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A SUPERCONVERGENT L^∞-ERROR ESTIMATE OF RT1 MIXED METHODS FOR ELLIPTIC CONTROL PROBLEMS WITH AN INTEGRAL CONSTRAINT*

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Abstract In this paper, we investigate the superconvergence property of mixed finite element methods for a linear elliptic control problem with an integral constraint. The state and co-state are approximated by the order $k = 1$ Raviart-Thomas mixed finite element spaces and the control variable is approximated by piecewise constant functions. A superconvergent approximation of the control variable u will be constructed by a projection of the discrete adjoint state. It is proved that this approximation have convergence order h^2 in L^∞-norm. Finally, a numerical example is given to demonstrate the theoretical results.

Keywords Elliptic equations, optimal control problems, superconvergence, mixed finite element methods, postprocessing.

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1. Introduction

As far as we know, the finite element approximation plays an important role in the numerical treatment of optimal control problems. There have been extensive studies in convergence and superconvergence of finite element approximations for optimal control problems, see, for example, [1, 6, 11–16, 19–21]. A systematic introduction of finite element methods for PDEs and optimal control problems can be found in, for example, [9, 18]. Note that all the above papers aim at the standard finite element methods for optimal controls.

In the recent years, convergence and superconvergence properties of mixed finite elements for optimal control problems have been done in [3–5]. In [4], the author used the postprocessing projection operator, which was defined by Meyer and Rösch (see [19]) to prove a quadratic superconvergence of the control by mixed finite element methods. Recently, the authors derived convergence and superconvergence of mixed methods for convex optimal control problems in [5]. Since the analysis was restricted by the low regularity of the control, the convergence order is $h^{\frac{3}{2}}$.

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The goal of this paper is to derive the superconvergence property of mixed finite element approximation for a linear elliptic control problem with an integral constraint. Firstly, we derive the superconvergence property between average L^2 projection and the approximation of the control variable, the convergence order is h^2. Then, after solving a fully discretized optimal control problem, a control $\hat{u}$ is calculated by the projection of the adjoint state z_h in a postprocessing step. Although the approximation of the discretized solution is only of order h in L^∞-norm, we will show that this postprocessing step improves the convergence order to h^2. Finally, we present a numerical experiment to demonstrate the practical side of the theoretical results about superconvergence.

We consider the following linear optimal control problems for the state variable y, and the control u with an integral constraint:

$$\min_{u\in U_{ad}} \left\{ \frac{1}{2}\|y-y_d\|^2 + \frac{\nu}{2}\|u\|^2 \right\} \tag{1.1}$$

subject to the state equation

$$-\operatorname{div}(A(x)\mathbf{grad}y) + a_0 y = u, \quad x \in \Omega, \tag{1.2}$$

which can be written in the form of the first order system

$$\operatorname{div}\boldsymbol{p} + a_0 y = u, \quad \boldsymbol{p} = -A(x)\mathbf{grad}y, \quad x \in \Omega, \tag{1.3}$$

and the boundary condition

$$y = 0, \quad x \in \partial\Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^2$. U_{ad} denotes the admissible set of the control variable, defined by

$$U_{ad} = \left\{ u \in L^\infty(\Omega) : \int_\Omega u dx \geq 0 \right\}. \tag{1.5}$$

We assume that $y_d \in H^1(\Omega)$ and $0 \leq a_0 \in W^{2,\infty}(\Omega)$. ν is a fixed positive number. The coefficient $A(x) = (a_{ij}(x))_{2\times 2}$ is a symmetric matrix function with $a_{ij}(x) \in W^{2,\infty}(\Omega)$, which satisfies the ellipticity condition

$$c_*|\xi|^2 \leq \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j, \quad \forall\ (\xi, x) \in \mathbb{R}^2 \times \bar{\Omega}, \quad c_* > 0.$$

The plan of this paper is as follows. In Section 2, we construct the mixed finite element approximation scheme for the optimal control problem (1.1)–(1.4) and give its equivalent optimality conditions. The main results of this paper are stated in Section 3. In Section 3, we derive the superconvergence properties between the average L^2 projection and the approximation, as well as between the postprocessing solution and the exact control solution. In Section 4, we present a numerical example to demonstrate our theoretical results.

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p = \sum_{|\alpha|\leq m} \|D^\alpha v\|_{L^p(\Omega)}^p$, a semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. In addition C denotes a general positive constant independent of h, where h is the spatial mesh-size for the control and state discretization.

2. Mixed methods for optimal control problems

In this section we shall construct mixed finite element approximation scheme of the control problem (1.1)–(1.4). For sake of simplicity, we assume that the domain Ω is a convex polygon. Now, we introduce the co-state elliptic equation

$$-\operatorname{div}(A(x)\mathbf{grad}z) + a_0 z = y - y_d, \quad x \in \Omega, \tag{2.1}$$

which can be written in the form of the first order system

$$\operatorname{div}\boldsymbol{q} + a_0 z = y - y_d, \quad \boldsymbol{q} = -A(x)\mathbf{grad}z, \quad x \in \Omega, \tag{2.2}$$

and the boundary condition

$$z = 0, \quad x \in \partial\Omega. \tag{2.3}$$

The domain Ω is said to be H^{s+2}-regular if the Dirichlet problem

$$-\operatorname{div}(\mathbf{grad}\phi) + a_0\phi = \psi \quad \text{in } \Omega, \ \phi|_{\partial\Omega} = 0 \tag{2.4}$$

is uniquely solvable for $\psi \in L^2(\Omega)$ and if

$$\|\phi\|_{s+2} \le C\|\psi\|_s, \tag{2.5}$$

for all $\psi \in H^s(\Omega)$.

