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# ON HIGHER-ORDER ANISOTROPIC CAGINALP PHASE-FIELD SYSTEMS WITH POLYNOMIAL NONLINEAR TERMS

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**Abstract** Our aim in this paper is to study higher-order (in space) anisotropic Caginalp phase-field systems. In particular, we obtain well-posedness results, as well as the existence of the global attractor and exponential attractor.

**Keywords** Phase-field systems, higher-order systems, anisotropy, well-posedness, dissi-pativity, global attractor, exponential attractor.

**MSC(2010)** 35K55, 35B41, 35B45.

## 1. Introduction

The Caginalp phase-field system,

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = T, \tag{1.1}$$

$$\frac{\partial T}{\partial t} - \Delta T = -\frac{\partial u}{\partial t}, \tag{1.2}$$

was proposed in [7] to model phase transition phenomena, such as melting-solidification phenomena. Here, $u$ is the order parameter, $T$ is the relative temperature and $f$ is the derivative of a double-well potential $F$ (a typical choice of potential is $F(s) = \frac{1}{4}(s^2-1)^2$, hence the usual cubic nonlinear term $f(s) = s^3 - s$). Furthermore, here and below, we set all physical parameters equal to one. This system has been much studied; we refer the reader to, e.g., [2–4].

These equations can be derived as follows: One introduces the (total Ginzburg-Landau) free energy

$$\Psi_{GL} = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + F(u) - uT - \frac{1}{2}T^2\right) \mathrm{d}x, \tag{1.3}$$

where $\Omega$ is the domain occupied by the system (we assume here that it is a bounded and regular domain of $\mathbb{R}^3$, with boundary $\Gamma$), and the enthalpy

$$H = u + T. \tag{1.4}$$

As far as the evolution for the order parameter is concerned, one postulates the relaxation dynamics (with relaxation parameter set equal to one)

$$\frac{\partial u}{\partial u} = -\frac{D\Psi_{GL}}{Du}, \tag{1.5}$$

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where $\frac{D}{Du}$ denotes a variational derivative with respect to $u$, which yields (1.1). Then, we have the energy equation

$$\frac{\partial H}{\partial t} = -divq, \tag{1.6}$$

where $q$ is the heat flux. Assuming finally the usual Fourier law for heat conduction,

$$q = -\nabla T, \tag{1.7}$$

we obtain (1.2)

In (1.3), the term $|\nabla u|^2$ models short-ranged interactions. It is, however, interesting to note that such a term is obtained by truncation of higher-order ones (see [9]); it can also be seen as a first- order approximation of a nonlocal term accounting for long-ranged interactions (see [14]).

Now, one essential drawback of the Fourier law is that it predicts that thermal signals propagate at an infinite speed, which violates causality (the so-called paradox of heat conduction). To overcome this drawback, or at least to account for more realistic features, several alternatives to the Fourier law, based, e.g., on the Maxwell-Cattaneo law or recent laws from thermomechanics, have been proposed and studied, in the context of the Caginalp phase-field system, in [20].

In the late 1960's, several authors proposed a heat conduction theory based on two temperatures (see [10, 23]). More precisely, one now considers the conductive temperature $T$ and the thermodynamic temperature $\theta$. In particular, for simple materials, these two temperatures are shown to coincide. However, for non-simple materials, they differ and are related as follows:

$$\theta = T - \Delta T. \tag{1.8}$$

The Caginalp system, based on this two temperatures theory and the usual Fourier law, was studied in [4].

Our aim in this paper is to study a variant of the Caginalp phase-field system based on the type III thermomechanics theory with two temperatures recently proposed in [20].

In that case, the free energy reads, in terms of the (relative) thermodynamic temperature $\theta$,

$$\Psi_{GL} = \int_\Omega \left( \frac{1}{2}|\nabla u|^2 + F(u) - u\theta - \frac{1}{2}\theta^2 \right) \mathrm{d}x \tag{1.9}$$

and (1.5) yields, in view of (1.8), the following evolution equation for the order parameter:

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = T - \Delta T. \tag{1.10}$$

Furthermore, the enthalpy now reads

$$H = u + \theta = u + T - \Delta T, \tag{1.11}$$

which yields, owing to (1.6), the energy equation

$$\frac{\partial T}{\partial t} - \Delta \frac{\partial T}{\partial t} + divq = -\frac{\partial u}{\partial t}. \tag{1.12}$$

Finally, the heat flux is given, in the type III theory with two temperatures, by (see [15,20])

$$q = -\nabla\alpha - \nabla T, \tag{1.13}$$

where

$$\alpha(t,x) = \int_0^t T(\tau,x)\mathrm{d}\tau + \alpha_0(x) \tag{1.14}$$

is the conductive thermal displacement. Noting that $T = \frac{\partial\alpha}{\partial t}$, we finally deduce from (1.10) and (1.12)–(1.13) the following variant of the Caginalp phase-field system (see [20]):

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}, \tag{1.15}$$

$$\frac{\partial^2\alpha}{\partial t^2} - \Delta\frac{\partial^2\alpha}{\partial t^2} - \Delta\frac{\partial\alpha}{\partial t} - \Delta\alpha = -\frac{\partial u}{\partial t}. \tag{1.16}$$

Caginalp and Esenturk recently proposed in [8] (see also [5,21]) higher-order phase-field models in order to account for anisotropic interfaces (see also [6,17] for other approaches which, however, do not provide an explicit way to compute the anisotropy). More precisely, these authors proposed the following modified (total) free energy

$$\Psi_{HOGL} = \int_\Omega \left(\frac{1}{2}\sum_{i=1}^{k}\sum_{|\beta|=i} a_\beta|\mathcal{D}^\beta u|^2 + F(u) - u\theta - \frac{1}{2}\theta^2\right)\mathrm{d}x, \tag{1.17}$$

where, for $\beta = (k_1,k_2,k_3) \in (\mathbb{N}\cup\{0\})^3$,

$$|\beta| = k_1 + k_2 + k_3$$

and, for $\beta \neq (0,0,0)$,

$$\mathcal{D}^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}}$$

(we agree that $\mathcal{D}^{(0,0,0)}v = v$). This then yields the following evolution equation for the order parameter $u$:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{k}(-1)^i\sum_{|\beta|=i} a_\beta\mathcal{D}^{2\beta}u + f(u) = \frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}. \tag{1.18}$$

Our aim in this paper was to study the model consisting of the higher-order anisotropic equation (1.18) and the temperature equation (1.16). In particular, we obtain the existence and uniqueness of solutions, as well as the existence of the global attractor and exponential attractors.

## 2. Setting of the problem

We consider the following initial and boundary value problem, for $k \in \mathbb{N}$:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{k} (-1)^i \sum_{|\beta|=i} a_\beta \mathcal{D}^{2\beta} u + f(u) = \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}, \tag{2.1}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t}, \tag{2.2}$$

$$\mathcal{D}^\beta u = \alpha = 0 \quad \text{on} \quad \Gamma, \quad |\beta| \leqslant k-1, \tag{2.3}$$

$$u|_{t=0} = u_0, \quad \alpha|_{t=0} = \alpha_0, \quad \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1. \tag{2.4}$$

We assume that

$$a_\beta > 0, \quad |\beta| = k, \tag{2.5}$$

and we introduce the elliptic operator $A_k$ defined by

$$\langle A_k v, w \rangle_{H^{-k}(\Omega), H_0^k(\Omega)} = \sum_{|\beta|=k} a_\beta \left( (\mathcal{D}^\beta v, \mathcal{D}^\beta w) \right), \tag{2.6}$$

where $H^{-k}(\Omega)$ is the topological dual of $H_0^k(\Omega)$. Furthermore, $((.,.))$ denotes the usual $L^2$-scalar product, with associated norm $\|.\|$; more generally, we denote by $\|.\|_X$ the norm on the Banach space X. We can note that

$$(v, w) \in H_0^k(\Omega)^2 \mapsto \sum_{|\beta|=k} a_\beta \left( (\mathcal{D}^\beta v, \mathcal{D}^\beta w) \right)$$

is bilinear, symmetric, continuous and coercive, so that

$$A_k : H_0^k(\Omega) \to H^{-k}(\Omega)$$

is indeed well defined. It then follows from elliptic regularity results for linear elliptic operators of order $2k$ (see [1]) that $A_k$ is a strictly positive, self-adjoint and unbounded linear operator with compact inverse, with domain

$$D(A_k) = H^{2k}(\Omega) \cap H_0^k(\Omega),$$

where, for $v \in D(A_k)$,

$$A_k v = (-1)^k \sum_{|\beta|=k} a_\beta \mathcal{D}^{2\beta} v.$$

We further note that $D(A_k^{\frac{1}{2}}) = H_0^k(\Omega)$ and, for $(v, w) \in D(A_k^{\frac{1}{2}})^2$,

$$\left( \left( A_k^{\frac{1}{2}} v, A_k^{\frac{1}{2}} w \right) \right) = \sum_{|\beta|=k} a_\beta \left( (\mathcal{D}^\beta v, \mathcal{D}^\beta w) \right).$$

We finally note that (see, e.g., [24]) $\|A_k.\|$ (resp., $\|A_k^{\frac{1}{2}}.\|$) is equivalent to the usual $H^{2k}$-norm (resp., $H^k$-norm) on $D(A_k)$ (resp., $D(A_k^{\frac{1}{2}})$).

