1-\mathrm{i}x_2+\mathrm{i}x_3+y_1(-1))-b_{16}(x_1+x_2+x_3+y_1(0))\left(-\frac{x_1}{a_1}+y_2(0)\right),\\
F_3^1 =& b_{17}(x_1+x_2+x_3+y_1(0))^2\left(-\frac{x_1}{a_1}+y_2(0)\right),\\
F_2^2 =& \left(\frac{(b_1-2)a_1\vartheta\alpha}{\alpha+d_2}\mu_2-\frac{\pi}{2}\mu_1\right)\left(-\frac{x_1}{a_1}+y_2(0)\right)\\
&+\mu_2\vartheta\alpha\left(\frac{b_1-2}{\alpha+d_2}x_1+y_3(0)\right)+b_{11}(x_1+x_2+x_3+y_1(0))\left(-\frac{x_1}{a_1}+y_2(0)\right),\\
F_3^2 =& -b_{12}(x_1+x_2+x_3+y_1(0))^2\left(-\frac{x_1}{a_1}+y_2(0)\right),\\
F_2^3 =& \mu_2a_1(b_1-2)\left(-\frac{x_1}{a_1}+y_2(0)\right)-\mu_2(\alpha+d_2)\left(\frac{b_1-2}{\alpha+d_2}x_1+y_3(0)\right)\\
&+b_{13}(x_1+x_2+x_3+y_1(0))\left(-\frac{x_1}{a_1}+y_2(0)\right),\\
F_3^3 =& b_{14}(x_1+x_2+x_3+y_1(0))^2\left(-\frac{x_1}{a_1}+y_2(0)\right),\\
b_{11} =& \frac{\pi a_2}{2}(b_1+Nb_2),\quad b_{12}=\frac{\pi a_3}{2}(b_1-b_2-b_3),\quad b_{13}=\frac{\pi a_2}{2}(2-b_1-Nb_2),\\
b_{14} =& \frac{-\pi a_3}{2}(2-b_1+b_2+b_3),\quad b_{15}=\frac{\pi}{2N},\quad b_{16}=\frac{\pi a_2}{2},\quad b_{17}=\frac{\pi a_3}{2}.
\end{aligned}
$$

We know that $(\mathrm{Im}(M_2^1))^c$ is spanned by the following elements (see also Wang etc [22, p12]):

$$\{x_1^2e_1, x_2x_3e_1, x_1\mu_ie_1, \mu_1\mu_2e_1, x_1x_2e_2, x_2\mu_ie_2, x_1x_3e_3, x_3\mu_ie_3\},\ i=1,2$$

with $e_1=(1,0,0)^T, e_2=(0,1,0)^T, e_3=(0,0,1)^T$. Thus the normal form of (4.5) on the center manifold of the origin near $(\mu_1,\mu_2)=0$ has the form

$$\dot{x}=Bx+\frac{1}{2!}g_2^1(\tilde{x},0,\mu)+\cdots$$

with $g_2^1(\tilde{x},0,\mu)=\mathrm{proj}_{(\mathrm{Im}(M_2^1))^c}f_2^1(\tilde{x},0,\mu)$.

To find the third order normal form of the Hopf-zero singularity, we need to find the center manifold $\tilde{y}=h(\tilde{x})$ at first (hence, neglecting the impact of small parameters μ_1 and μ_2 in the third order).

Let $V_2^3(\mathbb{C}^3\times\mathrm{Ker}(G))$ be the space of homogeneous polynomials of degree 2 in the variables $\tilde{x}=(x_1,x_2,x_3)^T$ with coefficients in $\mathbb{C}^3\times\mathrm{Ker}(G)$. Define the operator

$$
\begin{aligned}
&M_2: V_2^3(\mathbb{C}^3\times\mathrm{Ker}(G))\to V_2^3(\mathbb{C}^3\times\mathrm{Ker}(G)),\\
&M_2(p,h)=(M_2^1p, M_2^2h)\triangleq(D_{\tilde{x}}p(\tilde{x})B\tilde{x}-Bp(\tilde{x}), D_{\tilde{x}}h(\tilde{x})B\tilde{x}-A_{Q^1}(h(\tilde{x})),
\end{aligned}
$$

where $p(x)\in V_2^3(\mathbb{C}^3)$, $h(x)\in V_2^3(Q^1)$.

From decompositions (27) in Jiang etc [15, p3633], the projection associated with the preceding decomposition of $V_2^3(\mathbb{C}^3)\times V_2^3(\mathrm{G})$ over $\mathrm{Im}(M_2^1)\times\mathrm{Im}(M_2^2)$ is denoted by $P_{I,2} = (P^1_{I,2}, P^2_{I,2})$. Set

$$U_2(x) = (U_2^1, U_2^2)^T = M_2^{-1} P_{I,2} f_2(x, 0, 0),$$

where $U_2^1/2! \in \mathrm{Ker}(M_2^1)^c$, and $U_2^2/2! = h(\tilde{x}) = (h_1(\tilde{x}), h_2(\tilde{x}), h_3(\tilde{x}))^T$ is the unique solution in $V_2^3(Q^1)$ of the equation

$$M_2^2 U_2^2(\tilde{x}) = f_2^2(\tilde{x}, 0, 0).$$

Thus,

$$D_{\tilde{x}} h(\tilde{x}) B\tilde{x} - \dot{h}(\tilde{x})(\theta) + X_0[\dot{h}(\tilde{x})(0) - L(h(\tilde{x}))] = f_2^2/2, \tag{4.9}$$

where $\dot{h}$ denotes the derivative of $h(x)(\theta)$ with respect to θ. By a transformation of variables $(\tilde{x}, \tilde{y}) = (\hat{\tilde{x}}, \hat{\tilde{y}}) + U_2(\hat{\tilde{x}})/2!$, the first equation of (4.5) becomes, after dropping the hats,

$$\dot{\tilde{x}} = B\tilde{x} + \frac{1}{2!} g_2^1(\tilde{x}, 0, \mu) + \frac{1}{3!} \bar{f}_3^1(\tilde{x}, 0, 0) + \cdots,$$

where

$$\bar{f}_3^1(\tilde{x}, 0, 0) = f_3^1(\tilde{x}, 0, 0) + \frac{1}{3!}[(Df_2^1(\tilde{x}, y, 0))_{y=0} U_2(\tilde{x}) - DU_2^1(\tilde{x}) g_2^1(\tilde{x}, 0, 0)].$$

