Journal of Applied Analysis and Computation
Volume 7, Number 2, May 2017, 745–769
Website:http://jaac-online.com/
DOI:10.11948/2017047

COMPLEX DYNAMICS OF A SIMPLE 3D AUTONOMOUS CHAOTIC SYSTEM WITH FOUR-WING*

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Abstract The present paper revisits a three dimensional (3D) autonomous chaotic system with four-wing occurring in the known literature [Nonlinear Dyn (2010) 60(3): 443–457] with the entitle "A new type of four-wing chaotic attractors in 3-D quadratic autonomous systems" and is devoted to discussing its complex dynamical behaviors, mainly for its non-isolated equilibria, Hopf bifurcation, heteroclinic orbit and singularly degenerate heteroclinic cycles, etc. Firstly, the detailed distribution of its equilibrium points is formulated. Secondly, the local behaviors of its equilibria, especially the Hopf bifurcation, are studied. Thirdly, its such singular orbits as the heteroclinic orbits and singularly degenerate heteroclinic cycles are exploited. In particular, numerical simulations demonstrate that this system not only has four heteroclinic orbits to the origin and other four symmetry equilibria, but also two different kinds of infinitely many singularly degenerate heteroclinic cycles with the corresponding two-wing and four-wing chaotic attractors nearby.

Keywords Four-wing chaotic system, Hopf bifurcation, heteroclinic orbit, singularly degenerate heteroclinic cycle.

MSC(2010) 34C45, 34C37, 34D20.

1. Introduction

The reclamation of chaos field is said to date back to the year as early as 1961 [1]. Just then, the graduate student Yoshisuke Ueda discovered the phenomenon of chaos when he studied the Duffing–van der Pol oscillator and other combinations. However, his result was not discouraged until his professor published his observations of chaos after many years. Later in 1963, by simplifying a twelve-dimensional system of differential equations to model atmospheric convection, the meteorologist Edward Lorenz at the Massachusetts Institute of Technology eventually obtained a three-dimensional autonomous dissipative system with two quadratic nonlinearities. This simple model was found to possess sensitive dependence on initial conditions and exhibits a seemingly stochastic behavior, in particular, its phase illustrates a butterfly-shaped singular attractor under some certain parameters and initial conditions. All of the interesting results [22] were reported in the known literature "Journal of the Atmospheric Sciences". From then on, the system, i.e., the Lorenz

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*The authors are partly supported by NSF of China(61473340) and NSF of Zhejiang University of Science and Technology(F701108G13 and F701108G14).

system named by others later, is considered the landmark of study of chaos theory. Because it not only sheds light on revealing the nature of chaos [23, 58] but also motivates other researchers to propose and study new chaotic systems related to it from both theory [3, 4, 24–39, 55, 57, 59–63, 74] and applications [2, 5, 53, 64, 65].

The results from the present investigations suggest that chaos has such interesting properties [8] as stemming from nonlinear systems, having at least one positive Lyapunov exponent in the discussed model [14, 15, 40], extremely sensitive dependence on initial conditions, deterministicness, unpredictability, self-excited and hidden attractors [16–19, 41–46, 51, 66–71, 75–77], Yin attractors [9, 10], extensive existence in nature and so on.

However, the forming mechanism of chaos is not very clear up to now. Hence, researchers who come from different disciplines, for example mathematics, physics, biology, electronics, astronomy, sentics, engineering, even the inter-discipline among them and so on, are devoting to giving their distinct and unique visual angles to chaos, aiming to cover all aspects of chaos.

Among this effort, Huang et al [12, 13] investigated the stochastic chaotic systems, i.e., stochastic Lorenz family of chaotic systems with jump and stochastic Lorenz-Stenflo system; Ding and Jiang [7] studied the double Hopf bifurcation and chaos of Liu system with delayed feedback; Liu and Chen [47] explored the chaotic behavior in a new fractional-order love triangle system with competition, etc. Although these novel models open up some new windows to watch the chaos, their complex structures make one a bit confused to reveal the nature of chaos in their own rights.

In this awkward position, Sprott [59] proposed an interesting theme on "Elegant chaos" in the sense of showing that interesting and realistic behaviors can result from such extremely simple models. Such kind of models are of importance in applications. For example, simple structure of any chaotic system is not only easier to design circuit, but also more accessible to the populace. However, almost all systems listed in [59] only generate the kind of single wing or scroll chaotic attractor. Therefore, constructing simple models displaying multiple–wing or multiple–scroll chaotic attractor is a desirable and challenging task.

In the endeavor, Wang et al recently in [72] proposed and considered the following two 3D autonomous chaotic systems

$$\begin{cases} \dot{x}_1 = a_1 x_1 + c_1 y_1 z_1, \\ \dot{y}_1 = b_1 x_1 + d_1 y_1 - x_1 z_1, \\ \dot{z}_1 = e_1 z_1 + f_1 x_1 y_1, \end{cases} \tag{1.1}$$

and

$$\begin{cases} \dot{x}_2 = a_2 x_2 + b_2 y_2 + c_2 x_2 z_2, \\ \dot{y}_2 = d_2 y_2 - x_2 z_2, \\ \dot{z}_2 = e_2 z_2 + f_2 x_2 y_2, \end{cases} \tag{1.2}$$

where a_i, b_i, c_i, d_i, $e_i \in \mathbb{R}$, $i = 1, 2$, $f_1 \in \mathbb{R}^-$ and $f_2 \in \mathbb{R}^+$ are all constants and x_i, y_i and z_i, $i = 1, 2$, are the state variables.

Both systems (1.1) and (1.2) are the simpler 3D smooth autonomous chaotic ones which can generate four-wing butterfly-shape chaotic attractors in the sense of the less number of both liner and nonlinear terms in the right hand of such dynamical systems, compared with the ones [6, 48, 56, 73, 78]. They are mainly investigated in view of point of their equilibria, chaotic phase portraits, Poincaré mapping, bifurcation diagram, Lyapunov exponent, spectrum versus parameters and so on. Although some good work has been done in [72], there still are some problems in that paper.

First, one can not help asking: what are the necessary conditions on their parameters in order to make both systems (1.1) and (1.2) produce four-wing butterfly-shaped chaotic attractors?

Let's first consider the first equation and the third equation of system (1.1), i.e.

$$\dot{x}_1 = a_1x_1 + c_1y_1z_1, \tag{1.3}$$

and

$$\dot{z}_1 = e_1z_1 + f_1x_1y_1. \tag{1.4}$$

By multiplying both sides of equations (1.3) and (1.4) by f_1x_1 and c_1z_1, respectively, one gets

$$f_1x_1\dot{x}_1 = a_1f_1x_1^2 + c_1f_1x_1y_1z_1, \tag{1.5}$$

and

$$c_1z_1\dot{z}_1 = c_1e_1z_1^2 + c_1f_1x_1y_1z_1. \tag{1.6}$$

Subtracting both sides of equations (1.5) and (1.6) leads to

$$f_1x_1\dot{x}_1 - c_1z_1\dot{z}_1 = a_1f_1x_1^2 - c_1e_1z_1^2, \tag{1.7}$$

which is equivalent to

$$\frac{d(f_1x_1^2 - c_1z_1^2)}{dt} = 2a_1(f_1x_1^2 - c_1z_1^2) + 2c_1(a_1 - e_1)z_1^2. \tag{1.8}$$

Solving the above equation (1.8) yields

$$(f_1x_1^2 - c_1z_1^2)e^{-2a_1t} - (f_1x_{1_0}^2 - c_1z_{1_0}^2)e^{-2a_1t_0} = \int_{t_0}^{t} 2c_1(a_1 - e_1)z_1^2(\tau)e^{-2a_1\tau}d\tau, \tag{1.9}$$

where x_{1_0} and z_{1_0} are the initial states of system (1.1).

So, in order to discuss the ergodicity, dissipativity and invariance of system (1.1), it suffices to analyze the function $(f_1x_1^2(t) - c_1z_1^2(t))e^{-2a_1t}$. It follows from equation (1.9) that $(f_1x_1^2(t) - c_1z_1^2(t))e^{-2a_1t}$ is increasing in $t \in [t_0, \infty)$ for $a_1 > e_1$ *and* $c_1 > 0$ or $a_1 < e_1$ *and* $c_1 < 0$. Therefore, these give some links for the necessary conditions on the parameters to make system (1.1) produce chaotic attractors.

