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NONTRIVIAL SOLUTIONS OF NONLOCAL BOUNDARY VALUE PROBLEMS FOR NONLINEAR HIGHER-ORDER SINGULAR FRACTIONAL DIFFERENTIAL EQUATIONS*

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Abstract This paper deals with the existence and multiplicity of nontrivial solutions of nonlocal boundary value problems for nonlinear higher-order singular fractional differential equations with sign-changing nonlinear term. The main tool used in the proof is topological degree theory. Some examples explain that our results cannot be obtained by the method of cone theory.

Keywords Singular fractional differential equation, nontrivial solution, topological degree, Riemann-Liouville fractional derivative.

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1. Introduction

We consider the existence and multiplicity of nontrivial solutions for nonlinear higher-order singular fractional differential equations with fractional nonlocal boundary conditions:

$$\begin{cases} D_t^\alpha x(t) + h(t)f(t,x(t)) = 0,\ t \in (0,1), \\ x^{(i)}(0) = 0,\ 0 \le i \le n-2, \\ D_t^\beta x(1) = \sum\limits_{j=1}^{\infty} a_j D_t^\gamma x(\xi_j), \end{cases} \tag{1.1}$$

where $D_t^\alpha, D_t^\beta, D_t^\gamma$ are the standard Riemann-Liouville fractional derivative operator, $\alpha \in (n-1,n], \beta \in [1,n-2], \gamma \in [1,n-3]$ for $n \ge 4$ and $n \in \mathbb{N}^+ = \{1,2,3,\cdots\}$, $\gamma \le \beta, f \in C([0,1]\times\mathbb{R},\mathbb{R}), \mathbb{R} = (-\infty,+\infty)$, $h \in C((0,1),\mathbb{R}^+)$ and $h(t)$ may be singular at $t=0$ and/or $t=1$, $a_j \in (0,1), \sum_{j=1}^{\infty} a_j \xi_j^{\alpha-\gamma-1} < 1$ and

$$0 < \xi_1 < \xi_2 < \cdots < \xi_j < \cdots < \xi_m < \cdots < 1.$$

In view of fractional differential equations modeling capabilities in engineering, science, economy and other fields, in the last few decades, the theory of fractional differential equation has a rapid development, see the books [13, 15, 19]. This may

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explain the reason why the last few decades have witnessed an overgrowing interest in the research of such problems, with many papers in this direction published. We refer the interested reader to [10, 11, 16, 17, 29, 34] and the references therein.

Recently, the existence and uniqueness of a solution for the nonlinear fractional differential equations have been researched by means of the Schauder fixed-point theorem or coincidence degree theory or the lattice structure, see [1,3,6,12,22,25,26] and the references therein. For instance, authors of [6] studied the existence of a nontrivial solution for nonlinear higher-order fractional differential equations with multi-point boundary conditions:

$$\begin{cases} D_t^{\alpha} u(t) + f(t, u(t)) = 0,\ \alpha \in (n-1, n](n \geq 2), \\ u^{(i)}(0) = 0,\ 0 \leq i \leq n-2,\ u(1) = \sum\limits_{i=1}^{m-2} a_i u(\xi_i). \end{cases} \tag{1.2}$$

In [27], by using topological degree theory, Wu and Zhang obtained the existence results of a nontrivial solution for superlinear fractional boundary value problems:

$$\begin{cases} D_t^{\alpha} u(t) + p(t) f\big(t, u(t), D_t^{\mu_1} u(t), \cdots, D_t^{\mu_{n-1}} u(t)\big) = 0,\ t \in (0,1), n \geq 3, \\ D_t^{\mu_i} u(0) = 0, 1 \leq i \leq n-1, D_t^{\mu_{n-1}+1} u(0) = 0, D_t^{\mu_{n-1}} u(1) = \sum\limits_{i=1}^{m-2} a_i D_t^{\mu_{n-1}} u(\xi_i). \end{cases} \tag{1.3}$$

With the aid of some inequalities associated with the Green's function, authors of [30] obtained the existence of a nontrivial solution for superlinear and sublinear fractional boundary value problems:

$$\begin{cases} D_t^{\alpha} u(t) + f(t, u(t)) = 0,\ \alpha \in (2, 3], \\ u(0) = u'(0) = u'(1) = 0. \end{cases} \tag{1.4}$$

In [23, 24], Sun and Zhang obtained some existence results of nontrivial solutions for singular superlinear and sublinear Sturm-Liouville boundary value problems by using topological degree theory, respectively. However, in [4,23–26,30], the nonlinear term $f(t, u)$ in the equation (1.1) permits sign-changing, but it is required to be bounded from below.

In most works, the nonlinear term $f(t, u)$, which appears in the right-hand side of the equation (1.1), is required to be nonnegative to obtain the existence of positive solutions by using fixed point theorem on a cone, see [2, 9, 14, 16, 28, 32, 35] and the references therein. Generally, the operator A generated by nonnegative function $f(t, u)$ is a cone mapping. In this paper, the nonlinear term $f(t, u)$ may be a sign-changing function, and consequently, the operator A is not necessary to be a cone mapping, thus the theory of fixed point index on a cone becomes invalid, and in order to obtain the existence of nontrivial solution we make use of topological degree theory which is not confined in a cone.

Motivated by the papers [23,24,30], this article discusses the existence and multiplicity of nontrivial solutions for the singular problem (1.1) by using the topological degree theory. The nonlinear term $f(t, u)$ of (1.1) is sign-changing and unbounded from below. Finally, some examples show that our results can't be obtained by the method of cone theory.

2. Preliminaries

Definition 2.1 ([21]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

provided the right side is pointwise defined on $(0, +\infty)$. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$D_t^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds,$$

where $n = [\alpha] + 1, [\alpha]$ denotes the largest integer not greater than α, provided the right side is pointwise defined on $(0, +\infty)$. We stipulate that $D_t^0 u(t) = u(t)$ if $\alpha = 0$.