The value of $U_2(\tilde{x})$ is found in the Appendix. Let M_3 be the operator defined in $V_3^3(\mathbb{C}^3 \times \mathrm{Ker}(G)$, with $M_3^1 : V_3^3(\mathbb{C}^3) \to V_3^3(\mathbb{C}^3)$ and $(M_3^1 p)(\tilde{x}) = D_{\tilde{x}} p(\tilde{x}) B\tilde{x} - Bp(\tilde{x})$, where $V_3^3(\mathbb{C}^3)$ denotes the linear space of the third order homogeneous polynomials in three variables (x_1, x_2, x_3) with coefficients in $\mathbb{C}^3$. Then one may select the decomposition

$$V_3^3(\mathbb{C}^3) = \mathrm{Im}(M_3^1) \oplus \mathrm{Im}(M_3^1)^c$$

with the complementary space $(\mathrm{Im}(M_3^1))^c$ spanned by the elements

$$\{x_1^3 e_1, x_1 x_2 x_3 e_1, x_1^2 x_2 e_2, x_2^2 x_3 e_2, x_1^2 x_3 e_3, x_2 x_3^2 e_3\}.$$

The normal form of (4.5) on the center manifold of the origin near $(\mu_1, \mu_2) = 0$ up to the third order is

$$\dot{\tilde{x}} = B\tilde{x} + \frac{1}{2!} g_2^1(\tilde{x}, 0, \mu) + \frac{1}{3!} g_3^1(\tilde{x}, 0, 0) + \cdots$$

with $g_3^1(\tilde{x}, 0, 0) = \mathrm{proj}_{(\mathrm{Im}(M_3^1))^c} \bar{f}_3^1(\tilde{x}, 0, 0)$. Following Theorem 2.16 in Faria etc [7, p189], we know that the flow on the center manifold of equation (4.2) tangent to P near $X_t = 0$ is governed by the following normal form

$$\begin{aligned}
\dot{x}_1 = {} & n_2 \mu_1 x_1 + b_2^0 x_1^2 + e x_1 x_2 x_3 + f x_1^3 + h.o.t. \\
\dot{x}_2 = {} & \tfrac{\pi}{2} \mathrm{i} x_2 + (m_1^0 + \mathrm{i} p_1^0) \mu_2 x_2 + (a_1^0 + \mathrm{i} a_2^0) x_1 x_2 \\
& + (c_1^0 + \mathrm{i} c_2^0) x_2^2 x_3 + (d_1^0 - \mathrm{i} d_2^0) x_1^2 x_2 + h.o.t. \\
\dot{x}_3 = {} & -\tfrac{\pi}{2} \mathrm{i} x_3 + (m_1^0 - \mathrm{i} p_1^0) \mu_2 x_3 + (a_1^0 - \mathrm{i} a_2^0) x_1 x_3 \\
& + (c_1^0 - \mathrm{i} c_2^0) x_2 x_3^2 + (d_1^0 + \mathrm{i} d_2^0) x_1^2 x_3 + h.o.t..
\end{aligned} \tag{4.10}$$

where

$$
\begin{aligned}
&n_2 = \frac{\pi\psi_{12}}{2a_1},\ b_2^0 = -\frac{A_{11}}{a_1},\ m_1^0 + \mathrm{i}p_1^0 = \mathrm{i}\psi_{21},\ a_1^0 + \mathrm{i}a_2^0 = A_{22},\\
&e = \frac{2A_{12}}{a_1} + \frac{2A_{11}}{\pi}\left(-\frac{h_{1011}(0)}{a_1} + h_{2011}(0) + h_{2101}(0) + h_{2110}(0)\right),\\
&f = \frac{A_{12}}{a_1} + \frac{2A_{11}}{\pi}\left(-\frac{h_{1200}(0)}{a_1} + h_{2200}(0)\right) + \frac{2\mathrm{i}A_{11}}{\pi a_1}(A_{31} - A_{21}),\\
&c_1^0 + \mathrm{i}c_2^0 = \frac{2}{\pi}\{-\frac{2\mathrm{i}}{3}\psi_{21}\psi_{31}b_{15}^2 - \psi_{21}b_{15}(-\mathrm{i}h_{1011}(0) + \mathrm{i}h_{1020}(0) + h_{1011}(-1)\\
&\qquad + h_{1020}(-1)) + (-\psi_{21}b_{16} + \psi_{22}b_{11} + \psi_{23}b_{13})(h_{2011}(0) + h_{2020}(0))\},\\
&d_1^0 - \mathrm{i}d_2^0 = \frac{2(\psi_{21}b_{17} + \psi_{22}b_{12} - \psi_{23}b_{14})}{a_1} + \frac{2}{\pi}\{\frac{2\mathrm{i}A_{11}A_{21}}{a_1} - \frac{\mathrm{i}A_{23}A_{32}}{2}\\
&\qquad + A_{21}(-2\psi_{21}b_{15} + h_{1110}(0)) - \psi_{21}b_{15}(-\mathrm{i}h_{1200}(0) + h_{1110}(-1)\\
&\qquad + h_{1200}(-1)) + (-\psi_{21}b_{16} + \psi_{22}b_{11} + \psi_{23}b_{13})(h_{2110}(0) + h_{2200}(0)))\}.
\end{aligned}
$$

(see Appendix for the detailed calculation process)

5. Bifurcation analysis and numerical simulations

To further analyze the bifurcation situation in (4.10), we use the coordinate transformation $x_1 = Z, x_2 = r\cos\theta + \mathrm{i}r\sin\theta, x_3 = r\cos\theta - \mathrm{i}r\sin\theta$, $r > 0$. Then we obtain the normal form of truncated to the third order in the cylindrical coordinates

$$
\begin{aligned}
\dot{r} &= m_1^0\mu_2 r + a_1^0 rZ + c_1^0 r^3 + d_1^0 rZ^2,\\
\dot{Z} &= n_2\mu_1 Z + b_2^0 Z^2 + eZr^2 + fZ^3,\\
\dot{\theta} &= \frac{\pi}{2} + p_1^0\mu_2 + a_2^0 Z + c_2^0 r^2 - d_2^0 Z^2.
\end{aligned} \tag{5.1}
$$

Removing the azimuthal term, system (5.1) becomes

$$
\begin{aligned}
\dot{r} &= r(m_1^0\mu_2 + a_1^0 Z + c_1^0 r^2 + d_1^0 Z^2),\\
\dot{Z} &= Z(n_2\mu_1 + b_2^0 Z + er^2 + fZ^2).
\end{aligned} \tag{5.2}
$$