Let

$$\boldsymbol{V} = H(\operatorname{div};\Omega) = \left\{\boldsymbol{v} \in (L^2(\Omega))^2, \operatorname{div}\boldsymbol{v} \in L^2(\Omega)\right\}, \quad W = L^2(\Omega). \tag{2.6}$$

We recast (1.1)–(1.4) as the following weak form: find $(\boldsymbol{p}, y, u) \in \boldsymbol{V} \times W \times U_{ad}$ such that

$$\min_{u \in U_{ad}} \left\{\frac{1}{2}\|y - y_d\|^2 + \frac{\nu}{2}\|u\|^2\right\} \tag{2.7}$$

$$(A^{-1}\boldsymbol{p}, \boldsymbol{v}) - (y, \operatorname{div}\boldsymbol{v}) = 0, \ \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.8}$$

$$(\operatorname{div}\boldsymbol{p}, w) + (a_0 y, w) = (u, w), \ \forall\, w \in W. \tag{2.9}$$

It follows from [18] that the optimal control problem (2.7)–(2.9) has a unique solution $(\boldsymbol{p}, y, u)$, and that a triplet $(\boldsymbol{p}, y, u)$ is the solution of (2.7)–(2.9) if and only if there is a co-state $(\boldsymbol{q}, z) \in \boldsymbol{V} \times W$ such that $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ satisfies the following optimality conditions:

$$(A^{-1}\boldsymbol{p}, \boldsymbol{v}) - (y, \operatorname{div}\boldsymbol{v}) = 0, \ \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.10}$$

$$(\operatorname{div}\boldsymbol{p}, w) + (a_0 y, w) = (u, w), \ \forall\, w \in W, \tag{2.11}$$

$$(A^{-1}\boldsymbol{q}, \boldsymbol{v}) - (z, \operatorname{div}\boldsymbol{v}) = 0, \ \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.12}$$

$$(\operatorname{div}\boldsymbol{q}, w) + (a_0 z, w) = (y - y_d, w), \ \forall\, w \in W, \tag{2.13}$$

$$(\nu u + z, \tilde{u} - u) \ge 0, \ \forall\, \tilde{u} \in U_{ad}, \tag{2.14}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$.

In [8], the expression of the control variable is given. Here, we adopt the same method to derive the following operator.

$$u = (\max\{0, \bar{z}\} - z)/\nu, \tag{2.15}$$

where $\bar{z} = \int_\Omega z / \int_\Omega 1$ denotes the integral average on Ω of the function z.

Let $\mathcal{T}_h$ denote a regular triangulation of the polygonal domain Ω, h_T denotes the diameter of T and $h = \max_{T\in\mathcal{T}_h}\{h_T\}$. Let $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ denotes the order $k = 1$ Raviart-Thomas mixed finite element space [10, 22], namely,

$$\forall\ T \in \mathcal{T}_h,\quad \boldsymbol{V}(T) = \boldsymbol{P}_1(T) \oplus \mathrm{span}(xP_1(T)),\quad W(T) = P_1(T),$$

where $P_1(T)$ denote polynomials of total degree at most 1, $\boldsymbol{P}_1(T) = (P_1(T))^2$, $x = (x_1, x_2)$, which is treated as a vector, and

$$\boldsymbol{V}_h := \{\boldsymbol{v}_h \in \boldsymbol{V} : \forall\ T \in \mathcal{T}_h, \boldsymbol{v}_h|_T \in \boldsymbol{V}(T)\}, \tag{2.16}$$

$$W_h := \{w_h \in W : \forall\ T \in \mathcal{T}_h, w_h|_T \in W(T)\}. \tag{2.17}$$

The approximated space of control is given by

$$U_h := \{\tilde{u}_h \in U_{ad} : \forall\ T \in \mathcal{T}_h,\ \tilde{u}_h|_T = \mathrm{constant}\}. \tag{2.18}$$

Before the mixed finite element scheme is given, we introduce two operators. Firstly, we define the standard $L^2(\Omega)$-projection [10] $P_h :\ W \to W_h$, which satisfies: for any $\phi \in W$

$$(P_h\phi - \phi, w_h) = 0,\quad \forall\ w_h \in W_h, \tag{2.19}$$

$$\|\phi - P_h\phi\|_{0,\rho} \leq Ch^r\|\phi\|_{r,\rho},\ 1 \leq \rho \leq \infty,\ \forall\ \phi \in W^{r,\rho}(\Omega),\quad r = 1, 2, \tag{2.20}$$

$$\|\phi - P_h\phi\|_{-1} \leq Ch^3|\phi|_2,\ \forall\ \phi \in H^2(\Omega). \tag{2.21}$$

Next, recall the Fortin projection (see [2] and [10]) $\Pi_h :\ \boldsymbol{V} \to \boldsymbol{V}_h$, which satisfies: for any $\boldsymbol{q} \in \boldsymbol{V}$

$$(\mathrm{div}(\Pi_h\boldsymbol{q} - \boldsymbol{q}), w_h) = 0,\quad \forall\ w_h \in W_h, \tag{2.22}$$

$$\|\boldsymbol{q} - \Pi_h\boldsymbol{q}\| \leq Ch^r\|\boldsymbol{q}\|_r,\quad \forall\ \boldsymbol{q} \in (H^r(\Omega))^2,\quad r = 1, 2, \tag{2.23}$$

$$\|\mathrm{div}(\boldsymbol{q} - \Pi_h\boldsymbol{q})\| \leq Ch^r\|\mathrm{div}\boldsymbol{q}\|_r,\quad \forall\ \mathrm{div}\boldsymbol{q} \in H^r(\Omega),\quad r = 1, 2. \tag{2.24}$$

We have the commuting diagram property

$$\mathrm{div} \circ \Pi_h = P_h \circ \mathrm{div} : \boldsymbol{V} \to W_h \quad \text{and} \quad \mathrm{div}(I - \Pi_h)\boldsymbol{V} \perp W_h, \tag{2.25}$$

where and after, I denote the identity operator.

