Journal of Applied Analysis and Computation
Volume 7, Number 3, August 2017, 1022–1036
Website:http://jaac-online.com/
DOI:10.11948/2017064

ON MEASURABLE FUNCTIONAL SPACES BASED ON PSEUDO-ADDITION DECOMPOSABLE MEASURES*

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Abstract In this paper, we mainly discuss the measurable functional spaces based on strict pseudo-additions. Particularly, we obtained the some important theorems for the measurable functional spaces based on a strict pseudo-addition. Furthermore, we got that the some properties of the sequence of a.e. convergence and convergence in $\oplus$-measure, and the relationship between a.e. convergence and convergence in $\oplus$-measure on the measurable functional spaces based on a strict pseudo-addition.

Keywords Strict pseudo-additions, measurable functional spaces, decomposable measures.

MSC(2010) 28E10, 26E50.

1. Introduction

Originally functional analysis could be understood as a unifying abstract treatment of important aspects of linear mathematical models for problems in science, but the latter receded more and more into the background during the intensive theoretical investigations. Numerous questions in physics, chemistry, biology, and economics lead to nonlinear problems. Thus nonlinear functional analysis is an important branch of modern mathematics.

The classical measure theory is one of the most important theories in mathematics and based on countable additive measures [7, 26]. Although the additive measures are widely used, they do not allow modeling many phenomena involving interaction between criteria. For this reason, the fuzzy measure proposed by Sugeno as an extension of classical measure in which the additivity is replaced by a weaker condition, i.e., monotonicity [26]. So far, there have been many different fuzzy measures, such as the decomposable measure, the λ-additive measure, the belief measure,the possibility measure and the plausibility measure, etc. Among the fuzzy measure mentioned before, the decomposable measure was independently introduced by Dubois and Prade [5] and Weber [27], because of the close relation with the classical measure theory. Further developments of decomposable measures and related integrals have been extensive [4, 19–21, 23]. Decomposable measures include several well-known fuzzy measures such as the λ-additive measure and probability

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*The authors were supported by National Natural Science Foundations of China (Grant no.11671001 and 61472056).

and possibility measures, and they are a natural setting for relaxing probabilistic assumptions regarding the modeling of uncertainty [6,22]. Decomposable measures and the corresponding integrals are very useful in decision theory and the theory of nonlinear differential and integral equations [12,25].

Based on the above these, they also play an important role in theories of non-additive measures [6,26] and the notions of σ-$\oplus$-decomposable measure (pseudo-additive measure) and corresponding integral (pseudo-integral) based on this measure were introduced [13,24]. The families of the pseudo-operations based on generated g turn out to be solutions of well-known nonlinear functional equations [25]. In many problems with uncertainty as in the theory of probabilistic metric spaces [22], multi-valued logics, general measures [7,26] often we work with many operations different from the usual addition and multiplication of reals. Some of them are triangular norms, triangular conorms, pseudo-additions, pseudo-multiplications, etc. [12].

In this paper, we will discuss the measurable functional spaces based on strict pseudo-additions. Particularly, we will generalize the classical some theorems to the measurable functional space based on a strict pseudo-addition. Furthermore, we will obtain that the some properties of the sequence of a.e. convergence and convergence in $\oplus$-measure, and the relationship between a.e. convergence and convergence in $\oplus$-measure on the measurable functional spaces based on a strict pseudo-addition.

2. Preliminaries

Let $[a, b]$ be a closed subinterval of $\mathbb{R}$ (in some cases we will also take semiclosed subintervals). The total order on $[a, b]$ will be denoted by $\preceq$. This can be the usual order of the real line, but it can also be another order. We shall denote by M is maximum element on $[a, b]$ (usually M is either a or b) with respect to this total order.

Definition 2.1 ([19]). Let $\{x_n\}_{n\geq 1}$ be a sequence from $[a, b]$.

(1) If $x_m \preceq x_n$ whenever $n > m$, then we say that the sequence $\{x_n\}_{n\geq 1}$ is an increasing sequence;

(2) If $x_m \prec x_n$ whenever $n > m$, then we say that the sequence $\{x_n\}_{n\geq 1}$ is a strict increasing sequence;

(3) If $x_n \preceq x_m$ whenever $n > m$, then we say that the sequence $\{x_n\}_{n\geq 1}$ is a decreasing sequence;

(4) If $x_n \prec x_m$ whenever $n > m$, then we say that the sequence $\{x_n\}_{n\geq 1}$ is a strict decreasing sequence.

Let X be a non-empty set, we shall denote by $\mathscr{S}$, $\mathscr{A}$ and $\mathcal{B}_X$ are algebra, σ-algebra and Borel σ-algebra of subsets of a set X, respectively.

Denote by $\mathscr{F}(X)$ is the set of all functionals from X to $[a, b]$. For each $\lambda \in [a, b]$ the constant functional in $\mathscr{F}(X)$ with value λ will also be denoted by λ. It will be clear from the context which usage is intended. A functional $f \in \mathscr{F}(X)$ is said to be pseudo-finite if $f(x) \prec M$ for all $x \in X$. The functional $f \in \mathscr{F}(X)$ is said to be elementary if the set of values $f(X)$ of f is a finite subset of $[a, b]$ and the set of such elementary functionals will be denoted by $\mathscr{E}(X)$.

Let f and h be two functions defined on X and with values in $[a, b]$ and $*$ be arbitrary binary operation on $[a, b]$. Then, we define for any $x \in X$

$$(f * h)(x) = f(x) * h(x),$$

and for any $\lambda \in [a,b]$, $(\lambda * f)(x) = \lambda * f(x)$. Let $\mathcal{A}$ be a subset of $\mathscr{F}(X)$. If $f * h \in \mathcal{A}$ for all $f, h \in \mathcal{A}$, then $\mathcal{A}$ is $*$-closed. The total order $\preceq$ on $[a,b]$ induces a partial order $\preceq$ on $\mathscr{F}(X)$ defined pointwise by stipulating that $f \preceq h$ if and only if $f(x) \preceq h(x)$ for all $x \in X$. Thus $(\mathscr{F}(X), \preceq)$ is a poset, and whenever we consider $\mathscr{F}(X)$ as a poset then it will always be with respect to this partial order. Let $\mathcal{S}[\lambda \prec f] = \{x | \ x \in X, \ \lambda \prec f(x), f \in \mathscr{F}(X)\}$.