Having this, we rewrite (2.1) as

$$\frac{\partial u}{\partial t} + A_k u + B_k u + f(u) = \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}, \tag{2.7}$$

where $B_1 = 0$ and, for $k \geqslant 2$,

$$B_k v = \sum_{i=1}^{k-1} (-1)^i \sum_{|\beta|=i} a_\beta \mathcal{D}^{2\beta} v.$$

As far as the nonlinear term $f$ is concerned, we assume that

$$f \in C^1(\mathbb{R}), \quad f(0) = 0, \tag{2.8}$$

$$f' \geqslant -c_0, \quad c_0 \geqslant 0, \tag{2.9}$$

$$f(s)s \geqslant c_1 F(s) - c_2 \geqslant -c_3, \quad c_1 > 0, \quad c_2, \quad c_3 \geqslant 0, \quad s \in \mathbb{R}, \tag{2.10}$$

$$F(s) \geqslant c_4 s^4 - c_5, \quad c_4 > 0, \quad c_5 \geqslant 0, \quad s \in \mathbb{R}, \tag{2.11}$$

where $F(s) = \int_0^s f(\tau)\mathrm{d}\tau$. In particular, the usual cubic nonlinear term $f(s) = s^3 - s$ satisfies these assumptions.

Throughout the paper, the same letters $c$, $c'$ and $c''$ denote (generally positive) constants which may vary from line to line. Similary, the same letter $Q$ denotes (positive) monotone increasing (with respect to each argument) and continuous functions which may vary from line to line.

## 3. A Priori Estimates

We multiply (2.7) by $\frac{\partial u}{\partial t}$ and have, integrating over $\Omega$ and by parts

$$\frac{d}{dt}\left(\|A_k^{\frac{1}{2}} u\|^2 + B_k^{\frac{1}{2}}[u] + 2\int_\Omega F(u)\mathrm{d}x\right) + 2\|\frac{\partial u}{\partial t}\|^2 = 2\left(\left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)\right). \tag{3.1}$$

We then multiply (2.2) by $\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}$ and obtain

$$\begin{aligned} &\frac{d}{dt}\left(\|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 + \|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2\right) + 2\|\nabla\frac{\partial \alpha}{\partial t}\|^2 + 2\|\Delta\frac{\partial \alpha}{\partial t}\|^2 \\ &= -2\left(\left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)\right). \end{aligned} \tag{3.2}$$

Summing (3.1) and (3.2), we find the differential equality

$$\begin{aligned} &\frac{d}{dt}\left(\|A_k^{\frac{1}{2}} u\|^2 + B_k^{\frac{1}{2}}[u] + 2\int_\Omega F(u)\mathrm{d}x + \|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 + \|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2\right) \\ &+ 2\|\frac{\partial u}{\partial t}\|^2 + 2\|\nabla\frac{\partial \alpha}{\partial t}\|^2 + 2\|\Delta\frac{\partial \alpha}{\partial t}\|^2 = 0, \end{aligned} \tag{3.3}$$

where $B_1^{\frac{1}{2}}[u] = 0$ and, for $k \geqslant 2$,

$$B_k^{\frac{1}{2}}[u] = \sum_{i=1}^{k-1} \sum_{|\beta|=i} a_\beta \|\mathcal{D}^\beta u\|^2 \tag{3.4}$$

(note that $B_k^{\frac{1}{2}}[u]$ is not necessarily nonnegative). We can note that, owing to the interpolation inequality

$$\|(-\Delta)^{\frac{i}{2}}v\| \leqslant c(i)\|(-\Delta)^{\frac{m}{2}}v\|^{\frac{i}{m}}\|v\|^{1-\frac{i}{m}}, \qquad (3.5)$$
$$v \in H^m(\Omega), \quad i \in \{1,...,m-1\}, \quad m \in \mathbb{N}, \quad m \geqslant 2,$$

there holds

$$|B_k^{\frac{1}{2}}[u]| \leqslant \frac{1}{2}\|A_k^{\frac{1}{2}}u\|^2 + c\|u\|^2. \qquad (3.6)$$

This yields, employing (2.11)

$$\|A_k^{\frac{1}{2}}u\|^2 + B_k^{\frac{1}{2}}[u] + 2\int_\Omega F(u)\mathrm{d}x \geqslant \frac{1}{2}\|A_k^{\frac{1}{2}}u\|^2 + \int_\Omega F(u)\mathrm{d}x + c\|u\|^4_{L^4(\Omega)} - c'\|u\|^2 - c'',$$

whence

$$\|A_k^{\frac{1}{2}}u\|^2 + B_k^{\frac{1}{2}}[u] + 2\int_\Omega F(u)\mathrm{d}x \geqslant c\left(\|u\|^2_{H^k(\Omega)} + \int_\Omega F(u)\mathrm{d}x\right) - c', \quad c > 0, \quad (3.7)$$

nothing that, owing to Young's inequality,

$$\|u\|^2 \leqslant \epsilon\|u\|^4_{L^4(\Omega)} + c(\epsilon), \quad \forall\epsilon > 0. \qquad (3.8)$$

We then multiply (2.7) by $u$ and have, owing to (2.10) and the interpolation inequality (3.5),

$$\frac{d}{dt}\|u\|^2 + c\left(\|u\|^2_{H^k(\Omega)} + \int_\Omega F(u)\mathrm{d}x\right) \leqslant c'\left(\|u\|^2 + \|\frac{\partial\alpha}{\partial t}\|^2 + \|\Delta\frac{\partial\alpha}{\partial t}\|^2\right) + c'';$$

hence, proceeding as above and employing, in particular, (2.11),

$$\frac{d}{dt}\|u\|^2 + c\left(\|u\|^2_{H^k(\Omega)} + \int_\Omega F(u)\mathrm{d}x\right) \leqslant c'\left(\|\frac{\partial\alpha}{\partial t}\|^2 + \|\Delta\frac{\partial\alpha}{\partial t}\|^2\right) + c'', \quad c > 0. \quad (3.9)$$

Summing (3.3) and $\delta_1$ times (3.9), where $\delta_1 > 0$ is small enough, we obtain a differential inequality of the form

$$\frac{d}{dt}E_1 + c\left(\|u\|^2_{H^k(\Omega)} + \int_\Omega F(u)\mathrm{d}x + \|\frac{\partial u}{\partial t}\|^2 + \|\nabla\frac{\partial\alpha}{\partial t}\|^2 + \|\Delta\frac{\partial\alpha}{\partial t}\|^2\right) \leqslant c', \quad (3.10)$$

where

$$E_1 = \|A_k^{\frac{1}{2}}u\|^2 + B_k^{\frac{1}{2}}[u] + 2\int_\Omega F(u)\mathrm{d}x + \|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 + \|\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\|^2 + \delta_1\|u\|^2$$

satisfies, owing to (3.7)

$$E_1 \geqslant c\left(\|u\|^2_{H^k(\Omega)} + \int_\Omega F(u)\mathrm{d}x + \|\alpha\|^2_{H^2(\Omega)} + \|\frac{\partial\alpha}{\partial t}\|^2_{H^2(\Omega)}\right) - c', \quad c > 0. \quad (3.11)$$

Multiplying (2.2) by $-\Delta\alpha$, we then obtain

$$\begin{aligned}&\frac{d}{dt}\left(\|\Delta\alpha\|^2 - 2\left(\left(\frac{\partial\alpha}{\partial t}, \Delta\alpha\right)\right) + 2\left(\left(\Delta\frac{\partial\alpha}{\partial t}, \Delta\alpha\right)\right)\right) + \|\Delta\alpha\|^2 \\ &\leqslant \|\frac{\partial u}{\partial t}\|^2 + 2\|\nabla\frac{\partial\alpha}{\partial t}\|^2 + 2\|\Delta\frac{\partial\alpha}{\partial t}\|^2.\end{aligned} \qquad (3.12)$$