Furthermore, we also define the standard L^2-orthogonal projection $Q_h : U_{ad} \to U_h$, which satisfies: for any $u \in U_{ad}$

$$(u - Q_hu, u_h) = 0,\quad \forall\ u_h \in U_h. \tag{2.26}$$

We have the approximation property:

$$\|u - Q_hu\|_{-s,r} \leq Ch^{1+s}|\phi|_{1,r},\ s = 0, 1,\ \forall\ u \in W^{1,r}(\Omega). \tag{2.27}$$

Then the mixed finite element discretization of (2.7)–(2.9) is as follows: find $(\boldsymbol{p}_h, y_h, u_h) \in \boldsymbol{V}_h \times W_h \times U_h$ such that

$$\min_{u_h\in U_h}\left\{\frac{1}{2}\|y_h - y_d\|^2 + \frac{\nu}{2}\|u_h\|^2\right\} \tag{2.28}$$

$$(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \mathrm{div}\boldsymbol{v}_h) = 0,\ \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.29}$$

$$(\mathrm{div}\boldsymbol{p}_h, w_h) + (a_0y_h, w_h) = (u_h, w_h),\ \forall\ w_h \in W_h. \tag{2.30}$$

The optimal control problem (2.28)–(2.30) again has a unique solution $(\boldsymbol{p}_h, y_h, u_h)$, and that a triplet $(\boldsymbol{p}_h, y_h, u_h)$ is the solution of (2.28)–(2.30) if and only if there is a co-state $(\boldsymbol{q}_h, z_h) \in \boldsymbol{V}_h \times W_h$ such that $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \mathrm{div}\boldsymbol{v}_h) = 0,\ \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.31}$$
$$(\mathrm{div}\boldsymbol{p}_h, w_h) + (a_0 y_h, w_h) = (u_h, w_h),\ \forall\ w_h \in W_h, \tag{2.32}$$
$$(A^{-1}\boldsymbol{q}_h, \boldsymbol{v}_h) - (z_h, \mathrm{div}\boldsymbol{v}_h) = 0,\ \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.33}$$
$$(\mathrm{div}\boldsymbol{q}_h, w_h) + (a_0 z_h, w_h) = (y_h - y_d, w_h),\ \forall\ w_h \in W_h, \tag{2.34}$$
$$(\nu u_h + z_h, \tilde{u}_h - u_h) \geq 0,\ \forall\ \tilde{u}_h \in U_h. \tag{2.35}$$

As in [7], for the variational inequality (2.35) we have

$$u_h = Q_h\left(-\frac{z_h}{\nu} + \max\left\{0, \frac{\overline{z_h}}{\nu}\right\}\right), \quad \overline{z_h} = \frac{\int_\Omega z_h}{\int_\Omega 1}. \tag{2.36}$$

In the rest of the paper, we shall use some intermediate variables. For any control function $\tilde{u} \in U_{ad}$, we first define the state solution $(\boldsymbol{p}(\tilde{u}), y(\tilde{u}), \boldsymbol{q}(\tilde{u}), z(\tilde{u})) \in (\boldsymbol{V} \times W)^2$ associated with $\tilde{u}$ that satisfies

$$(A^{-1}\boldsymbol{p}(\tilde{u}), \boldsymbol{v}) - (y(\tilde{u}), \mathrm{div}\boldsymbol{v}) = 0, \quad \forall\ \boldsymbol{v} \in \boldsymbol{V}, \tag{2.37}$$
$$(\mathrm{div}\boldsymbol{p}(\tilde{u}), w) + (a_0 y(\tilde{u}), w) = (\tilde{u}, w), \quad \forall\ w \in W, \tag{2.38}$$
$$(A^{-1}\boldsymbol{q}(\tilde{u}), \boldsymbol{v}) - (z(\tilde{u}), \mathrm{div}\boldsymbol{v}) = 0, \quad \forall\ \boldsymbol{v} \in \boldsymbol{V}, \tag{2.39}$$
$$(\mathrm{div}\boldsymbol{q}(\tilde{u}), w) + (a_0 z(\tilde{u}), w) = (y(\tilde{u}) - y_d, w), \quad \forall\ w \in W. \tag{2.40}$$

Then, we define the discrete state solution $(\boldsymbol{p}_h(\tilde{u}), y_h(\tilde{u}), \boldsymbol{q}_h(\tilde{u}), z_h(\tilde{u})) \in (\boldsymbol{V}_h \times W_h)^2$ associated with $\tilde{u}$ that satisfies

$$(A^{-1}\boldsymbol{p}_h(\tilde{u}), \boldsymbol{v}_h) - (y_h(\tilde{u}), \mathrm{div}\boldsymbol{v}_h) = 0, \quad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.41}$$
$$(\mathrm{div}\mathbf{p}_h(\tilde{u}), w_h) + (a_0 y_h(\tilde{u}), w_h) = (\tilde{u}, w_h), \quad \forall\ w_h \in W_h, \tag{2.42}$$
$$(A^{-1}\boldsymbol{q}_h(\tilde{u}), \boldsymbol{v}_h) - (z_h(\tilde{u}), \mathrm{div}\boldsymbol{v}_h) = 0, \quad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.43}$$
$$(\mathrm{div}\boldsymbol{q}_h(\tilde{u}), w_h) + (a_0 z_h(\tilde{u}), w_h) = (y_h(\tilde{u}) - y_d, w_h), \quad \forall w_h \in W_h. \tag{2.44}$$

Thus, as we defined before, the exact solution and its approximation can be written in the following way:

$$(\boldsymbol{p}, y, \boldsymbol{q}, z) = (\boldsymbol{p}(u), y(u), \boldsymbol{q}(u), z(u)),$$
$$(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) = (\boldsymbol{p}_h(u_h), y_h(u_h), \boldsymbol{q}_h(u_h), z_h(u_h)).$$

3. Superconvergence and postprocessing

In this section, we will give a detailed superconvergence analysis.

Now, we are in the position of deriving the estimates for $\|P_h y(u_h) - y_h\|_{-1}$ and $\|P_h z(u_h) - z_h\|$.