Definition 2.2 ([11]). A binary operation $\oplus : [a,b] \times [a,b] \to [a,b]$ is called a pseudo-addition, if it satisfies the following conditions, for all $x, y, z, w \in [a,b]$:

(1) $\mathbf{0} \oplus x = x$, where $\mathbf{0}$ is a zero element (usually $\mathbf{0}$ is either a or b);(boundary condition)

(2) $x \oplus z \preceq y \oplus w$ whenever $x \preceq y$ and $z \preceq w$; (monotonicity)

(3) $x \oplus y = y \oplus x$; (commutativity)

(4) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$. (associativity)

A pseudo-addition $\oplus$ is said to be continuous if it is a continuous function in $[a,b]^2$. The following are examples of pseudo-additions: $x \vee_{\oplus} y = y$ if and only if $x \preceq y$; $x \oplus y = g^{-1}(g(x) + g(y))$, where $g : [a,b] \to [0, \infty]$ is a strictly monotone and continuous generator surjective function and $x \preceq y$ if and only if $g(x) \leq g(y)$, this pseudo-addition also called strict pseudo-addition. It is obvious that $M \oplus x = M$ for all $x \in [a,b]$.

In order to ensure that $g(\mathbf{0}) = 0$ for pseudo-addition. In this paper, we will consider that for all $x \in [a,b]$, it satisfies $\mathbf{0} \preceq x$.

Definition 2.3 ([11]). A binary operation $\odot : [a,b] \times [a,b] \to [a,b]$ is called a pseudo-multiplication, if it satisfies the following conditions, for all $x, y, z, w \in [a,b]$:

(1) $\mathbf{1} \odot x = x$, where $\mathbf{1} \in [a,b]$ is an unit element; (boundary condition)

(2) $x \odot z \preceq y \odot w$ whenever $x \preceq y$ and $z \preceq w$; (monotonicity)

(3) $x \odot y = y \odot x$; (commutativity)

(4) $(x \odot y) \odot z = x \odot (y \odot z)$. (associativity)

A pseudo-multiplication $\odot$ is said to be continuous if it is a continuous function in $[a,b]^2$. The following are examples of pseudo-multiplications: $x \wedge_{\odot} y = x$ if and only if $x \preceq y$; $x \odot y = g^{-1}(g(x) \cdot g(y))$, where $g : [a,b] \to [0, \infty]$ is a strictly monotone and continuous generator surjective function and $x \preceq y$ if and only if $g(x) \leq g(y)$. It is obvious that $g(\mathbf{0}) = 0$.

We assume also $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z).$$

The structure $([a,b], \oplus, \odot)$ is called a real semiring.

Because of the associative property of the pseudo-addition $\oplus$, it can be extended by induction to n-ary operation by setting

$$\bigoplus_{i=1}^{n} x_i = \left(\bigoplus_{i=1}^{n-1} x_i \right) \oplus x_n.$$

Due to monotonicity, for each sequence $\{x_i\}_{i \in \mathbb{N}}$ of elements of $[a,b]$, the following limit can be considered:

$$\bigoplus_{i=1}^{\infty} x_i = \lim_{n \to \infty} \bigoplus_{i=1}^{n} x_i.$$

Definition 2.4 ([18]). Let A be a non-empty set and $\oplus$ a pseudo-addition. A binary operation $d_\oplus : A \times A \to [a,b]$ is called a pseudo-metric on A, if it satisfies the following conditions, for all $x, y, z \in A$:

(1) $d_\oplus(x,y) = \mathbf{0}$ if and only if $x = y$;

(2) $d_\oplus(x,y) = d_\oplus(y,x)$;

(3) there exists $\lambda \in [a,b]$ such that

$$d_\oplus(x,y) \preceq \lambda \odot (d_\oplus(x,z) \oplus d_\oplus(z,y)),$$

where $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$.

Let $\{x_n\}_{n\geq 1}$ be a sequence from $[a,b]$. The sequence $\{x_n\}_{n\geq 1}$ is said to be convergent, if for any $\mathbf{0} \prec \varepsilon$, there exists positive integer $N(\varepsilon)$, such that $d_\oplus(x_n, x) \prec \varepsilon$ for all $n \geq N(\varepsilon)$, denote by $x = \lim\limits_{n\to\infty} x_n$, and x is said to be the limit of the sequence $\{x_n\}_{n\geq 1}$.

$$\underline{\lim}_{n\to\infty} x_n = \bigvee_{n=1}^{\infty}{}_{\oplus} (\bigwedge_{k\geq n}{}_{\odot}\, x_k)$$

is said to be the lower-limit of the sequence $\{x_n\}_{n\geq 1}$;

$$\overline{\lim}_{n\to\infty} x_n = \bigwedge_{n=1}^{\infty}{}_{\odot} (\bigvee_{k\geq n}{}_{\oplus}\, x_k)$$

is said to be the upper-limit of the sequence $\{x_n\}_{n\geq 1}$. It is obvious that $\underline{\lim}_{n\to\infty} x_n \preceq \overline{\lim}_{n\to\infty} x_n$. Let $\{f_n\}_{n\geq 1}$ be a sequence from $\mathscr{F}(X)$.

(1) The sequence $\{f_n\}_{n\geq 1}$ is said to be convergent, if for any $\mathbf{0} \prec \varepsilon$, and for each point $x_0 \in X$, there exists positive integer $N(\varepsilon, x_0)$, such that $d_\oplus(f_n(x_0), f(x_0)) \prec \varepsilon$ for all $n \geq N(\varepsilon, x_0)$;

(2) The sequence $\{f_n\}_{n\geq 1}$ is said to be uniform convergent, if for any $\mathbf{0} \prec \varepsilon$, there exists positive integer $N(\varepsilon)$, such that $\sup\limits_{x\in X} d_\oplus(f_n(x), f(x)) \prec \varepsilon$ for all $n \geq N(\varepsilon)$;

Let $\mathcal{A}$ be a subset of $\mathscr{F}(X)$. The poset $\mathcal{A}$ is said to be upper-complete if $\lim\limits_{n\to\infty} f_n \in \mathcal{A}$ for each increasing sequence $\{f_n\}_{n\geq 1}$ from $\mathcal{A}$; the poset $\mathcal{A}$ is said to be lower-complete if $\lim\limits_{n\to\infty} f_n \in \mathcal{A}$ for each decreasing sequence $\{f_n\}_{n\geq 1}$ from $\mathcal{A}$; the poset $\mathcal{A}$ is said to be complete if $\lim\limits_{n\to\infty} f_n \in \mathcal{A}$ for each sequence $\{f_n\}_{n\geq 1}$ from $\mathcal{A}$, where the limit of the functional sequence $\{f_n\}_{n\geq 1}$ is given by $(\lim\limits_{n\to\infty} f_n)(x) = \lim\limits_{n\to\infty} f_n(x)$ for all $x \in X$.

For any continuous pseudo-addition $\oplus$ and $x, y \in [a,b]$ with $x \preceq y$, there exists at least one point $z \in [a,b]$ such that $y = x \oplus z$. If pseudo-addition $\oplus$ is strict, then there exists only one point $z \in [a,b]$ such that $y = x \oplus z$ for all $x, y \in [a,b]$ with $x \prec M$. Thus we have the following concepts.