Next, one can easily see that system (1.1) with $c_1 > 0$ is topologically equivalent to system (1.2) with $c_2 > 0$. (This indicates Remark 1 [72, p. 447] is not precise.)

In fact, on the one hand, the homothetic transformation

$$x_2 = y_1, \quad y_2 = x_1, \quad z_2 = -c_1z_1,$$

changes system (1.1) into

$$\begin{cases} \dot{x}_2 = d_1x_2 + b_1y_2 + \frac{1}{c_1}y_2z_2, \\ \dot{y}_2 = a_1y_2 - x_2z_2, \\ \dot{z}_2 = e_1z_2 - c_1f_1x_2y_2, \end{cases} \tag{1.10}$$

which is a special case of system (1.2).

On the other hand, the corresponding inverse transformation

$$x_1 = y_2, \quad y_1 = x_2, \quad z_1 = -\frac{1}{c} z_2,$$

brings system (1.2) into

$$\begin{cases} \dot{x}_1 = d_2 x_1 + c_1 y_1 z_1, \\ \dot{y}_1 = b_2 x_1 + a_2 y_1 - c_1 c_2 x_1 z_1, \\ \dot{z}_1 = e_2 z_1 - \frac{f_2}{c_1} x_1 y_1, \end{cases} \tag{1.11}$$

which reads that system (1.2) is only a special case of system (1.1).

Hence, for the two systems (1.1) and (1.2), it suffices to only consider system (1.1). Having carried out a great many times of numerical simulations, one finds that the solutions of system (1.1) tend to infinity for $c_1 < 0$. Hence, one only studies system (1.1) with $c_1 > 0$. Noticing that the homothetic transformation

$$x = \pm\sqrt{-f_1} x_1, \quad y = \pm\sqrt{-c_1 f_1} y_1, \quad z = \pm\sqrt{c_1} z_1,$$

changes system (1.1) into

$$\begin{cases} \dot{x} = a_1 x + yz, \\ \dot{y} = \pm b_1 \sqrt{c_1} x + d_1 y - xz, \\ \dot{z} = e_1 z - xy, \end{cases} \tag{1.12}$$

for convenience of writing, system (1.12) may be re-written as

$$\begin{cases} \dot{x} = ax + yz, \\ \dot{y} = bx + dy - xz, \\ \dot{z} = -cz - xy, \end{cases} \tag{1.13}$$

where $a, b, c, d \in \mathbb{R}$.

So, in the sequel, we mainly consider system (1.13). Anyway, some of other rich dynamics of system (1.13), i.e., system (1.1), for example, for its Hopf bifurcation, heteroclinic orbit, singularly degenerate heteroclinic cycle, etc, have not been considered yet at all. So, our main aim in this article is, by some deeper investigations and combining some numerical simulations, to formulate some new theoretical results of system (1.13), mainly for its different kind of equilibria, Hopf bifurcation and singular orbits.

The rest of this paper is organized as follows. The local dynamical behaviors of system (1.13), such as different equilibrium points and their stability and bifurcation, are discussed in Section 2. In Section 3, combining with the technique of numerical simulation, we explore the existence of some important singular orbits, including the singularly degenerate heteroclinic cycles and the heteroclinic orbits. Finally, some conclusions are drawn in Section 4.

2. Local dynamical behaviors of system (1.13)

In this section one considers the local dynamical behaviors of system (1.13) according to the following subsections.

2.1. Distribution of equilibrium point of system (1.13)

By some simple analysis, one can easily derive the following conclusion.

Theorem 2.1. *For the distribution of equilibrium point of system (1.13), the following statements hold.* $E_0 = (0,0,0)$ *is always its an equilibrium point.*

1. *When* $a = 0$*, if* $c = 0$*, then these points* $E_z = (0,0,z)$ $(z \in \mathbb{R})$ *in the z-axis and these points* $E_x^b = (x,0,b)$ *are the non-isolated equilibria of system (1.13); if* $d = 0$*, then these points* $E_y = (0,y,0)$ $(y \in \mathbb{R})$ *in the y-axis are the non-isolated equilibria of system (1.13); if* $cd \neq 0$*, then, for* $b = 0$*, these points* $E_x = (x,0,0)$ $(x \in \mathbb{R})$ *in the x-axis are the non-isolated equilibria of system (1.13); for* $b \neq 0$*,* $E_0 = (0,0,0)$ *is a unique isolate equilibrium point of system (1.13).*
2. *When* $ac < 0$*,* $d \neq 0$ *or* $ac \neq 0, b^2 - 4ad < 0$*,* $E_0 = (0,0,0)$ *is the single equilibrium point of system (1.13); when* $ac < 0$*,* $d = 0$*, these points* $E_y = (0,y,0)$ $(y \in \mathbb{R})$ *in the y-axis are the non-isolated equilibria of system (1.13).*
3. *When* $ac > 0$*, system (1.13) has two symmetric equilibria*
$$E_{1,2} = (\pm\frac{b}{2a}\sqrt{ac}, \mp\sqrt{ac}, \frac{b}{2})$$
for $b^2 - 4ad = 0$*, while, for* $b^2 - 4ad > 0$*, two pairs of symmetric ones*
$$E_{3,4} = (\pm\frac{b+\sqrt{b^2-4ad}}{2a}\sqrt{ac}, \mp\sqrt{ac}, \frac{b+\sqrt{b^2-4ad}}{2})$$
and
$$E_{5,6} = (\pm\frac{b-\sqrt{b^2-4ad}}{2a}\sqrt{ac}, \mp\sqrt{ac}, \frac{b-\sqrt{b^2-4ad}}{2}).$$

2.2. Local dynamical behaviors of E_0

Employing the linearized analysis and the center manifold theory [20], one derives the local dynamical behaviors of E_0.

At E_0, the characteristic equation of the Jacobian matrix of system (1.13) is

$$(\lambda - a)(\lambda - d)(\lambda + c) = 0 \tag{2.1}$$

with three roots $\lambda_1 = a$, $\lambda_2 = d$ and $\lambda_3 = -c$.

Consider two cases: 1. $acd \neq 0$; 2. $acd = 0$.

• 2.2.1 $acd \neq 0$.

Then E_0 is a hyperbolic stable or unstable node or saddle.

• 2.2.2 $acd = 0$.

Subcase (i) $a = 0$, $d \neq 0$ and $c \neq 0$.

Obviously, E_0 is unstable when $d > 0$ or $c < 0$. While $d < 0$ and $c > 0$, one can discuss its stability by Center Manifold Theorem [20]. It may be divided into the following two subcases: (A) $b \neq 0$ and (B) $b = 0$.

The subcase (A) $b \neq 0$ implies that Eq. (2.1) has three eigenvalues $\lambda_1 = 0$, $\lambda_2 = d$ and $\lambda_3 = -c$ with the corresponding eigenvectors $v_1 = (d, -b, 0)$, $v_2 = (0, 1, 0)$ and $v_3 = (0, 0, 1)$. Therefore, some bifurcations may occur at the non-hyperbolic equilibrium E_0.

First, we study the fold bifurcation at the origin E_0 using the bifurcation theory [20]. The following statement holds.

Theorem 2.2. *If $a = 0$, $b \neq 0$, $d < 0$ and $c > 0$, system (1.13) undergoes a degenerate fold bifurcation at the origin E_0.*

Proof. Since Eq. (2.1) has a zero root $\lambda_1 = 0$ when $a = 0$, the fold bifurcation may occur at E_0. Performing some simple algebra computations, one can easily obtain

$$p = \frac{1}{d}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} d \\ -b \\ 0 \end{pmatrix} \tag{2.2}$$

to satisfy

$$Aq = 0, \quad A^T p = 0, \quad \langle p, q \rangle = \sum_{i=1}^{3} \bar{p}_i q_i = 1,$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^3$.

The restriction of system (1.13) to the 1D center manifold has the form

$$\dot{X} = \sigma X^2 + O(|X^3|), \quad X \in \mathbb{R}, \tag{2.3}$$

where the coefficient σ can be computed by the formula

$$\sigma = \frac{1}{2}\langle p, B(q, q) \rangle. \tag{2.4}$$

If $\sigma \neq 0$, Eq. (2.3) is locally topologically equivalent to the normal form

$$\dot{X} = \beta_1 + \sigma X^2,$$

where β_1 is the unfolding parameter.

Substituting (2.2) into (2.4) yields $\sigma = 0$. So, the fold bifurcation is degenerate. □

Next, by employing bifurcation theory [11, 20], one may derive the result on the pitchfork bifurcation at E_0 as follows.