Let $(\boldsymbol{p}(u_h), y(u_h), \boldsymbol{q}(u_h), z(u_h)) \in (\boldsymbol{V} \times W)^2$ and $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) \in (\boldsymbol{V}_h \times W_h)^2$ be the solutions of (2.37)–(2.40) and (2.41)–(2.44) with $\tilde{u} = u_h$ respectively. We can easily obtain the following error equations

$$(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\boldsymbol{v}_h)-(y(u_h)-y_h,\mathrm{div}\boldsymbol{v}_h)=0, \tag{3.1}$$
$$(\mathrm{div}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),w_h)+(a_0(y(u_h)-y_h),w_h)=0, \tag{3.2}$$
$$(A^{-1}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),\boldsymbol{v}_h)-(z(u_h)-z_h,\mathrm{div}\boldsymbol{v}_h)=0, \tag{3.3}$$
$$(\mathrm{div}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),w_h)+(a_0(z(u_h)-z_h),w_h)=(y(u_h)-y_h,w_h), \tag{3.4}$$

for any $\boldsymbol{v}_h\in\boldsymbol{V}_h$ and $w_h\in W_h$.

As a result of (2.19), we can rewrite (3.1)–(3.4) as

$$(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\boldsymbol{v}_h)-(P_hy(u_h)-y_h,\mathrm{div}\boldsymbol{v}_h)=0, \tag{3.5}$$
$$(\mathrm{div}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),w_h)+(a_0(y(u_h)-y_h),w_h)=0, \tag{3.6}$$
$$(A^{-1}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),\boldsymbol{v}_h)-(P_hz(u_h)-z_h,\mathrm{div}\boldsymbol{v}_h)=0, \tag{3.7}$$
$$(\mathrm{div}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),w_h)+(a_0(z(u_h)-z_h),w_h)=(P_hy(u_h)-y_h,w_h), \tag{3.8}$$

for any $\boldsymbol{v}_h\in\boldsymbol{V}_h$ and $w_h\in W_h$.

For sake of simplicity, we now denote

$$\tau=P_hy(u_h)-y_h,\qquad e=P_hz(u_h)-z_h. \tag{3.9}$$

Lemma 3.1. *Let* $(\boldsymbol{p}(u_h),y(u_h),\boldsymbol{q}(u_h),z(u_h))\in(\boldsymbol{V}\times W)^2$ *and* $(\boldsymbol{p}_h,y_h,\boldsymbol{q}_h,z_h)\in(\boldsymbol{V}_h\times W_h)^2$ *be the solutions of (2.37)-(2.40) and (2.41)-(2.44) with* $\tilde{u}=u_h$ *respectively. Assume that the domain* Ω *is* H^{s+2}*-regular* $(0\le s\le 1)$*, then we have*

$$\|P_hy(u_h)-y_h\|_{-1}+h\|P_hy(u_h)-y_h\|\le Ch^3(\|u\|+\|Q_hu-u_h\|). \tag{3.10}$$

Proof. As we can see,

$$\|\tau\|_{-1}=\sup_{\psi\in H^1(\Omega),\psi\neq 0}\frac{(\tau,\psi)}{\|\psi\|_1}, \tag{3.11}$$

we then need to bound (τ,ψ) for $\psi\in H^1(\Omega)$. Let $\phi\in H^3(\Omega)\cap H_0^1(\Omega)$ be the solution of (2.4). We can see from (2.22) and (3.5)

$$\begin{aligned}(\tau,\psi)&=(\tau,-\mathrm{div}(A\nabla\phi))+(\tau,a_0\phi)\\&=-(\tau,\mathrm{div}(\Pi_h(A\nabla\phi)))+(\tau,a_0\phi)\\&=-(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\Pi_h(A\nabla\phi))+(\tau,a_0\phi).\end{aligned} \tag{3.12}$$

Note that

$$(\mathrm{div}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\phi)+(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),A\nabla\phi)=0. \tag{3.13}$$

Thus, from (3.6), (3.12) and (3.13), we derive

$$\begin{aligned}(\tau,\psi)=&(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),A\nabla\phi-\Pi_h(A\nabla\phi))\\&+(\mathrm{div}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\phi-P_h\phi)+(a_0\tau,\phi-P_h\phi)\\&+(a_0(y(u_h)-P_h(y(u_h))),\phi-P_h\phi)-(a_0(y(u_h)-P_h(y(u_h))),\phi).\end{aligned} \tag{3.14}$$

From (2.23), we have

$$(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),A\nabla\phi-\Pi_h(A\nabla\phi))\le Ch^2\|A\|_{2,\infty}\|\boldsymbol{p}(u_h)-\boldsymbol{p}_h\|\cdot\|\phi\|_3. \tag{3.15}$$

Let $\tilde{u} = u_h$ and $w = \mathrm{div}\boldsymbol{p}(u_h) + a_0 y(u_h) - u_h$ in (2.38), we can find that

$$\mathrm{div}\boldsymbol{p}(u_h) + a_0 y(u_h) - u_h = 0. \tag{3.16}$$

Similarly, by (2.19) and (2.32), it is easy to see that

$$\mathrm{div}\boldsymbol{p}_h = u_h - P_h(a_0 y_h). \tag{3.17}$$

By (3.16)–(3.17) and (2.19)–(2.20), we have

$$\begin{aligned}(\mathrm{div}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \phi - P_h\phi) &= (P_h(a_0 y_h) - a_0 y(u_h), \phi - P_h\phi)\\ &= (P_h(a_0 y(u_h)) - a_0 y(u_h), \phi - P_h\phi)\\ &\le C\|P_h(a_0 y(u_h)) - a_0 y(u_h)\| \cdot \|\phi - P_h\phi\|\\ &\le Ch^3\|a_0\|_{1,\infty}\|y(u_h)\|_1\|\phi\|_2. \end{aligned} \tag{3.18}$$

For the third and the fourth terms on the right side of (3.14), using (2.20), we get

$$(a_0\tau, \phi - P_h\phi) \le Ch^2\|a_0\|_{0,\infty}\|\tau\| \cdot \|\phi\|_2, \tag{3.19}$$

$$(a_0(y(u_h) - P_h(y(u_h))), \phi - P_h\phi) \le Ch^3\|a_0\|_{0,\infty}\|y(u_h)\|_1\|\phi\|_2. \tag{3.20}$$