If $a^0 = b_2^0 = 0$, then Eq.(5.2) has 12 distinct types of unfolding by linear transformation, according to the framework by Guckenheimer etc [11, p399]; if $a^0 \neq 0$ and $b_2^0 \neq 0$, then we further discuss the bifurcation with parameters μ_1 and μ_2. In Eq. (5.2), $M_0 = (r, Z) = (0, 0)$ is always an equilibrium and the other equilibria are

$$
\begin{aligned}
M_1 &= \left(\sqrt{-\frac{m_1^0\mu_2}{c_1^0}}, 0\right) \quad \text{for} \quad \frac{m_1^0\mu_2}{c_1^0} < 0,\\
M_2^{\pm} &= \left(0, \frac{-b_2^0 \pm \sqrt{(b_2^0)^2 - 4fn_2\mu_1}}{2f}\right) \quad \text{for} \quad (b_2^0)^2 - 4fn_2\mu_1 > 0,\\
M_3^{\pm} &= (r_*, Z_*),
\end{aligned}
$$

where

$$\begin{aligned}
Z_* &= \frac{a_1^0 e - b_2^0 c_1^0 \pm \sqrt{\Theta}}{2(c_1^0 f - d_1^0 e)},\\
r_* &= \sqrt{-\frac{d_1^0 (Z_*)^2 + a_1^0 Z_* + m_1^0 \mu_2}{c_1^0}}, \quad \frac{d_1^0 (Z_*)^2 + a_1^0 Z_* + m_1^0 \mu_2}{c_1^0} < 0,\\
\Theta &= (b_2^0 c_1^0 - a_1^0 e)^2 - 4(c_1^0 f - d_1^0 e)(c_1^0 n_2 \mu_1 - e m_1^0 \mu_2) > 0.
\end{aligned}$$

In the normal form (5.2), it is quite difficult to estimate the sign of c_1^0, n_2, m_1^0, thus we set up some assumptions, which are $c_1^0 < 0, n_2 < 0, m_1^0 > 0$ to more conveniently analyze the existence of equilibria $M_1, M_2^{\pm}, M_3^{\pm}$. In addition, we briefly obtain the following results:

Theorem 5.1. *If $c_1^0 < 0$, $n_2 < 0$ and $m_1^0 > 0$ hold, then*
(i) System (5.2) undergoes two pitchfork bifurcations and a transcritical bifurcation at the trivial equilibrium M_0, respectively, on the curves

$$L_0 = \{(\mu_1, \mu_2) : \mu_1 = \frac{(b_2^0)^2}{4 f n_2}, \mu_2 \neq 0\},$$

$$L_1 = \{(\mu_1, \mu_2) : \mu_2 = 0, \mu_1 \neq 0\} \text{ and } L_2 = \{(\mu_1, \mu_2) : \mu_1 = 0, \mu_2 < 0\};$$

(ii) System (5.2) undergoes a pitchfork bifurcation at the half-trivial equilibrium M_1 and $M_2^{\pm}$, respectively, on the curves

$$L_3 = \{(\mu_1, \mu_2) : \mu_2 = \frac{c_1^0 n_2}{e m_1^0}\mu_1, \mu_2 > 0\} \text{ and } L_4^{\pm} = \{(\mu_1, \mu_2) : \mu_2 = \frac{-d_1^0 (Z_*)^2 - a_1^0 Z_*}{m_1^0}\}.$$

Noticing that curve $L_4^{\pm}$ includes two curves L_4^+ and L_4^-, described as

$$L_4^{\pm}: \quad \mu_2 = \frac{d_1^0 n_2}{m_1^0 f}\mu_1 + \frac{d_1^0 b_2^0 - a_1^0 f}{2 m_1^0 f^2}(-b_2^0 \pm \sqrt{(b_2^0)^2 - 4 f n_2 \mu_1}).$$

According to the center manifold theory [3, p119], Eq.(5.1) on the center manifold determines the asymptotic behavior of solutions of the entire equations (1.1) when there exist no unstable manifolds containing the trivial solution. In addition, the bifurcation analysis for the three-dimensional system (5.1) is based on rotational symmetry. Since system (5.1) rotates around the Z-axis, the correspondences between 2-dimensional flows for (5.2) and 3-dimensional flows for (5.1) can be established. Thus for (5.1), the equilibrium on the Z-axis in (5.2) remain a equilibrium, while the equilibria outside the Z-axis in (5.2) become a periodic orbit.

Based on Theorem 5.1, numerical simulation results in Wang etc [19, p1546] and experimental results in [1,2], we give an example and perform numerical simulations to show the relation between the obtained normal form (5.2) with the original parameters and the original system (1.1).

Example 5.1. If we choose $N = 2.025$, $m = 1.2$, $\sigma = 20$, $\eta = 0.5$, $d_1 = 1.6066 + \mu_1$, $\vartheta = 0.2$, $\alpha = 0.8$, $d_2 = 0.05$ and $\tau = \pi/2 + \mu_2$ of system (1.1), then system (1.1) undergoes a Hopf-zero bifurcation at the equilibrium $(2.025, 0, 0)$, when $(\mu_1, \mu_2) = (0, 0)$.

By direct calculation, we obtain $n_2 = -1.5708$, $b_2^0 = 0.4120$, $m_1^0 + p_1^0\mathrm{i} = 0.4530 + 0.2884\mathrm{i}$, $a_1^0 + a_2^0\mathrm{i} = 0.0150 + 0.3401\mathrm{i}$, $c_1^0 + c_2^0\mathrm{i} = -0.0522 - 0.0911\mathrm{i}$, $d_1^0 - d_2^0\mathrm{i} = 0.0928 - 4.0675\mathrm{i}$, $e = -0.2724$, $f = -0.2058$, and

$$\begin{aligned}\dot{r} =& 0.4530\mu_2 r + 0.0150rZ - 0.0522r^3 + 0.0929rZ^2,\\ \dot{Z} =& -1.5708\mu_1 Z + 0.4120Z^2 - 0.2724Zr^2 - 0.2058Z^3.\end{aligned} \tag{5.3}$$

Clearly, parameters c_1^0, n_2 and m_1^0 in Eq.(5.3) satisfy the conditions of Theorem 5.1, The (d_1, τ) plane (according to(μ_1, μ_2) plane) is divided into eight regions (see Fig.1) by five lines $L_0 - L_4^{\pm}$:

$L_0 : d_1 = 1.7379$, i.e. $\mu_1 = 0.1313, \mu_2 \neq 0$;
$L_1 : \tau = \pi/2,\ d_1 \neq 1.6066$, i.e. $\mu_2 = 0, \mu_1 \neq 0$;
$L_2 : d_1 = 1.6066, \tau < \pi/2$, i.e. $\mu_1 = 0, \quad \mu_2 < 0$;
$L_3 : \tau = -0.6645(d_1 - 1.6066) + \pi/2, \quad \tau > \pi/2$, i.e. $\mu_2 = -0.6645\mu_1, \quad \mu_2 > 0$;
$L_4^{\pm} : \tau = -0.4437 + 1.5636(d_1 - 1.6066) \pm 1.0768\sqrt{0.1697 - 1.2931(d_1 - 1.6066)}$,
i.e. $\mu_2 = -0.4437 + 1.5636\mu_1 \pm 1.0768\sqrt{0.1697 - 1.2931\mu_1}$.