Summing (3.10) and $\delta_2$ times (3.12), where $\delta_2>0$ is small enough, we obtain a differential inequality of the form

$$\frac{dE_2}{dt}+c\left(E_2+\|\frac{\partial u}{\partial t}\|^2\right)\leqslant c',\quad c>0, \tag{3.13}$$

where

$$E_2=E_1+\delta_2\left(\|\Delta\alpha\|^2-2\left(\left(\frac{\partial\alpha}{\partial t},\Delta\alpha\right)\right)+2\left(\left(\Delta\frac{\partial\alpha}{\partial t},\Delta\alpha\right)\right)\right)$$

satisfies

$$E_2\geqslant c\left(\|u\|^2_{H^k(\Omega)}+\int_\Omega F(u)\mathrm{d}x+\|\alpha\|^2_{H^2(\Omega)}+\|\frac{\partial\alpha}{\partial t}\|^2_{H^2(\Omega)}\right)-c',\quad c>0. \tag{3.14}$$

It particular, if follows from (3.13) – (3.14) and Gronwall's lemma that

$$\begin{aligned}&\|u(t)\|^2_{H^k(\Omega)}+\|\alpha(t)\|^2_{H^2(\Omega)}+\|\frac{\partial\alpha}{\partial t}(t)\|^2_{H^2(\Omega)}\\ \leqslant& ce^{-c't}\left(\|u_0\|^2_{H^k(\Omega)}+\int_\Omega F(u_0)\mathrm{d}x+\|\alpha_0\|^2_{H^2(\Omega)}+\|\alpha_1\|^2_{H^2(\Omega)}\right)+c'',c'>0,t\geqslant 0,\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\int_t^{t+r}\|\frac{\partial u}{\partial t}\|^2\mathrm{d}s\leqslant& ce^{-c't}\Bigg(\|u_0\|^2_{H^k(\Omega)}+\int_\Omega F(u_0)\mathrm{d}x\\ &+\|\alpha_0\|^2_{H^2(\Omega)}+\|\alpha_1\|^2_{H^2(\Omega)}\Bigg)+c''(r),\quad c'>0,\quad t\geqslant 0,\end{aligned} \tag{3.16}$$

$r>0$, given.

Our aim is now to obtain higher-order estimates. To do so, we will distinguish between the cases $k=1$ and $k\geqslant 2$.

**First case: k=1**

Setting $A=A_1=-\sum_{i=1}^3 a_i\dfrac{\partial^2}{\partial x_i^2}$, $a_i>0$, $i=1,2$ and 3, we then consider the initial and boundary value problem

$$\frac{\partial u}{\partial t}+Au+f(u)=\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}, \tag{3.17}$$

$$\frac{\partial^2\alpha}{\partial t^2}-\Delta\frac{\partial^2\alpha}{\partial t^2}-\Delta\frac{\partial\alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t}, \tag{3.18}$$

$$u=\alpha=0\quad\text{on}\quad\Gamma, \tag{3.19}$$

$$u|_{t=0}=u_0,\quad \alpha|_{t=0}=\alpha_0,\quad \frac{\partial\alpha}{\partial t}|_{t=0}=\alpha_1. \tag{3.20}$$

Multipying (3.17) by $Au$, we have

$$\frac{d}{dt}\|A^{\frac12}u\|^2+2\|Au\|^2+2\sum_{i=1}^3 a_i\int_\Omega f'(u)\Big|\frac{\partial u}{\partial x_i}\Big|^2\mathrm{d}x=2\left(\left(\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t},Au\right)\right),$$

which yields, owing to (2.9),

$$\frac{d}{dt}\|A^{\frac{1}{2}}u\|^2 + c\|u\|^2_{H^2(\Omega)} \leqslant c'\left(\|u\|^2_{H^1(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}\|^2\right), \quad c>0. \tag{3.21}$$

Summing (3.13) (for $k=1$) and $\delta_3$ times (3.21), where $\delta_3>0$ is small enough, we obtain a differential inequality of the form

$$\frac{dE_3}{dt} + c\left(E_3 + \|u\|^2_{H^2(\Omega)} + \|\frac{\partial u}{\partial t}\|^2\right) \leqslant c', \quad c>0, \tag{3.22}$$

where

$$E_3 = E_2 + \delta_3\|A^{\frac{1}{2}}u\|^2$$

satisfies

$$E_3 \geqslant c\left(\|u\|^2_{H^1(\Omega)} + \int_\Omega F(u)\mathrm{d}x + \|\alpha\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}\right) - c', \quad c>0. \tag{3.23}$$

We then differentiate (3.17) with respect to time and to find, owing to (2.2),

$$\frac{\partial}{\partial t}\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial t} = \Delta\frac{\partial \alpha}{\partial t} + \Delta\alpha - \frac{\partial u}{\partial t}, \tag{3.24}$$

$$\frac{\partial u}{\partial t} = 0 \quad \text{on} \quad \Gamma, \tag{3.25}$$

$$\frac{\partial u}{\partial t}|_{t=0} = -Au_0 - f(u_0) + \alpha_1 - \Delta\alpha_1. \tag{3.26}$$

In particular, if $u_0 \in H^2(\Omega)\cap H^1_0(\Omega)$ and $\alpha_1 \in H^2(\Omega)\cap H^1_0(\Omega)$, then $\frac{\partial u}{\partial t}(0) \in L^2(\Omega)$ and

$$\|\frac{\partial u}{\partial t}(0)\| \leqslant Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^2(\Omega)}). \tag{3.27}$$

Indeed, if follows from the continuity of $f$ and the continuous embedding $H^2(\Omega) \subset C(\bar{\Omega})$ that

$$\|f(u_0)\| \leqslant Q(\|u_0\|_{H^2(\Omega)}). \tag{3.28}$$

Multiplying (3.24) by $\frac{\partial u}{\partial t}$, we have, owing to (2.9)

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|^2 + c\|\frac{\partial u}{\partial t}\|^2_{H^1(\Omega)} \leqslant c'\left(\|\frac{\partial u}{\partial t}\|^2 + \|\alpha\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}\right), \quad c>0, \tag{3.29}$$

whence, owing to Gronwall's lemma and employing (3.15) and (3.16),

$$\|\frac{\partial u}{\partial t}(t)\| \leqslant e^{ct}Q\left(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^2(\Omega)}\right), t \geqslant 0. \tag{3.30}$$

We can also note that it then follows from (3.15), (3.29) (both for $k=1$) and the uniform Gronwall's lemma (say, for $r=1$; see, e.g., [24]) that

$$\begin{aligned}&\|\frac{\partial u}{\partial t}(t)\|^2 \\ \leqslant & ce^{-c't}Q\left(\|u_0\|^2_{H^1(\Omega)} + \int_\Omega F(u_0)\mathrm{d}x + \|\alpha_0\|^2_{H^2(\Omega)} + \|\alpha_1\|^2_{H^2(\Omega)}\right) + c'', c'>0, t \geqslant 1.\end{aligned} \tag{3.31}$$

Coming back to (3.31), we can note that if follows from the continuity of $F$ and the continuous embedding $H^2(\Omega) \subset C(\bar{\Omega})$ again that

$$\|\frac{\partial u}{\partial t}(t)\| \leqslant e^{-ct}Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^2(\Omega)}) + c', \quad c > 0, \quad t \geqslant 1. \quad (3.32)$$

So that, employing (3.30) for $t \in [0,1]$,

$$\|\frac{\partial u}{\partial t}(t)\| \leqslant e^{-ct}Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^2(\Omega)}) + c', \quad c > 0, \quad t \geqslant 0. \quad (3.33)$$

We finally rewrite (3.17) as an elliptic equation, for $t > 0$ fixed,

$$Au + f(u) = -\frac{\partial u}{\partial t} + \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}, \quad u = 0 \quad \text{on} \quad \Gamma. \quad (3.34)$$

Multiplying (3.34) by $Au$, we have

$$\|Au\|^2 + 2\sum_{i=1}^{3} a_i \int_\Omega f'(u)\Big|\frac{\partial u}{\partial x_i}\Big|^2 \mathrm{d}x \leqslant c\left(\|\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}\right),$$

which yields, owing to (2.9)

$$\|u\|^2_{H^2(\Omega)} \leqslant c\left(\|u\|^2_{H^1(\Omega)} + \|\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}\right). \quad (3.35)$$