Lemma 2.1 ([13]). *Let $x \in L^p(0,1)$ $(1 \leq p \leq +\infty)$, $\rho > \sigma > 0$.*

(i) $D_{0+}^{\sigma} I_{0+}^{\rho} x(t) = I_{0+}^{\rho-\sigma} x(t)$, $D_{0+}^{\sigma} I_{0+}^{\sigma} x(t) = x(t)$, $I_{0+}^{\rho} I_{0+}^{\sigma} x(t) = I_{0+}^{\rho+\sigma} x(t)$ hold at almost every point $t \in (0,1)$. If $\rho + \sigma > 1$, then the above third equation holds at any point of $[0,1]$;

(ii) $D_{0+}^{\sigma} t^{\rho-1} = \Gamma(\rho) t^{\rho-\sigma-1} / \Gamma(\rho - \sigma), t > 0$.

Lemma 2.2 ([13]). *Let $y_1 \in C(0,1) \cap L^1(0,1)$, $\alpha > 0$, then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} y_1(t) = y_1(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_1, c_2, \cdots, c_n$ are arbitrary real constants, n is the smallest integer greater than or equal to α.

Lemma 2.3. *Let $\sum_{j=1}^{\infty} a_j \xi_j^{\alpha-\gamma-1} \in [0,1), \gamma \leq \beta$ for $\gamma \in [1, n-3], \beta \in [1, n-2]$ and $n \geq 4$, then for any $y \in L^1[0,1]$, the unique solution of the fractional nonlocal boundary value problem:*

$$\begin{cases} D_t^{\alpha} u(t) + y(t) = 0, \ 0 < t < 1, \\ u^{(i)}(0) = 0, \ 0 \leq i \leq n-2, \\ D_t^{\beta} u(1) = \sum\limits_{j=1}^{\infty} a_j D_t^{\gamma} u(\xi_j), \end{cases} \tag{2.1}$$

is given by

$$u(t) = \int_0^1 G(t,s) y(s) ds, \tag{2.2}$$

where the Green's function

$$G(t,s) = g(t,s) + \frac{t^{\alpha-1}}{d} \sum_{j=1}^{\infty} a_j k(\xi_j, s), \tag{2.3}$$

$$d = \Gamma(\alpha-\gamma) - \Gamma(\alpha-\beta) \sum_{j=1}^{\infty} a_j \xi_j^{\alpha-\gamma-1},$$

$$g(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.4}$$

$$k(t,s)=\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\begin{cases}t^{\alpha-\gamma-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-\gamma-1}, & 0\le s\le t\le 1,\\ t^{\alpha-\gamma-1}(1-s)^{\alpha-\beta-1}, & 0\le t\le s\le 1.\end{cases}\tag{2.5}$$

$\Gamma(\alpha), \Gamma(\alpha-\gamma)$ and $\Gamma(\alpha-\beta)$ are the Gamma function.

Proof. By using Lemma 2.1 and Lemma 2.2, the solutions of the equation (2.1) are

$$u(t)=-I^{\alpha}_{0+}y(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n},$$

where $c_1,c_2,\cdots,c_n$ are arbitrary real constants. By $u(0)=0$, we have $c_n=0$. Then

$$u(t)=-I^{\alpha}_{0+}y(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_{n-1}t^{\alpha-n+1}.\tag{2.6}$$

Differentiating (2.6), we get

$$u'(t)=-I^{\alpha-1}_{0+}y(t)+c_1(\alpha-1)t^{\alpha-2}+\cdots+c_{n-1}(\alpha-n+1)t^{\alpha-n}.\tag{2.7}$$

By (2.7) and $u'(0)=0$, we get $c_{n-1}=0$. Similarly, we have $c_2=c_3=\cdots=c_{n-2}=0$. Hence,

$$u(t)=\frac{-1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds+c_1t^{\alpha-1},\tag{2.8}$$

$$D^{\gamma}_tu(t)=\frac{1}{\Gamma(\alpha-\gamma)}\Big[c_1\Gamma(\alpha)t^{\alpha-\gamma-1}-\int_0^t(t-s)^{\alpha-\gamma-1}y(s)ds\Big].$$

By $D^{\beta}_tu(1)=\sum\limits_{j=1}^{\infty}a_jD^{\gamma}_tu(\xi_j)$ and Lemma 2.1, we get

$$c_1=\frac{1}{d\Gamma(\alpha)}\Big[\Gamma(\alpha-\gamma)\int_0^1(1-s)^{\alpha-\gamma-1}y(s)ds-\Gamma(\alpha-\beta)\sum_{j=1}^{\infty}a_j\int_0^{\xi_j}(\xi_j-s)^{\alpha-\gamma-1}y(s)ds\Big].$$

Substituting c_1 into (2.8), we get that the unique solution of the problem (2.1) is

$$\begin{aligned}u(t)=&\frac{-1}{\Gamma(\alpha)}\textstyle\int_0^t(t-s)^{\alpha-1}y(s)ds+\dfrac{t^{\alpha-1}}{d}\\&\Big[\Gamma(\alpha-\gamma)\textstyle\int_0^1(1-s)^{\alpha-\beta-1}y(s)ds-\Gamma(\alpha-\beta)\sum\limits_{j=1}^{\infty}a_j\int_0^{\xi_j}(\xi_j-s)^{\alpha-\gamma-1}y(s)ds\Big]\\=&\frac{1}{\Gamma(\alpha)}\Big[\textstyle\int_0^t\big[t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}\big]y(s)ds\\&+\textstyle\int_t^1t^{\alpha-1}(1-s)^{\alpha-\beta-1}y(s)ds+\dfrac{\Gamma(\alpha-\gamma)-d}{d}\displaystyle\int_0^1t^{\alpha-1}(1-s)^{\alpha-\beta-1}y(s)ds\\&-\frac{t^{\alpha-1}}{d}\Gamma(\alpha-\beta)\textstyle\sum\limits_{j=1}^{\infty}a_j\int_0^{\xi_j}(\xi_j-s)^{\alpha-\gamma-1}h(s)y(s)ds\Big]\\=&\int_0^1g(t,s)y(s)ds+\frac{\Gamma(\alpha-\beta)t^{\alpha-1}}{d\Gamma(\alpha)}\sum_{j=1}^{\infty}a_j\Big[\int_{\xi_j}^1\xi_j^{\alpha-\gamma-1}(1-s)^{\alpha-\beta-1}y(s)ds\\&+\int_0^{\xi_j}\big[\xi_j^{\alpha-\gamma-1}(1-s)^{\alpha-\beta-1}-(\xi_j-s)^{\alpha-\gamma-1}\big]y(s)ds\Big]\\=&\int_0^1g(t,s)y(s)ds+\frac{t^{\alpha-1}}{d}\sum_{j=1}^{\infty}a_j\int_0^1k(\xi_j,s)y(s)ds\\=&\textstyle\int_0^1G(t,s)y(s)ds.\end{aligned}$$