Moreover, by (2.21), we find that

$$\begin{aligned}(a_0(y(u_h) - P_h(y(u_h))), \phi) =&(y(u_h) - P_h(y(u_h)), a_0\phi)\\ \le& C\|y(u_h) - P_h(y(u_h))\|_{-1}\|a_0\phi\|_1\\ \le& Ch^3\|a_0\|_{1,\infty}\|y(u_h)\|_2\|\phi\|_1. \end{aligned} \tag{3.21}$$

By (3.11), (3.14)–(3.15) and (3.18)–(3.21), we derive

$$\|P_h y(u_h) - y_h\|_{-1} \le Ch^2(\|\boldsymbol{p}(u_h) - \boldsymbol{p}_h\| + \|\tau\|) + Ch^3\|y(u_h)\|_2. \tag{3.22}$$

Choosing $\boldsymbol{v}_h = \Pi_h\boldsymbol{p}(u_h) - \boldsymbol{p}_h$ in (3.5) and $w_h = P_h y(u_h) - y_h$ in (3.6), respectively. Then adding the two equations to get

$$\begin{aligned}&(A^{-1}(\Pi_h\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \Pi_h\boldsymbol{p}(u_h) - \boldsymbol{p}_h) + (a_0(P_h y(u_h) - y_h), P_h y(u_h) - y_h)\\ =& -(A^{-1}(\boldsymbol{p}(u_h) - \Pi_h\boldsymbol{p}(u_h)), \Pi_h\boldsymbol{p}(u_h) - \boldsymbol{p}_h) - (a_0(y(u_h) - P_h y(u_h)), P_h y(u_h) - y_h). \end{aligned} \tag{3.23}$$

Using (3.23), (2.20), (2.23) and the assumptions on A and a_0, we find that

$$\|\Pi_h\boldsymbol{p}(u_h) - \boldsymbol{p}_h\| + \|\tau\| \le Ch(\|\boldsymbol{p}(u_h)\|_1 + \|y(u_h)\|_1). \tag{3.24}$$

Substituting (3.24) into (3.22), using (2.23), for sufficiently small h, we have

$$\|P_h y(u_h) - y_h\|_{-1} \le Ch^3(\|\boldsymbol{p}(u_h)\|_1 + \|y(u_h)\|_2). \tag{3.25}$$

Since the domain Ω is H^2-regular, we have

$$\|\boldsymbol{p}(u_h)\|_1 + \|y(u_h)\|_2 \le C\|y(u_h)\|_2 \le C\|u_h\| \le C(\|u\| + \|Q_h u - u_h\|). \tag{3.26}$$

From (3.25)–(3.26), we derive the first part of (3.10).

Similarly, we can derive

$$\|\tau\| \le Ch^2(\|u\| + \|Q_h u - u_h\|). \tag{3.27}$$

Thus, we complete the proof. □

Lemma 3.2. *Let* $(\boldsymbol{p}(u_h), y(u_h), \boldsymbol{q}(u_h), z(u_h)) \in (\boldsymbol{V} \times W)^2$ *and* $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) \in (\boldsymbol{V}_h \times W_h)^2$ *be the solutions of (2.37)-(2.40) and (2.41)-(2.44) with* $\tilde{u} = u_h$ *respectively. Assume that the domain* Ω *is* H^{s+2}*-regular* $(0 \le s \le 1)$*, then we have*

$$\|P_h z(u_h) - z_h\| \le Ch^3(\|y_d\|_1 + \|u\| + \|Q_h u - u_h\|). \tag{3.28}$$

Proof. Since

$$\|e\| = \sup_{\psi \in L^2(\Omega), \psi \ne 0} \frac{(e, \psi)}{\|\psi\|}, \tag{3.29}$$

we then need to bound (e, ψ) for $\psi \in L^2(\Omega)$. From (2.4), (2.22) and (3.7), we can see that

$$\begin{aligned}(e, \psi) =&(e, -\mathrm{div}(A\nabla\phi)) + (e, a_0\phi)\\ =& - (e, \mathrm{div}(\Pi_h(A\nabla\phi))) + (e, a_0\phi)\\ =& - (A^{-1}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), \Pi_h(A\nabla\phi)) + (e, a_0\phi) - (\boldsymbol{p}(u_h) - \boldsymbol{p}_h, \Pi_h(A\nabla\phi)).\end{aligned} \tag{3.30}$$

Note that

$$(\mathrm{div}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), \phi) + (A^{-1}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), A\nabla\phi) = 0. \tag{3.31}$$

Thus, from (3.8), (3.30) and (3.31), we derive

$$\begin{aligned}(e, \psi) =&(A^{-1}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), A\nabla\phi - \Pi_h(A\nabla\phi)) + (\mathrm{div}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), \phi - P_h\phi)\\ &+ (a_0 e, \phi - P_h\phi) + (a_0(z(u_h) - P_h(z(u_h))), \phi - P_h\phi)\\ &- (a_0(z(u_h) - P_h(z(u_h))), \phi) - (\tau, P_h\phi) =: \sum_{i=1}^{6} I_i.\end{aligned} \tag{3.32}$$