Theorem 2.3. *If $|a| \ll 1$, $b \neq 0$ and $c \neq 0$, system (1.13) undergoes a non-degenerate pitchfork bifurcation at E_0.*

Proof. Let $a = a_0 + \delta = 0 + \delta = \delta$. Then system (1.13) can be changed into the following system

$$\begin{cases} \dot{x} = \delta x + yz, \\ \dot{y} = bx + dy - xz, \\ \dot{z} = -cz - xy, \end{cases} \tag{2.5}$$

where δ is a small parameter.

Making the linear transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d & 0 & 0 \\ -b & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \tag{2.6}$$

the system (2.5) is converted into the one

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{z}_1 \end{pmatrix} = \begin{pmatrix} \delta & 0 & 0 \\ b\delta & d & 0 \\ 0 & 0 & -c \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} \frac{1}{d} z_1(-bx_1 + y_1) \\ -[(d + \frac{b^2}{d})x_1 - \frac{b}{d} y_1] z_1 \\ -dx_1(-bx_1 + y_1) \end{pmatrix}. \tag{2.7}$$

Let

$$W^c_{loc} = \left\{(x_1, y_1, z_1) \times \delta \in \mathbb{R}^3 \times \mathbb{R} | y_1 = h_1(x_1, \delta), z_1 = h_2(x_1, \delta), |x_1| < \varepsilon, |\delta| < \bar{\varepsilon}\right\},$$

where $h_1(0,0) = 0$, $h_2(0,0) = 0$, $\frac{\partial h_1}{\partial x_1}(0,0) = 0$, $\frac{\partial h_1}{\partial \delta}(0,0) = 0$, $\frac{\partial h_2}{\partial x_1}(0,0) = 0$, $\frac{\partial h_2}{\partial \delta}(0,0) = 0$ with ε, $\bar{\varepsilon}$ sufficiently small.

Assume

$$\begin{aligned} y_1 = h_1(x_1, \delta) = A_1 x_1^2 + A_2 x_1 \delta + A_3 \delta^2 + h.o.t., \\ z_1 = h_2(x_1, \delta) = B_1 x_1^2 + B_2 x_1 \delta + B_3 \delta^2 + h.o.t., \end{aligned}$$

where the high-order terms (h.o.t.) are of the orders $O(x_1^k \delta^{3-k})$, $k = 1, 2, 3$. Using the procedures as in [11,20], we obtain the vector field reduced to the center manifold

$$\dot{x_1} = h(x_1, \delta) = \delta x_1 - \frac{b^2}{c}(1 + \frac{\delta}{d}) x_1^3 + h.o.t. \tag{2.8}$$

Since $|a| \ll 1$, $b \neq 0$ and $c \neq 0$, the so-called transversality and non-degeneracy conditions

$$\begin{aligned} \frac{\partial^3 h(x_1, \delta)}{\partial x_1^3}|_{(0,0)} = -\frac{6b^2}{c} \neq 0, \\ \frac{\partial^2 h(x_1, \delta)}{\partial x_1 \partial \delta}|_{(0,0)} = 1 \neq 0 \end{aligned} \tag{2.9}$$

of generic conditions hold for pitchfork bifurcation [11, 20]. Hence, system (1.13) undergoes a non-degenerate pitchfork bifurcation when parameter a passes through the critical value $a_0 = 0$.

Moreover, when $a = 0$ and $bd \neq 0$, there is a one-dimensional center manifold, and the stability of E_0 on this center manifold is determined by the first non-zero term $-\frac{b^2}{c}$ of the right side of (2.8). Thus, E_0 is stable (resp. unstable) on its one-dimensional center manifold for $c > 0$ (resp. < 0). □

The subcase (B) implies that E_0 is more degenerate.

Subcase (ii) $a \neq 0$, $d \neq 0$ and $c = 0$. Then the following result may be easily derived.

Lemma 2.1. *Assume that $a \neq 0$, $d \neq 0$ and $c = 0$, then the center manifold of E_0 is unique and coincides with the z–axis.*

Table 1. The dynamical behavior of equilibrium E_0 of system (1.13).

a	d	c	b	Type of E_0	Property of E_0
	<0	<0		saddle	a 2D W^s_{loc} and a 1D W^u_{loc}
	<0	$=0$		non–hyperbolic	a 2D W^s_{loc} and a 1D W^c_{loc}
	<0	>0		sink	a 3D W^s_{loc}
	$=0$	<0		non–hyperbolic	a 1D W^s_{loc}, a 1D W^c_{loc} and a 1D W^u_{loc}
	$=0$	$=0$		non–hyperbolic	a 1D W^s_{loc} and a 2D W^c_{loc}
<0	$=0$	>0		non–hyperbolic	a 2D W^s_{loc} and a 1D W^c_{loc}
	>0	<0		saddle	a 1D W^s_{loc} and a 2D W^u_{loc}
	>0	$=0$		non–hyperbolic	a 1D W^s_{loc}, a 1D W^c_{loc} and a 1D W^u_{loc}
	>0	>0		saddle	a 2D W^s_{loc} and a 1D W^u_{loc}
	<0	<0	$=0$	non–hyperbolic	a 1D W^s_{loc}, a 1D W^c_{loc} and a 1D W^u_{loc}
	<0	<0	$\neq 0$	non–hyperbolic	a 1D W^s_{loc} and a 2D W^u_{loc}
	<0	$=0$		non–hyperbolic	a 1D W^s_{loc} and a 2D W^c_{loc}
	<0	>0	$=0$	non–hyperbolic	a 2D W^s_{loc} and a 1D W^c_{loc}
	<0	>0	$\neq 0$	non–hyperbolic	a 3D W^s_{loc}
	$=0$	<0		non–hyperbolic	a 2D W^c_{loc} and a 1D W^u_{loc}
$=0$	$=0$	$=0$		non–hyperbolic	a 3D W^c_{loc}
	$=0$	>0		non–hyperbolic	a 1D W^s_{loc} and a 2D W^c_{loc}
	>0	<0	$=0$	non–hyperbolic	a 1D W^c_{loc} and a 2D W^u_{loc}
	>0	<0	$\neq 0$	non–hyperbolic	a 3D W^u_{loc}
	>0	$=0$		non–hyperbolic	a 2D W^c_{loc} and a 1D W^u_{loc}
	>0	>0	$=0$	non–hyperbolic	a 1D W^s_{loc}, a 1D W^c_{loc} and a 1D W^u_{loc}
	>0	>0	$\neq 0$	non–hyperbolic	a 2D W^s_{loc} and a 1D W^u_{loc}
	<0	<0		saddle	a 1D W^s_{loc} and a 2D W^u_{loc}
	<0	$=0$		non–hyperbolic	a 1D W^s_{loc}, a 1D W^c_{loc} and a 1D W^u_{loc}
	<0	>0		saddle	a 2D W^s_{loc} and a 1D W^u_{loc}
	$=0$	<0		non–hyperbolic	a 1D W^c_{loc} and a 2D W^u_{loc}
	$=0$	$=0$		non–hyperbolic	a 2D W^c_{loc} and a 1D W^u_{loc}
>0	$=0$	>0		non–hyperbolic	a 1D W^s_{loc}, a 1D W^c_{loc} and a 1D W^u_{loc}
	>0	<0		source	a 3D W^u_{loc}
	>0	$=0$		non–hyperbolic	a 1D W^c_{loc} and a 2D W^u_{loc}
	>0	>0		saddle	a 1D W^s_{loc} and a 2D W^u_{loc}

Subcase (iii) Other cases. They may be similarly discussed. All cases can be summarized as follows.

Theorem 2.4. *The local dynamical behaviors of equilibrium E_0 of system (1.13) are totally summarized in the Table 1 when a, b, c, $d \in \mathbb{R}^4$.*

2.3. Behaviors of non-isolated equilibria E_z, E_x^b, E_y and E_x

In this subsection, one studies the dynamics of non-isolated equilibria E_z, E_x^b, E_y and E_x.