i.e., (2.2) holds.

Conversely, if $u \in C[0,1]$ is a solution of the integral equation (2.2), from Lemma 2.1 we easily see that u satisfies the equation and boundary conditions of (2.1). □

Remark 2.1. If $\beta = \gamma \in [1, n-3]$ $(n \geq 4)$, the equation (2.5) can be written as

$$k(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-\gamma-1}(1-s)^{\alpha-\gamma-1} - (t-s)^{\alpha-\gamma-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-\gamma-1}(1-s)^{\alpha-\gamma-1}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Hence, the boundary condition of the problem (1.1) is wider than that of [12,27,35].

Lemma 2.4. *Under the assumption of Lemma 2.3, functions $g(t,s)$ and $k(t,s)$ defined by* (2.4) *and* (2.5) *have the following properties:*

(*i*) *$g(t,s) \geq 0$ is continuous on $[0,1] \times [0,1]$ and $g(t,s) > 0$ for $t, s \in (0,1)$.*

(*ii*) *$t^{\alpha-1} g(1,s) \leq g(t,s) \leq \max_{t\in[0,1]} g(t,s) = g(1,s)$ for $t, s \in [0,1]$, where*

$$g(1,s) = \frac{1}{\Gamma(\alpha)} \left[(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1} \right].$$

(*iii*) *$k(t,s) \geq 0$ is continuous on $[0,1] \times [0,1]$.*

Proof. For the proof of (i) and (ii), respectively, see Theorem 3.2.6 in [9] and Lemma 2.7 in [14].

(iii) it is clear that $k(t,s) \in C([0,1] \times [0,1])$. Since $\gamma \leq \beta$ for $\gamma \in [1, n-3], \beta \in [1, n-2]$ and $n \geq 4$, then $0 < \alpha - \beta - 1 \leq \alpha - \gamma - 1$, and

$$\begin{aligned} & t^{\alpha-\gamma-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-\gamma-1} \\ &= t^{\alpha-\gamma-1} \left[(1-s)^{\alpha-\beta-1} - \left(1 - \frac{s}{t}\right)^{\alpha-\gamma-1} \right] \\ &\geq t^{\alpha-\gamma-1} \left[(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-\gamma-1} \right] \\ &\geq t^{\alpha-\gamma-1} \left[(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-\beta-1} \right] = 0, \ \ 0 \leq s \leq t \leq 1. \end{aligned}$$

Hence $k(t,s) \geq 0$ for $t, s \in [0,1]$. □

We give the following assumption to be used in the rest of this paper.

(H_1) $h \in C((0,1), \mathbb{R}^+), h(t) \not\equiv 0$ on any subinterval of $(0,1)$ and

$$0 < \int_0^1 (1-t) h(t) dt < +\infty.$$

Lemma 2.5. *Under the assumption of Lemma 2.3, the Green's function $G(t,s)$ defined by* (2.3) *has the following properties:*

(1) *$G(t,s) \geq 0$ is continuous on $[0,1] \times [0,1]$ and $G(t,s) > 0$ for $t, s \in (0,1)$.*

(2) *$t^{\alpha-1} G(1,s) \leq G(t,s) \leq \max_{t\in[0,1]} G(t,s) = G(1,s)$ for $t, s \in [0,1]$, where*

$$G(1,s) = g(1,s) + \frac{1}{d} \sum_{j=1}^{\infty} a_j k(\xi_j, s) \leq \frac{\Gamma(\alpha-\gamma)}{d\Gamma(\alpha)} (1-s)^{\alpha-\beta-1}, \ \ s \in [0,1].$$

(3) *Let $\omega(t) = G(1,t), t \in [0,1]$. If the condition ($H_1$) holds, then*

$$k_1 \omega(s) \leq \int_0^1 G(t,s) h(t) \omega(t) dt \leq k_2 \omega(s), \ \ s \in [0,1], \tag{2.9}$$

where $0 < k_1 = \int_0^1 t^{\alpha-1} \omega(t) h(t) dt \leq k_2 = \int_0^1 \omega(t) h(t) dt$.

Proof. From Lemma 2.4 we know that (1) and (2) are true. Since $\alpha-\beta-1>1$,

$$\int_0^1 \omega(t)h(t)dt \le \int_0^1 \frac{\Gamma(\alpha-\gamma)}{d\Gamma(\alpha)}h(t)(1-t)^{\alpha-\beta-1}dt \le \int_0^1 \frac{h(t)}{d\Gamma(\alpha)}(1-t)dt < +\infty.$$

By simple computation, we arrive at the inequality (2.9) immediately. □

Let $E=C[0,1]$ be a real Banach space endowed with the norm $\|u\|=\max_{t\in J}|u(t)|$, and $P=\{u\in C[0,1]: u(t)\ge 0, t\in[0,1]\}$, then P is a total cone in E. For fixed $r>0$, let $\Omega_r=\{u\in E:\|u\|<r\}$.