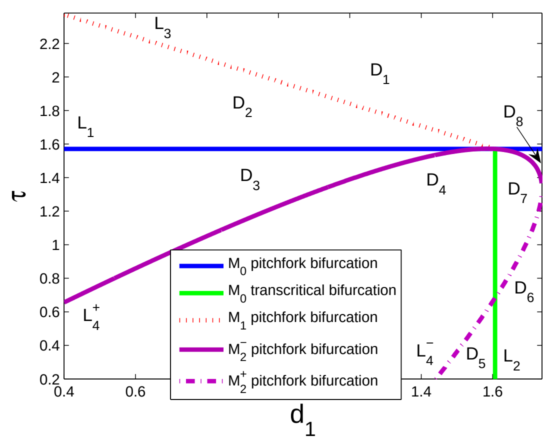

Figure 1. Bifurcation sets around $(d_1, \tau) = (1.6066, \pi/2)$ with parameters near $(\mu_1, \mu_2) = (0, 0)$.

According to the expressions of $M_1, M_2^{\pm}, M_3^{\pm}$, and note that in Fig.2 an equilibrium with the component $r \neq 0$ corresponds to the nonconstant periodic solution of system (1.1), we know that L_0 is a pitchfork bifurcation curve of system (5.3) and corresponds to a pitchfork bifurcation curve of system (1.1); L_1 is a pitchfork bifurcation curve of system (5.3) and corresponds to a Hopf bifurcation curve of system (1.1); L_2 is a transcritical bifurcation curve of system (5.3) and corresponds to a fold bifurcation curve of system (1.1); L_3 is a pitchfork bifurcation curve of an equilibrium in system (5.3) and corresponds to a pitchfork bifurcation curve of limit cycles in system (1.1); $L_4^{\pm}$ is a pitchfork bifurcation curve of system (5.3) corresponds to a Hopf bifurcation curve of system (1.1). According to the expressions of $M_1, M_2^{\pm}, M_3^{\pm}$ and the bifurcation curves $L_0 - L_4^{\pm}$, we choose eight groups of parameter values $(\mu_1, \mu_2) = (-0.0066, 0.0292)$, $(-1.2066, 0.0292)$, $(-1.2066, -0.8708)$, $(-1.2066, -0.9408)$, $(-0.0266, -1.0708)$, $(0.0234, -0.8708)$, $(0.0734, -0.1208)$, and $(0.0154, -0.0008)$,which belong to the regions D_1,D_2,D_3,D_4,D_5,D_6,D_7, and D_8, to theoretically analyze and numerically simulate the phase portrait with pplane7

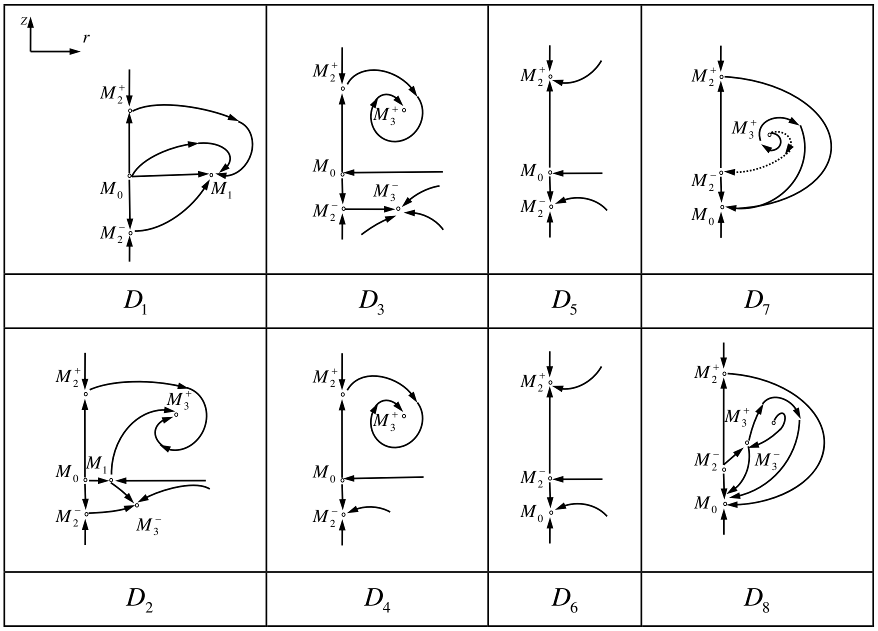

Figure 2. Phase portraits of system (5.3) with parameter (μ_1, μ_2) around $(0, 0)$ in D_1-D_8 .

program of Matlab, and obtain the phase portraits (Fig.2) of system (5.3) with parameter(μ_1, μ_2) around $(0, 0)$ in D_1-D_8, respectively.In $D_i(i = 1, 2, ..., 8.)$ of Fig.2, the vertical axis represents the Z-axis in Eq. (5.3), while the horizontal axis represents the r-axis in Eq. (5.3). Therefore, the detailed dynamics of system (1.1) in $D_1 - D_8$ near the original parameters $(d_1, \tau) = (1.6066, \pi/2)$ around $(\mu_1, \mu_2) = (0, 0)$ without curve L_0 are as follows:

According Proposition2.1, system (1.1) has no coexistence equilibrium in D_6, D_7, and D_8. In D_5, $\mu_1 < 0, \mu_2 < -0.4437 + 1.5636\mu_1 - 1.0768\sqrt{0.1697 - 1.2931\mu_1}$, system (1.1) has an unstable equilibrium (corresponding to M_0) and two stable equilibriums (corresponding to M_2^- and M_2^+) , which are bifurcated from the equilibrium (corresponding to M_0) when parameters μ_1 and μ_2 pass through the pitchfork bifurcation curve L_0. In D_4, $\mu_1 < 0, -0.4437 + 1.5636\mu_1 - 1.0768\sqrt{0.1697 - 1.2931\mu_1} < \mu_2 < -0.4437 + 1.5636\mu_1 + 1.0768\sqrt{0.1697 - 1.2931\mu_1}$ a stable periodic orbit (corresponding to M_3^+) in system (1.1) appears when parameters μ_1 and μ_2 pass through the Hopf bifurcation curve L_4^- , a stable equilibrium (corresponding to M_2^+) becomes unstable, and other two equilibriums (corresponding to M_0 and M_2^-) remain at the original states: one is stable and the other is unstable . In D_3, $\mu_1 < 0, -0.4437 + 1.5636\mu_1 + 1.0768\sqrt{0.1697 - 1.2931\mu_1} < \mu_2 < 0$, another stable periodic orbit (corresponding to M_3^-) in system (1.1) appears when parameters μ_1 and μ_2 pass through the Hopf bifurcation curve L_4^+. System (1.1) has two stable periodic orbits (corresponding to M_3^+ and M_3^-) and three unstable equilibriums (corresponding to M_0,M_2^-, and M_2^+). In D_2, $\mu_1 < 0, 0 < \mu_2 < -0.6645\mu_1$, an unstable periodic orbit (corresponding to M_1) in system (1.1) appears when parameters μ_1 and μ_2 pass through the Hopf bifurcation curve L_1, the two stable periodic orbits (corresponding to M_3^+ and M_3^-) in D_3 remain stable, and three equilibriums (corresponding to M_0, M_2^-,and M_2^+) remain unstable in system (1.1). In D_1, $\mu_1 < 0, 0.6645\mu_1 < \mu_2$, two stable periodic orbits (corresponding to M_3^+ and M_3^-) in system (1.1) disappear when parameters μ_1 and μ_2 pass through the Hopf bifurcation curve L_3, and the unstable periodic orbit (corresponding to M_1) in D_2 becomes stable. In fact , we can also regard our approach like this : in D_1, a stable periodic orbit (corresponding to M_1) in system (1.1) appears when parameters μ_1 and μ_2 pass through the Hopf bifurcation curve L_1. Then the stable periodic orbit (corresponding to M_1) in D_1 bifurcates two stable periodic orbits (corresponding to M_3^+ and M_3^-) in D_2, when parameters μ_1 and μ_2 pass through the pitchfork bifurcation (of periodic solution) curve L_3, and the periodic orbit (corresponding to M_1) becomes unstable.

We select three groups of parameter values: $(\mu_1, \mu_2) = (-1.2066, -0.8708)$ in D_3, $(\mu_1, \mu_2) = (-1.2066, -0.9408)$ in D_4, and $(\mu_1, \mu_2) = (-0.0266, -1.0708)$ in D_5 to simulate systems (1.1) (see Figs. 3, 4, and 5). In Fig.3, there are two locally sable periodic orbits corresponding to M_3^+ and M_3^-, respectively. In Fig.4, there is a locally stable equilibrium corresponding to M_2^- and a locally stable periodic orbit corresponding to M_3^+ in D_4. In Fig.5, there are two locally sable equilibriums corresponding to M_2^+ and M_2^- respectively.

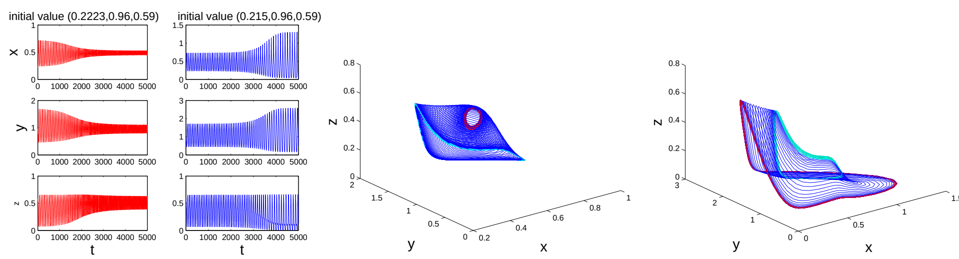

Figure 3. Waveform plots and phase planes of example 5.1 with the initial values $(0.2223, 0.96, 0.59)$ and $(0.215, 0.96, 0.59)$ when $(\mu_1, \mu_2) = (-1.2066, -0.8708)$. There are two locally stable periodic orbits.

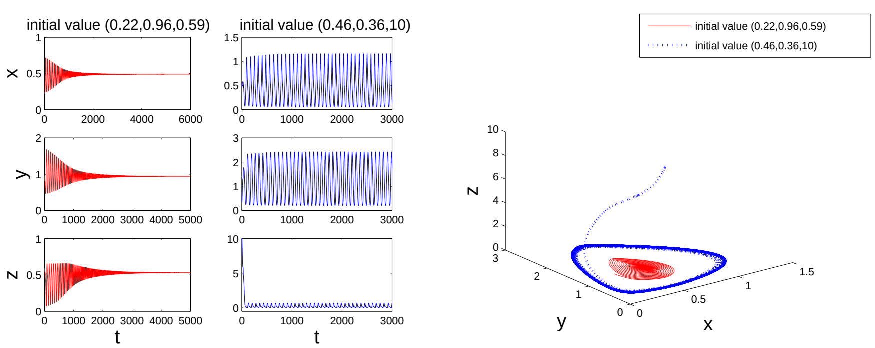

Figure 4. Waveform plots and phase plane of example 5.1 with the initial values $(0.22, 0.96, 0.59)$ and $(0.46, 0.36, 10)$ when $(\mu_1, \mu_2) = (-1.2066, -0.9408)$. There is a locally stable equilibrium and a locally stable periodic orbit.

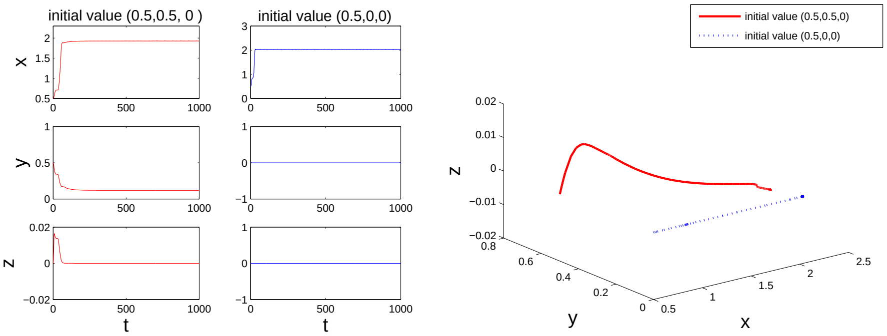

Figure 5. Waveform plots and phase plane of example 5.1 with the initial values $(0.5, 0.5, 0)$ and $(0.5, 0, 0)$ when $(\mu_1, \mu_2) = (-0.0266, -1.0708)$. There are two locally stable equilibriums.