2.3.1. Behaviors of E_z

When $a = c = 0$, system (1.13) has non-isolated equilibria $E_z = (0, 0, z)$ for any $z \in \mathbb{R}$ with the eigenvalues $\lambda_{1,2} = \frac{d \pm \sqrt{d^2 + 4z(b-z)}}{2}$ and $\lambda_3 = 0$. It is easy to derive

Table 2. The behavior of non-isolated equilibria E_z of system (1.13).

d	$z(b-z)$	Property of E_z
<0	<0	a 2D W^s_{loc} and a 1D W^c_{loc}
	$=0$	a 1D W^s_{loc} and a 2D W^c_{loc}
	>0	a 1D W^s_{loc}, a 1D W^c_{loc} and a 1D W^u_{loc}
$=0$	<0	zero-Hopf bifurcation may occur
	$=0$	a 3D W^c_{loc}
	>0	a 1D W^s_{loc}, a 1D W^c_{loc} and a 1D W^u_{loc}
>0	<0	a 1D W^c_{loc} and a 2D W^u_{loc}
	$=0$	a 2D W^c_{loc} and a 1D W^u_{loc}
	>0	a 1D W^s_{loc}, a 1D W^c_{loc} and a 1D W^u_{loc}

the following consequence.

Theorem 2.5. *Assume that $a = c = 0$. Then system (1.13) has non-isolated equilibria E_z. Moreover, the local dynamical behaviors of any one are formulated in the Table 2.*

2.3.2. Behaviors of E^b_x

It follows from Theorem 2.1 that system (1.13) has the non-isolated equilibria $E^b_x = (x, 0, b)$ (for any $x, b \in \mathbb{R}$) when $a = c = 0$. The corresponding eigenvalues are $\lambda_{1,2} = \frac{d\pm\sqrt{d^2+4x^2}}{2}$ and $\lambda_3 = 0$. Hence, the dynamical properties for non-isolated equilibria $E^b_x = (x, 0, b)$ may be deduced as follows:

(i) For $d = x = 0$, there is a 3D W^c_{loc} at the neighbourhood of $E^b_x = (x, 0, b)$ for any $b \in \mathbb{R}$.

(ii) For $d > (<)0$, $x = 0$, there is a 1D W^u_{loc} (W^s_{loc}) and a 2D W^c_{loc} at the neighbourhood of $E^b_x = (x, 0, b)$ for any $b \in \mathbb{R}$.

(iii) For $x \neq 0$, there is a 1D W^s_{loc}, a 1D W^c_{loc} and a 1D W^u_{loc} at the neighbourhood of $E^b_x = (x, 0, b)$ for any $b \in \mathbb{R}$.

2.3.3. Behaviors of E_y

According to Theorem 2.1, system (1.13) has the non-isolated equilibria $E_y = (0, y, 0)$ (for any $y \in \mathbb{R}$) for the following two cases.

(i) $d = 0$, $a = 0$ and $c \neq 0$

At this time, the eigenvalues for any one of E_y are $\lambda_{1,2} = \frac{-c\pm\sqrt{c^2-4y^2}}{2}$ and $\lambda_3 = 0$. Since $c \neq 0$, $\lambda_{1,2} < 0$ or $Re(\lambda_{1,2}) < 0$ for $c > 0$, and $\lambda_{1,2} > 0$ or $Re(\lambda_{1,2}) > 0$ for $c < 0$.

(ii) $d = 0$ and $ac < 0$

It is easy to obtain the corresponding eigenvalues $\lambda_{1,2} = \frac{(a-c)\pm\sqrt{(a-c)^2+4(ac-y^2)}}{2}$ and $\lambda_3 = 0$. Since $ac < 0$, $\lambda_{1,2} < 0$ or $Re(\lambda_{1,2}) < 0$ for $a - c < 0$, and $\lambda_{1,2} > 0$ or $Re(\lambda_{1,2}) > 0$ for $a - c > 0$.

Table 3. The behavior of non-isolated equilibria E_x of system (1.13).

$d-c$	$cd+x^2$	Property of E_x
<0	<0	a 2D W^s_{loc} and a 1D W^c_{loc}
	$=0$	a 1D W^s_{loc} and a 2D W^c_{loc}
	>0	a 1D W^s_{loc}, a 1D W^c_{loc} and a 1D W^u_{loc}
$=0$	>0	a 1D W^s_{loc}, a 1D W^c_{loc} and a 1D W^u_{loc}
>0	<0	a 1D W^c_{loc} and a 2D W^u_{loc}
	$=0$	a 2D W^c_{loc} and a 1D W^u_{loc}
	>0	a 1D W^s_{loc}, a 1D W^c_{loc} and a 1D W^u_{loc}

2.3.4. Behaviors of E_x

It follows Theorem 2.1 that system (1.13) has the non-isolated equilibria $E_x = (x,0,0)$ (any $x \in \mathbb{R}$) for $a=b=0$ and $cd \neq 0$. All the eigenvalues for any one of E_x are $\lambda_{1,2} = \frac{d-c\pm\sqrt{(d-c)^2+4(cd+x^2)}}{2}$ and $\lambda_3 = 0$.

Therefore, one may derive the following consequence.

Theorem 2.6. *Assume that $a=b=0$ and $cd \neq 0$. Then system (1.13) has non-isolated equilibria E_x. Moreover, the local dynamical behaviors of any one are formulated in the Table 3.*

2.4. Behaviors of $E_{1,2}$

In this subsection, one studies the dynamics of isolated equilibria $E_{1,2}$. Notice that $ac>0$ and $b^2=4ad$ at this time. The characteristic equation of the Jacobian matrix of system (1.13) at equilibria $E_{1,2}$ is

$$\lambda^3 + (c-a-d)\lambda^2 - 2cd\lambda = 0$$

with the corresponding eigenvalues $\lambda_1 = 0$ and $\lambda_{2,3} = \frac{a+d-c\pm\sqrt{(a+d-c)^2+8cd}}{2}$. So, the following statements are valid.

Theorem 2.7. *Assume that $ac>0$ and $b^2=4ad$. Then system (1.13) has a pair of non-hyperbolic equilibria $E_{1,2}$. For $b=0$, at their neighbourhood there exist as least a two-dimensional center. For $b\neq 0$, at their neighbourhood there exist a 1D W^s_{loc}, a 1D W^c_{loc} and a 1D W^u_{loc}.*

Furthermore, one can study the fold and pitchfork bifurcation at $E_{1,2}$ as well as at E_0. The following conclusion can be obtained.

Theorem 2.8. *Assume that $ac>0$ and $b^2=4ad$. Then system (1.13) undergoes a non-degenerate fold bifurcation but no pitchfork bifurcations at $E_{1,2}$.*

Remark 2.1. It follows Theorem 2.8 that two pairs of equilibria $E_{3,4}$ and $E_{5,6}$ are produced through the fold bifurcations occurring at the twin equilibria $E_{1,2}$.

2.5. Behavior of $E_{3,4}$ and $E_{5,6}$

In this subsection, by invoking the Routh–Hurwitz stability criterion [54] and bifurcation theory [20], we mainly study the stability and Hopf bifurcation of $E_{3,4}$ and

$E_{5,6}$.

The characteristic equation of the Jacobian matrix of system (1.13) at any one of the above four equilibria is

$$\lambda^3 + (c - a - d)\lambda^2 - \frac{bcz}{a}\lambda + 2bcz - 4acd = 0, \tag{2.10}$$

where $z = \frac{b+\sqrt{b^2-4ad}}{2}$ for $E_{3,4}$ and $z = \frac{b-\sqrt{b^2-4ad}}{2}$ for $E_{5,6}$.

For convenience of discussion in the sequel, the Routh–Hurwitz stability criterion [54] is stated as follows:

Lemma 2.2. *The polynomial $P(\lambda) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3$ with real coefficients p_1, p_2 and p_3 has all roots with negative real parts if and only if the numbers p_1, p_2 and p_3 are positive and the inequality $p_1p_2 > p_3$ is satisfied.*

Remember that the previous condition for the existence of $E_{3,4}$ and $E_{5,6}$ is the parameters a, b, c, d lie in the set $W = \{(a, b, c, d) \in \mathbb{R}^4, ac > 0, b^2 - 4ad > 0\}$. For convenience of discussion in the sequel, divide the set W into W_1 and W_2 as follows:

$$W_1 = \{(a, b, c, d) \in W : a > 0, c > 0, b^2 - 4ad > 0\}$$

and

$$W_2 = \{(a, b, c, d) \in W : a < 0, c < 0, b^2 - 4ad > 0\}.$$

Then $W = W_1 \cup W_2$.