By (H_1), we define three integral operators $A, L, L^*: E\to E$ by

$$Lu(t)=\int_0^1 G(t,s)h(s)u(s)ds,\ \ L^*u(s)=\int_0^1 G(t,s)h(t)u(t)dt.$$

$$Au(t)=\int_0^1 G(t,s)h(s)f(s,u(s))ds,\ \ t\in[0,1], \tag{2.10}$$

Similar to the proof of Lemma 2.15 in [27], we can prove that $L, L^*: E\to E$ are completely continuous linear operators with the spectral radius $r(L)>0$ and the first eigenvalue $\lambda_1=r^{-1}(L)$, satisfying $L(P)\subset P$. Then there are $\varphi,\psi\in P\setminus\{0\}$ such that

$$\begin{aligned} L\varphi(t)&=\textstyle\int_0^1 G(t,s)h(s)\varphi(s)ds=r(L)\varphi(t),\\ L^*\psi(s)&=\textstyle\int_0^1 G(t,s)h(t)\psi(t)dt=r(L)\psi(s).\end{aligned} \tag{2.11}$$

Since $G(t,0)=G(t,1)=0, t\in[0,1]$, it follows the second equation in (2.11) from $\psi(0)=\psi(1)=0$, which implies $\psi'(0)>0,\psi'(1)<0$ (see [20]). Define a function $\mathcal{X}$ on $[0,1]$ by

$$\mathcal{X}(s)=\begin{cases}\psi'(0), & s=0,\\ \dfrac{\psi(s)}{s(1-s)}, & 0<s<1,\\ -\psi'(1), & s=1.\end{cases}$$

So $\mathcal{X}$ is continuous on $[0,1]$ and $\mathcal{X}(s)>0$ for all $s\in[0,1]$. Then there exist $\delta_1,\delta_2>0$ such that $\delta_1\le\mathcal{X}(s)\le\delta_2$ for all $s\in[0,1]$. Thus

$$\delta_1 s(1-s)\le\psi(s)\le\delta_2 s(1-s),\ \ s\in[0,1].$$

From this and (H_1) we yield

$$\int_0^1\psi(s)h(s)u(s)ds\le\delta_2\|u\|\int_0^1 h(s)(1-s)ds<+\infty,\ \ u\in P.$$

Let

$$P_1=\{u\in P:\int_0^1\psi(t)h(t)u(t)dt\ge k_1\|u\|\},$$

where k_1 is given by Lemma 2.5. It is easy to verify that P_1 is a cone in E.

Lemma 2.6. *Under the assumption of Lemma 2.3 and (H_1), we can get the following conclusions,*

1) $k_1\le r(L)\le k_2$,
2) $L(P)\subset P_1$.

Proof. 1) Multiply by $h(t)\omega(t)$ the first equation in (2.11) and integrate over [0,1], and use the inequality (2.9) to obtain

$$k_1 \int_0^1 h(s)\omega(s)\varphi(s)ds \le \int_0^1 h(s)\varphi(s)ds \int_0^1 G(t,s)\omega(t)h(t)dt$$
$$= r(L)\int_0^1 \omega(t)h(t)\varphi(t)dt \le k_2 \int_0^1 \omega(s)h(s)\varphi(s)ds.$$

Since $\int_0^1 \omega(t)h(t)\varphi(t)dt > 0, k_1 \le r(L) \le k_2$.

2) For $u \in P, t \in [0,1]$, we have

$$Lu(t) = \int_0^1 G(t,s)h(s)u(s)ds \le \int_0^1 G(1,s)h(s)u(s)ds.$$

On the other hand, we have by the second equation in (2.11)

$$\int_0^1 \psi(t)h(t)Lu(t)dt \ge \int_0^1 t^{\alpha-1}\psi(t)h(t)dt \int_0^1 G(1,s)h(s)u(s)ds$$
$$\ge k_1 \int_0^1 G(t,s)h(s)u(s)ds = k_1 Lu(t).$$

Hence $\int_0^1 \psi(t)h(t)Lu(t)dt \ge \delta\|Lu\|$, and so $L(P) \subset P_1$. □

It is similar to the proof of Lemma 3 in [23], we get the following lemma.

Lemma 2.7. *Assume that the condition (H_1) is satisfied, then $A: E \to E$ is a completely continuous operator.*

Lemma 2.8 ([5]). *Let X be a Banach space and Ω be a bounded open set. Assume that $A: \overline{\Omega} \to X$ is a completely continuous operator. If there is $u_0 \ne 0$ such that $u \ne Au + \lambda u_0$ for any $\lambda \ge 0, u \in \partial\Omega$, then the topological degree $deg(I - A, \Omega, 0) = 0$.*

Lemma 2.9 ([5]). *Let X be a Banach space and $\Omega \subset X$ be a bounded open set with $0 \in \Omega$. Assume that $A: \overline{\Omega} \to X$ is a completely continuous operator and $Au \ne \mu u$ for any $\mu \ge 1, u \in \partial\Omega$ $(Particularly, if\ \|Au\| \le \|u\|, Au \ne u, \forall\ u \in \partial\Omega)$, then the topological degree $deg(I - A, \Omega, 0) = 1$.*

Let X be a Banach space and W be a convex closed set in X. W is called a wedge if it satisfies the following conditions,

(1) $\lambda u \in W$ for any $u \in W$ and $\lambda \ge 0$,

(2) there is $y \in W$ such that $-y \notin W$.

Lemma 2.10 ([18]). *Let X and Y_i be Banach spaces, $P_i \subset Y_i$ be a cone for each $i = 1, 2, \cdots, n$, $\Omega \subset X$ be a bounded open set. Assume that $A: \overline{\Omega} \to X$ is a condensing operator, which has no fixed point on $\partial\Omega$. If there are linear operators $T: W \to W, N_i: W \to P_i$ $(i = 1, 2, \cdots, n)$ and $x_0 \in W \setminus \{0\}$ such that*

(1) $N_i x_0 \ne 0$ $(i = 1, 2, \cdots, n)$,

(2) there is $u^* \in W$ such that $A(\partial\Omega) \subset W(u^*) =: \{x \in X : x + u^* \in W\}$,

(3) $N_i T x = N_i x$ for any $x \in W, i = 1, 2, \cdots, n$,

(4) for any $x \in \partial\Omega \cap W(u^*)$, there is $i_0 = i_0(x)$ such that $N_{i_0} Ax \ge N_{i_0} Tx$.