We finally deduce from (3.15) (for $k = 1$), (3.31), (3.33) and (3.35) that

$$\begin{aligned} &\|u(t)\|^2_{H^2(\Omega)} \\ &\leqslant ce^{-c't}\left(\|u_0\|^2_{H^1(\Omega)} + \int_\Omega F(u_0)\mathrm{d}x + \|\alpha_0\|^2_{H^2(\Omega)} + \|\alpha_1\|^2_{H^2(\Omega)}\right) + c'', c' > 0, t \geqslant 1 \end{aligned} \quad (3.36)$$

and

$$\|u(t)\|^2_{H^2(\Omega)} \leqslant e^{-ct}Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^2(\Omega)}) + c', \quad c > 0, \quad t \geqslant 0. \quad (3.37)$$

**Second case:** $k \geqslant 2$

We multiply (2.7) by $A_k u$ and obtain, owing to the the interpolation inequality (3.5)

$$\frac{d}{dt}\|A_k^{\frac{1}{2}}u\|^2 + c\|u\|^2_{H^{2k}(\Omega)} + \leqslant \left(\|u\|^2 + \|f(u)\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}\right). \quad (3.38)$$

It follows from the continuity of $f$ and $F$, the continuous embedding $H^k(\Omega) \subset C(\bar{\Omega})$ (recall that $k \geqslant 2$) and (3.15) that

$$\begin{aligned} &\|u\|^2 + \|f(u)\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}\|^2, \\ \leqslant &Q(\|u_0\|_{H^k(\Omega)}) + \|\frac{\partial \alpha}{\partial t}\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)} \\ \leqslant &e^{-ct}Q(\|u_0\|_{H^k(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^2(\Omega)}) + c', \quad c > 0, \quad t \geqslant 0, \end{aligned} \quad (3.39)$$

so that

$$\begin{aligned}&\frac{d}{dt}\|A_k^{\frac{1}{2}}u\|^2+c\|u\|^2_{H^{2k}(\Omega)}\\&\leqslant e^{-c't}Q(\|u_0\|_{H^k(\Omega)},\|\alpha_0\|_{H^2(\Omega)},\|\alpha_1\|_{H^2(\Omega)})+c'',\quad c,\quad c'>0,\quad t\geqslant 0.\end{aligned}\tag{3.40}$$

Summing (3.13) and (3.40), we find a differential inequality of the form

$$\begin{aligned}&\frac{dE_4}{dt}+c\left(E_4+\|u\|^2_{H^{2k}(\Omega)}+\|\frac{\partial u}{\partial t}\|^2\right)\\&\leqslant e^{c't}Q(\|u_0\|_{H^k(\Omega)},\|\alpha_0\|_{H^2(\Omega)},\|\alpha_1\|_{H^2(\Omega)})+c'',\quad c,\quad c'>0,\quad t\geqslant 0,\end{aligned}\tag{3.41}$$

where

$$E_4=E_2+\|A_k^{\frac{1}{2}}u\|^2$$

satisfies

$$E_4\geqslant c\left(\|u\|^2_{H^k(\Omega)}+\int_\Omega F(u)\mathrm{d}x+\|\alpha\|^2_{H^2(\Omega)}+\|\frac{\partial\alpha}{\partial t}\|^2_{H^2(\Omega)}\right)-c',\quad c>0.\tag{3.42}$$

We then rewrite (2.7) as an elliptic equation, for $t>0$ fixed,

$$A_ku=-\frac{\partial u}{\partial t}-B_ku-f(u)+\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t},\quad \mathcal{D}^\beta u=0\quad\text{on}\quad\Gamma,\quad|\beta|\leqslant k-1.\tag{3.43}$$

Multiplying (3.43) by $A_ku$, we have, owing to the interpolation inequality (3.5),

$$\|A_ku\|^2\leqslant c\left(\|u\|^2+\|f(u)\|^2+\|\frac{\partial u}{\partial t}\|^2+\|\frac{\partial\alpha}{\partial t}\|^2+\|\frac{\partial\alpha}{\partial t}\|^2_{H^2(\Omega)}\right),\tag{3.44}$$

hence, owing to (3.39),

$$\begin{aligned}&\|u(t)\|^2_{H^{2k}(\Omega)}\\&\leqslant c\left(e^{-c't}Q(\|u_0\|_{H^k(\Omega)},\|\alpha_0\|_{H^2(\Omega)},\|\alpha_1\|_{H^2(\Omega)})+\|\frac{\partial u}{\partial t}\|^2\right)+c'',\quad c'>0.\end{aligned}\tag{3.45}$$

Next, we differentiate (2.7) with respect to time and obtain

$$\frac{\partial}{\partial t}\frac{\partial u}{\partial t}+A_k\frac{\partial u}{\partial t}+B_k\frac{\partial u}{\partial t}+f'(u)\frac{\partial u}{\partial t}=\Delta\frac{\partial\alpha}{\partial t}+\Delta\alpha-\frac{\partial u}{\partial t},\tag{3.46}$$

$$\mathcal{D}^\beta\frac{\partial u}{\partial t}=0\quad\text{on}\quad\Gamma,\quad|\beta|\leqslant k-1,\tag{3.47}$$

$$\frac{\partial u}{\partial t}|_{t=0}=-A_ku_0-B_ku_0-f(u_0)+\alpha_1-\Delta\alpha_1.\tag{3.48}$$

We can note that, if $u_0\in H^{2k}(\Omega)\cap H^k_0(\Omega)$ and $\alpha_1\in H^2(\Omega)\cap H^1_0(\Omega)$, then $\frac{\partial u}{\partial t}(0)\in L^2(\Omega)$ and

$$\|\frac{\partial u}{\partial t}(0)\|\leqslant Q(\|u_0\|_{H^{2k}(\Omega)},\|\alpha_1\|_{H^2(\Omega)}).\tag{3.49}$$

We multiply (3.46) by $\frac{\partial u}{\partial t}$ and find, owing to (2.9) and the interpolation inequality (3.3),

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|^2+c\|\frac{\partial u}{\partial t}\|^2_{H^k(\Omega)}\leqslant c'\left(\|\frac{\partial u}{\partial t}\|^2+\|\alpha\|^2_{H^2(\Omega)}+\|\frac{\partial\alpha}{\partial t}\|^2_{H^2(\Omega)}\right),\quad c>0.\tag{3.50}$$

It follows from (3.15) (both for $r = 1$), (3.50) and the uniform Gronwall's lemma that

$$\|\frac{\partial u}{\partial t}(t)\|^2 \leqslant e^{-ct}Q(\|u_0\|_{H^k(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^2(\Omega)}) + c', \quad c > 0, \quad t \geqslant 1, \tag{3.51}$$

an from (3.41), (3.42), (3.50) and Gronwall's lemma that

$$\|\frac{\partial u}{\partial t}(t)\|^2 \leqslant e^{ct}Q(\|u_0\|_{H^{2k}(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^2(\Omega)}), \quad t \geqslant 0. \tag{3.52}$$

We finally deduce from (3.45), (3.51) and (3.52) that

$$\|u(t)\|_{H^{2k}(\Omega)} \leqslant e^{-ct}Q(\|u_0\|_{H^k(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^2(\Omega)}) + c', \quad c > 0, \quad t \geqslant 1, \tag{3.53}$$

and

$$\|u(t)\|_{H^{2k}(\Omega)} \leqslant e^{-ct}Q(\|u_0\|_{H^{2k}(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^2(\Omega)}) + c', \quad c > 0, \quad t \geqslant 0. \tag{3.54}$$

## 4. The dissipative semigroup

We first have the following theorem.

