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SINGULAR PERIODIC WAVES OF AN INTEGRABLE EQUATION FROM SHORT CAPILLARY-GRAVITY WAVES*

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Abstract The effects of parabola singular curves in the integrable nonlinear wave equation are studied by using the bifurcation theory of dynamical system. We find new singular periodic waves for a nonlinear wave equation from short capillary-gravity waves. The new periodic waves possess weaker singularity than the periodic peakon. It is shown that the second derivatives of the new singular periodic wave solutions do not exist in countable number of points but the first derivatives exist. We show that there exist close connection between the new singular periodic waves and parabola singular curve in phase plane of traveling wave system for the first time.

Keywords Bifurcation theory of dynamical systems, peakon, periodic peakon, periodic wave, parabola singular curve.

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1. Introduction

Mathematical modeling of dynamical processes in a great variety of natural phenomena leads in general to nonlinear partial differential equations. There is a particular class of solutions for these nonlinear equations that are of considerable interest. They are the traveling wave solutions. Such a wave is a special solution of the governing equations, that may be localized or periodic, which does not change its shape and which propagates at constant speed.

Directly seeking for exact traveling wave solutions of nonlinear partial differential equations to describe many important phenomena in physics, biology, chemistry, etc., has become a central theme of perpetual interest in recent decades. Many powerful methods have been presented for finding the solutions, such as the Bäcklund transformation [6], tanh-coth method [21], Riccati equation method [8], bilinear method [14], symbolic computation method [7], Lie group analysis method [12],

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and so on. Furthermore, a great amount of activity has been concentrated on the various extensions and applications of the methods to simplify the routine of calculation. The basic idea of the above-mentioned approaches is that, by introducing different types of Ansatz, the original partial differential equations can be transformed into a set of algebraic equations through balancing the same order of the Ansatz, which yields explicit expressions for the waves. However, only part of the special form of the waves can be derived by using most of these methods. In order to obtain all possible forms of the waves and analyze qualitative behaviors of solutions, recently bifurcation theory has been introduced to study the evolution of wave patterns with variation of the parameters [5, 9–11, 19, 22, 25, 26, 28].

To study the traveling wave solutions of a nonlinear equation

$$\Phi(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \cdots) = 0, \tag{1.1}$$

let $\xi = x - ct, u(x,t) = \varphi(\xi)$, where c is the wave speed. Substituting them into (1.1), we have

$$\Phi_1(\varphi, \varphi', \varphi'', \cdots) = 0. \tag{1.2}$$

Here, we consider the case that (1.2) can be reduced to the following planar dynamical system:

$$\frac{d\varphi}{d\xi} = y, \quad \frac{dy}{d\xi} = F(\varphi, y), \tag{1.3}$$

by integrals and let $\varphi' = y$, that is to say, (1.3) is the corresponding traveling wave system of the nonlinear equation (1.1). That means that to study the traveling wave solutions of the nonlinear equation (1.1) we only need to study the corresponding traveling wave system (1.3).

We start with the well-known Camassa-Holm equation. The Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1.4}$$

arises as a model for nonlinear waves in cylindrical axially symmetric hyperelastic rods, with $u(x,t)$ representing the radial stretch relative to a prestressed state [2]. Camassa and Holm showed that Eq. (1.4) has peakon of the form $u(x,t) = ce^{-|x-ct|}$. Among the nonanalytic entities, the peakon, a soliton with a finite discontinuity in gradient at its crest, is perhaps the weakest nonanalyticity observable by the eye [18]. The Camassa-Holm equation has also periodic peakon [23]. The first derivatives of the peakons and periodic peakons do not exist in a finite or countable number of points.

The Camassa-Holm equation and almost all integrable nonlinear dispersion equations have the same class of traveling wave systems with vertical straight line. It has been pointed out that traveling waves sometimes lose their smoothness during the propagation due to the existence of singular curves within the solution surfaces of the wave equation. The relationships between the traveling waves of the nonlinear equations with a singular straight line and the orbits of the corresponding traveling wave systems are well known [13, 17, 20, 24, 27, 28]. But till now there have been few works on the integrable nonlinear equations with other types of singular curves.