2.5.1. Behavior of $E_{3,4}$

Due to the symmetry between E_3 and E_4, it suffices to consider E_3. The characteristic equation of the Jacobian matrix of system (1.13) at E_3 is

$$\lambda^3 + (c - a - d)\lambda^2 - \frac{bc(b + \sqrt{b^2 - 4ad})}{2a}\lambda + c\sqrt{b^2 - 4ad}(b + \sqrt{b^2 - 4ad}) = 0. \tag{2.11}$$

(In fact, it follows from (2.10) that the equilibria E_3 and E_4 have the same characteristic equation, so, it is indeed sufficient to consider the equilibrium point E_3.) Now divide the set W_1 into the union of the subsets W_{11} and W_{12}, where

$$W_{11} = \{(a, b, c, d) \in W_1 : b < 0, d < 0\},$$
$$W_{12} = \{(a, b, c, d) \in W_1 : b > 0 \quad \text{or} \quad b = 0,\ d < 0 \quad \text{or} \quad b < 0,\ d \geq 0\}.$$

For $(a, b, c, d) \in W_1$, define $c_0^1 = a + d - \frac{2a\sqrt{b^2-4ad}}{b}$. Denote W_{11} as

$$W_{11} = W_{111} \bigcup W_{112} \bigcup W_{113} \bigcup W_{114},$$

where

$$W_{111} = \{(a, b, c, d) \in W_{11} : c > c_0^1 > 0\},$$
$$W_{112} = \{(a, b, c, d) \in W_{11} : c = c_0^1 > 0\},$$
$$W_{113} = \{(a, b, c, d) \in W_{11} : 0 < c < c_0^1\},$$
$$W_{114} = \{(a, b, c, d) \in W_{11} : c_0^1 \leq 0\}.$$

Using Lemma 2.2, the following consequences may be easily derived.

Theorem 2.9. *The equilibrium E_3 of system (1.13) is unstable for $(a,b,c,d) \in W_2 \cup W_{12} \cup W_{113}$ whereas E_3 is asymptotically stable for $(a,b,c,d) \in W_{111} \bigcup W_{114}$.*

From the above Theorem 2.9, one can see that there will be a bifurcation occurrence for $(a,b,c,d) \in W_{112}$. Then, what kind of bifurcation is it ? Next, we will give the answer. The following lemma is true.

Lemma 2.3. *For $(a,b,c,d) \in W_{112}$, system (1.13) undergoes a Andronov-Hopf bifurcation at E_3.*

Proof. For $(a,b,c,d) \in W_{112}$, it follows that Eq.(2.11) has one negative real root $\lambda_1 = \frac{2a\sqrt{b^2-4ad}}{b}$ and a pair of conjugate purely imaginary roots $\lambda_{2,3} = \pm\omega i$ with $\omega = \sqrt{\frac{2a\sqrt{b^2-4ad}}{-b}(a+d-\frac{2a\sqrt{b^2-4ad}}{b})}$. Taking into account that $Re(\lambda_2) = 0$ at $c = c_0^1$, one obtains

$$\left.\frac{dRe(\lambda_2)}{dc}\right|_{c=c_0^1} = -\frac{\omega^2}{2[\omega^2+\lambda_1^2]} < 0.$$

Hence, the transversal condition holds. Also, $\lambda_1 < 0$. Therefore, all conditions for Hopf bifurcation [20] to occur are met. So, the Hopf bifurcation happens at E_3. The proof for this lemma is over. □

Therefore, one has the following result.

Theorem 2.10. *The equilibria $E_{3,4}$ of system (1.13) are unstable for $(a,b,c,d) \in W_2 \cup W_{12} \cup W_{113}$ whereas asymptotically stable for $(a,b,c,d) \in W_{111} \cup W_{114}$, and system (1.13) undergoes Andronov-Hopf bifurcations at the equilibria $E_{3,4}$ for (a,b,c,d)$\in W_{112}$.*

For the numerical simulations, see Fig.1.

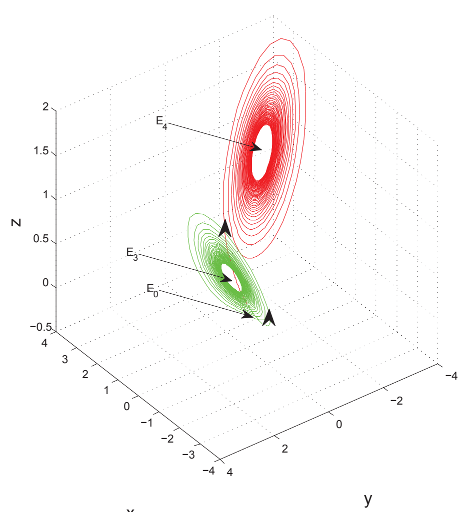


Figure 1. Phase portraits of system (1.13) for $(a,b,d,c) = (1,-1,-2,5.0)$ and initial values $(x_0,y_0,z_0) = (\pm 0.314 \times 1e-2, \pm 0.382 \times 1e-2, 0)$, illustrating that system (1.13) undergoes Hopf bifurcations at $E_{3,4}$.

2.5.2. Behavior of $E_{5,6}$

It suffices to consider E_5. Similar to dealing with the equilibrium point E_3, divide the set W_1 into the union of the subsets W_{21} and W_{22}, where

$$W_{21} = \{(a,b,c,d) \in W_1 : b > 0, d < 0\},$$
$$W_{22} = \{(a,b,c,d) \in W_1 : b < 0 \quad or \quad b = 0,\ d < 0 \quad or \quad b > 0,\ d \geq 0\}.$$

For $(a,b,c,d) \in W_{21}$, define $c_0^2 = a + d + \frac{2a\sqrt{b^2-4ad}}{b}$. Denote W_{21} as

$$W_{21} = W_{211} \bigcup W_{212} \bigcup W_{213} \bigcup W_{214},$$

where

$$W_{211} = \{(a,b,c,d) \in W_{11} : c > c_0^2 > 0\},$$
$$W_{212} = \{(a,b,c,d) \in W_{11} : c = c_0^2 > 0\},$$
$$W_{213} = \{(a,b,c,d) \in W_{11} : 0 < c < c_0^2\},$$
$$W_{214} = \{(a,b,c,d) \in W_{11} : c_0^2 \leq 0\}.$$

Then we can obtain the results for the stability and Hopf bifurcation of $E_{5,6}$ as follows.

Theorem 2.11. *The equilibria $E_{5,6}$ of system (1.13) are unstable for $(a,b,c,d) \in W_2 \cup W_{22} \cup W_{213}$ whereas $E_{5,6}$ are asymptotically stable when $(a,b,c,d) \in W_{211} \cup W_{214}$, and system (1.13) undergoes Andronov-Hopf bifurcations at equilibria $E_{5,6}$ for $(a,b,c,d) \in W_{212}$.*

For the numerical simulations, see Fig.2.

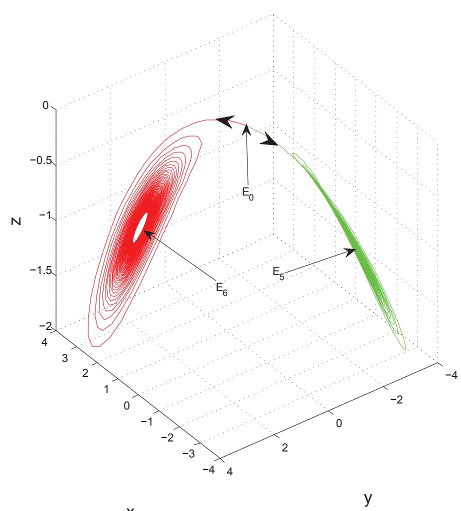
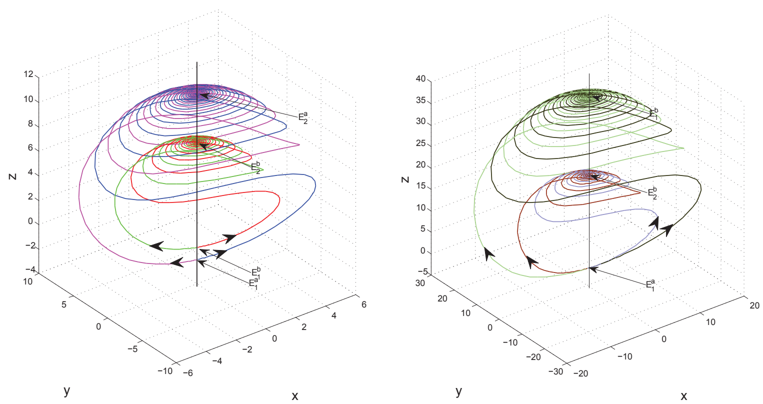


Figure 2. Phase portraits of system (1.13) when $(a,b,d,c) = (1,1,-2,5.0)$ and initial values $(x_0, y_0, z_0) = (\pm 0.314 \times 1e-2, \pm 0.382 \times 1e-2, 0)$, which implies that system (1.13) undergoes Hopf bifurcations at $E_{5,6}$.