Let $\tilde{u} = u_h$ and $w = \mathrm{div}\boldsymbol{q}(u_h) + a_0 z(u_h) - y(u_h) + y_d$ in (2.40), we can find that

$$\mathrm{div}\boldsymbol{q}(u_h) + a_0 z(u_h) = y(u_h) - y_d. \tag{3.33}$$

Similarly, by (2.19) and (2.34), it is easy to see that

$$\mathrm{div}\boldsymbol{q}_h = y_h - P_h y_d - P_h(a_0 z_h). \tag{3.34}$$

By (2.19)–(2.20) and (3.33)–(3.34), we have

$$\begin{aligned}I_2 =&(P_h(a_0 z_h) - a_0 z(u_h), \phi - P_h\phi) + (P_h y_d - y_d, \phi - P_h\phi)\\ &+ (y(u_h) - P_h y(u_h), \phi - P_h\phi) + (P_h y(u_h) - y_h, \phi - P_h\phi)\\ =&(P_h(a_0 z(u_h)) - a_0 z(u_h), \phi - P_h\phi) + (P_h y_d - y_d, \phi - P_h\phi)\\ &+ (y(u_h) - P_h y(u_h), \phi - P_h\phi)\\ \le& Ch^3(\|a_0\|_{1,\infty}\|z(u_h)\|_1 + \|y_d\|_1 + \|y(u_h)\|_1)\|\phi\|_2.\end{aligned} \tag{3.35}$$

Similar to the estimates (3.15) and (3.18)–(3.21), we estimate I_1, I_3, I_4 and I_5 as follows

$$I_1 \le Ch\|\boldsymbol{q}(u_h) - \boldsymbol{q}_h\| \cdot \|\phi\|_2, \tag{3.36}$$

$$I_3 \le Ch\|e\| \cdot \|\phi\|_1, \tag{3.37}$$

$$I_4 \le Ch^3\|z(u_h)\|_1\|\phi\|_2, \tag{3.38}$$

$$I_5 \le Ch^3\|a_0\|_{1,\infty}\|z(u_h)\|_2\|\phi\|_1. \tag{3.39}$$

For I_6, by using of (2.19) and (3.10), we get

$$I_6 = -(\tau, \phi) \le C\|\tau\|_{-1}\|\phi\|_1 \le Ch^3(\|u\| + \|Q_h u - u_h\|)\|\phi\|_1. \tag{3.40}$$

Substituting the estimates I_1-I_6 in (3.32), for sufficiently small h, by (3.29), we derive

$$\begin{aligned}\|P_h z(u_h) - z_h\| \le & Ch^3(\|z(u_h)\|_2 + \|y(u_h)\|_1 + \|y_d\|_1 + \|u\| + \|Q_h u - u_h\|) \\ & + Ch\|\boldsymbol{q}(u_h) - \boldsymbol{q}_h\|. \end{aligned} \tag{3.41}$$

Next, using (2.22), we rewrite (3.7)–(3.8) as

$$\begin{aligned} & (A^{-1}(\Pi_h \boldsymbol{q}(u_h) - \boldsymbol{q}_h), \boldsymbol{v}_h) - (P_h z(u_h) - z_h, \mathrm{div}\boldsymbol{v}_h) \\ = & -(A^{-1}(\boldsymbol{q}(u_h) - \Pi_h \boldsymbol{q}(u_h)), \boldsymbol{v}_h), \ \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h, \end{aligned} \tag{3.42}$$

$$\begin{aligned} & (\mathrm{div}(\Pi_h \boldsymbol{q}(u_h) - \boldsymbol{q}_h), w_h) + (a_0(P_h z(u_h) - z_h), w_h) \\ = & -(a_0(z(u_h) - P_h z(u_h)), w_h) + (P_h y(u_h) - y_h, w_h), \ \forall\ w_h \in W_h. \end{aligned} \tag{3.43}$$

Similar to (3.24), we can get

$$\|\Pi_h \boldsymbol{q}(u_h) - \boldsymbol{q}_h\| + \|e\| \le Ch^2(\|\boldsymbol{q}(u_h)\|_2 + \|z(u_h)\|_1) + C\|\tau\|. \tag{3.44}$$

Substituting (3.44) into (3.41), using (2.23) and (3.27), for sufficiently small h, we have

$$\begin{aligned} & \|P_h z(u_h) - z_h\| \\ \le & Ch^3(\|\boldsymbol{q}(u_h)\|_2 + \|z(u_h)\|_2 + \|y(u_h)\|_1 + \|y_d\|_1 + \|u\| + \|Q_h u - u_h\|). \end{aligned} \tag{3.45}$$

Since the domain Ω is H^3-regular, we have

$$\|\boldsymbol{q}(u_h)\|_2 + \|z(u_h)\|_2 \le C\|z(u_h)\|_3 \le C(\|y(u_h)\|_1 + \|y_d\|_1). \tag{3.46}$$

Thus, using (3.26) and (3.45)–(3.46), we complete the proof. □

Lemma 3.3. *Let $(\boldsymbol{p}(Q_h u), y(Q_h u), \boldsymbol{q}(Q_h u), z(Q_h u))$ and $(\boldsymbol{p}(u), y(u), \boldsymbol{q}(u), z(u))$ be the solutions of (2.37)–(2.40) with $\tilde{u} = Q_h u$ and $\tilde{u} = u$, respectively. Assume that $u \in H^1(\Omega)$ and the domain Ω is H^2-regular, then we have*

$$\|z(u) - z(Q_h u)\|_{0,\infty} \le Ch^2. \tag{3.47}$$

Proof. First, we choose $\tilde{u} = Q_h u$ and $\tilde{u} = u$ in (2.37)–(2.40) respectively, then we obtain the following error equations

$$(A^{-1}(\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u)), \boldsymbol{v}) - (y(Q_h u) - y(u), \mathrm{div}\boldsymbol{v}) = 0, \tag{3.48}$$

$$(\mathrm{div}(\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u)), w) + (a_0(y(Q_h u) - y(u)), w) = (Q_h u - u, w), \tag{3.49}$$

for any $\boldsymbol{v} \in \boldsymbol{V}$ and $w \in W$.