In [3, 16], the authors studied the existence of the "W/M"-shape solitary waves of several nonlinear wave equations. For these equations, it is not difficult to find that the corresponding traveling wave system (1.3) posses hyperbola singular curves in the phase plane. Obviously, there exist close connection between the "W/M"-shape solitary waves and hyperbola singular curves in phase plane of traveling wave system.

For some nonlinear wave equations, one found that the corresponding traveling wave system (1.3) posses also elliptic singular curves in the phase plane. For example, Bi [1] obtained some new types of solitary waves with peaks by considering the effects of elliptic singular curves. Recently, Chen and Wen [4] found the Olver-Rosenau compactons are induced by a singular elliptic rather than a singular straight line.

What kinds of traveling wave solution will appear with the appearance of the parabola singular curves for a given nonlinear wave equation still needs a further study.

In 2003, Manna and Neveu [15] derived a new integrable model equation from asymptotic dynamics of a short capillary-gravity wave, namely

$$u_{xt} = \frac{3g(1-3\theta)}{2vh}u - \frac{1}{2}uu_{xx} - \frac{1}{4}u_x^2 + \frac{3h^2}{4v}u_{xx}u_x^2. \tag{1.5}$$

Here $u(x,t)$ is the fluid velocity on the surface, x and t are space and time variables. For the Eq. (1.5), we find that the corresponding traveling wave system (1.3) posses parabola singular curves in the phase plane. In this paper, some new singular periodic waves are obtained. To the best of our knowledge, this is the first time that this type of results have been obtained.

The paper is organized as follows. In Section 2, we introduce periodic peakon of the Camassa-Holm equation. In Section 3, we obtain new singular periodic waves of the Eq.(1.5). A short conclusion is given in Section 4.

2. Singular straight line and periodic peakon

The existence of periodic peakons is of interest for the nonlinear integrable equations since they are relatively new periodic waves (for most models the periodic waves are quite smooth). The first derivatives of the periodic peakons do not exist in countable number of points. The periodic peakons are called also periodic cusp waves [16]. To compare the difference between the effects of singular straight lines and effects of parabola singular curves in the integrable nonlinear wave equations, we recall the occurrence of the periodic peakons by using phase space analytical technique(see for more details [16]).

Let $u(x,t) = \varphi(\xi)$ with $\xi = x - ct$ be the solution of Eq. (1.4), then it follows that

$$-c\varphi' + c\varphi''' + 3\varphi\varphi' = 2\varphi'\varphi'' + \varphi\varphi'''. \tag{2.1}$$

Integration yields

$$-c\varphi + c\varphi'' + \frac{3}{2}\varphi^2 = \varphi\varphi'' + \frac{1}{2}(\varphi')^2 + g, \tag{2.2}$$

where g is integral constant. Clearly, Eq. (2.2) is equivalent to the two-dimensional system

$$\frac{d\varphi}{d\xi} = y, \frac{dy}{d\xi} = \frac{\frac{3}{2}\varphi^2 - c\varphi - g - \frac{1}{2}y^2}{\varphi - c}, \tag{2.3}$$

which has the first integral

$$H(\varphi, y) = (\varphi - c)\left(\frac{1}{2}y^2 - \frac{1}{2}(\varphi - c)^2 - c(\varphi - c) - \frac{c^2 - 2g}{2}\right). \tag{2.4}$$

Let $d\xi = (\varphi - c)d\zeta$, then system (2.3) become

$$\frac{d\varphi}{d\zeta} = (\varphi - c)y, \frac{dy}{d\zeta} = \frac{3}{2}\varphi^2 - c\varphi - g - \frac{1}{2}y^2. \tag{2.5}$$