3. Existence of singularly degenerate heteroclinic cycle

As stated in [21, 49, 50, 52, 62, 63, 69, 70, 74] and the references therein, the importance of singularly degenerate heteroclinic cycle lies in that chaotic attractors can be created when the singularly degenerate heteroclinic cycle disappears for certain parameter values of the system. For system (1.13), Wang et.al [72] did not study the existence of singularly degenerate heteroclinic cycle at all. Up to now, this has not been discussed in other literature as well as one knows. Since system (1.13) is considered as the simplest chaotic system which can produce some four-wing butterfly-shape chaotic attractors, it is interesting to study the existence of singularly degenerate heteroclinic cycle, which may be helpful to understand the mechanism of forming this kind of strange attractor. Some similar mechanics has been verified in the Rabinovich system [49]. Whether or not can various chaotic attractors, especially the interesting four-wing attractors, be bifurcated from singularly degenerate heteroclinic cycle of system (1.13)? We will give a positive answer in this section.



(a) (b)

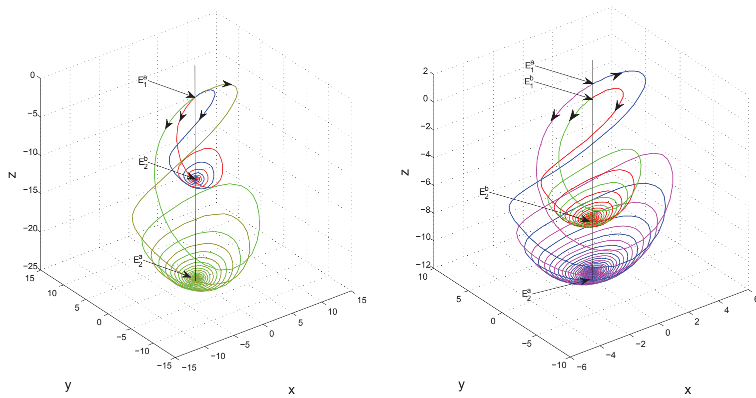
Figure 3. Phase portraits of system (1.13) when (a) $(a, b, d, c) = (2, -4, -3, 0)$ and initial values $E_1^a = (\pm 0.314 \times 1e-2, \pm 0.382 \times 1e-2, -2)$, and $E_1^b = (\pm 0.314 \times 1e-2, \pm 0.382 \times 1e-2, -1)$, (b) $E_1^a = (\pm 0.314 \times 1e-2, \pm 0.382 \times 1e-2, -1)$, and $(a, b, d, c) = (6, -3, -9, 0)$, $(a, b, d, c) = (4, -3, -7, 0)$. These figures illustrate that system (1.13) has infinitely many degenerate heteroclinic cycles when $a > 0$, $d < 0$, $b < 0$, $c = 0$ and $a + d < 0$.

First, combining the dynamics of E_z and some suitable choice of parameters a, b, c, d, one may derive the following conclusion.

Numerical Result. 3.1. For $a > 0$, $d < 0$, $c = 0$ and $a + d < 0$, the 1D unstable manifold $W^u(E_1)$ of each normally hyperbolic saddle-like (see Figs. 3–4) $E_1 = (0, 0, z_1)$ tends to one of the normal hyperbolic stable focus-like $E_2 = (0, 0, z_2)$ given in Theorem 2.5 as $t \to \infty$, forming singularly degenerate heteroclinic cycles.

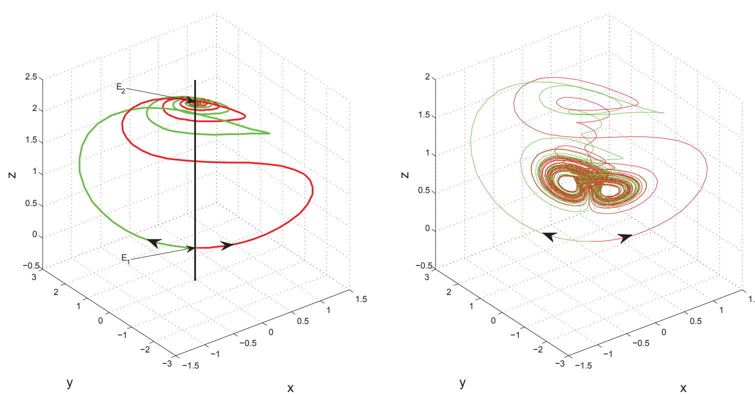
As suggested in [21, 30, 50, 52, 62, 63, 74], such the two-wing strange attractors as shown in Figs. 5–6 can be created in a neighborhood of the families of singularly degenerate heteroclinic cycles.

Now considering the similarity of system (1.13) and Rabinovich system [49] and choosing some suitable parameters a, b, c, d, one may derive the following conclusion.



(a) (b)

Figure 4. Phase portraits of system (1.13) when (a) $E_1^a = (\pm 0.314 \times 1e-2, \pm 0.382 \times 1e-2, -1)$, and $(a, b, d, c) = (6, 3, -9, 0)$, $(a, b, d, c) = (4, 3, -7, 0)$, (b) $(a, b, d, c) = (2, 4, -3, 0)$, initial values $E_1^a = (\pm 0.314 \times 1e-2, \pm 0.382 \times 1e-2, 2)$, and $E_1^b = (\pm 0.314 \times 1e-2, \pm 0.382 \times 1e-2, 1)$. These figures illustrate that system (1.13) has infinitely many degenerate heteroclinic cycles when $a > 0$, $d < 0$, $b > 0$, $c = 0$ and $a + d < 0$.



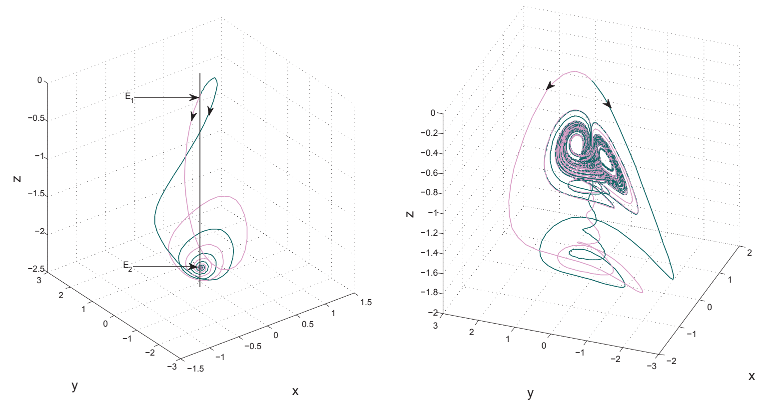
(a) (b)

Figure 5. Phase portraits of system (1.13) when $E_1 = (\pm 0.314 \times 1e-2, \pm 0.382 \times 1e-2, 0)$, (a) $(a, b, d, c) = (2, -8, -3, 0.0)$, (b) $(a, b, d, c) = (2, -8, -3, 0.1)$. The figures demonstrate that system (1.13) has some two-wing chaotic attractors which can be produced near the corresponding singularly degenerate heteroclinic cycles for $c > 0$.

Numerical Result. 3.2. For $a = b = 0$ and $d = -c \neq 0$, the 1D unstable manifold $W^u(E_1)$ of each normally hyperbolic saddle-like (see Figs. 7–8) $E_1 = (x_1, 0, 0)$ tends to one of the normal hyperbolic stable focus-like $E_2 = (x_2, 0, 0)$ in the planes $\{y = z\}$ and $\{y = -z\}$ as $t \to \infty$, forming singularly degenerate heteroclinic cycles.