Then the topological degree $deg(I - A, \Omega, 0) = 0$.

3. Existence of a nontrivial solution

Theorem 3.1. *Assume that (H_1) holds and the following conditions are satisfied,*

(H_2) *there is a constant* $\beta_1 > k_1^{-1}$ *such that* $\liminf\limits_{u\to+\infty} \min_{t\in[0,1]} \frac{f(t,u)}{u} \geq \beta_1$,

(H_3) *there is a constant* $\beta_2 \in (0, k_2^{-1})$ *such that* $\limsup\limits_{u\to-\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u} \leq \beta_2$,

(H_4) *there is a constant* $b \in [0, k_2^{-1})$ *such that* $\limsup\limits_{u\to 0} \max_{t\in[0,1]} \frac{|f(t,u)|}{|u|} \leq b$.

Then the singular problem (1.1) *has at least one nontrivial solution.*

Proof. By $(H_2), (H_3)$, there are $\varepsilon_1 \in (0, \frac{\beta_1-\beta_2}{2})$ and $C_1 > 0$ such that $(\beta_1-\varepsilon_1) > k_1^{-1}$, $(\beta_2+\varepsilon_1) < k_2^{-1}$, and

$$f(t,u) \geq (\beta_1-\varepsilon_1)u - C_1,\ t\in[0,1], u\geq 0, \tag{3.1}$$

$$f(t,u) \geq (\beta_2+\varepsilon_1)u - C_1,\ t\in[0,1], u\leq 0. \tag{3.2}$$

By (3) of Lemma 2.5, we have $\beta_1-\varepsilon_1 > \beta_2+\varepsilon_1$, then (3.1) and (3.2) yield

$$f(t,u) \geq (\beta_1-\varepsilon_1)u - C_1,\ t\in[0,1], u\in\mathbb{R}, \tag{3.3}$$

$$f(t,u) \geq (\beta_2+\varepsilon_1)u - C_1,\ t\in[0,1], u\in\mathbb{R}. \tag{3.4}$$

Now we prove that

$$\Omega = \{u \in E : u = Au + \lambda\varphi \text{ for some } \lambda \geq 0\}$$

is a bounded set, where $\varphi \in P\setminus\{0\}$ is determined by (2.11). Indeed, for any $u\in\Omega$, there is a $\lambda\geq 0$ such that $u = Au+\lambda\varphi$. From this and (3.3), we have

$$u(t) \geq Au(t) \geq (\beta_1-\varepsilon_1)\int_0^1 G(t,s)h(s)u(s)ds - C_1\int_0^1 G(t,s)h(s)ds. \tag{3.5}$$

Multiply by $\psi(t)h(t)$ both sides of the inequality (3.5) and integrate over $[0,1]$, we get

$$\int_0^1 \psi(t)h(t)u(t)dt \geq (\beta_1-\varepsilon_1)r(L)\int_0^1 \psi(t)h(t)u(t)dt - C_2, \tag{3.6}$$

where $C_2 = C_1 r(L)\int_0^1 \psi(t)h(t)dt$. Since $(\beta_1-\varepsilon_1)r(L) \geq (\beta_1-\varepsilon_1)k_1 > 1$, (3.6) yields

$$\int_0^1 \psi(t)h(t)u(t)dt \leq \frac{C_2}{(\beta_1-\varepsilon_1)r(L)-1}. \tag{3.7}$$

Similarly, noting $(\beta_2+\varepsilon_1)r(L) \leq (\beta_2+\varepsilon_1)k_2 < 1$, we have

$$\int_0^1 \psi(t)h(t)u(t)dt \geq \frac{-C_2}{1-(\beta_2+\varepsilon_1)r(L)}. \tag{3.8}$$

Since $u = Au+\lambda\varphi$,

$$u - (\beta_2+\varepsilon_1)Lu + C_1Lu_0 = L\big(\mathbf{f}u - (\beta_2+\varepsilon_1)u + C_1u_0 + \lambda r^{-1}(L)\varphi\big), \tag{3.9}$$

where $u_0 \in P, u_0(t)\equiv 1$, $Au = L\mathbf{f}u$, $\mathbf{f} : E\to E$ is the Nemytskii operator, $\mathbf{f}u(t) = f(t,u(t))$. From (3.4), (3.9) and Lemma 2.6 we obtain that $u-(\beta_2+\varepsilon_1)Lu+C_1Lu_0 \in P_1$. Consequently,

$$\begin{aligned} k_1\|u-(\beta_2+\varepsilon_1)Lu+C_1Lu_0\| &\leq \textstyle\int_0^1 \big[u(t)-(\beta_2+\varepsilon_1)Lu(t)+C_1Lu_0(t)\big]\psi(t)h(t)dt \\ &= [1-(\beta_2+\varepsilon_1)r(L)]\textstyle\int_0^1 \psi(t)h(t)u(t)dt + C_2. \end{aligned} \tag{3.10}$$