6. Conclusion

In this paper, we take two system variables of (1.1) as two bifurcation parameters. One is the feedback time delay of prey growth leading to the occurrence of Hopf bifurcation to explain the existence of periodic orbits in the system; the other is the mortality rates of the active predator leading to the occurrence of fold bifurcation to explain the coexistence equilibrium and other coexistence of stable states. By applying the equilibrium characteristic equation, we deduce that system (1.1) undergoes Hopf-zero bifurcation at the boundary equilibrium under certain conditions. Using the delay τ and the mortality rate d_1 as two parameters, we obtain the occurrence condition of the Hopf-zero bifurcation and a corresponding third order normal form. Moreover, according to the specific parameter situation in Wang etc [19, p1546], we give the bifurcation set and the phase diagram in Section 5, and obtain that the Hopf-zero bifurcation can lead to the coexistence of three periodic orbits , as well as the coexistence of a locally stable coexistence equilibrium and a locally stable periodic orbit. In addition, we find another bistable state, that is the coexistence of two locally stable equilibriums. The different aspects between the bistable state and the monostable state are the different degrees of sensitivity property of the system. That is to say, system (1.1) with the bistable state is extra sensitive to the initial value and the variables of time delay τ and mortality rate d_1. Numerical simulation results show that under the same numerical parameters, different initial values have different dynamic behavior. Biologically, if the feedback time delay of prey growth is near $\pi/2$ and the mortality rates of the active predator is near the critical value, the states of the predator-prey system are determined sensitively by the different initial densities of system (1.1). By Proposition 2.1 we know that, when the mortality rates of active predator d_1 is greater than $g(N)/(\alpha + d_2)$, the predator will tend to die, that is, the corresponding equilibrium $(N, 0, 0)$ is stable; when the mortality rates of active predator d_1 is less than $g(N)/(\alpha + d_2)$, there is the persistent coexistence state of the predator and prey corresponding to the existence of coexistence equilibrium (x^*, y^*, z^*). In D_4, the persistent coexistence state of the predator and prey in D_5 are destroyed, the population of the predator and prey periodically oscillates to maintain persistent coexistence corresponding to the stable periodic orbit, because when the parameter value of τ increases and the parameter value of d_1 reduces, which means that the predator quickly eats up prey. Due to the lack of food, the number of the predator population reduces, which leads the prey population to slowly recover, and this cycle goes on continuously, thus the number of the predator population and that of the prey population will periodically change. The immigration of preys or predators can lead to different amplitude population changes of preys or predators, which corresponds to the theoretical results in D_3. The sensitivity properties of the system reflects that the population of predator and prey is affected by the outside condition and the season. That is to say, temperature, oxygen conditions, light, and location are the most important factors controlling egg development and dormancy in organisms, which induces the complexity of dynamics of the predator-prey system.

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Appendix: Derivation of third-order normal form

(1) Computation of $U_2^1/2! \in \mathrm{Ker}(M_2^1)^c$. From (4.5) it follows that

$$p_{I,2}^1(\frac{1}{2!}f_2) = \begin{pmatrix} -\frac{A_{11}}{a_1}x_1x_2 - \frac{A_{11}}{a_1}x_1x_3 \\ A_{21}x_1^2 + \mathrm{i}\psi_{21}b_{15}x_2^2 - \mathrm{i}\psi_{21}b_{15}x_3^2 + A_{23}x_1x_3 \\ A_{31}x_1^2 + \mathrm{i}\psi_{31}b_{15}x_2^2 - \mathrm{i}\psi_{31}b_{15}x_3^2 + A_{32}x_1x_2 \end{pmatrix}.$$

Therefore, according to $U_2^1/2! = M_2^{-1}[p_{I,2}^1(\frac{1}{2!}f_2)]$, we have

$$U_2^1/2! = \frac{2}{\pi}\begin{pmatrix} \mathrm{i}\frac{A_{11}}{a_1}x_1x_2 - \mathrm{i}\frac{A_{11}}{a_1}x_1x_3 \\ \mathrm{i}A_{21}x_1^2 + \psi_{21}b_{15}x_2^2 + \frac{1}{3}\psi_{21}b_{15}x_3^2 + \frac{\mathrm{i}}{2}A_{23}x_1x_3 \\ -\mathrm{i}A_{31}x_1^2 + \frac{1}{3}\psi_{31}b_{15}x_2^2 + \psi_{31}b_{15}x_3^2 - \frac{\mathrm{i}}{2}A_{32}x_1x_2 \end{pmatrix},$$

where

$$\begin{aligned} A_{11} =&\psi_{12}b_{11} + \psi_{13}b_{13}; \quad A_{12} = \psi_{12}b_{12} - \psi_{13}b_{14}, \\ A_{21} =& \left[-\psi_{21}b_{15} - \frac{1}{a_1}(-\psi_{21}b_{16} + \psi_{22}b_{11} + \psi_{23}b_{13})\right], \\ A_{22} =&A_{21} + \mathrm{i}\psi_{21}b_{15}; \quad A_{23} = A_{21} - \mathrm{i}\psi_{21}b_{15}, \\ A_{31} =& \left[-\psi_{31}b_{15} - \frac{1}{a_1}(-\psi_{31}b_{16} + \psi_{32}b_{11} + \psi_{33}b_{13})\right], \\ A_{32} =&A_{31} + \mathrm{i}\psi_{31}b_{15}; \quad A_{33} = A_{31} - \mathrm{i}\psi_{31}b_{15}. \end{aligned}$$

(2) Computation of $U_2^2/2! = h(\tilde{x}) = (h_1(\tilde{x}), h_2(\tilde{x}), h_3(\tilde{x}))^T \in V_2^3(Q^1)$.

From Eq.(4.9), we have

$$\frac{1}{2!}f_2^2(\tilde{x},0,0)=(I-G)Y_0F_2^2=(Y_0-\Phi(\theta)\Psi(0))F_2^2=$$

$$\left(Y_0-\begin{pmatrix}\mathrm{e}^{\frac{\pi}{2}\mathrm{i}\theta}\psi_{21}+\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\theta}\psi_{31} & \psi_{12}+\mathrm{e}^{\frac{\pi}{2}\mathrm{i}\theta}\psi_{22}+\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\theta}\psi_{32} & \psi_{13}+\mathrm{e}^{\frac{\pi}{2}\mathrm{i}\theta}\psi_{23}+\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\theta}\psi_{33}\\ 0 & -\frac{\psi_{12}}{a_1} & -\frac{\psi_{13}}{a_1}\\ 0 & \frac{\psi_{12}(b_1-2)}{\alpha+d_2} & \frac{\psi_{13}(b_1-2)}{\alpha+d_2}\end{pmatrix}\right)\cdot$$

$$\begin{pmatrix}-b_{15}(x_1+\mathrm{e}^{\frac{-\pi}{2}\mathrm{i}}x_2+\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}x_3)(x_1+x_2+x_3)-b_{16}(x_1+x_2+x_3)(-\frac{x_1}{a_1})\\ b_{11}(x_1+x_2+x_3)(-\frac{x_1}{a_1})\\ b_{13}(x_1+x_2+x_3)(-\frac{x_1}{a_1})\end{pmatrix}.$$