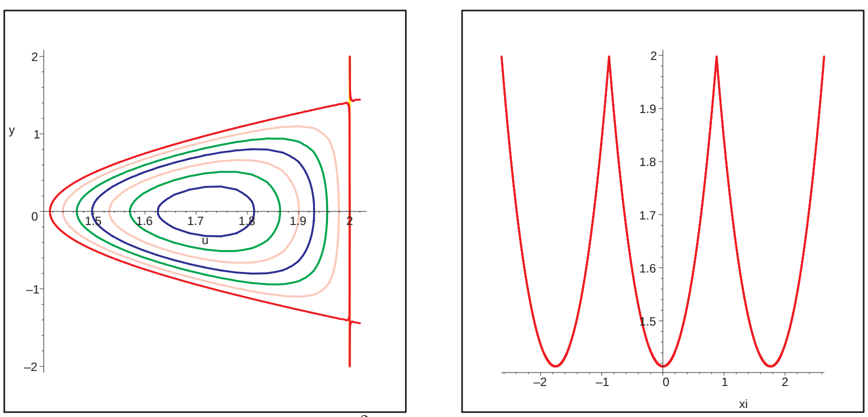
If $c > 0$ and $0 < g < \frac{c^2}{2}$, there are a family of periodic orbits enclosing the center point $(\frac{1}{3}(c + \sqrt{c^2 + 6g}), 0)$, and the family of periodic orbits are surrounded by two boundary curves consisting of a segment of $\varphi = c$ and a arch curve connecting the saddle points $(c, \sqrt{c^2 - 2g})$ and $(0, -\sqrt{c^2 - 2g})$(see Fig. 1 (1-1)). The heteroclinic orbit defined by $H(\varphi, y) = 0$ has two intersection points with the singular line $\varphi = c$. We have the algebraic equation of this orbit

$$y^2 = \varphi^2 - 2g. \tag{2.6}$$

Thus we obtain the parametric representations of the periodic peakon

$$\varphi(\xi) = \sqrt{2g}\cosh(\xi - 2nT), \quad (2n - 1)T \le \xi \le (2n + 1)T, \tag{2.7}$$

where $T = \cosh^{-1}(\frac{c\sqrt{2g}}{2g})$. Obviously, the first derivatives of the peakon solution $\varphi(\xi)$ do not exist if $\xi = (2n + 1)T, n = 0, \pm 1, \pm 2, \cdots$. The profiles of periodic peakon waves are shown in Fig. 1 (1-2).



(1-1) The phase portrait for $0 < g < \frac{c^2}{2}, c > 0$.

(1-2) The profile of periodic peakon.

Figure 1. The phase portrait of system (2.5) and the profile of periodic peakon.

3. Parabola singular curve and singular periodic wave

As mentioned in the last section, most of works are concentrated on the class of nonlinear wave equations with a vertical singular straight line. But till now there have not been works on the integrable nonlinear equations with parabola singular curves. For the Eq. (1.5), the corresponding traveling wave system (1.3) posses parabola singular curves in the phase plane.

To investigated the traveling wave solutions of Eq. (1.5), substituting $u = u(x-ct) = \varphi(\xi)$ into Eq. (1.5), we obtain

$$\left(\frac{1}{2}\varphi - c - \frac{3h^2}{4\nu}(\varphi')^2\right)\varphi'' = \frac{3g(1-3\theta)}{2\nu h}\varphi - \frac{1}{4}(\varphi')^2. \tag{3.1}$$

We rewrite (3.1) as a two-dimensional system

$$\begin{cases} \dfrac{d\varphi}{d\xi} = y, \\ \dfrac{dy}{d\xi} = \dfrac{\frac{3g(1-3\theta)}{2\nu h}\varphi - \frac{1}{4}y^2}{\frac{1}{2}\varphi - c - \frac{3h^2}{4\nu}y^2}. \end{cases} \tag{3.2}$$