Setting $\boldsymbol{v} = \boldsymbol{p}(Q_h u) - \boldsymbol{p}(u)$ and $w = y(Q_h u) - y(u)$ in (3.48) and (3.49) respectively and adding the two equations to get

$$\begin{aligned} & (A^{-1}(\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u)), \boldsymbol{p}(Q_h u) - \boldsymbol{p}(u)) + (a_0(y(Q_h u) - y(u)), y(Q_h u) - y(u)) \\ = & (Q_h u - u, y(Q_h u) - y(u)). \end{aligned} \tag{3.50}$$

Then, we estimate the right side of (3.50). Note that $\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u) = -A\nabla(y(Q_h u) - y(u))$, by (2.27) and Poincaré's inequality, we have

$$\begin{aligned}(Q_h u - u, y(Q_h u) - y(u)) &\le C\|Q_h u - u\|_{-1}\|y(Q_h u) - y(u))\|_1 \\ &\le Ch^2\|u\|_1\|\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u)\|. \end{aligned} \tag{3.51}$$

It follows from the assumptions on A and a_0, (3.50) and (3.51) that

$$\|\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u)\| \le Ch^2. \tag{3.52}$$

By the Poincaré's inequality again, we have

$$\|y(Q_h u) - y(u)\| \le C\|\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u)\| \le Ch^2. \tag{3.53}$$

From (2.1), we obtain the following equation

$$-\mathrm{div}(A\mathbf{grad}(z(Q_h u) - z(u))) + a_0(z(Q_h u) - z(u)) = y(Q_h u) - y(u). \tag{3.54}$$

Using (2.5), (3.53) and the classical imbedding theorem, we can see that

$$\begin{aligned}\|z(Q_h u) - z(u)\|_{0,\infty} \le& C\|z(Q_h u) - z(u)\|_2 \\ \le& C\|y(Q_h u) - y(u)\| \\ \le& Ch^2. \end{aligned} \tag{3.55}$$

Thus, we complete the proof. □

Lemma 3.4. *Let u be the solution of (2.10)–(2.14) and u_h be the solution of (2.31)–(2.35), respectively. Assume that $u \in H^1(\Omega)$ and all the assumptions in previous Lemmas 3.1–3.3 hold. Then, we have*

$$\|Q_h u - u_h\| \le Ch^2. \tag{3.56}$$

Proof. We choose $\tilde{u} = u_h$ in (2.14) and $\tilde{u}_h = Q_h u$ in (2.35) to get the following two inequalities:

$$(\nu u + z, u_h - u) \ge 0 \tag{3.57}$$

and

$$(\nu u_h + z_h, Q_h u - u_h) \ge 0. \tag{3.58}$$

Note that $u_h - u = u_h - Q_h u + Q_h u - u$. Adding the two inequalities (3.57) and (3.58), we have

$$(\nu u_h + z_h - \nu u - z, Q_h u - u_h) + (\nu u + z, Q_h u - u) \ge 0. \tag{3.59}$$

Thus, by (3.59), we find that

$$\begin{aligned}\nu\|Q_h u - u_h\|^2 =& \nu(Q_h u - u_h, Q_h u - u_h) \\ =& \nu(Q_h u - u, Q_h u - u_h) + \nu(u - u_h, Q_h u - u_h) \\ \le& (z_h - z, Q_h u - u_h) + (\nu u + z, Q_h u - u). \end{aligned} \tag{3.60}$$

Observe that

$$(z_h - z, Q_h u - u_h) = (z_h - z(u_h), Q_h u - u_h) + (z(u_h) - z(Q_h u), Q_h u - u_h) + (z(Q_h u) - z(u), Q_h u - u_h). \tag{3.61}$$

By (2.19), Lemma 3.2 and Lemma 3.3, we arrive at

$$\begin{aligned}(z_h - z(u_h), Q_h u - u_h) =& (z_h - P_h z(u_h), Q_h u - u_h) \\ \leq& Ch^6 + \frac{\nu}{4}\|Q_h u - u_h\|^2 + Ch^3\|Q_h u - u_h\|^2\end{aligned} \tag{3.62}$$

and

$$(z(Q_h u) - z(u), Q_h u - u_h) \leq Ch^4 + \frac{\nu}{4}\|Q_h u - u_h\|^2. \tag{3.63}$$

Moreover, we can prove that

$$(z(u_h) - z(Q_h u), Q_h u - u_h) = -\|y(Q_h u) - y(u_h)\|^2 \leq 0. \tag{3.64}$$

From (2.15), we know that

$$\nu u + z = \max\{0, \bar{z}\} = \text{constant}. \tag{3.65}$$

Thus, we have

$$(\nu u + z, Q_h u - u) = (\nu u + z)\int_\Omega (Q_h u - u) = 0. \tag{3.66}$$

Combining (3.60)–(3.64) with (3.66), for sufficiently small h, we derive (3.56). □

Let $(\boldsymbol{p}(u_h), y(u_h), \boldsymbol{q}(u_h), z(u_h))$ and $(\boldsymbol{p}(Q_h u), y(Q_h u), \boldsymbol{q}(Q_h u), z(Q_h u))$ be the solutions of (2.37)-(2.40) with $\tilde{u} = u_h$ and $\tilde{u} = Q_h u$, respectively. Then we have the following error equations

$$(A^{-1}(\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u_h)), \boldsymbol{v}) - (y(Q_h u) - y(u_h), \text{div}\boldsymbol{v}) = 0, \tag{3.67}$$

$$(\text{div}(\boldsymbol{p}(Q_h u) - \boldsymbol{p}(u_h)), w) + (a_0(y(Q_h u) - y(u_h)), w) = (Q_h u - u_h, w), \tag{3.68}$$

$$(A^{-1}(\boldsymbol{q}(Q_h u) - \boldsymbol{q}(u_h)), \boldsymbol{v}) - (z(Q_h u) - z(u_h), \text{div}\boldsymbol{v}) = 0, \tag{3.69}$$

$$(\text{div}(\boldsymbol{q}(Q_h u) - \boldsymbol{q}(u_h)), w) + (a_0(z(Q_h u) - z(u_h)), w) = (y(Q_h u) - y(u_h), w), \tag{3.70}$$

for any $\boldsymbol{v} \in \boldsymbol{V}$ and $w \in W$.

Similar to Lemma 3.3, using Lemma 3.4, we can prove the following estimate.