Definition 2.5 ([19]). For any continuous pseudo-addition $\oplus$ and $x, y \in [a,b]$ with $x \preceq y$, the pseudo-subtraction set $y -_\oplus x$ is a nonempty set of all points z such that $y = x \oplus z$. For any $x, y \in [a,b]$, define pseudo-absolute value set as

$$|y -_\oplus x| = \begin{cases} y -_\oplus x, & if \quad x \preceq y, \\ x -_\oplus y, & if \quad y \prec x. \end{cases}$$

Example 2.1. Let total order $\preceq$ on $[0,1]$ be the usual reverse order of the real line and pseudo-addition $\oplus$ be the usual multiplication of the real numbers. It is obvious that zero element is 1. If $x = 0$, then $y = 0$ and $y -_{\oplus} x = [0,1]$. If $x \neq 0$, then for any $0 \leq y < x$, we have $y -_{\oplus} x = \{y/x\} \subseteq [0,1]$.

Definition 2.6. For any continuous pseudo-addition $\oplus$, if $f, h \in \mathscr{F}(X)$, then define the pseudo-absolute value functional set $|f -_{\oplus} h|$ as the set of all those functionals φ such that

$$\varphi(x) \in |f(x) -_{\oplus} h(x)|$$

for all $x \in X$.

Example 2.2. Let total order $\preceq$ on $[0,1]$ be the usual reverse order of the real line and pseudo-addition $\oplus$ be the usual multiplication of the real numbers. It is obvious that zero element is 1. Let $X = \mathbb{R} = \bigcup_{i=1}^{4} A_i$, where

$$A_1 = \{x \in \mathbb{R} | f(x) = 0 \text{ and } h(x) = 0\}; A_2 = \{x \in \mathbb{R} | f(x) = 0 \text{ and } h(x) \neq 0\};$$
$$A_3 = \{x \in \mathbb{R} | f(x) \neq 0 \text{ and } h(x) = 0\}; A_4 = \{x \in \mathbb{R} | f(x) \neq 0 \text{ and } h(x) \neq 0\}.$$

For any $x \in \mathbb{R}$, define

$$\varphi_1(x) = \begin{cases} 1, x \notin A_1, \\ c, x \in A_1, \end{cases} \varphi_2(x) = \begin{cases} 1, x \notin A_2 \cup A_3, \\ 0, x \in A_2 \cup A_3, \end{cases}$$

$$\varphi_3(x) = \begin{cases} 1, & x \notin A_4, \\ \min\{f(x)/h(x), h(x)/f(x)\}, & x \in A_4, \end{cases}$$

where $c \in [0,1]$ is arbitrary number. Then

$$|f -_{\oplus} h| = \{\varphi | \varphi(x) = \{\min\{\varphi_1(x), \varphi_2(x), \varphi_3(x)\}\} \subseteq \mathscr{F}(\mathbb{R}).$$

Definition 2.7 ([19]). For strict pseudo-addition $\oplus$ and $x, y \in [a,b]$ with $x \preceq y$, the pseudo-subtraction $y -'_{\oplus} x$ is defined as

$$y -'_{\oplus} x = \begin{cases} g^{-1}(g(y) - g(x))), & \text{if } x \prec M, \\ \mathbf{0}, & \text{if } x = M. \end{cases}$$

For any $x, y \in [a,b]$, define pseudo-absolute value as

$$|y -'_{\oplus} x| = \begin{cases} y -'_{\oplus} x, & \text{if } x \preceq y, \\ x -'_{\oplus} y, & \text{if } y \prec x. \end{cases}$$

Definition 2.8. For strict pseudo-addition $\oplus$, if $f, h \in \mathscr{F}(X)$, then define the pseudo-absolute value functional $|f -'_{\oplus} h|$ pointwise as

$$|f -'_{\oplus} h|(x) = |f(x) -'_{\oplus} h(x)|$$

for all $x \in X$.

Definition 2.9 ([19])**.** For any pseudo-addition $\oplus$, a non-empty subset $\mathcal{K}$ of $\mathscr{F}(X)$ is said to be a functional space with respect to $\oplus$, denoted by $(\mathcal{K}, \oplus)$, if $(\lambda \odot f) \oplus (\mu \odot h) \in \mathcal{K}$ for all $f, h \in \mathcal{K}$ and $\lambda, \mu \in [a, b]$, where $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$.

It is clear that $(\mathscr{F}(X), \oplus)$ is the greatest functional space with respect to any pseudo-addition $\oplus$. Thus the functional space $(\mathcal{K}, \oplus)$ with $\mathcal{K} \subseteq \mathscr{F}(X)$ is also called a subspace of $(\mathscr{F}(X), \oplus)$. If $(\mathcal{K}, \oplus)$ is a functional space with respect to $\oplus$, then we just write $\mathcal{K}$ instead of $(\mathcal{K}, \oplus)$ whenever $\oplus$ can be determined from the context.

Definition 2.10 ([19])**.** For each subset $\mathcal{A}$ of $\mathscr{F}(X)$ the upper-closure of $\mathcal{A}$, denoted by $\hat{\mathcal{A}}$, is the set of all elements of $\mathscr{F}(X)$ having the form $\lim\limits_{n\to\infty} f_n$ for some increasing sequence $\{f_n\}_{n\geq 1}$ from $\mathcal{A}$.

It follows from Definition 2.10 that $\mathcal{A} \subseteq \hat{\mathcal{A}}$ and $\mathcal{A} = \hat{\mathcal{A}}$ if and only if $\mathcal{A}$ is upper-complete.

Definition 2.11. For any continuous pseudo-addition $\oplus$, a subspace $(\mathcal{K}, \oplus)$ will be called para-complemented if $|f -_{\oplus} h| \subseteq \mathcal{K}$ for all $f, h \in \mathcal{K}$; For strict pseudo-addition $\oplus$, a subspace $(\mathcal{K}, \oplus)$ will be called complemented if $|f -'_{\oplus} h| \in \mathcal{K}$ for all $f, h \in \mathcal{K}$.

Definition 2.12. For any continuous pseudo-addition $\oplus$, a para-complemented subspace $(\mathcal{K}, \oplus)$ is regular if it contains $\mathbf{1}$ and is closed under $\vee_{\oplus}$; For strict pseudo-addition $\oplus$, a complemented subspace $(\mathcal{K}, \oplus)$ is normal if it contains $\mathbf{1}$ and is closed under $\vee_{\oplus}$.

Note that $(f \vee_{\oplus} h) \oplus (f \wedge_{\odot} h) = f \oplus h$ for all $f, h \in \mathscr{F}(X)$ and thus a para-complemented subspace of $\mathscr{F}(X)$ is $\wedge_{\odot}$-closed if and only if it is $\vee_{\oplus}$-closed. It is obvious that regular and normal are closed under $\wedge_{\odot}$.

Definition 2.13 ([2])**.** The pseudo-characteristic function of a set $E \subseteq X$ is defined with:

$$I_E(x) = \begin{cases} \mathbf{0}, & x \notin E, \\ \mathbf{1}, & x \in E, \end{cases}$$

where $\mathbf{0}$ is zero element for $\oplus$ and $\mathbf{1}$ is unit element for $\odot$. It is obvious that $I_E \in \mathscr{E}(X)$, for all $E \subseteq X$.