The system (3.2) has a parabola singular curves $\varphi = \frac{3h^2}{2\nu}y^2 + 2c$. system (3.2) can be reduced to the following Hamilton system

$$\begin{cases} \dfrac{d\varphi}{d\zeta} = y(2\nu\varphi - 4\nu c - 3h^2y^2) = -\dfrac{\partial H}{\partial y}, \\ \dfrac{dy}{d\zeta} = \dfrac{6g(1-3\theta)}{h}\varphi - \nu y^2 = \dfrac{\partial H}{\partial \varphi} \end{cases} \tag{3.3}$$

with Hamiltonian

$$H(\varphi, y) = \frac{3g(1-3\theta)}{h}\varphi^2 - \nu\varphi y^2 + 2\nu c y^2 + \frac{3h^2}{4}y^4 = E. \tag{3.4}$$

If $(1-3\theta) > 0$, then origin is a center of the system (3.3). In addition, from Eq. (3.4), we have

$$E_0 = H(0,0) = 0, \quad E_c = H(2c, 0) = \frac{12gc^2(1-3\theta)}{h}. \tag{3.5}$$

Assuming all parameters of Eq. (3.1) are positive and $\theta < \frac{1}{3}$, the periodic orbit defined by $H(\varphi, y) = E(E \in (0, E_c))$ has not intersection point with the parabola $\varphi = \frac{3h^2}{2\nu}y^2 + 2c$(see Fig. 2). Thus, Eq. (1.5) has a family of smooth periodic wave solutions. We have the algebraic equations of periodic orbits

$$y^2 = \frac{1}{3h^2}\left(2\nu\varphi - 4\nu c \pm 2\sqrt{(\nu\varphi - 2\nu c)^2 - 3h(3g(1-3\theta)\varphi^2 - hE)}\right). \tag{3.6}$$

The signs before the term $2\sqrt{(\nu\varphi - 2\nu c)^2 - 3h(3g(1-3\theta)\varphi^2 - hE)})$ are dependent on the interval of φ. Under the condition $h > 0, \nu > 0, c > 0, g > 0, \theta < \frac{1}{3}$, for $\varphi \in (-2c, 2c)$, we need to take positive before the term. Therefore, the periodic orbits surrounding the center $O(0,0)$ can be expressed as

$$y = \pm\sqrt{\frac{1}{3h^2}(2\nu\varphi - 4\nu c + 2\sqrt{(\nu\varphi - 2\nu c)^2 - 3h(3g(1-3\theta)\varphi^2 - hE)})}, \tag{3.7}$$

which intersect the $\varphi-$axis at two points with the $\varphi-$coordinates $\pm\varphi_m$ respectively. From (3.7) and the first equation of (3.2), we obtain the parametric representation for the corresponding periodic orbits as

$$\int_{\varphi_m}^{\varphi} \frac{d\varphi}{\sqrt{2\nu\varphi - 4\nu c + 2\sqrt{(\nu\varphi - 2\nu c)^2 - 3h(3g(1-3\theta)\varphi^2 - hE)}}} = \frac{1}{\sqrt{3}h}|\xi - 2nT_1|, \tag{3.8}$$

where $|\xi - 2nT_1| \le T_1$ and

$$T_1 = \int_{-\varphi_m}^{\varphi_m} \frac{\sqrt{3}hd\varphi}{\sqrt{2\nu\varphi - 4\nu c + 2\sqrt{(\nu\varphi - 2\nu c)^2 - 3h(3g(1-3\theta)\varphi^2 - hE)}}}. \tag{3.9}$$