By (3.7), (3.8) and (3.10), there is $C_3 > 0$ such that $\|u-(\beta_2+\varepsilon_1)Lu\| \le C_3$. Hence,

$$-C_3u_0 \le u - (\beta_2+\varepsilon_1)Lu \le C_3u_0.$$

Since $(\beta_2+\varepsilon_1)r(L) \le (\beta_2+\varepsilon_1)k_2 < 1$, it follows from that $-\overline{u} \le u \le \overline{u}$, where $\overline{u} = [I-(\beta_2+\varepsilon_1)L]^{-1}C_3u_0$. This shows Ω is bounded. Then there exists a sufficiently large constant $K > 0$ such that

$$u \ne Au + \lambda\varphi,\ \forall\ u \in \partial\Omega_K, \lambda \ge 0.$$

Lemma 2.8 yields

$$deg(I-A, \Omega_K, 0) = 0. \tag{3.11}$$

On the other hand, by (H_4), there are sufficiently small constants $\varepsilon_2 > 0, \rho > 0$ such that $b+\varepsilon_2 < k_2^{-1}$, and

$$|f(t,u)| \le (b+\varepsilon_2)|u|,\ t \in [0,1], |u| \le \rho.$$

Consequently,

$$\begin{aligned}\|Au\| &\le (b+\varepsilon_2)\textstyle\int_0^1 G(1,s)h(s)|u(s)|ds\\ &< k_2^{-1}\textstyle\int_0^1 \omega(s)h(s)ds\|u\| = \|u\|,\ \forall\ u \in \overline{\Omega}_\rho.\end{aligned}$$

(3.12) and Lemma 2.9 yield

$$deg(I-A, \Omega_\rho, 0) = 1. \tag{3.12}$$

We have by (3.11) and (3.12)

$$deg(I-A, \Omega_K \setminus \overline{\Omega}_\rho, 0) = deg(I-A, \Omega_K, 0) - deg(I-A, \Omega_\rho, 0) = -1.$$

Then A has a fixed point on $\Omega_K \setminus \overline{\Omega}_\rho$. This means that the singular problem (1.1) has at least one nontrivial solution. □

Theorem 3.2. *Assume that* (H_1) *holds and the following conditions are satisfied,*

(H_5) *there is* $\eta_1 > k_1^{-1}$ *such that* $\liminf\limits_{u\to 0+} \min_{t\in[0,1]} \frac{f(t,u)}{u} \ge \eta_1$,

(H_6) *there is* $\eta_2 \in (0, k_2^{-1})$ *such that* $\limsup\limits_{u\to 0-} \max_{t\in[0,1]} \frac{f(t,u)}{u} \le \eta_2$,

(H_7) *there is* $d \in [0, k_2^{-1})$ *such that* $\limsup\limits_{|u|\to+\infty} \max_{t\in[0,1]} \frac{|f(t,u)|}{|u|} \le d$.

Then the singular problem (1.1) *has at least one nontrivial solution.*

Proof. By (H_5) and (H_6), there are $\varepsilon_3 \in (0, \frac{\eta_1-\eta_2}{2})$ and sufficiently small constant $\sigma > 0$ such that $\eta_1 - \varepsilon_3 > k_1^{-1}, \eta_2 + \varepsilon_3 < k_2^{-1}$, and

$$f(t,u) \ge (\eta_1 - \varepsilon_3)u,\ t \in [0,1], u \in [0,\sigma], \tag{3.13}$$

$$f(t,u) \ge (\eta_2 + \varepsilon_2)u,\ t \in [0,1], u \le [-\sigma, 0]. \tag{3.14}$$

Since $\eta_1 - \varepsilon_3 > \eta_2 + \varepsilon_3$, (3.13) and (3.14) yield

$$f(t,u) \ge (\eta_1 - \varepsilon_3)u,\ t \in [0,1], u \in [-\sigma, \sigma], \tag{3.15}$$

$$f(t,u) \ge (\eta_2 + \varepsilon_3)u,\ t \in [0,1], u \le [-\sigma, \sigma]. \tag{3.16}$$

Now we prove that

$$M_1 = \{u \in E : u = Au + \lambda\varphi \text{ for some } \lambda \geq 0\} = \emptyset,$$

where $\varphi \in P \setminus \{0\}$ is determined by (2.11). If not, there are $\lambda_2 \geq 0, u_2 \in \partial\Omega_\sigma \cap P$ such that $u_2 = Au_2 + \lambda_2\varphi$, then

$$u_2 = Au_2 + \lambda_2\varphi = L\mathbf{f}u_2 + \lambda_2\varphi. \tag{3.17}$$

Multiply by $\psi(t)h(t)$ both sides of (3.17) and integrate over $[0, 1]$, and use (3.15) to obtain

$$\int_0^1 \psi(t)h(t)u_2(t)dt \geq (\eta_1 - \varepsilon_3)r(L) \int_0^1 \psi(s)h(s)u_2(s)ds. \tag{3.18}$$

Since $(\eta_1 - \varepsilon_3)r(L) \geq (\eta_1 - \varepsilon_3)k_1 > 1$, (3.18) yields that $\int_0^1 \psi(s)h(s)u_2(s)ds \leq 0$. Similarly, noting $(\eta_2+\varepsilon_3)r(L) \leq (\eta_2+\varepsilon_3)k_2 < 1$, from (3.16) and (3.17) we get that $\int_0^1 \psi(s)h(s)u_2(s)ds \geq 0$. Hence, $\int_0^1 \psi(s)h(s)u_2(s)ds = 0$. (3.17) can be written as

$$u_2 - (\eta_2 + \varepsilon_3)Lu_2 = L\big(\mathbf{f}u_2 - (\eta_2 + \varepsilon_3)u_2 + \lambda_2 r^{-1}(L)\varphi\big). \tag{3.19}$$

It is similar to the proof of Theorem 3.1, we obtain from (3.16), (3.19) and Lemma 2.6 that $u_2 - (\eta_2 + \varepsilon_3)Lu_2 \in P_1$. Consequently,

$$\begin{aligned} k_1\|u_2 - (\eta_2 + \varepsilon_3)Lu_2\| &\leq \textstyle\int_0^1 \big[u_2(t) - (\eta_2 + \varepsilon_3)Lu_2(t)\big]\psi(t)h(t)dt \\ &= [1 - (\eta_2 + \varepsilon_3)r(L)] \textstyle\int_0^1 \psi(s)h(s)u_2(s)ds = 0, \end{aligned}$$