Definition 2.14 ([12])**.** A set function $m : \mathscr{A} \to [a, b]$ (or semiclosed interval) is called a σ-$\oplus$-decomposable measure if it satisfies the following conditions:

(1) $m(\emptyset) = \mathbf{0}$;

(2) $m(E \cup F) = m(E) \oplus m(F)$ for all $E, F \in \mathscr{A}$ and $E \cap F = \emptyset$;

(3) $m(\bigcup\limits_{i=1}^{\infty} E_i) = \bigoplus\limits_{i=1}^{\infty} m(E_i)$ for any sequence $\{E_i\}_{i\geq 1}$ of pairwise disjoint sets from $\mathscr{A}$.

A pair $(X, \mathscr{A})$ consisting of a non-empty set X and a σ-algebra of subsets of X is called a measurable space. A functional $f : X \to [a, b]$ is said to be a measurable functional if $f^{-1}(\mathcal{B}_{[a,b]}) \subseteq \mathscr{A}$. Let $\mathcal{M}(\mathscr{A})$ be the set of all measurable mappings from $(X, \mathscr{A})$ to $([a, b], \mathscr{B}_{[a,b]})$, i.e.,

$$\mathcal{M}(\mathscr{A}) = \{f \in \mathscr{F}(X) |\ f^{-1}(\mathscr{B}_{[a,b]}) \subseteq \mathscr{A}\}.$$

Then $\mathscr{E}(\mathscr{S})$ will denote the set of those elements $f \in \mathscr{E}(X)$ for which $f^{-1}(\lambda) = \{x \in X | f(x) = \lambda\} \in \mathscr{S}$ for each $\lambda \in f(X)$. In particular, this means that $\mathscr{E}(\mathscr{A}) = \mathcal{M}(\mathscr{A}) \cap \mathscr{E}(X)$.

Definition 2.15. Let $\oplus$ be a pseudo-addition and $m : \mathscr{A} \to [a, b]$ a σ-$\oplus$-decomposable measure. Let $\{f_n\}_{n\geq 1}$ be a sequence of measurable functionals of a.e. pseudo-finite on X. If there exists a measurable functional f of a.e. pseudo-finite on X, such that

$$\lim_{n\to\infty} m\mathcal{S}[\sigma \preceq d_\oplus(f_n, f)] = \mathbf{0}$$

for arbitrary $\mathbf{0} \prec \sigma \prec M$, then the functionals sequence $\{f_n\}_{n\geq 1}$ is said to be convergent to f with respect to $\oplus$-measure, denote by $f_n \Rightarrow f$. If the functionals sequence $\{f_n\}_{n\geq 1}$ does not converge to f with respect to $\oplus$-measure, denote by $f_n \not\Rightarrow f$.

Definition 2.16 ([20]). A set function $m : \mathscr{A} \to [a, b]$ (or semiclosed interval) is monotone if

$$m(E) \preceq m(F)$$

whenever $E, F \in \mathscr{A}$ and $E \subset F$.

3. Main results

In this section we will give some of important properties for the measurable functional space based on a strict pseudo-addition.

Theorem 3.1 ([3]). *The σ-$\oplus$-decomposable measure $m : \mathscr{A} \to [a, b]$ is monotone.*

Lemma 3.1. *Let $\oplus$ be a strict pseudo-addition. The function $d_\oplus : [a, b]^2 \to [a, b]$ given by*

$$d_\oplus(x, y) = |x -'_\oplus y|$$

Then $d_\oplus$ is a pseudo-metric on $[a, b]$ and $\lambda = \mathbf{1}$.

Proof. It is obvious that the function $d_\oplus$ satisfies the conditions (1), (2) and (3) of the Definition 2.4, that is $d_\oplus$ is a pseudo-metric on $[a, b]$. We easy to get

$$d_\oplus(x, y) \preceq d_\oplus(x, z) \oplus d_\oplus(z, y) = \mathbf{1} \odot (d_\oplus(x, z) \oplus d_\oplus(z, y)).$$

□

Theorem 3.2. *Let $\oplus$ be a strict pseudo-addition and $\{x_n\}_{n\geq 1}$ a sequence from $[a, b]$. Then $\underline{\lim}_{n\to\infty} x_n = \overline{\lim}_{n\to\infty} x_n$ if and only if the sequence $\{x_n\}_{n\geq 1}$ is convergence.*

Proof. Let

$$y = \underline{\lim}_{n\to\infty} x_n = \bigvee_{\oplus\, n=1}^{\infty} (\bigwedge_{\odot\, k\geq n} x_k) \quad and \quad z = \overline{\lim}_{n\to\infty} x_n = \bigwedge_{\odot\, n=1}^{\infty} (\bigvee_{\oplus\, k\geq n} x_k).$$

Put $y_n = \wedge_{\odot k\geq n} x_k$, then $\{y_n\}$ is an increasing sequence and $y = \vee_{\oplus n=1}^{\infty} y_n$. Thus for $\mathbf{0} \prec \varepsilon = \lambda_i$, where $\cdots \prec \lambda_i \prec \cdots \prec \lambda_2 \prec \lambda_1 \prec y$ and $\lim_{i\to\infty} \lambda_i = \mathbf{0}$, there exists n_i such that $y -'_\oplus \varepsilon \prec y_{n_i}$ and there exists $k_i > n_i$ such that $x_{k_i} \prec y_{n_i} \oplus \varepsilon \preceq y \oplus \varepsilon$. Since $y_{n_i} \preceq x_{k_i}$, we have $y -'_\oplus \varepsilon \prec x_{k_i}$. Thus we have $|y -'_\oplus x_{k_i}| \prec \varepsilon$ which implies that the subsequence $\{x_{k_i}\}$ converges to y. Similarly, we can obtain that there exists $\{x_{k_j}\}$ converges to z.

Suppose $\lim_{n\to\infty} x_n = x$. For any $\mathbf{0} \prec \varepsilon$, there exists positive integer $N(\varepsilon)$, such that $|x_n -'_{\oplus} x| \prec \varepsilon$ for all $n \geq N(\varepsilon)$. For any subsequence $\{x_{n_k}\}$ of $\{x_n\}$, if $n_k \geq n \geq N(\varepsilon)$, then $|x_{n_k} -'_{\oplus} x| \prec \varepsilon$. Thus the subsequence $\{x_{n_k}\}$ converges to x which implies that $x = y = z$.