If $E = E_c$, the periodic orbit is tangent to the parabola $\varphi = \frac{3h^2}{2\nu}y^2 + 2c$ at point $(2c, 0)$. The corresponding periodic wave solution satisfies

$$\left(\frac{d\varphi}{d\xi}\right)^2 = \frac{2}{3h^2}(\nu(\varphi - 2c) + \sqrt{\nu^2(\varphi - 2c)^2 - 9gh(1-3\theta)(\varphi + 2c)(\varphi - 2c)}) \tag{3.10}$$

and

$$\left(\frac{d^2\varphi}{d\xi^2}\right)^2 = \frac{4\nu^2(\frac{3g(1-3\theta)}{2\nu h}\varphi - \frac{1}{6h^2}(\nu(\varphi - 2c) + \sqrt{\nu^2(\varphi - 2c)^2 - 9gh(1-3\theta)(\varphi + 2c)(\varphi - 2c)}))^2}{\nu^2(\varphi - 2c)^2 - 9gh(1-3\theta)(\varphi + 2c)(\varphi - 2c)}. \tag{3.11}$$

Consequently, along this orbit when $\varphi \to 2c$, $\frac{d\varphi}{d\xi} \to 0$, $\frac{d^2\varphi}{d\xi^2} \to \pm\infty$. Thus when $E \to E_c$, the smooth periodic wave evolves into a singular periodic wave. The process just is simulated by Maple and is shown in Figs. 3 (3-1)–(3-3).

The singular periodic wave can be expressed as

$$\int_{-2c}^{\varphi} \frac{d\varphi}{\sqrt{2\nu(\varphi - 2c) + 2\sqrt{\nu^2(\varphi - 2c)^2 - 9gh(1-3\theta)(\varphi^2 - 4c^2)}}} = \frac{1}{\sqrt{3}h}|\xi - 2nT_2|, \tag{3.12}$$

where $|\xi - 2nT_2| \le T_2$ and

$$T_2 = \int_{-2c}^{2c} \frac{\sqrt{3}hd\varphi}{\sqrt{2\nu(\varphi - 2c) + 2\sqrt{\nu^2(\varphi - 2c)^2 - 9gh(1-3\theta)(\varphi^2 - 4c^2)}}}. \tag{3.13}$$

Specially, when $\nu^2 = 9gh(1-3\theta)$, setting $\psi^2 = c(2c - \varphi)$, we have $y^2 = \frac{1}{3h^2}(4\nu\psi - \frac{2\nu}{c}\psi^2)$. By the first equation of system (3.2), we have

$$\int_{2c}^{\psi} \frac{\psi d\psi}{\sqrt{\psi(2c - \psi)}} = -\frac{\sqrt{2c\nu}}{2\sqrt{3}h}|\xi - 2nT_2|. \tag{3.14}$$

Therefore we obtain the following exact parametric representations of singular periodic wave solutions of Eq. (1.5)

$$\varphi(\xi) = 2c - \frac{\psi^2(\xi)}{c} \tag{3.15}$$

and

$$\sqrt{\psi(2c - \psi)} + \arctan(\frac{c - \psi}{\sqrt{\psi(2c - \psi)}}) = \frac{\sqrt{2c\nu}}{2\sqrt{3}h}|\xi - 2nT_2| - \frac{c\pi}{2}. \tag{3.16}$$

Remark 3.1. Let $\xi_n = (2n+1)T$, then the periodic peakon $\varphi(\xi)$ defined by (2.7) for the Camassa-Holm equation satisfies

$$\varphi(\xi_n) = c, \quad \lim_{\xi \uparrow \xi_n} \varphi_\xi(\xi) = \sqrt{c^2 - 2g}, \quad \lim_{\xi \downarrow \xi_n} \varphi_\xi(\xi) = -\sqrt{c^2 - 2g}.$$

Therefore the first derivatives of the periodic peakons do not exist in countable number of points. In comparison with the periodic peakon of the Camassa-Holm equation, the singular periodic wave $\varphi(\xi)$ defined by (3.12) has the following properties:

$$\varphi(\xi_n) = 2c, \quad \varphi_\xi(\xi_n) = 0, \quad \lim_{\xi \uparrow \xi_n} \varphi_{\xi\xi}(\xi) = -\infty, \quad \lim_{\xi \downarrow \xi_n} \varphi_{\xi\xi}(\xi) = +\infty,$$

where $\xi_n = (2n+1)T_2$. Thus the second derivatives of the new singular periodic wave solutions do not exist in countable number of points but the first derivatives exist. **In fact, the singular traveling wave solutions that has continuous first-order derivative but discontinuous second-order derivative at some points also are given in [24].**