Lemma 3.5. *Assume that all the assumptions in Lemma 3.4 are hold. Then we have*

$$\|z(Q_h u) - z(u_h)\|_{0,\infty} \leq Ch^2. \tag{3.71}$$

Lemma 3.6. *Assume that all the assumptions in Lemma 3.4 hold and $u \in W^{1,\infty}(\Omega)$. Let u and u_h be the solutions of (2.10)–(2.14) and (2.31)–(2.35), respectively. Then we have*

$$\|u - u_h\|_{0,\infty} \leq Ch. \tag{3.72}$$

Proof. By (2.27) and the inverse inequality, we arrive at

$$\begin{aligned}\|u-u_h\|_{0,\infty} &\le C(\|u-Q_hu\|_{0,\infty}+\|Q_hu-u_h\|_{0,\infty})\\ &\le C(h\|u\|_{1,\infty}+h^{-1}\|Q_hu-u_h\|).\end{aligned} \tag{3.73}$$

Gathering (3.73) and Lemma 3.4, we derive (3.72). □

Moreover, in order to improve the accuracy of the control approximation on a global scale, similar to the case in [19], we construct the following a postprocessing projection operator of the discrete co-state to the admissible set

$$\hat{u} = (\max\{0, \overline{z_h}\} - z_h)/\nu. \tag{3.74}$$

Now, we can prove the following global superconvergence result.

Theorem 3.1. *Assume that all the assumptions in previous Lemmas hold. Let u be the solution of (2.10)–(2.14) and $\hat{u}$ be the function constructed in (3.74). Then we have*

$$\|u-\hat{u}\|_{0,\infty} \le Ch^2. \tag{3.75}$$

Proof. By use of (2.20), Lemma 3.2, Lemma 3.3, Lemma 3.5 and the inverse estimate, we find that

$$\begin{aligned}\|z-z_h\|_{0,\infty} \le &\|z-z(Q_hu)\|_{0,\infty}+\|z(Q_hu)-z(u_h)\|_{0,\infty}\\ &+\|z(u_h)-P_hz(u_h)\|_{0,\infty}+\|P_hz(u_h)-z_h\|_{0,\infty}\\ \le &Ch^2.\end{aligned} \tag{3.76}$$

From (2.15) and (3.74), we arrive at

$$|u-\hat{u}| \le C|z-z_h| + C|\bar{z}-\overline{z_h}|. \tag{3.77}$$

By (3.76) and (3.77), we have

$$\|u-\hat{u}\|_{0,\infty} \le C\|z-z_h\|_{0,\infty} \le Ch^2, \tag{3.78}$$

which yields to (3.75). □

4. Numerical experiments

In this section, we present below an example to illustrate the theoretical results. The optimization problems were solved numerically by projected gradient methods, with codes developed based on AFEPack [17]. The discretization was already described in previous sections: the control function u was discretized by piecewise constant functions, whereas the state $(y, \boldsymbol{p})$ and the co-state $(z, \boldsymbol{q})$ were approximated by the order $k = 1$ Raviart-Thomas mixed finite element functions. In the following example, we choose the domain $\Omega = [0,1]\times[0,1]$, $a_0 = 0$, $\nu = 1$ and A is a unit matrix.

Example 4.1. We consider the following two-dimensional elliptic control problem

$$\min_{u\in U_{ad}} \left\{\frac{1}{2}\|y-y_d\|^2+\frac{1}{2}\|u\|^2\right\} \tag{4.1}$$

subject to the state equation

$$\text{div}\boldsymbol{p} = f + u, \quad \boldsymbol{p} = -\textbf{grad}y, \tag{4.2}$$

where

$$\begin{aligned} &y = \sin(\pi x_1)\sin(\pi x_2), \quad z = \sin(2\pi x_1)\sin(2\pi x_2), \\ &u = \max(0, \bar{z}) - z, \quad f = 2\pi^2 y - u, \quad y_d = y - 8\pi^2 z. \end{aligned} \tag{4.3}$$

In the numerical implementation, we choose the solution u which satisfies $\int_\Omega u dx = 0$. In Table 1, the errors $\|u - u_h\|_{0,\infty}$, $\|Q_h u - u_h\|$ and $\|u - \hat{u}\|_{0,\infty}$ obtained on a sequence of uniformly refined meshes are shown. In Figure 1, the profile of the numerical solution of u on the 64×64 mesh grid is plotted. Moreover, in Figure 2, we show the convergence orders by slopes. In the Figure 2, we denote $\hat{u}$ by u_{proj}. The theoretical results can be observed clearly from the data.

Table 1. The errors of Example on a sequential uniform refined meshes.

h	$\|u - u_h\|_{0,\infty}$	$\|Q_h u - u_h\|$	$\|u - \hat{u}\|_{0,\infty}$
1/16	9.4173e-02	1.2837e-04	3.5792e-02
1/32	4.7042e-02	2.5802e-05	8.9386e-03
1/64	2.3767e-02	6.3516e-06	2.2261e-03
1/128	1.1893e-02	1.5769e-06	5.5875e-04

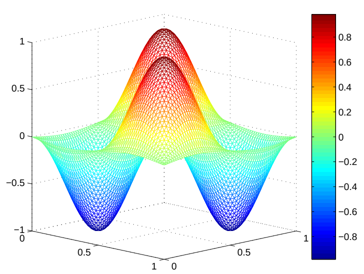

Figure 1. The The profile of the numerical solution of Example on 64×64 triangle mesh.

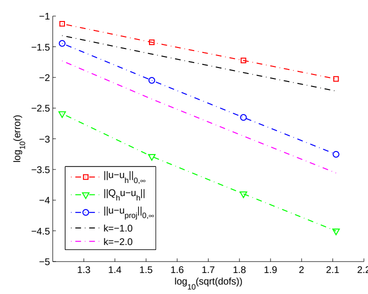

Figure 2. Convergence orders of $u - u_h$, $Q_h u - u_h$ and $u - u_{proj}$ in different norms.

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