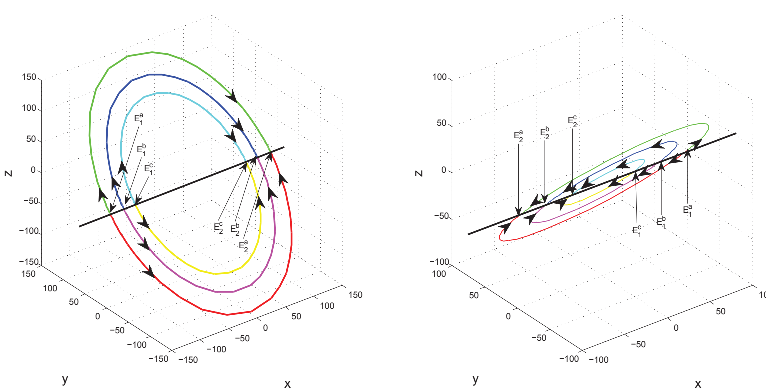
Numerical Result. 3.3. For $a = b = 0$, $d < 0$, $c > 0$ and $d - c < 0$, the 1D unstable manifold $W^u(E_1)$ of each normally hyperbolic saddle-like (see Fig. 10) $E_1 = (x_1, 0, 0)$ tends to one of the normal hyperbolic stable focus-like $E_2 = (x_2, 0, 0)$ as $t \to \infty$, forming singularly degenerate heteroclinic cycles.

Therefore, it follows from **Numerical Result. 3.2** and [49, Theorem 4, p.275210-4] that the following statements hold.



(a) (b)

Figure 6. Phase portraits of system (1.13) when $E_1 = (\pm 0.314 \times 1e - 2, \pm 0.382 \times 1e - 2, 0)$, (a) $(a, b, d, c) = (2, 8, -3, 0.0)$, (b) $(a, b, d, c) = (2, 8, -3, 0.1)$. The figures display that system (1.13) has some two-wing chaotic attractors which can be produced near the corresponding singularly degenerate heteroclinic cycles for $c > 0$.

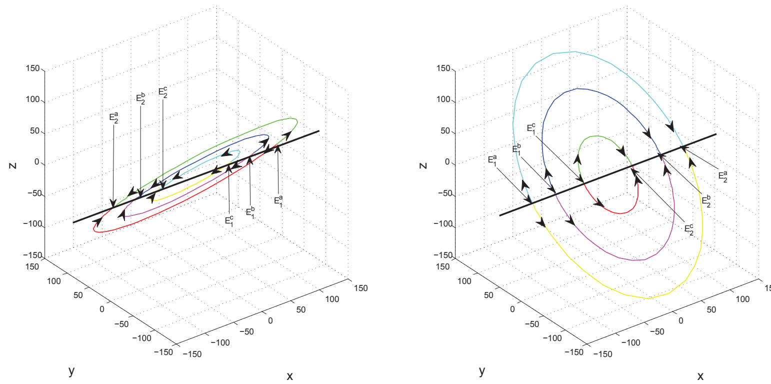


(a) (b)

Figure 7. Phase portraits of system (1.13) when $(a, b, d, c) = (0, 0, -2, 2)$ and (a) $E_1^a = (-144, \pm 0.382 \times 1e - 2, \pm 0.382 \times 1e - 2)$, $E_1^b = (-120, \pm 0.382 \times 1e - 2, \pm 0.382 \times 1e - 2)$, $E_1^c = (-100, \pm 0.382 \times 1e - 2, \pm 0.382 \times 1e - 2)$, (b) $E_1^a = (100, \pm 0.382 \times 1e - 2, \mp 0.382 \times 1e - 2)$, $E_1^b = (70, \pm 0.382 \times 1e - 2, \mp 0.382 \times 1e - 2)$, $E_1^c = (40, \pm 0.382 \times 1e - 2, \mp 0.382 \times 1e - 2)$. The figures imply that system (1.13) has two families of singularly degenerate heteroclinic cycles when $a = b = 0$ and for any $-d = c > 0$, one of which is contained in the plane $\{y = z\}$ and the other in the plane $\{y = -z\}$.

Theorem 3.1. *For the parameter values $a = b = 0$ and for any $d = -c \neq 0$, system (1.13) has another two families of singularly degenerate heteroclinic cycles. One of the families is contained in the plane $\{y = z\}$ and the other in the plane $\{y = -z\}$. Moreover each family contains an infinite set of such degenerate cycles such that they accumulate together at a heteroclinic cycle on the sphere of infinity when these cycles run to infinity.*

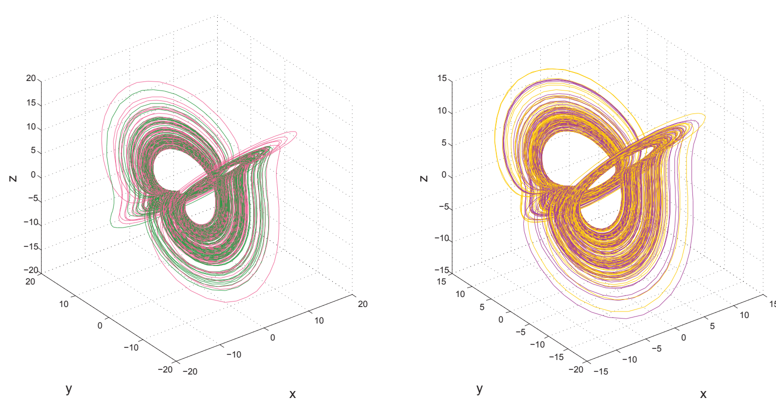
Except for the two-wing chaotic attractors illustrated in Figs. 5–6 occuring in a neighborhood of the families of singularly degenerate heteroclinic cycles which are shown in Figs. 3–4, one also detects the attractors of four wings type near the



(a) (b)

Figure 8. Phase portraits of system (1.13) when $(a, b, d, c) = (0, 0, 2, -2)$ and (a) $E_1^a = (140, \pm 0.382 \times 1e-2, \pm 0.382 \times 1e-2)$, $E_1^b = (100, \pm 0.382 \times 1e-2, \pm 0.382 \times 1e-2)$, $E_1^c = (60, \pm 0.382 \times 1e-2, \pm 0.382 \times 1e-2)$, (b) $E_1^a = (-130, \pm 0.382 \times 1e-2, \mp 0.382 \times 1e-2)$, $E_1^b = (-90, \pm 0.382 \times 1e-2, \mp 0.382 \times 1e-2)$, $E_1^c = (-40, \pm 0.382 \times 1e-2, \mp 0.382 \times 1e-2)$. The figures demonstrate that system (1.13) has two families of singularly degenerate heteroclinic cycles when $a = b = 0$ and for any $-d = c < 0$, one of which is contained in the plane $\{y = z\}$ and the other in the plane $\{y = -z\}$.

singularly degenerate heteroclinic cycles displayed in Figs. 9, 11, especially the ones which lies in the plane $\{y = z\}$ and the others in the plane $\{y = -z\}$, see Fig. 9.

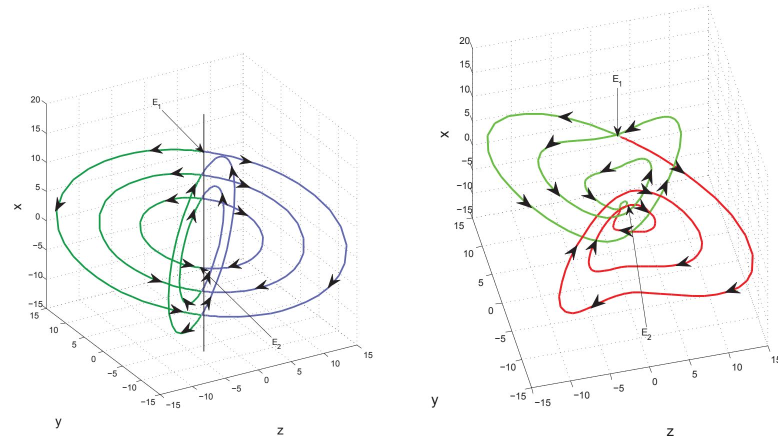


(a) (b)

Figure 9. Phase portraits of system (1.13) when $(a, b, d, c) = (1.2, 1.1, -2, 3)$ and (a) $(x_0, y_0, z_0) = (3, \pm 0.382 \times 1e-2, \pm 0.382 \times 1e-2)$, (b) $(x_0, y_0, z_0) = (3, \pm 0.382 \times 1e-2, \mp 0.382 \times 1e-2)$. The figures illustrate that system (1.13) has some four-wing chaotic attractors near two families of singularly degenerate heteroclinic cycles, one of which is contained in the plane $\{y = z\}$ and the other in the plane $\{y = -z\}$.