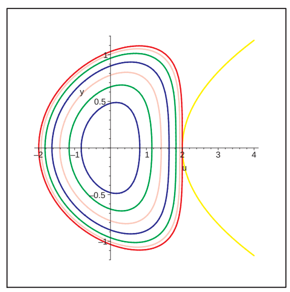
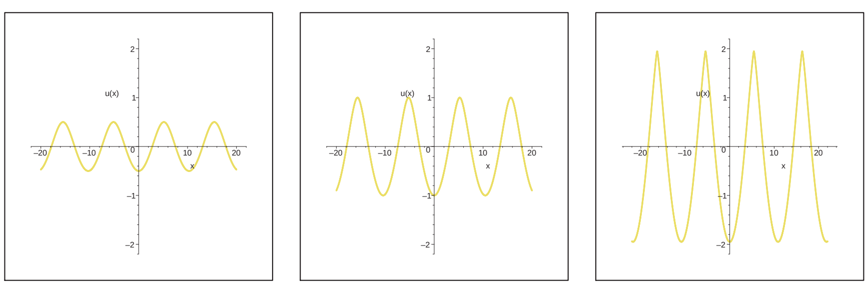


Figure 2. The phase portrait of the system (3.3) for $0 < \theta < \frac{1}{3}$.



(3-1) $E = 0.1875$ **(3-2)** $E = 0.75$ **(3-3)** $E = 3$

Figure 3. As E from 0 tends to E_c, the smooth periodic waves evolve into a singular periodic wave.

4. Conclusion

In this paper, we investigated the nonlinear wave equation from short capillary-gravity waves (1.5) by using the bifurcation theory of dynamical system. We find new singular periodic waves. The new periodic waves possess weaker singularity than the periodic peakon. It is shown that the second derivatives of the new singular periodic wave solutions do not exist in countable number of points but the first derivatives exist. We show that there exist close connection between the new singular periodic waves and parabola singular curve in phase plane of traveling wave system for the first time.

References

[1] Q. Bi, *Singular solitary waves associated with homoclinic orbits,* Phys. Lett. A., 2006, 352(3), 227–232.

[2] R. Camassa and D. D. Holm, *An integrable shallow wave equation with peaked solitons,* Phys. Rev. Lett., 1993, 71, 1661–1664.

[3] A. Chen, J. Li, C. Li and Y. Zhang, *From bell-shaped solitary wave to W/M-shaped solitary wave solutions in an integrable nonlinear wave equation*, Pramana-Journal of Physics, 2010, 74(1), 19–26.

[4] A. Chen and S. Wen, *Double compactons in the Olver-Rosenau equation*, Pramana-Journal of Physics, 2013, 80(3), 471–478.

[5] A. Chen, S. Wen, S. Tang, W. Huang and Z. Qiao, *Effects of Quadratic Singular Curves in Integrable Equations,* Studies in Applied Mathematics, 2015, 134(1), 24–61.

[6] X. Hu and J. Zhao, *Commutativity of Pfaffianization and Bäcklund transformations: the KP equation*, Inverse Problems, 2005, 21(4), 1461–1472.

[7] S. Kun, B. Tian, W. Liu, M. Li, Q. Qu and Y. Jiang, *Symbolic computation study on the (2+1)-dimensional dispersive long wave system*, SIAM J. Appl. Math., 2010, 70(7), 2259–2272.

[8] X. Li, J. Han and F. Wang, *The extended Riccati equation method for travelling wave solutions of ZK equation,* Journal of Applied Analysis and Computation, 2012, 2(4), 423–430.

[9] J. Li and Z. Qiao, *Explicit soliton solutions of the Kaup-kupershmidt equation through the dynamical system approach,* Journal of Applied Analysis and Computation, 2011, 1(2), 243–250.