Suppose $y = z$. Put $z_n = \vee_{\oplus k\geq n} x_k$, then $\{z_n\}$ is a decreasing sequence and $z = \wedge_{\odot}{}_{n=1}^{\infty} z_n$. Thus for any $\mathbf{0} \prec \varepsilon$, there exists positive integer N, such that

$$y \prec y_N \oplus \varepsilon \preceq y_n \oplus \varepsilon = \wedge_{\odot k\geq n} x_k \oplus \varepsilon \preceq x_n \oplus \varepsilon$$

and

$$x_n \preceq \vee_{\oplus k\geq n} x_k = z_n \preceq z_N \prec z \oplus \varepsilon = y \oplus \varepsilon,$$

for all $n \geq N$. Hence, $|y -'_{\oplus} x_n| \prec \varepsilon$, which implies that $y = \lim_{n\to\infty} x_n$. □

Theorem 3.3. *Let $\oplus$ be a strict pseudo-addition and $\{E_n\}_{n\geq 1} \subset \mathscr{A}(X)$ a decreasing sequence. If there exists at least one $l \in \mathbb{N}$ such that $m(E_l) \prec M$, then*

$$m\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} m\left(E_n\right).$$

Proof. Suppose $m(E_l) \prec M$ for some $l \in \mathbb{N}$. If $\{E_n\}_{n\geq 1}$ is a decreasing sequence, then $\{E_l - E_n\}_{n\geq 1}$ is a increasing sequence. By Theorem 3.2 in [20], we have

$$m\left(E_l - \lim_{n\to\infty} E_n\right) = m\left(\lim_{n\to\infty} (E_l - E_n)\right) = \lim_{n\to\infty} m(E_l - E_n).$$

But

$$m(E_l) = m(E_l - E_n) \oplus m(E_n)$$

and

$$m(E_l) = m\left(E_l - \lim_{n\to\infty} E_n\right) \oplus m\left(\lim_{n\to\infty} E_n\right).$$

Thus, we have

$$\begin{aligned} m(E_l) &= \lim_{n\to\infty} (m(E_l - E_n) \oplus m(E_n)) \\ &= \lim_{n\to\infty} m(E_l - E_n) \oplus \lim_{n\to\infty} m(E_n) \\ &= m\left(E_l - \lim_{n\to\infty} E_n\right) \oplus \lim_{n\to\infty} m(E_n). \end{aligned}$$

Hence, we obtain that $m\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} m\left(E_n\right)$, because $m(E_l) \prec M$. □

Lemma 3.2 ([19]). *Let $\oplus$ be a strict pseudo-addition. Then $(\mathcal{M}(\mathscr{A}), \oplus)$ are both upper-complete and lower-complete.*

Lemma 3.3 ([19]). *Let $\oplus$ be a strict pseudo-addition. Then $(\mathcal{M}(\mathscr{A}), \oplus)$ is an upper-complete normal subspace of $(\mathscr{F}(X), \oplus)$.*

Theorem 3.4. *Let $\oplus$ be a strict pseudo-addition and $\{f_n\}_{n\geq 1}$ a sequence from $(\mathcal{M}(\mathscr{A}), \oplus)$. Then $\varphi,\ \psi \in \mathcal{M}(\mathscr{A})$, where $\varphi(x) = \bigwedge_{\odot}{}_{n=1}^{\infty} f_n(x)$ and $\psi(x) = \bigvee_{\oplus}{}_{n=1}^{\infty} f_n(x)$.*

Proof. Let
$$\varphi_n = f_1 \wedge_\odot f_2 \wedge_\odot \cdots \wedge_\odot f_n$$
and
$$\psi_n = f_1 \vee_\oplus f_2 \vee_\oplus \cdots \vee_\oplus f_n.$$
Then $\{\varphi_n\}_{n\geq 1}$ with $\lim\limits_{n\to\infty} \varphi_n = \varphi$ and $\{\psi_n\}_{n\geq 1}$ with $\lim\limits_{n\to\infty} \psi_n = \psi$ are decreasing sequence and increasing sequence from $\mathcal{M}(\mathscr{A})$. By Lemma 3.3, we have $\mathcal{M}(\mathscr{A})$ are both $\vee_\oplus$-closed and $\wedge_\odot$-closed. Hence, from Lemma 3.2, we have $\varphi,\ \psi \in \mathcal{M}(\mathscr{A})$. $\square$

Theorem 3.5. *Let $\oplus$ be a strict pseudo-addition and $\{f_n\}_{n\geq 1}$ a sequence from $(\mathcal{M}(\mathscr{A}), \oplus)$. Then $\varphi,\ \psi \in \mathcal{M}(\mathscr{A})$, where $\varphi = \varliminf\limits_{n\to\infty} f_n$ and $\psi = \varlimsup\limits_{n\to\infty} f_n$. In particular, if there exists $f = \lim\limits_{n\to\infty} f_n$, then $f \in \mathcal{M}(\mathscr{A})$.*

Proof. Since
$$\varliminf_{n\to\infty} f_n = \mathop{\vee_\oplus}_{n=1}^{\infty} (\mathop{\wedge_\odot}_{m\geq n} f_m) \quad \textit{and} \quad \varlimsup_{n\to\infty} f_n = \mathop{\wedge_\odot}_{n=1}^{\infty} (\mathop{\vee_\oplus}_{m\geq n} (f_m)),$$
by Theorem 3.4, we have $\varphi,\ \psi \in \mathcal{M}(\mathscr{A})$. If there exists $f = \lim\limits_{n\to\infty} f_n$, then by Theorem 3.2, we have $f = \varphi = \psi$, i.e., $f \in \mathcal{M}(\mathscr{A})$. $\square$

Corollary 3.1. *Let $\oplus$ be a strict pseudo-addition. Then $(\mathcal{M}(\mathscr{A}), \oplus)$ is an complete normal subspace of $(\mathscr{F}(X), \oplus)$.*

Proof. By Lemma 3.3, we have $(\mathcal{M}(\mathscr{A}), \oplus)$ is a normal subspace of $(\mathscr{F}(X), \oplus)$. Consequently, by Theorem 3.5, we have $(\mathcal{M}(\mathscr{A}), \oplus)$ is an complete. $\square$

Theorem 3.6. *Let $\oplus$ be a strict pseudo-addition and $\{f_n\}_{n\geq 1}$ a sequence of measurable functionals of a.e. pseudo-finite on X. If $m(X) \prec M$, $\lim\limits_{n\to\infty} f_n = f$ a.e. and $f \prec M$ a.e. on X, then for any positive integer n and $\mathbf{0} \prec \varepsilon \prec M$,*
$$\lim_{n\to\infty} m(X - \mathcal{S}[n,\varepsilon]) = \mathbf{0},$$
where $\mathcal{S}[n,\varepsilon] = \mathcal{S}[|f_k -'_\oplus f| \prec \varepsilon,\ k \geq n]$.