$u_2 = (\eta_2 + \varepsilon_3)Lu_2$. From $(\eta_2 + \varepsilon_3)r(L) \leq (\eta_2 + \varepsilon_3)k_2 < 1$ We obtain that $u_2 = 0$, which contradicts $\|u_2\| = \sigma$. Hence $M_1 = \emptyset$. According to the property of the lack of direction of the Leray-Schauder degree, we get

$$deg(I - A, \Omega_\sigma, 0) = 0. \tag{3.20}$$

On the other hand, by (H_7), there are $\varepsilon_4 > 0, C_4 > 0$ such that $(d + \varepsilon_4) < k_2^{-1}$, and

$$|f(t, u)| \leq (d + \varepsilon_4)|u| + C_4,\ t \in [0, 1], u \in \mathbb{R}.$$

Consequently,

$$|Au(t)| \leq (d + \varepsilon_4) \int_0^1 G(t, s)h(s)|u(s)|ds + C_4Lu_0,\ t \in [0, 1], u \in E. \tag{3.21}$$

We claim that the set

$$M_2 = \{u \in E : Au = \mu u \text{ for some } \mu \geq 1\}$$

is bounded. Indeed, for any $u \in M_2$, there is $\mu_2 \geq 1$ such that $Au = \mu_2 u$. (3.21) yields

$$|u(t)| \leq (d + \varepsilon_4)L|u(t)| + C_5u_0,\ t \in [0, 1],$$

where $C_5 = C_4 \int_0^1 G(1, s)ds$. Due to $(d + \varepsilon_4)r(L) \leq (d + \varepsilon_4)k_2 < 1$, this yields that $|u(t)| \leq [I - (d + \varepsilon_4)L]^{-1}C_4u_0, t \in [0, 1]$. This shows M_2 is bounded. Then there is a sufficiently large constant $G > 0$ such that

$$Au \neq \mu u, \forall\ u \in \partial\Omega_G, \mu \geq 1,$$

that is, $I-A$ and I are homotopic on $\partial\Omega_G$. From the homotopic invariant property of the Leray-Schauder degree, we yield

$$deg(I-A,\Omega_G,0)=1. \tag{3.22}$$

We have by (3.20) and (3.22)

$$deg(I-A,\Omega_G\setminus\overline{\Omega}_\sigma,0)=deg(I-A,\Omega_G,0)-deg(I-A,\Omega_\sigma,0)=1.$$

Then A has a fixed point on $\Omega_G\setminus\overline{\Omega}_\sigma$. This means that the singular problem (1.1) has at least one nontrivial solution. □

Theorem 3.3. *Assume that $(H_1),(H_7)$ hold and the following condition is satisfied,*

(H_8) *there is $r>0$ such that $f(t,u)\geq k_1^{-1}u$ for any $t\in[0,1],|u|\leq r$.*

Then the singular problem (1.1) has at least one nontrivial solution.

Proof. Take $n=1, Y_1=\mathbb{R}, P_1=\mathbb{R}^+, W=P_1$ and $u^*(t)=x_0(t)=k_1^{-1}r\int_0^1 G(t,s)h(s)ds$ in Lemma 2.9. Linear operators $T:W\to E, N:W\to\mathbb{R}$ are defined by

$$Tu=\lambda_1 Lu,\ \ N_1u=\int_0^1\psi(t)h(t)u(t)dt,$$

where $\lambda_1=r^{-1}(L),\psi\in P\setminus\{0\}$ is given by (2.11). Clearly, $N_1x_0>0, N_1(W)\subset P_1$. It is similar to the proof of 2) in Lemma 2.6, we can prove that $T(W)\subset W$, and use (2.11) to obtain

$$\begin{aligned}\int_0^1\psi(t)h(t)x_0(t)dt &\geq k_1^{-1}r\int_0^1 t^{\alpha-1}\psi(t)h(t)dt\int_0^1 G(1,s)h(s)ds\\ &= r\int_0^1 G(1,s)h(s)ds=k_1\|x_0\|,\end{aligned}$$

so $x_0\in W\setminus\{0\}$. Now we verify that the conditions of Lemma 2.9 are satisfied in $\overline{\Omega}_r$. Without loss of generality, we may assume that A has no fixed point on $\partial\Omega_r$. For any $u\in W$, we have by (2.11),

$$N_1T(u)=\lambda_1\int_0^1\psi(t)h(t)dt\int_0^1 G(t,s)h(s)u(s)ds=\int_0^1\psi(s)h(s)u(s)ds=N_1u.$$

From (H_8) we obtain that

$$u^*(t)+Au(t)=\int_0^1 G(t,s)h(s)[k_1^{-1}r+f(s,u(s))]ds\geq 0,\ \ t\in[0,1],u\in\overline{\Omega}_r, \tag{3.23}$$

$$\begin{aligned}&\int_0^1\psi(t)h(t)[u^*(t)+Au(t)]dt\\ &=\int_0^1\psi(t)h(t)dt\int_0^1 G(t,s)h(s)[k_1^{-1}r+f(s,u(s))]ds\\ &\geq\int_0^1\psi(t)h(t)t^{\alpha-1}dt\int_0^1 G(1,s)h(s)[k_1^{-1}r+f(s,u(s))]ds\\ &\geq k_1\int_0^1 G(t,s)h(s)[k_1^{-1}r+f(s,u(s))]ds\\ &=k_1[u^*(t)+Au(t)],\ \ t\in[0,1].\end{aligned}$$

This, together with (3.23), we have

$$\int_0^1\psi(t)h(t)[u^*(t)+Au(t)]dt\geq k_1\|u^*+Au\|.$$

Hence $u^* + Au \in W$. Further, for any $u \in \partial\Omega_r \cap W(u^*)$, (H_8) yields

$$\begin{aligned} N_1 A(u) &\geq k_1^{-1} \int_0^1 h(t)\psi(t) \int_0^1 G(t,s)h(s)u(s)dsdt \\ &= k_1^{-1} r(L) \int_0^1 \psi(s)h(s)u(s)ds \\ &\geq \int_0^1 \psi(s)h(s)u(s)ds = N_1 u. \end{aligned}$$