**Theorem 4.1.** *(i) We assume that $(u_0, \alpha_0, \alpha_1) \in H_0^k(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega))$, with $\int_\Omega F(u_0)\mathrm{d}x < +\infty$ when $k = 1$. Then,* (2.1)–(2.4) *possesses a unique solution $(u, \alpha, \frac{\partial \alpha}{\partial t})$ such that, $\forall T > 0$,*

$$\begin{aligned}
&u(0) = u_0, \quad \alpha(0) = \alpha_0, \quad \frac{\partial \alpha}{\partial t}(0) = \alpha_1,\\
&u \in L^\infty(\mathbb{R}^+; H_0^k(\Omega)) \cap L^2(0, T; H^{2k}(\Omega) \cap H_0^k(\Omega)),\\
&\frac{\partial u}{\partial t} \in L^2(0, T; L^2(\Omega)),\\
&\alpha \in L^\infty(\mathbb{R}^+; H^2(\Omega) \cap H_0^1(\Omega)),\\
&\frac{\partial \alpha}{\partial t} \in L^\infty(\mathbb{R}^+; H^2(\Omega) \cap H_0^1(\Omega))
\end{aligned}$$

*and*

$$\begin{aligned}
&\frac{d}{dt}\left((u, v)\right) + \left(\left(A^{\frac{1}{2}}u, A^{\frac{1}{2}}v\right)\right) + B_k^{\frac{1}{2}}[u, v] + ((f(u), v))\\
=&\frac{d}{dt}\left(((\alpha, v)) + ((\nabla\alpha, \nabla v))\right), \forall v \in C_c^\infty(\Omega),\\
&\frac{d}{dt}\left(\left(\left(\frac{\partial \alpha}{\partial t}, w\right)\right) + \left(\left(\nabla\frac{\partial \alpha}{\partial t}, \nabla w\right)\right) + ((\nabla\alpha, \nabla w))\right) + ((\nabla\alpha, \nabla w))\\
=&-\frac{d}{dt}\left((u, w)\right), \forall w \in C_c^\infty(\Omega),
\end{aligned}$$

*where $B_1^{\frac{1}{2}}[u, v] = 0$ and, for $k \geqslant 2$,*

$$B_k^{\frac{1}{2}}[u, v] = \sum_{i=1}^{k-1}\sum_{|\beta|=i} a_\beta\left(\left(\mathcal{D}^\beta u, \mathcal{D}^\beta v\right)\right).$$

*(ii) If we further assume that* $(u_0,\alpha_0,\alpha_1)\in(H^{2k}(\Omega)\cap H_0^k(\Omega))\times(H^2(\Omega)\cap H_0^1(\Omega))\times(H^2(\Omega)\cap H_0^1(\Omega))$*, then*

$$u\in L^\infty(\mathbb{R}^+;H^{2k}(\Omega)\cap H_0^k(\Omega)),$$
$$\frac{\partial u}{\partial t}\in L^2\left(0,T;L^2(\Omega)\right),$$
$$\alpha\in L^\infty(\mathbb{R}^+;H^2(\Omega)\cap H_0^1(\Omega))$$

*and*

$$\frac{\partial\alpha}{\partial t}\in L^\infty(\mathbb{R}^+;H^2(\Omega)\cap H_0^1(\Omega)).$$

*(iii) If we further assume that*

$$|f'(s)|\leqslant c_6|s|^2+c_7,\quad c_6,c_7\geqslant 0, s\in\mathbb{R}, \tag{4.1}$$

*when* $k=1$*, then we have the continuous dependence with respect to the initial data in the* $H^k\times H^2\times H^2$*-norm.*

**Proof.**

a) **Existence :**

The proofs of existence and regulararity in (i) and (ii) follow from the a priori estimates derived in the previous section and, e.g., a standard Galerkin scheme.

b) **Uniqueness :**

Let now $\left(u^{(1)},\alpha^{(1)},\frac{\partial\alpha^{(1)}}{\partial t}\right)$ and $\left(u^{(2)},\alpha^{(2)},\frac{\partial\alpha^{(2)}}{\partial t}\right)$ be two solutions to (2.1)–(2.3) with initial data $\left(u_0^{(1)},\alpha_0^{(1)},\alpha_1^{(1)}\right)$ and $\left(u_0^{(2)},\alpha_0^{(2)},\alpha_1^{(2)}\right)$, respectively. We set

$$\left(u,\alpha,\frac{\partial\alpha}{\partial t}\right)=\left(u^{(1)},\alpha^{(1)},\frac{\partial\alpha^{(1)}}{\partial t}\right)-\left(u^{(2)},\alpha^{(2)},\frac{\partial\alpha^{(2)}}{\partial t}\right)$$

and

$$(u_0,\alpha_0,\alpha_1)=\left(u_0^{(1)},\alpha_0^{(1)},\alpha_1^{(1)}\right)-\left(u_0^{(2)},\alpha_0^{(2)},\alpha_1^{(2)}\right).$$

Then, $(u,\alpha)$ satisfies

$$\frac{\partial u}{\partial t}+A_ku+B_ku+f(u^{(1)})-f(u^{(2)})=\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}, \tag{4.2}$$

$$\frac{\partial^2\alpha}{\partial t^2}-\Delta\frac{\partial^2\alpha}{\partial t^2}-\Delta\frac{\partial\alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t}, \tag{4.3}$$

$$\mathcal{D}^\beta u=\alpha=0\quad\text{on}\quad\Gamma,\quad|\beta|\leqslant k-1, \tag{4.4}$$

$$u|_{t=0}=u_0,\alpha|_{t=0}=\alpha_0,\frac{\partial\alpha}{\partial t}|_{t=0}=\alpha_1. \tag{4.5}$$

We multiply (4.2) by $u$, we obtain, owing to (2.9) and the interpolation inequality (3.5),

$$\frac{d}{dt}\|u\|^2+c\|u\|^2_{H^k(\Omega}\leqslant c'\left(\|u\|^2+\|\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\|^2\right),c>0. \tag{4.6}$$

Next, multiplying (4.3) by $u+\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}$, we find

$$\frac{d}{dt}\left(\|\nabla\alpha\|^2+\|\Delta\alpha\|^2+\|u+\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t}\|^2\right)+c\left(\|\frac{\partial\alpha}{\partial t}\|^2_{H^1(\Omega)}+\|\frac{\partial\alpha}{\partial t}\|^2_{H^2(\Omega)}\right)$$
$$\leqslant c'(\|u\|^2_{H^1(\Omega}+\|\alpha\|^2_{H^1(\Omega)}). \tag{4.7}$$

Summing then (4.6) and $\delta_4$ times (4.7), where $\delta_4 > 0$ is small enough, we have a differential inequality of the form (note that $k \geqslant 1$)

$$\frac{d}{dt}E_5 \leqslant E_5, \tag{4.8}$$

where

$$E_5 = \|u\|^2 + \delta_4 \left( \|\nabla \alpha\|^2 + \|\Delta \alpha\|^2 + \|u + \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\|^2 \right)$$

satisfies

$$E_5 \geqslant c \left( \|u\|^2 + \|\alpha\|^2_{H^2(\Omega} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega} \right), c > 0. \tag{4.9}$$

It follows from (4.8), (4.9) and Gronwall's lemma that

$$\begin{aligned} &\|u(t)\|^2 + \|\alpha(t)\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}(t)\|^2_{H^2(\Omega)} \\ \leqslant & ce^{c't} \left( \|u_0\|^2 + \|\alpha_0\|^2_{H^2(\Omega)} + \|\alpha_1\|^2_{H^2(\Omega)} \right), t \geqslant 0, \end{aligned} \tag{4.10}$$

hence the uniqueness, as well as the continuity (with respect to the $L^2(\Omega) \times H^2(\Omega)^2$-norm) with respect to the initial data.

We finally turn to the proof of (iii).

We multiply (4.2) by $\frac{\partial u}{\partial t}$ and obtain

$$\begin{aligned} &\frac{d}{dt}\left( \|A_k^{\frac{1}{2}} u\|^2 + B_k^{\frac{1}{2}}[u] \right) + 2\|\frac{\partial u}{\partial t}\|^2 \\ =&2\left( \left( \frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} \right) \right) - \left( \left( f(u^{(1)}) - f(u^{(2)}), \frac{\partial u}{\partial t} \right) \right). \end{aligned} \tag{4.11}$$

We first assume that $k = 1$. We then find, owing to (4.1) and employing Hlder's inequality,

$$\begin{aligned} \left| \left( \left( f(u^{(1)}) - f(u^{(2)}), \frac{\partial u}{\partial t} \right) \right) \right| &\leqslant c \int_\Omega (|u^{(1)}|^2 + |u^{(2)}|^2 + 1)|u| \left| \frac{\partial u}{\partial t} \right| \mathrm{d}x \\ &\leqslant c(\|u^{(1)}\|^2_{L^6(\Omega)} + \|u^{(2)}\|^2_{L^6(\Omega)} + 1)\|u\|_{L^6(\Omega)} \|\frac{\partial u}{\partial t}\| \end{aligned}$$

hence, owing to the continuous embedding $H^1(\Omega) \subset L^6(\Omega)$ and (3.15) (for $k = 1$),

$$\begin{aligned} &\left| \left( \left( f(u^{(1)}) - f(u^{(2)}), \frac{\partial u}{\partial t} \right) \right) \right| \\ \leqslant & Q(\|u_0^{(1)}\|_{H^1(\Omega)}, \|u_0^{(2)}\|_{H^1(\Omega)}, \|\alpha_0^{(1)}\|_{H^2(\Omega)}, \|\alpha_0^{(2)}\|_{H^2(\Omega)}, \|\alpha_1^{(1)}\|_{H^2(\Omega)}, \|\alpha_1^{(2)}\|_{H^2(\Omega)}) \\ & \times \|u\|_{H^1(\Omega)} \|\frac{\partial u}{\partial t}\|. \end{aligned} \tag{4.12}$$