Proof. If $\{f_n\}_{n\geq 1} \subseteq \mathcal{M}(\mathscr{A})$, then by Theorem 3.5, we have that f is a measurable functional, i.e., $f \in \mathcal{M}(\mathscr{A})$, and therefore by Lemma 3.3, we have $|f_k -'_\oplus f| \in \mathcal{M}(\mathscr{A})$, i.e., $\mathcal{S}[|f_k -'_\oplus f| \prec \varepsilon] \in \mathscr{A}$ for all $k \geq n$. Thus, we have $\mathcal{S}[n,\varepsilon] \in \mathscr{A}$, because
$$\mathcal{S}[n,\varepsilon] = \bigcap_{k=n}^{\infty} \mathcal{S}[|f_k -'_\oplus f| \prec \varepsilon].$$
By the definitions of f_n and f, we have $m(X - \mathcal{S}[f_n \to pseudo-finite\ f]) = \mathbf{0}$. But
$$\begin{aligned}\mathcal{S}[f_n \to pseudo-finite\ f] &\subset \varliminf_{n\to\infty} \mathcal{S}[|f_n -'_\oplus f| \prec \varepsilon] \\ &= \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} \mathcal{S}[|f_n -'_\oplus f| \prec \varepsilon] \\ &= \bigcup_{n=1}^{\infty} \mathcal{S}[n,\varepsilon] = \lim_{n\to\infty} \mathcal{S}[n,\varepsilon],\end{aligned}$$

and $\{X - \mathcal{S}[n, \varepsilon]\}$ is a decreasing sequence with respect to n, because $\{\mathcal{S}[n, \varepsilon]\}$ is a increasing sequence with respect to n. Hence, by theorem 3.1, we have

$$m(\mathcal{S}[f_n \to pseudo - finite\ f]) \preceq m\left(\lim_{n\to\infty} \mathcal{S}[n, \varepsilon]\right).$$

By Theorem 3.3, we get that

$$m\left(X - \lim_{n\to\infty} \mathcal{S}[n, \varepsilon]\right) = m\left(\lim_{n\to\infty} (X - \mathcal{S}[n, \varepsilon])\right) = \lim_{n\to\infty} m(X - \mathcal{S}[n, \varepsilon]).$$

Thus we have

$$\begin{aligned} m(X) &= m\left(X - \lim_{n\to\infty} \mathcal{S}[n, \varepsilon]\right) \oplus m\left(\lim_{n\to\infty} \mathcal{S}[n, \varepsilon]\right) \\ &= \lim_{n\to\infty} m(X - \mathcal{S}[n, \varepsilon]) \oplus m\left(\lim_{n\to\infty} \mathcal{S}[n, \varepsilon]\right). \end{aligned}$$

But

$$m(X) = m(X - \mathcal{S}[f_n \to pseudo - finite\ f]) \oplus m(\mathcal{S}[f_n \to pseudo - finite\ f]).$$

Hence, we obtain that

$$\lim_{n\to\infty} m(X - \mathcal{S}[n, \varepsilon]) \preceq m(X - \mathcal{S}[f_n \to pseudo - finite\ f]) = \mathbf{0},$$

because $m(X) \prec M$. $\square$

Corollary 3.2. *Let $\oplus$ be a strict pseudo-addition and $\{f_n\}_{n\geq 1}$ a sequence of measurable functionals on X. If $m(X) \prec M$, $\lim_{n\to\infty} f_n = f$ a.e. and $f \prec M$ a.e. on X, then for any $\mathbf{0} \prec \varepsilon \prec M$,*

$$\lim_{n\to\infty} m(X - \mathcal{S}[|f_n -'_{\oplus} f| \prec \varepsilon]) = \mathbf{0}.$$

Proof. Since $\mathcal{S}[n, \varepsilon] \subset \mathcal{S}[|f_n -'_{\oplus} f| \prec \varepsilon]$, where $\mathcal{S}[n, \varepsilon]$ is the same as in Theorem 3.6, we have $X - \mathcal{S}[|f_n -'_{\oplus} f| \prec \varepsilon] \subset X - \mathcal{S}[n, \varepsilon]$. By Theorem 3.1 and 3.6, we obtain that

$$\lim_{n\to\infty} m(X - \mathcal{S}[|f_n -'_{\oplus} f| \prec \varepsilon]) \preceq \lim_{n\to\infty} m(X - \mathcal{S}[n, \varepsilon]) = \mathbf{0},$$

which implies that $\lim_{n\to\infty} m(X - \mathcal{S}[|f_n -'_{\oplus} f| \prec \varepsilon]) = \mathbf{0}$. $\square$

Theorem 3.7. *Let $\oplus$ be a strict pseudo-addition and $\{f_n\}_{n\geq 1}$ a sequence of measurable functionals on X. If $m(X) \prec M$, $\lim_{n\to\infty} f_n = f$ a.e. and $f \prec M$ a.e. on X, then for any $\mathbf{0} \prec \delta \prec M$, there exists $E_\delta \subset X$, such that $\{f_n\}_{n\geq 1}$ is uniform convergence on E_δ and*

$$m(X - E_\delta) \prec \delta.$$

Proof. For a sequence $\{n_i\}_{i\geq 1}$, where n_i is a positive integer for all $i \in \mathbb{N}$, let

$$\mathcal{S}[\{n_i\}_{i\geq 1}] = \bigcap_{i=1}^{\infty} \mathcal{S}[n_i, \lambda_i],$$

where $\mathcal{S}[n_i, \lambda_i]$ is the same as in Theorem 3.6, $\cdots \lambda_j \prec \cdots \prec \lambda_2 \prec \lambda_1 = \mathbf{1}$ and $\lim_{j\to\infty} \lambda_j = \mathbf{0}$. Then $\{f_n\}_{n\geq 1}$ is uniform convergence to f on $\mathcal{S}[\{n_i\}_{i\geq 1}]$. In fact, For any given $\mathbf{0} \prec \varepsilon \prec M$, there exists i_0 such that $\lambda_{i_0} \prec \varepsilon$. Then if $n \geq n_{i_0}$, we have

$$(|f_n -'_{\oplus} f|)(x) \prec \lambda_{i_0} \prec \varepsilon,$$

for all $x \in \mathcal{S}[\{n_i\}_{i\geq 1}] \subset \mathcal{S}[n_{i_0}, \lambda_{i_0}]$. Thus, for any given $\mathbf{0} \prec \delta \prec M$, if there exists a sequence $\{n_i\}_{i\geq 1}$ such that

$$m(X - \mathcal{S}[\{n_i\}_{i\geq 1}]) \prec \delta,$$

then let $E_\delta = \mathcal{S}[\{n_i\}_{i\geq 1}] \subset X$.