On the other hand, from $(M_2^2U_2^2)=f_2^2(\tilde{x},0,0)$, Eq.(4.9) is equivalent to the following equations

$\dot{h}_{1200}(\tilde{x})(\theta)=A_{21}\mathrm{e}^{\frac{\pi}{2}\mathrm{i}\theta}+A_{31}\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\theta}-\frac{A_{11}}{a_1}$,
$\dot{h}_{1020}(\tilde{x})(\theta)-\pi\mathrm{i}h_{1020}(\tilde{x})(\theta)=\mathrm{i}b_{15}(\psi_{21}\mathrm{e}^{\frac{\pi}{2}\mathrm{i}\theta}+\psi_{31}\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\theta})$,
$\dot{h}_{1002}(\tilde{x})(\theta)+\pi\mathrm{i}h_{1002}(\tilde{x})(\theta)=-\mathrm{i}b_{15}(\psi_{21}\mathrm{e}^{\frac{\pi}{2}\mathrm{i}\theta}+\psi_{31}\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\theta})$,
$\dot{h}_{1110}(\tilde{x})(\theta)-\frac{\pi}{2}\mathrm{i}h_{1110}(\tilde{x})(\theta)=A_{22}\mathrm{e}^{\frac{\pi}{2}\mathrm{i}\theta}+A_{32}\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\theta}-\frac{A_{11}}{a_1}$,
$\dot{h}_{1101}(\tilde{x})(\theta)+\frac{\pi}{2}\mathrm{i}h_{1101}(\tilde{x})(\theta)=A_{23}\mathrm{e}^{\frac{\pi}{2}\mathrm{i}\theta}+A_{33}e^{-\frac{\pi}{2}\mathrm{i}\theta}-\frac{A_{11}}{a_1}$,
$\dot{h}_{1011}(\tilde{x})(\theta)=0$; $\dot{h}_{2200}(\tilde{x})(\theta)=\frac{A_{11}}{a_1^2}$; $\dot{h}_{2020}(\tilde{x})(\theta)-\pi\mathrm{i}h_{2020}(\tilde{x})(\theta)=0$,
$\dot{h}_{2002}(\tilde{x})(\theta)+\pi\mathrm{i}h_{2002}(\tilde{x})(\theta)=0$; $\dot{h}_{2110}(\tilde{x})(\theta)-\frac{\pi}{2}\mathrm{i}h_{2110}(\tilde{x})(\theta)=\frac{A_{11}}{a_1^2}$,
$\dot{h}_{2101}(\tilde{x})(\theta)+\frac{\pi}{2}\mathrm{i}h_{2101}(\tilde{x})(\theta)=\frac{A_{11}}{a_1^2}$;$\dot{h}_{2011}(\tilde{x})(\theta)=0$;$\dot{h}_{3200}(\tilde{x})(\theta)=\frac{2-b_1}{(\alpha+d_2)a_1}A_{11}$,
$\dot{h}_{3020}(\tilde{x})(\theta)-\pi\mathrm{i}h_{3020}(\tilde{x})(\theta)=0$;$\dot{h}_{3002}(\tilde{x})(\theta)+\pi\mathrm{i}h_{3002}(\tilde{x})(\theta)=0$; $\dot{h}_{3011}(\tilde{x})(\theta)=0$,
$\dot{h}_{3110}(\tilde{x})(\theta)-\frac{\pi}{2}\mathrm{i}h_{3110}(\tilde{x})(\theta)=\frac{2-b_1}{(\alpha+d_2)a_1}A_{11}$;$\dot{h}_{3101}(\tilde{x})(\theta)+\frac{\pi}{2}\mathrm{i}h_{3101}(\tilde{x})(\theta)=\frac{2-b_1}{(\alpha+d_2)a_1}A_{11}$,
with boundary conditions:
$\dot{h}_{1200}(\tilde{x})(0)+\frac{\pi a_1}{2}h_{2200}(\tilde{x})(0)+\frac{\pi}{2}h_{1200}(\tilde{x})(-1)=-b_{15}+\frac{b_{16}}{a_1}$,
$\dot{h}_{1020}(\tilde{x})(0)+\frac{\pi a_1}{2}h_{2020}(\tilde{x})(0)+\frac{\pi}{2}h_{1020}(\tilde{x})(-1)=b_{15}\mathrm{i}$,
$\dot{h}_{1002}(\tilde{x})(0)+\frac{\pi a_1}{2}h_{2002}(\tilde{x})(0)+\frac{\pi}{2}h_{1002}(\tilde{x})(-1)=-b_{15}\mathrm{i}$,
$\dot{h}_{1110}(\tilde{x})(0)+\frac{\pi a_1}{2}h_{2110}(\tilde{x})(0)+\frac{\pi}{2}h_{1110}(\tilde{x})(-1)=-b_{15}(1-\mathrm{i})+\frac{b_{16}}{a_1}$,
$\dot{h}_{1101}(\tilde{x})(0)+\frac{\pi a_1}{2}h_{2101}(\tilde{x})(0)+\frac{\pi}{2}h_{1101}(\tilde{x})(-1)=-b_{15}(1+\mathrm{i})+\frac{b_{16}}{a_1}$,
$\dot{h}_{1011}(\tilde{x})(0)+\frac{\pi a_1}{2}h_{2011}(\tilde{x})(0)+\frac{\pi}{2}h_{1011}(\tilde{x})(-1)=0$,
$\dot{h}_{2200}(\tilde{x})(0)-\frac{\pi(b_1-2)a_1\vartheta\alpha}{2(\alpha+d_2)}h_{2200}(\tilde{x})(0)-\frac{\pi\vartheta\alpha}{2}h_{3200}(\tilde{x})(0)=-\frac{b_{11}}{a_1}$,
$\dot{h}_{2110}(\tilde{x})(0)-\frac{\pi(b_1-2)a_1\vartheta\alpha}{2(\alpha+d_2)}h_{2110}(\tilde{x})(0)-\frac{\pi\vartheta\alpha}{2}h_{3110}(\tilde{x})(0)=-\frac{b_{11}}{a_1}$,
$\dot{h}_{2101}(\tilde{x})(0)-\frac{\pi(b_1-2)a_1\vartheta\alpha}{2(\alpha+d_2)}h_{2101}(\tilde{x})(0)-\frac{\pi\vartheta\alpha}{2}h_{3101}(\tilde{x})(0)=-\frac{b_{11}}{a_1}$,
$\dot{h}_{2020}(\tilde{x})(0)-\frac{\pi(b_1-2)a_1\vartheta\alpha}{2(\alpha+d_2)}h_{2020}(\tilde{x})(0)-\frac{\pi\vartheta\alpha}{2}h_{3020}(\tilde{x})(0)=0$,
$\dot{h}_{2002}(\tilde{x})(0)-\frac{\pi(b_1-2)a_1\vartheta\alpha}{2(\alpha+d_2)}h_{2002}(\tilde{x})(0)-\frac{\pi\vartheta\alpha}{2}h_{3002}(\tilde{x})(0)=0$,
$\dot{h}_{2011}(\tilde{x})(0)-\frac{\pi(b_1-2)a_1\vartheta\alpha}{2(\alpha+d_2)}h_{2011}(\tilde{x})(0)-\frac{\pi\vartheta\alpha}{2}h_{3011}(\tilde{x})(0)=0$,
$\dot{h}_{3200}(\tilde{x})(0)-\frac{\pi a_1(2-b_1)}{2}h_{2200}(\tilde{x})(0)+\frac{\pi(\alpha+d_2)}{2}h_{3200}(\tilde{x})(0)=-\frac{b_{13}}{a_1}$,
$\dot{h}_{3110}(\tilde{x})(0)-\frac{\pi a_1(2-b_1)}{2}h_{2110}(\tilde{x})(0)+\frac{\pi(\alpha+d_2)}{2}h_{3110}(\tilde{x})(0)=-\frac{b_{13}}{a_1}$,
$\dot{h}_{3101}(\tilde{x})(0)-\frac{\pi a_1(2-b_1)}{2}h_{2101}(\tilde{x})(0)+\frac{\pi(\alpha+d_2)}{2}h_{3101}(\tilde{x})(0)=-\frac{b_{13}}{a_1}$,
$\dot{h}_{3020}(\tilde{x})(0)-\frac{\pi a_1(2-b_1)}{2}h_{2020}(\tilde{x})(0)+\frac{\pi(\alpha+d_2)}{2}h_{3020}(\tilde{x})(0)=0$,
$\dot{h}_{3002}(\tilde{x})(0)-\frac{\pi a_1(2-b_1)}{2}h_{2002}(\tilde{x})(0)+\frac{\pi(\alpha+d_2)}{2}h_{3002}(\tilde{x})(0)=0$,