Note indeed that (4.1) implies that

$$|F(s)| \leqslant cs^4 + c', \quad c, c' \in \mathbb{R}.$$

We now assume that $k \geqslant 2$. It then follows from the continuity of $f'$, the continuous embeddind $H^k \subset C(\bar{\Omega})$ and (3.15) again that

$$\begin{aligned}&\left| \left( \left( f(u^{(1)}) - f(u^{(2)}), \frac{\partial u}{\partial t} \right) \right) \right| \\ \leqslant & Q(\|u_0^{(1)}\|_{H^1(\Omega)}, \|u_0^{(2)}\|_{H^1(\Omega)}, \|\alpha_0^{(1)}\|_{H^2(\Omega)}, \|\alpha_0^{(2)}\|_{H^2(\Omega)}, \|\alpha_1^{(1)}\|_{H^2(\Omega)}, \|\alpha_1^{(2)}\|_{H^2(\Omega)}) \\ & \times \|u\|_{H^k(\Omega)} \|\frac{\partial u}{\partial t}\|. \end{aligned} \tag{4.13}$$

It thus from (4.11) to (4.13) that, in both cases,

$$\begin{aligned}&\frac{d}{dt}\left(\|A_k^{\frac{1}{2}} u\|^2 + B_k^{\frac{1}{2}}[u]\right) + \|\frac{\partial u}{\partial t}\|^2 \\ \leqslant & Q(\|u_0^{(1)}\|_{H^1(\Omega)}, \|u_0^{(2)}\|_{H^1(\Omega)}, \|\alpha_0^{(1)}\|_{H^2(\Omega)}, \|\alpha_0^{(2)}\|_{H^2(\Omega)}, \|\alpha_1^{(1)}\|_{H^2(\Omega)}, \\ & \|\alpha_1^{(2)}\|_{H^2(\Omega)}(\|u\|^2_{H^k(\Omega)} + \|\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\|^2). \end{aligned} \tag{4.14}$$

Next, we multiply (4.3) by $\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}$ and have

$$\frac{d}{dt}\left(\|\nabla \alpha\|^2 + \|\Delta \alpha\|^2 + \|\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\|^2\right) + c\left(\|\nabla \frac{\partial \alpha}{\partial t}\|^2 + \|\Delta \frac{\partial \alpha}{\partial t}\|^2\right) \leqslant \|\frac{\partial u}{\partial t}\|^2. \tag{4.15}$$

Summing (4.14) and $\delta_5$ times (4.15), where $\delta_5 > 0$ is small enough, we obtain a differential inequality of the form

$$\frac{d}{dt}E_6 + c\left(\|\frac{\partial u}{\partial t}\|^2 + \|\nabla \frac{\partial \alpha}{\partial t}\|^2 + \|\Delta \frac{\partial \alpha}{\partial t}\|^2\right) \leqslant QE_6, \quad c > 0, \tag{4.16}$$

where

$$E_6 = \|A_k^{\frac{1}{2}} u\|^2 + \delta_5\left(\|\nabla \alpha\|^2 + \|\Delta \alpha\|^2 + \|\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\|^2\right)$$

satisfies, owing to the interpolation inequality (3.5),

$$E_6 \geqslant c\left(\|u\|^2_{H^k(\Omega)} + \|\alpha\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}\right) - c'\|u\|^2, c > 0, \quad c' \geqslant 0. \tag{4.17}$$

Summing finally (4.16) and $c'$ times (4.6), where $c'$ is the same constant as in (4.16), we find a differential inequality of the form

$$\frac{d}{dt}E_7 \leqslant QE_7, \tag{4.18}$$

where

$$E_7 = E_6 + c'\|u\|^2$$

satisfies

$$E_7 \geqslant c\left(\|u\|^2_{H^k(\Omega)} + \|\alpha\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}\right), c > 0. \tag{4.19}$$

If thus follows from (4.18), (4.19) and Gronwall's lemma that

$$\begin{aligned}&\|u(t)\|^2_{H^k(\Omega)} + \|\alpha(t)\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}(t)\|^2_{H^2(\Omega)} \\ \leqslant & ce^{Qt}(\|u_0\|^2_{H^k(\Omega)} + \|\alpha_0\|^2_{H^2(\Omega)} + \|\alpha_1\|^2_{H^2(\Omega)}), \quad t \geqslant 0, \end{aligned} \tag{4.20}$$

hence the continuous dependence with respect to the initial data in the $H^k(\Omega) \times H^2(\Omega)^2$-norm.

$\square$

**Remark 4.1.** We can note that (4.1) is satisfied by the usual cubic nonlinear term $f(s) = s^3 - s$.

We assume that (4.1) holds when $k = 1$. It follows from Theorem 4.1 that we can define the continuous semigroup $S(t) : \Phi \to \Phi, (u_0, \alpha_0, \alpha_1) \mapsto (u(t), \alpha(t), \frac{\partial \alpha}{\partial t}(t))$, $t \geqslant 0$ (i.e., $S(0) = I$, $S(t+\tau) = S(t) \circ S(\tau)$, $t$, $\tau \geqslant 0$), where $\Phi = H_0^k(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))^2$. Furthermore, $S(t)$ is dissipative in $\Phi$, owing to (3.15), in the sens that it possesses a bounded absording set $\mathcal{B}_0 \subset \Phi$ (i.e., $\forall B \subset \Phi$ bounded, $\exists t_0 = t_0(B) \geqslant 0$ such that $t \geqslant t_0 \Longrightarrow S(t)B \subset \mathcal{B}_0$).

**Remark 4.2.** We can also prove the continuous depence with respect to the initial data in the $H^{2k}(\Omega) \times H^2(\Omega)^2$-norm, without any growth restriction on $f'$ when $k = 1$, and it then follows from (3.37), (3.51) and (3.55) that $S(t)$ is defined, continuous and dissipative in $H_0^{2k}(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))^2$.

Actually, it follows from (3.36) and (3.53) that $S(t)$ possesses a bounded absorbing set $\mathcal{B}_1$ such that $\mathcal{B}_1$ is bounded in $\Phi$ and compact in $H^{2k}(\Omega) \times H^2(\Omega)^2$. It thus follows from classical results (see, e.g., [19,22,24]) that we have the

**Theorem 4.2.** *The semigroup $S(t)$ possesses the global attractor $\mathcal{A}$ which is compact in $\Phi$ and bounded in $H^{2k}(\Omega) \times H^2(\Omega) \times H^2(\Omega)$.*

**Remark 4.3.** It follows from (4.10) that we can extend $S(t)$ (by continuity and in a unique way) to $L^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)$. Therefore, Theorem 4.1 also holds, without any growth restriction on $f'$ when $k = 1$, except that the attraction of the trajectories to the global attractor holds in the $L^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)$-norm, instead of the $H^k(\Omega) \times H^2(\Omega) \times H^2(\Omega)$-one.

**Remark 4.4.** (i) We recall that the global attroctor $\mathcal{A}$ is the smallest (for the inclusion) compact set of the phase space which is invariant by the flow (i.e., $S(t)\mathcal{A} = \mathcal{A}$, $\forall t \geqslant 0$) and attracts all bounded sets of initial data as times goes to infinity; it thus appears as a suitable object in view of the study of the asymptotic behavior of the system. We refer the reader to, e.g., [19,22,24] for more details and discussion on this.

(ii) We can also prove, based on standard arguments (see, e.g., [19,22,24]) that $\mathcal{A}$ has finite dimension, in the sense of covering dimensions such as the Hausdorff and the fractal dimensions. The finite-dimensionality means, very roughly speaking, that, even thought the initial phase space has infinite dimension, the reduced dynamics can be described by a finite number of parameters (we refer the interested reader to, e.g., [19,22,24] for discussion on this subject).

## 5. Existence of exponential attractors

First case: $k = 1$

We first derive a smoothing property on the difference of two solutions which is one of the key tools to construct exponential attractors (see [11–13,18]).