By Theorem 3.6, for each $\varepsilon_i = \lambda_i,\ i = 1, 2, \cdots$, where $\cdots \lambda_n \prec \cdots \prec \lambda_2 \prec \lambda_1 = \mathbf{1}$ and $\lim\limits_{n\to\infty} \lambda_n = \mathbf{0}$, there exists a sufficiently large n_i, respectively, such that

$$m(X - \mathcal{S}[n_i, \lambda_i]) \prec \mu_{2^i} \odot \delta,$$

where $\mathbf{0} \prec \mu_{2^i} \oplus \mu_{2^i} = \mu_{2^{i-1}},\ i = 1, 2, \cdots,\ \mu_1 = \mathbf{1}$ and $\lim\limits_{n\to\infty} \mu_{2^n} = \mathbf{0}$. Hence, the sequence $\{n_i\}_{i\geq 1}$ consisting of these $n_i,\ i = 1, 2, \cdots$ satisfies

$$\begin{aligned} m(X - \mathcal{S}[\{n_i\}_{i\geq 1}]) &= m\left(X - \bigcap_{i=1}^{\infty} \mathcal{S}[n_i, \lambda_i]\right) = m\left(\bigcup_{i=1}^{\infty} X - \mathcal{S}[n_i, \lambda_i]\right) \\ &\preceq \bigoplus_{i=1}^{\infty} m(X - \mathcal{S}[n_i, \lambda_i]) \prec \bigoplus_{i=1}^{\infty} (\mu_{2^i} \odot \delta) \\ &= \delta \odot \left(\bigoplus_{i=1}^{\infty} \mu_{2^i}\right) = \delta \odot \mathbf{1} = \delta, \end{aligned}$$

which implies that $m(X - E_\delta) \prec \delta$. □

Theorem 3.8. *Let $\oplus$ be a strict pseudo-addition and the sequence $\{f_n\}_{n\geq 1}$ converges to f with respect to $\oplus$-measure. Then there exists a subsequence $\{f_{n_i}\}_{i\geq 1}$ of $\{f_n\}_{n\geq 1}$ is a.e. converges to f on X.*

Proof. For any positive integer s, put $\delta = \varepsilon = \lambda_{2^s}$, where $\mathbf{0} \prec \lambda_{2^i} \oplus \lambda_{2^i} = \lambda_{2^{i-1}},\ i = 1, 2, \cdots,\ \lambda_1 = \mathbf{1}$ and $\lim\limits_{n\to\infty} \lambda_{2^n} = \mathbf{0}$. Since $f_n \Rightarrow f$, there exists positive integer n_s, such that

$$mE_s \prec \lambda_{2^s},\ s = 1, 2, \cdots,$$

where $E_s = \mathcal{S}[\lambda_{2^s} \preceq |f_{n_s} -'_{\oplus} f|]$. Assume $n_1 \leq n_2 \leq \cdots$, let

$$F_k = \bigcap_{s=k}^{\infty} (X - E_s),$$

which implies that

$$F_k = \mathcal{S}[|f_{n_s} -'_{\oplus} f| \prec \lambda_{2^s}, s \geq k],$$

because $X - E_s = \mathcal{S}[|f_{n_s} -'_{\oplus} f| \prec \lambda_{2^s}]$. It is obvious that the subsequence $\{f_{n_s}\}_{s\geq 1}$ converges to f on F_k. Let $F = \bigcup\limits_{k=1}^{\infty} F_k$. Then the subsequence $\{f_{n_s}\}_{s\geq 1}$ converges to f on F.

Now we show that $m(X - F) = \mathbf{0}$. Since $\bigoplus\limits_{s=1}^{n} \lambda_{2^s} \oplus \lambda_{2^n} = \lambda_2 \oplus \lambda_2 = \lambda_1 = \mathbf{1}$, we have

$$\mathbf{1} = \lim_{n\to\infty} \bigoplus_{s=1}^{n} \lambda_{2^s} \oplus \lim_{n\to\infty} \lambda_{2^n} = \bigoplus_{s=1}^{\infty} \lambda_{2^s} \oplus \mathbf{0} = \bigoplus_{s=1}^{\infty} \lambda_{2^s}.$$

Thus we get that

$$\bigoplus_{s=1}^{\infty} mE_s \prec \bigoplus_{s=1}^{\infty} \lambda_{2^s} = \mathbf{1}.$$

Since

$$X - F = X - (\bigcup_{k=1}^{\infty} F_k) = \bigcap_{k=1}^{\infty} (X - F_k) = \bigcap_{k=1}^{\infty} \bigcup_{s=k}^{\infty} E_s = \overline{\lim_{s\to\infty}} E_s,$$

by the definition of upper-limit, for any positive integer k, we have

$$\overline{\lim_{s\to\infty}} E_s \subset \bigcup_{s=k}^{\infty} E_s.$$

Hence, by Theorem 3.1, we have

$$m(\overline{\lim_{s\to\infty}} E_s) \preceq m(\bigcup_{s=k}^{\infty} E_s) \preceq \bigoplus_{s=k}^{\infty} mE_s \preceq \bigoplus_{s=k}^{\infty} \lambda_{2^s} \preceq \lambda_{2^{k-1}},$$

which implies that

$$m(X - F) = m(\overline{\lim_{s\to\infty}} E_s) = \mathbf{0}.$$

□

Example 3.1. Let $X = (0, 1]$ and $\mathscr{A}$ be a generated σ-algebra by all open subinterval of X. We define functions:

$$f_{n,k}(x) = \begin{cases} \mathbf{1}, & x \in (\frac{k-1}{2^n}, \frac{k}{2^n}], \\ \mathbf{0}, & x \notin (\frac{k-1}{2^n}, \frac{k}{2^n}], \end{cases}$$

$$n = 1, 2, \cdots ;\ k = 1, 2, \cdots, 2^n.$$

The order of the functional sequence $\{f_{n,k}\}$ as follows:

$$f_{1,1},\ f_{1,2},\ f_{2,1},\ f_{2,2},\ f_{2,3},\ f_{2,4},\ \cdots,\ f_{n,1},\ f_{n,2},\ \cdots,\ f_{n,2^n},\ \cdots,$$

where $f_{n,k}$ is the Nth ($N = 2^n + k - 2$) function. The σ-$\oplus$-decomposable measure m satisfies:

$$m((\frac{k-1}{2^n}, \frac{k}{2^n}]) = \lambda_{2^n},$$

where $\mathbf{0} \prec \lambda_{2^n} \oplus \lambda_{2^n} = \lambda_{2^{n-1}},\ n = 1, 2, \cdots,\ \lambda_1 = \mathbf{1}$ and $\lim_{n\to\infty} \lambda_{2^n} = \mathbf{0}$. Now we show that $\{f_{n,k}\}$ is convergence in $\oplus$-measure in $\mathbf{0}$ on X. For any $\mathbf{0} \prec \sigma$, we have

$$\mathcal{S}[\sigma \prec |f_{n,k} -'_{\oplus} \mathbf{0}|] = \begin{cases} \emptyset, & \mathbf{1} \prec \sigma, \\ (\frac{k-1}{2^n}, \frac{k}{2^n}], & \mathbf{0} \prec \sigma \preceq \mathbf{1}, \end{cases}$$

which implies that

$$m(\mathcal{S}[\sigma \prec |f_{n,k} -'_{\oplus} \mathbf{0}|]) \preceq \lambda_{2^n}.$$