[10] J. Li and G. Chen, *On a class of singular nonlinear traveling wave equations,* Int. J. Bifurcat. Chaos., 2007, 17(11), 4049–4065.

[11] J. Li, Y. Zhang and X. Zhao, *On a class of singular nonlinear traveling wave equations. II. An example of GCKdV equations*, Int. J. Bifurcat. Chaos., 2009, 19(6), 1995–2007.

[12] H. Liu and J. Li, *Lie symmetry analysis and exact solutions for the short pulse equation*, Nonlinear Analysis: Theory, Methods & Applications, 2009, 71(5–6), 2126–2133.

[13] Z. Liu, Q. Li and Q. Lin, *New bounded travelling waves of Camassa-Holm equation,* Int. J. Bifurcat. Chaos., 2004, 14(10), 3541–3556.

[14] W. Ma and Y. You, *Solving the Korteweg-de Vries equation by its bilinear form: Wronskian solutions*, Trans. Amer. Math. Soc., 2005, 357(5), 1753–1778.

[15] M. Manna and A. Neveu, *A singular integrable equation from short capillary-gravity waves,* preprint arXiv: physics/0303085 2003.

[16] Z. Qiao, *A new integrable equation with cuspons and W/M-shape peaks solitons*, J. Math. Physics, 2009, 50, 024101–2.

[17] Z. Qiao and G. Zhang, *On peaked and smooth solitons for the Camassa-Holm equation,* Europhys. Lett., 2006, 73(5), 657–663.

[18] P. Rosenau, *On nonanalytic solitary waves formed by a nonlinear dispersion,* Phys. Lett. A., 1997, 230(5–6), 305–318.

[19] W. Rui, B. He, Y. Long and C. Chen, *The integral bifurcation method and its application for solving a family of third-order dispersive PDEs*, Nonlinear Analysis, 2008, 69(4), 1256–1267.

[20] J. Shen and W. Xu, *Bifurcations of smooth and non-smooth travelling wave solutions in the generalized Camassa-Holm equation*, Chaos, Solitons and Fractals, 2005, 26(4), 1149–1162.

[21] A. M. Wazwaz, *The tanh method for travelling wave solutions to the Zhiber-Shabat equation and other related equations*, Commun Nonlinear Sci. Numer. Simul., 2008, 13(3), 584–592.

[22] M. Wei, X. Sun and S. Tang, *Single peak solitary wave solutions for the CH-KP$(2,1)$ equation under boundary condition*, J. Differential Equations, 2015, 259(2), 628–641.

[23] Z. Wen and Z. Liu, *Bifurcation of peakons and periodic cusp waves for the generalization of the Camassa-Holm equation*, Nonlinear Analysis, 2011, 12(3), 1698–1707.

[24] L. Zhang, L. Chen and X. Hou, *The effects of horizontal singular straight line in a generalized nonlinear Klein-Gordon model equation*, Nonlinear Dyn, 2013, 72(4), 789–801.

[25] H. Zhao, L. Qiao and S. Tang, *Peakon, pseudo-peakon, loop, and periodic cusp wave solutions of a three-dimensional $3DKP(2,2)$ equation with nonlinear dispersion,* Journal of Applied Analysis and Computation, 2015, 5(3), 301–312.

[26] L. Zhong, S. Tang, D. Li and H. Zhao, *Compacton, peakon, cuspons, loop solutions and smooth solitons for the generalized KP-MEW equation*, Comput. Math. Appl., 2014, 68(12), 1775–1786.

[27] , L. Zhong, S. Tang and L. Qiao, *Bifurcations and exact traveling wave solutions for a class of nonlinear fourth order partial differential equations*, Nonlinear Dyn, 2015, 80(1), 129–145.

[28] J. Zhou and L. Tian, *Soliton solution of the osmosis $K(2,2)$ equation,* Phys. Lett. A., 2008, 372(41), 6232–6234.