**Proposition 5.1.** *Let* $\left(u^{(1)},\alpha^{(1)},\frac{\partial\alpha^{(1)}}{\partial t}\right)$ *and* $\left(u^{(2)},\alpha^{(2)},\frac{\partial\alpha^{(2)}}{\partial t}\right)$ *be two solutions to* (2.1)–(2.3) *with initial data* $\left(u_0^{(1)},\alpha_0^{(1)},\alpha_1^{(1)}\right)$ *and* $\left(u_0^{(2)},\alpha_0^{(2)},\alpha_1^{(2)}\right)$*, respectively, belonging to the bounded and positively invariant aborbing set* $\mathcal{B}_0$ *constructed in the previous section, then*

$$\begin{aligned}&\|(u^{(1)}(t)-u^{(2)}(t)\|^2_{H^{2k}(\Omega)}+\|\alpha^{(1)}(t)-\alpha^{(2)}(t)\|^2_{H^3(\Omega)}\\&+\|\frac{\partial\alpha^{(1)}}{\partial t}(t)-\frac{\partial\alpha^{(2)}}{\partial t}(t)\|^2_{H^3(\Omega)}\\&\leqslant ce^{c't}\left(\|(u_0^{(1)}-u_0^{(2)}\|^2_{H^k(\Omega)}+\|\alpha_0^{(1)}-\alpha_0^{(2)}\|^2_{H^2(\Omega)}+\|\alpha_1^{(1)}-\alpha_1^{(2)}\|^2_{H^2(\Omega)}\right),\end{aligned}\tag{5.1}$$

*where the constant* $c$ *and* $c'$ *only depend on* $\mathcal{B}_0$*.*

**Proof.** We consider the following initial and boundary value problem:

$$\frac{\partial u}{\partial t}+A_ku+B_ku+f(u^{(1)})-f(u^{(2)})=\frac{\partial\alpha}{\partial t}-\Delta\frac{\partial\alpha}{\partial t},\tag{5.2}$$

$$\frac{\partial^2\alpha}{\partial t^2}-\Delta\frac{\partial^2\alpha}{\partial t^2}-\Delta\frac{\partial\alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t},\tag{5.3}$$

$$\mathcal{D}^\beta u=\alpha=0\quad\text{on}\quad\Gamma,\quad|\beta|\leqslant k-1,\tag{5.4}$$

$$u|_{t=0}=u_0,\alpha|_{t=0}=\alpha_0,\frac{\partial\alpha}{\partial t}|_{t=0}=\alpha_1.\tag{5.5}$$

We first deduce from (4.16) that

$$\begin{aligned}&\|u(t)\|^2_{H^k(\Omega)}+\|\alpha(t)\|^2_{H^2(\Omega)}+\|\frac{\partial\alpha}{\partial t}(t)\|^2_{H^2(\Omega)}\\&\leqslant ce^{c't}\left(\|u_0\|^2_{H^k(\Omega)}+\|\alpha_0\|^2_{H^2(\Omega)}+\|\alpha_1\|^2_{H^2(\Omega)}\right),\quad t\geqslant 0,\end{aligned}\tag{5.6}$$

where the constants only depend on $\mathcal{B}_0$.

It also follows from (4.16) that

$$\begin{aligned}&\int_0^t\left(\|\frac{\partial u}{\partial t}\|^2+\|\nabla\frac{\partial\alpha}{\partial t}\|^2+\|\Delta\frac{\partial\alpha}{\partial t}\|^2\right)\mathrm{d}\tau\\&\leqslant ce^{c't}\left(\|u_0\|^2_{H^k(\Omega)}+\|\alpha_0\|^2_{H^2(\Omega)}+\|\alpha_1\|^2_{H^2(\Omega)}\right),\quad t\geqslant 0,\end{aligned}\tag{5.7}$$

where the constants only depend on $\mathcal{B}_0$.

We now differentiate (5.2) with respect to time and have, owing to (5.3),

$$\frac{\partial}{\partial t}\frac{\partial u}{\partial t}+A_k\frac{\partial u}{\partial t}+B_k\frac{\partial u}{\partial t}+f'(u^{(1)})\frac{\partial u}{\partial t}+(f'(u^{(1)})-f'(u^{(2)}))\frac{\partial u^{(2)}}{\partial t}=\Delta\frac{\partial\alpha}{\partial t}+\Delta\alpha-\frac{\partial u}{\partial t}.\tag{5.8}$$

We multiply (5.8) by $t\frac{\partial u}{\partial t}$, and obtain, owing to (2.9)

$$\begin{aligned}&\frac{d}{dt}\left(t\|\frac{\partial u}{\partial t}\|^2\right)+2t\|A_k^{\frac12}\frac{\partial u}{\partial t}\|^2+2tB_k^{\frac12}\left[\frac{\partial u}{\partial t}\right]\\&\leqslant ct\|\frac{\partial u}{\partial t}\|^2+\|\frac{\partial u}{\partial t}\|^2-2t\left(\left(\nabla\alpha,\nabla\frac{\partial u}{\partial t}\right)\right)-2t\left(\left(\nabla\frac{\partial\alpha}{\partial t},\nabla\frac{\partial u}{\partial t}\right)\right)\\&\quad+2t\int_\Omega|f'(u^{(1)})-f'(u^{(2)})|\Big|\frac{\partial u}{\partial t}\Big|\Big|\frac{\partial u^{(2)}}{\partial t}\Big|\mathrm{d}x.\end{aligned}$$

Noting that

$$\int_\Omega |f'(u^{(1)}) - f'(u^{(2)})|\Big|\frac{\partial u}{\partial t}\Big|\Big|\frac{\partial u^{(2)}}{\partial t}\Big|\mathrm{d}x \leqslant c\int_\Omega |u|\Big|\frac{\partial u}{\partial t}\Big|\Big|\frac{\partial u^{(2)}}{\partial t}\Big|\mathrm{d}x$$
$$\leqslant c\|\nabla u\|\|\nabla \frac{\partial u}{\partial t}\|\|\frac{\partial u^{(2)}}{\partial t}\|,$$

we obtain, owing to a proper interpolation ineguality,

$$\frac{d}{dt}\left(t\|\frac{\partial u}{\partial t}\|^2\right) + t\|A_k^{\frac{1}{2}}\frac{\partial u}{\partial t}\|^2$$
$$\leqslant ct\left(\|\frac{\partial u}{\partial t}\|^2 + \|\nabla\alpha\|^2 + \|\nabla\frac{\partial\alpha}{\partial t}\|^2\right) + c'\|\nabla u\|\|\nabla\frac{\partial u}{\partial t}\|\|\frac{\partial u^{(2)}}{\partial t}\| + \|\frac{\partial u}{\partial t}\|^2. \tag{5.9}$$

We also multiply (5.3) by $t\Delta^2\frac{\partial\alpha}{\partial t}$ and obtain

$$\frac{d}{dt}\left(t\left(\|\nabla\Delta\alpha\|^2 + \|\Delta\frac{\partial\alpha}{\partial t}\|^2 + \|\nabla\Delta\frac{\partial\alpha}{\partial t}\|^2\right)\right) + t\|\nabla\Delta\frac{\partial\alpha}{\partial t}\|^2$$
$$\leqslant ct\|\nabla\frac{\partial u}{\partial t}\|^2 + \|\nabla\Delta\alpha\|^2 + \|\Delta\frac{\partial\alpha}{\partial t}\|^2 + \|\nabla\Delta\frac{\partial\alpha}{\partial t}\|^2. \tag{5.10}$$

Therefore, noting that it follows from (3.22), (3.29) and (3.30) (for $\left(u, \alpha, \frac{\partial\alpha}{\partial t}\right) = \left(u^{(2)}, \alpha^{(2)}, \frac{\partial\alpha^{(2)}}{\partial t}\right)$) that

$$\int_0^t \|\frac{\partial u^{(2)}}{\partial t}\|^2\mathrm{d}\tau \leqslant ce^{c't}, \quad t\geqslant 0,$$

where the constants only depend on $\mathcal{B}_0$, we find, combining (5.9) and (5.10) and owing to (5.6), (5.7) and Gronwall's lemma (applied over $(0,t)$),

$$\|\frac{\partial u}{\partial t}\|^2 + \|\alpha(t)\|^2_{H^3(\Omega)} + \|\frac{\partial\alpha}{\partial t}(t)\|^2_{H^3(\Omega)}$$
$$\leqslant ce^{c't}\left(\|u_0\|^2_{H^k(\Omega)} + \|\alpha_0\|^2_{H^2(\Omega)} + \|\alpha_1\|^2_{H^2(\Omega)}\right), \quad t\geqslant 0, \tag{5.11}$$

where the constants only depend on $\mathcal{B}_0$.