If $N \to \infty$ ($N = 2^n + k - 2,\ k = 1, 2, \cdots, 2^n$), then $n \to \infty$. Thus, we have

$$\lim_{N\to\infty} m(\mathcal{S}[\sigma \prec |f_{n,k} -'_{\oplus} \mathbf{0}|]) = \mathbf{0},$$

i.e., $f_{n,k} \Rightarrow \mathbf{0}$. But functional sequence $\{f_{n,k}\}$ is not convergence for all $x \in X$. In fact, for any $x_0 \in X$, no matter how large n, there exists $1 \geq k \geq 2^n$, such that $x_0 \in (\frac{k-1}{2^n}, \frac{k}{2^n}]$, i.e., $f_{n,k}(x_0) = \mathbf{1}$. Hence, we have $f_{n,k+1}(x_0) = \mathbf{0}$ or $f_{n,k-1}(x_0) = \mathbf{0}$. In other words, for any $x_0 \in X$, there exists two subsequence of $\{f_{n,k}(x_0)\}$, the function value of one subsequence is $\mathbf{1}$ in $x_0 \in X$, and the other is $\mathbf{0}$ in $x_0 \in X$. Hence, the functional sequence $\{f_{n,k}\}$ is not convergence for all $x \in X$.

Theorem 3.9. *Let $\oplus$ be a strict pseudo-addition and $\{f_n\}_{n\geq 1}$ a sequence of measurable functionals of a.e. pseudo-finite on X. If $m(X) \prec M$, $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. and $f(x) \prec M$ a.e., then $f_n \Rightarrow f$.*

Proof. By Corollary 3.2, we have that for any $\mathbf{0} \prec \varepsilon \prec M$,

$$\lim_{n\to\infty} m(X - \mathcal{S}[|f_n -'_{\oplus} f| \prec \varepsilon]) = \mathbf{0},$$

which implies that

$$\lim_{n\to\infty} m\mathcal{S}[\varepsilon \preceq |f_n -'_{\oplus} f|] = \mathbf{0}.$$

□

Example 3.2. Let $X = (0, +\infty)$ and $\mathscr{A}$ be a generated σ-algebra by all open subinterval of X. The σ-$\oplus$-decomposable measure $m : \mathscr{A} \to [a, b]$ satisfies:

$$m((0, n]) = \lambda_n \text{ and } m((n, +\infty)) = M, \ n = 1, 2, \cdots,$$

where $\mathbf{0} \prec \lambda_1 \prec \lambda_2 \prec \cdots \prec \lambda_n \prec \cdots$ and $\lim_{n\to\infty} \lambda_n = M$. We define the function sequence as

$$f_n(x) = \begin{cases} \mathbf{1}, & x \in (0, n], \\ \mathbf{0}, & x \in (n, +\infty), \end{cases}$$
$$n = 1, 2, \cdots.$$

It is obvious that $f_n \to \mathbf{1}$ $(n \to +\infty)$ on X. But for any $\mathbf{0} \prec \sigma \prec \mathbf{1}$,

$$m\mathcal{S}[\sigma \preceq |f_n -'_{\oplus} \mathbf{1}|] = m((n, +\infty)) = M,$$

which implies that the sequence $\{f_n\}_{n\geq 1}$ does not converge to $\mathbf{1}$ with respect to $\oplus$-measure, i.e., $f_n \not\Rightarrow \mathbf{1}$.

Theorem 3.10. *Let $\oplus$ be a strict pseudo-addition and $\{f_n\}_{n\geq 1}$ a measurable functional sequence on X. If $f_n \Rightarrow f$ and $f_n \Rightarrow h$, then $f = h$ a.e. on X.*

Proof. By Lemma 3.1, we have

$$|f -'_{\oplus} h| \preceq |f -'_{\oplus} f_k| \oplus |f_k -'_{\oplus} h|.$$

Then for any positive integer n,

$$\mathcal{S}[\lambda_n \preceq |f -'_{\oplus} h|] \subseteq \mathcal{S}[\lambda_{2n} \preceq |f -'_{\oplus} f_k|] \cup \mathcal{S}[\lambda_{2n} \preceq |f_k -'_{\oplus} h|],$$

where $\mathbf{0} \prec \cdots \prec \lambda_n \prec \cdots \prec \lambda_2 \prec \lambda_1 = \mathbf{1}$, $\lambda_{2n} \oplus \lambda_{2n} = \lambda_n$ and $\lim_{n\to\infty} \lambda_n = \mathbf{0}$. Thus, we get that

$$m\mathcal{S}[\lambda_n \preceq |f -'_{\oplus} h|] \preceq m\mathcal{S}[\lambda_{2n} \preceq |f -'_{\oplus} f_k|] \oplus m\mathcal{S}[\lambda_{2n} \preceq |f_k -'_{\oplus} h|],$$

which implies that

$$m\mathcal{S}[\lambda_n \preceq |f -'_{\oplus} h|] \preceq \lim_{k\to\infty} m\mathcal{S}[\lambda_{2n} \preceq |f -'_{\oplus} f_k|] \oplus \lim_{k\to\infty} m\mathcal{S}[\lambda_{2n} \preceq |f_k -'_{\oplus} h|] = \mathbf{0},$$

i.e. $m\mathcal{S}[\lambda_n \preceq |f -'_{\oplus} h|] = \mathbf{0}$. Hence, we obtain that $m\mathcal{S}[f \neq g] = \mathbf{0}$, because $\mathcal{S}[f \neq g] = \bigcup_{n=1}^{\infty} \mathcal{S}[\lambda_n \preceq |f -'_{\oplus} h|]$. □

4. Conclusions

In [18], E. Pap, M. Štrboja and I. Rudas gave a generalization of the classical L^p space in the frame work of pseudo-analysis as $L^p_\oplus$ space. They introduced three types of convergence and the relationships among these convergence concepts have been investigated for two important cases of the semiring (Case I and Case II) from an integral point of view. Comparing with the results in [18], we proved some algebraic properties of the measurable functional space $(\mathcal{M}(\mathscr{A}), \oplus)$ based on a strict pseudo-addition, which show $(\mathcal{M}(\mathscr{A}), \oplus)$ is an complete normal subspace of $(\mathscr{F}(X), \oplus)$. Furthermore, by introducing the pseudo-subtraction, we obtained that the some properties of the sequence of a.e. convergence and convergence in $\oplus$-measure, and the relationship between a.e. convergence and convergence in $\oplus$-measure on the measurable functional spaces $(\mathcal{M}(\mathscr{A}), \oplus)$. Because the concepts of pseudo-addition-decomposable measures and pseudo-addition-decomposable integrals [1, 2, 11, 13, 17, 18, 24] are very useful in the theory of nonlinear differential and integral equations [10, 14–16, 25], the relationships between nonlinear functional spaces based on pseudo-additions and those concepts will also be explored in our future research.

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