$\dot{h}_{3011}(\tilde{x})(0) - \frac{\pi a_1(2-b_1)}{2} h_{2011}(\tilde{x})(0) + \frac{\pi(\alpha+d_2)}{2} h_{3011}(\tilde{x})(0) = 0.$
Solving these equations, noting that $(\Psi(s), h_q(\theta)) = 0$ $(|q| = 3)$, we obtain
$h_{1200}(\tilde{x})(\theta) = -\frac{2}{\pi}\mathrm{i}A_{21}\mathrm{e}^{\frac{\pi}{2}\mathrm{i}\theta} + \frac{2}{\pi}\mathrm{i}A_{31}\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\theta} - \frac{A_{11}}{a_1}\theta + C_{1200},$
$h_{1020}(\tilde{x})(\theta) = -\frac{2}{\pi}b_{15}\psi_{21}\mathrm{e}^{\frac{\pi}{2}\mathrm{i}\theta} - \frac{2}{3\pi}b_{15}\psi_{31}\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\theta} + C_{1020}\mathrm{e}^{\pi\mathrm{i}\theta},$
$h_{1002}(\tilde{x})(\theta) = -\frac{2}{3\pi}b_{15}\psi_{21}\mathrm{e}^{\frac{\pi}{2}\mathrm{i}\theta} - \frac{2}{\pi}b_{15}\psi_{31}\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\theta} + C_{1002}\mathrm{e}^{-\pi\mathrm{i}\theta},$
$h_{1110}(\tilde{x})(\theta) = A_{22}\theta\mathrm{e}^{\frac{\pi}{2}\mathrm{i}\theta} + \frac{\mathrm{i}}{\pi}A_{32}\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\theta} - \frac{2A_{11}}{\pi a_1}\mathrm{i} + C_{1110}\mathrm{e}^{\frac{\pi}{2}\mathrm{i}\theta},$
$h_{1101}(\tilde{x})(\theta) = -\frac{\mathrm{i}}{\pi}A_{23}\mathrm{e}^{\frac{\pi}{2}\mathrm{i}\theta} + A_{33}\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\theta} + \frac{2}{\pi}\frac{A_{11}}{a_1}\mathrm{i} + C_{1101}\mathrm{e}^{\frac{-\pi}{2}\mathrm{i}\theta},$
$h_{1011}(\tilde{x})(\theta) = 0$; $h_{2200}(\tilde{x})(\theta) = \frac{A_{11}}{a_1^2}\theta$; $h_{2020}(\tilde{x})(\theta) = 0$;$h_{2002}(x)(\theta) = 0,$
$h_{2110}(\tilde{x})(\theta) = \frac{2A_{11}}{\pi a_1^2}\mathrm{i} + C_{2110}\mathrm{e}^{\frac{\pi}{2}\mathrm{i}\theta}$; $h_{2101}(\tilde{x})(\theta) = -\frac{2A_{11}}{\pi a_1^2}\mathrm{i} + C_{2101}\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\theta}$;
$h_{2011}(\tilde{x})(\theta) = 0,$
where
$C_{1200} = \frac{2}{\pi}[-b_{15} + \frac{b_{16}}{a_1} + (1-\frac{\pi}{2})\frac{A_{11}}{a_1}]$; $C_{1020} = \frac{4}{5\pi}(1-\frac{\mathrm{i}}{2})b_{15}$; $C_{1002} = \frac{4}{5\pi}(1+\frac{\mathrm{i}}{2})b_{15},$
$C_{1110} = 117A11 * (1-2/\pi)/a1;$
$C_{1101} = 117A11 * (1-2/\pi)/a1;$
$C_{2110} = \frac{2}{\pi a_1}[-b_{15}(1-\mathrm{i}) + \frac{b_{16}}{a_1} - A_{22}(1+\frac{\pi}{2}\mathrm{i})],$
$C_{2101} = \frac{2}{\pi a_1}[-b_{15}(1+\mathrm{i}) + \frac{b_{16}}{a_1} - A_{33}(1-\frac{\pi}{2}\mathrm{i})].$

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