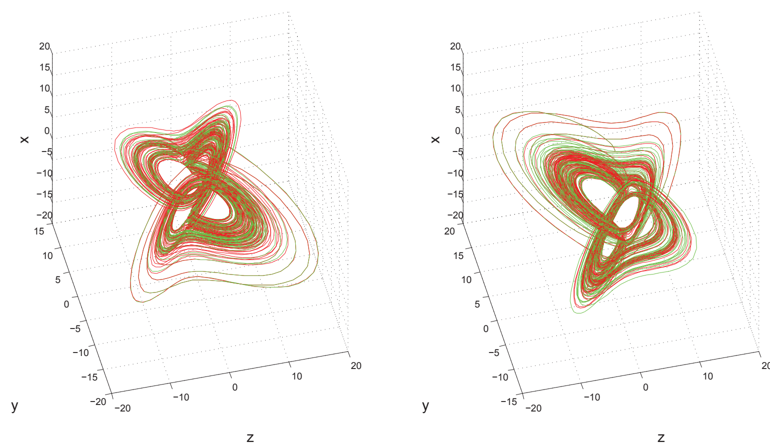
4. Existence of heteroclinic orbit

In order to discuss the herteroclinic orbit of system (1.13), let's review some facts as follows.



(a) (b)

Figure 10. Phase portraits of system (1.13) when $(x_0, y_0, z_0) = (15, \pm 0.314 \times 1e-2, \pm 0.382 \times 1e-2)$ and (a) $(a, b, d, c) = (0, 0, -1, 1)$, (b) $(a, b, d, c) = (0, 0, -1, 1.6)$. The figures demonstrate that system (1.13) has some singularly degenerate heteroclinic when $d < 0$, $c > 0$ and $d - c < 0$.



(a) (b)

Figure 11. Phase portraits of system (1.13) when $(x_0, y_0, z_0) = (15, \pm 0.314 \times 1e-2, \pm 0.382 \times 1e-2)$ and (a) $(a, b, d, c) = (0.7, -2, -1, 1.6)$, (b) $(a, b, d, c) = (0.7, 2, -1, 1.6)$. The figures suggest that system (1.13) has some four-wing chaotic attractors near singularly degenerate heteroclinic when $d < 0$, $c > 0$, $d - c < 0$, $a > 0$ and $b \neq 0$.

Fact 4.1. E_0 is a saddle but $E_{3,4}$ are locally asymptotically stable when $(a, b, c, d) \in W_{111} \cup W_{114}$ according to Theorem 2.10.

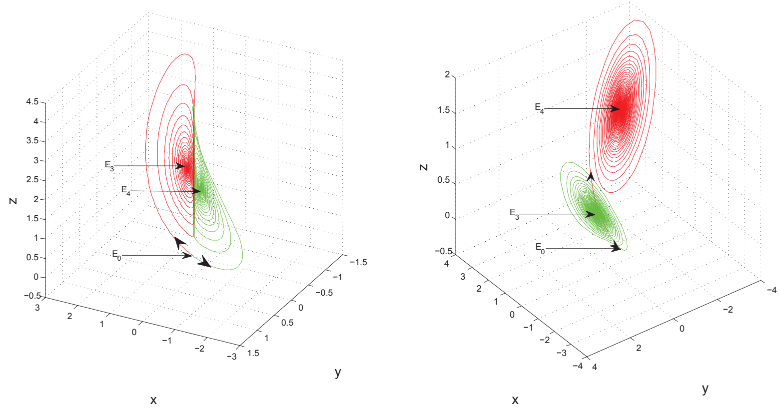
Fact 4.2. E_0 is a saddle while $E_{5,6}$ are locally asymptotically stable when $(a,b,c,d) \in W_{211} \cup W_{214}$ according to Theorem 2.11.

Heuristically, one has the following numerical simulations concerning with the herteroclinic orbit of system (1.13), see Figs. 12–13.

Numerical Result. 4.1. For $(a, b, c, d) \in W_{111} \cup W_{114}$, the 1D unstable manifold $W^u(E_0)$ of the saddle E_0 tends to the stable manifolds $W^s(E_{3,4})$ of the focus $E_{3,4}$

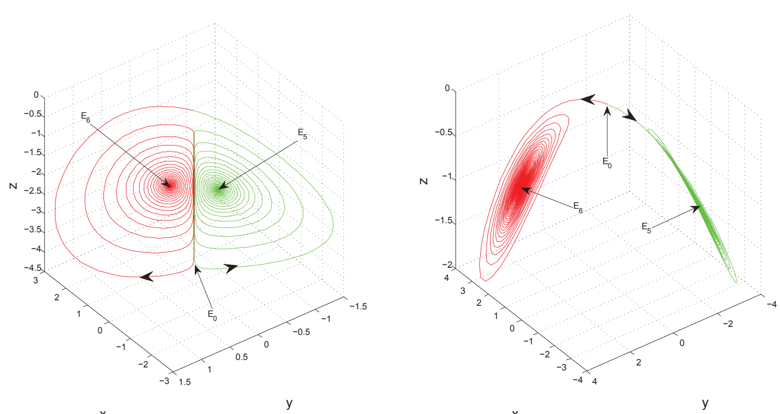
presented in Remark 4.1 as $t \to \infty$, forming two heteroclinic orbits to E_0 and $E_{3,4}$, see Fig. 12.

Numerical Result. 4.2. For $(a, b, c, d) \in W_{211} \cup W_{214}$, the 1D unstable manifold $W^u(E_0)$ of the saddle E_0 tends to the stable manifolds $W^s(E_{5,6})$ of the focus $E_{5,6}$ presented in Remark 4.2 as $t \to \infty$, forming two heteroclinic orbits to E_0 and $E_{5,6}$, see Fig. 13.



(a) (b)

Figure 12. Phase portraits of system (1.13) when $(x_0, y_0, z_0) = (\pm 0.314 \times 1e - 2, \pm 0.382 \times 1e - 2, 0)$ and parameters (a) $(a, b, d, c) = (1, -3, -10, 0.05)$, (b) $(a, b, d, c) = (1, -1, -2, 5.2)$. Both figures suggest that system (1.13) has two heteroclinic orbits to E_0 and $E_{3,4}$ for $(a, b, c, d) \in W_{111} \cup W_{114}$.



(a) (b)

Figure 13. Phase portraits of system (1.13) when $(x_0, y_0, z_0) = (\pm 0.314 \times 1e - 2, \pm 0.382 \times 1e - 2, 0)$ and parameters (a) $(a, b, d, c) = (1, 3, -10, 0.05)$, (b) $(a, b, d, c) = (1, 1, -2, 5.2)$. Both figures show that system (1.13) has two heteroclinic orbits to E_0 and $E_{5,6}$ when $(a, b, c, d) \in W_{211} \cup W_{214}$.

5. Conclusion

In this paper, we have revisited two so-called simplest 3D autonomous chaotic systems, i.e. system (20) and system (21) in [72, p. 447], which are able to generate four-wing butterfly-shaped chaotic attractors. By some linear transformations, we find, system (20) with $c_1 > 0$ and $f_1 < 0$ is topologically equivalent to system (21) with $c_2 > 0$ and $f_2 > 0$. Therefore, the two systems can be only reduced into one system (1.13) in this paper. Hence, we also point out that the comparison between system (20) and system (21) is unnecessary through numerical simulations illustrated in [72, Figs. 3–12, pp. 452–456].

After some other problems in [72] are pointed out and completely solved, some interesting and important dynamical behaviors of system (1.13), such as the Hopf bifurcation, four different kinds of non-isolated equilibria, the existence of different families of infinitely many degenerate heteroclinic cycles and the existence of heteroclinic orbits, etc., which are not studied in any known literature, are formulated in this paper.

By numerical simulations, we find that there exist two-wing, four-wing butterfly-shaped chaotic attractors that can be bifurcated from distinct singularly degenerate heteroclinic cycles, and four heteroclinic orbits to E_0, $E_{3,4}$ and $E_{5,6}$. It is worthwhile to further theoretically explore the mechanism for the occurrence of such chaotic attractors in the future. Hence, all of these complex dynamics demonstrate that system (1.13) deserves further considering.

It is hoped that our work will shed some lights on revealing the true geometrical structure of the amazing original Lorenz attractor, even the forming mechanism of chaos.

Acknowledgements

This work is partly supported by NSF of China (61473340) and NSF of Zhejiang University of Science and Technology (F701108G13 and F701108G14).

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