We rewrite (5.2) in the form

$$A_k u = h_u(t), \quad \mathcal{D}^\beta u = 0 \quad on \quad \Gamma, \quad |\beta|\leqslant k-1, \tag{5.12}$$

where

$$h_u(t) = -\frac{\partial u}{\partial t} - B_k u - (f(u^{(1)} - f(u^{(2)})) + \frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}, \tag{5.13}$$

satisfies, owing to (5.6) and (5.11),

$$\|h_u(t)\| \leqslant ce^{c't}\left(\|u_0\|^2_{H^k(\Omega)} + \|\alpha_0\|^2_{H^2(\Omega)} + \|\alpha_1\|^2_{H^2(\Omega)}\right), \quad t\geqslant 0, \tag{5.14}$$

where the constants only depend on $\mathcal{B}_0$.

Multiplying (5.12) by $A_k u$, we have

$$\|A_k u\| \leqslant \|h_u(t)\|, \quad t \geqslant 0,$$

hence

$$\|u(t)\|^2_{H^{2k}(\Omega)} \leqslant ce^{c't}\left(\|u_0\|^2_{H^k(\Omega)} + \|\alpha_0\|^2_{H^2(\Omega)} + \|\alpha_1\|^2_{H^2(\Omega)}\right), \quad t \geqslant 0, \tag{5.15}$$

where the constants only depend on $\mathcal{B}_0$.

We finally deduce from (5.11) and (5.15) that

$$\begin{aligned} &\|u(t)\|^2_{H^{2k}(\Omega)} + \|\alpha(t)\|^2_{H^3(\Omega)} + \|\frac{\partial \alpha}{\partial t}(t)\|^2_{H^3(\Omega)} \\ \leqslant& ce^{c't}\left(\|u_0\|^2_{H^k(\Omega)} + \|\alpha_0\|^2_{H^2(\Omega)} + \|\alpha_1\|^2_{H^2(\Omega)}\right), \quad t \geqslant 0, \end{aligned} \tag{5.16}$$

where the constants only depend on $\mathcal{B}_0$.

$\square$

We then have

**Proposition 5.2.** *There holds, for any solution to* (2.1)–(2.4) *with initial datum belonging to* $\mathcal{B}_0$ *and for any* $T > 0$,

$$\|S(t_1)(u_0, \alpha_0, \alpha_1) - S(t_2)(u_0, \alpha_0, \alpha_1)\|_\Phi \leqslant c|t_1 - t_2|^{\frac{1}{2}}, \quad t_1, t_2 \in [0, T].$$

**Proof.** In view of the estimate (3.13), we have

$$\begin{aligned} &\|S(t_1)(u_0, \alpha_0, \alpha_1) - S(t_2)(u_0, \alpha_0, \alpha_1)\|_\Phi \\ =&\|u(t_1) - u(t_2), \alpha(t_1) - \alpha(t_2), \frac{\partial \alpha}{\partial t}(t_1) - \frac{\partial \alpha}{\partial t}(t_2)\|_\Phi \\ \leqslant&\|u(t_1) - u(t_2)\|_{H^k(\Omega)} + \|\alpha(t_1) - \alpha(t_2)\|_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}(t_1) - \frac{\partial \alpha}{\partial t}(t_2)\|_{H^2(\Omega)} \\ =&\|\int_{t_2}^{t_1} \frac{\partial u}{\partial t}\mathrm{d}\tau\|_{H^k(\Omega)} + \|\int_{t_2}^{t_1} \frac{\partial \alpha}{\partial t}\mathrm{d}\tau\|_{H^2(\Omega)} + \|\int_{t_2}^{t_1} \frac{\partial^2 \alpha}{\partial t^2}\mathrm{d}\tau\|_{H^2(\Omega)} \\ \leqslant&\int_{t_2}^{t_1} \|\frac{\partial u}{\partial t}\|_{H^k(\Omega)}\mathrm{d}\tau + \int_{t_2}^{t_1} \|\frac{\partial \alpha}{\partial t}\|_{H^2(\Omega)}\mathrm{d}\tau + \int_{t_2}^{t_1} \|\frac{\partial^2 \alpha}{\partial t^2}\|_{H^2(\Omega)}\mathrm{d}\tau \\ \leqslant&c|t_1 - t_2|^{\frac{1}{2}} + \left(\int_{t_2}^{t_1} \|\frac{\partial^2 \alpha}{\partial t^2}\|_{H^2(\Omega)}\mathrm{d}\tau\right)^{\frac{1}{2}} |t_1 - t_2|^{\frac{1}{2}}. \end{aligned}$$

Then, multiplying (5.3) by $-\Delta\frac{\partial^2 \alpha}{\partial t^2}$, we obtain, proceeding as above,

$$\begin{aligned} &\frac{d}{dt}\left(\|\Delta\frac{\partial \alpha}{\partial t}\|^2 + 2\left(\left(\Delta\alpha, \Delta\frac{\partial \alpha}{\partial t}\right)\right)\right) + 2\|\nabla\frac{\partial^2 \alpha}{\partial t^2}\|^2 + \|\Delta\frac{\partial^2 \alpha}{\partial t^2}\|^2 \\ \leqslant& c\left(\|\frac{\partial u}{\partial t}\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}\|^2\right), \end{aligned} \tag{5.17}$$

where the constants $c$ depends only on $\mathcal{B}_0$. Therefore, owing to (3.13),

$$\left|\int_{t_2}^{t_1} \|\frac{\partial^2 \alpha}{\partial t^2}\|_{H^2(\Omega)}\mathrm{d}\tau\right| \leqslant c, \tag{5.18}$$

where the constants $c$ depends only on $\mathcal{B}_0$ and $T \geqslant 0$ such that $t_1, t_2 \in [0,T]$, so that

$$\|S(t_1)(u_0,\alpha_0,\alpha_1) - S(t_2)(u_0,\alpha_0,\alpha_1)\|_\Phi \leqslant c|t_1 - t_2|^{\frac{1}{2}}, \tag{5.19}$$

where the constants $c$ depends only on $\mathcal{B}_0$ and $T \geqslant 0$ such that $t_1, t_2 \in [0,T]$. □

We finally deduce the following result.

**Theorem 5.1.** *The semigroup $S(t)$ possesses an exponential attractor $\mathcal{M} \subset \mathcal{B}_0$, i.e.,*

*(i) $\mathcal{M}$ is compact in $L^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)$;*

*(ii) $\mathcal{M}$ is positively invariant, $S(t)\mathcal{M} \subset \mathcal{M}, \forall t \geqslant 0$;*

*(iii) $\mathcal{M}$ has a finite fractal dimension in $L^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)$ ;*

*(iv) $\mathcal{M}$ attracts exponentially fast the bounded subset of $\Phi$,*
*$\forall \mathcal{B} \subset \Phi$ bounded, $dist_{H^k(\Omega)\times H^2(\Omega)\times H^2(\Omega)}(S(t)\mathcal{B},\mathcal{M}) \leqslant Q(\|\mathcal{B}\|_\Phi)e^{-ct}$, $c > 0$, $t \geqslant 0$,*

*where the constant $c$ is independent of $\mathcal{B}$ and $dist_{L^2(\Omega)\times H^2(\Omega)\times H^2(\Omega)}$ denotes the Hausdorff semidistance between sets defined by*

$$dist_{L^2(\Omega)\times H^2(\Omega)\times H^2(\Omega)}(A,B) = \sup_{a\in A}\inf_{b\in B}\|a-b\|_{L^2(\Omega)\times H^2(\Omega)\times H^2(\Omega)}.$$

**Remark 5.1.** Setting $\tilde{\mathcal{M}} = S(1)\mathcal{M}$, we cane prove that $\tilde{\mathcal{M}}$ is exponential attractor for $S(t)$, but now in the topology of $\Phi$.

Since $\mathcal{M}$ (or $\tilde{\mathcal{M}}$) is a compact attracting set, we deduce from Theorem 5.1 and standard results the

**Corollary 5.1.** *The global attractor $\mathcal{A}$ (see theorem 4.2) is finite-dimensional.*

**Remark 5.2.** If $k \geqslant 2$, then we prove the same way the existence of an exponential attractor $\mathcal{M}' \subset \mathcal{B}_1$ .

**Remark 5.3.** Compared to the global attractor, an exponential attractor is expected to be more robust under perturbations. Indeed, the rate of attraction of trajectories to the global attractor may be slow and it is very difficult, if not impossible, to estimate this rate of attraction with respect to the physical parameters of the problem in general. As a consequence, global attractors may change drastically under small perturbations.

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