According to Lemma 2.10, we get

$$deg(I - A, \Omega_r, 0) = 0. \tag{3.24}$$

On the other hand, by (H_7), take $G > r$ such that (3.22) hold. (3.22) and (3.24) yield

$$deg(I-A,(\Omega_G \cap P)\setminus(\Omega_r \cap P),0) = deg(I-A,\Omega_G \cap P,0) - deg(I-A,\Omega_r \cap P,0) = 1.$$

Then A has a fixed point on $\Omega_G \setminus \overline{\Omega}_r$. This means that the singular problem (1.1) has at least one nontrivial solution. □

Theorem 3.4. *Assume that* $(H_1), (H_4)$ *and* (H_8) *hold, then the singular problem* (1.1) *has at least one nontrivial solution.*

Proof. We may take $r > \rho$ such that (3.12) and (3.24) hold, then

$$deg(I - A, \Omega_r \setminus \overline{\Omega}_\rho, 0) = deg(I - A, \Omega_r, 0) - deg(I - A, \Omega_\rho, 0) = -1.$$

Hence A has a fixed point on $\Omega_r \setminus \overline{\Omega}_\rho$. This means that the singular problem (1.1) has at least one nontrivial solution. □

4. Existence of multiple nontrivial solutions

Theorem 4.1. *Assume that* $(H_1), (H_2), (H_3), (H_4)$ *hold and*

$$uf(t,u) \geq 0, \ \forall\ t \in [0,1], u \in \mathbb{R}. \tag{4.1}$$

Then the singular problem (1.1) *has at least one positive solution and one negative solution.*

Proof. From (4.1) we know that $A(P) \subset P$. Similar to the proof of Theorem 3.1, from (3.11) and (3.12) we know that fixed point index

$$i(A, \Omega_K \cap P, 0) = deg(I - A, \Omega_K, 0) = 0,\ i(A, \Omega_\rho \cap P, 0) = deg(I - A, \Omega_\rho, 0) = 1.$$

Consequently,

$$i(A, (\Omega_K \cap P) \setminus (\Omega_\rho \cap P), 0) = i(A, \Omega_K \cap P, 0) - i(A, \Omega_\rho \cap P, 0) = -1,$$

so A has a fixed point in $(\Omega_K \cap P) \setminus (\Omega_\rho \cap P)$ and the singular problem (1.1) has at least one positive solution.

Let $f_1(t,u) = -f(t,-u),\ \forall\ t \in [0,1], u \in \mathbb{R}$ and define

$$A_1 u(t) = \int_0^1 G(t,s)h(s)f_1(s,u(s))ds,\ t \in [0,1].$$

Then $A_1(P) \subset P$. By the same method as above, we know that A_1 has a fixed point $v \in P \setminus \{0\}$, i.e., $A_1 v = v$. Consequently,

$$-v(t) = \int_0^1 G(t,s)h(s)f(s,-v(s))ds = A(-v(t)),\ t \in [0,1].$$

Hence $-v$ is the negative solution of the singular problem (1.1). $\square$

Similarly, we can prove the following theorem.

Theorem 4.2. *Assume that* (H_1), $(H_7), (H_8)$ *hold and*

$$uf(t,u) \geq 0,\ \forall\ t \in [0,1], u \in \mathbb{R}.$$

Then the singular problem (1.1) *has at least one positive solution and one negative solution.*

Theorem 4.3. *Assume that* $(H_1), (H_4), (H_7)$ *and* (H_8) *hold, then the singular problem* (1.1) *has at least two nontrivial solutions.*

Proof. We may take $G > r > \rho$ such that (3.12), (3.22) and (3.24) hold. Then

$$deg(I - A, \Omega_r \setminus \overline{\Omega}_\rho, 0) = deg(I - A, \Omega_r, 0) - deg(I - A, \Omega_\rho, 0) = -1,$$
$$deg(I - A, \Omega_G \setminus \overline{\Omega}_r, 0) = deg(I - A, \Omega_G, 0) - deg(I - A, \Omega_r, 0) = 1.$$

Hence A has a fixed point on $\Omega_G \setminus \overline{\Omega}_r$ and $\Omega_r \setminus \overline{\Omega}_\rho$, respectively. This means that the singular problem (1.1) has at least two nontrivial solutions. $\square$

Remark 4.1. Since $k_1 \leq r(L) \leq k_2$, our conditions $(H_2) - (H_8)$ are relatively easy to verify in many applications.

Remark 4.2. If $h(t) \equiv 1$ in the problem (1.1), all our conclusion is true for $\beta, \gamma \in [1, n-2], \gamma \leq \beta$ and $n \geq 3$.

5. Some examples

Example 5.1. Consider nonlocal boundary problems of nonlinear higher-order singular fractional differential equations:

$$\begin{cases} D_{0+}^{\frac{7}{2}} u(t) + \dfrac{f(t,u(t))}{1-t} = 0,\ t \in (0,1), \\ u(0) = u'(0) = u''(0) = 0, u'(1) = \sum\limits_{k=1}^{\infty} \dfrac{1}{2\sqrt{k}} u'\Big(\dfrac{1}{k}\Big), \end{cases} \tag{5.1}$$

where $\alpha = \dfrac{7}{2}, \beta = \gamma = 1, a_k = \dfrac{1}{2\sqrt{k}}, \xi_k = \dfrac{1}{k}$ $(k = 1, 2, 3, \cdots)$, $h(t) = (1 - t)^{-1}, f(t,u) = u^2 + b\sin u$ is usually sign-changing for $\dfrac{\pi^2}{4} < b < k_2^{-1}$. By simple calculation, we get $k_1 = \frac{5\sqrt{\pi}}{96d}, k_2 = \frac{16}{45d\sqrt{\pi}}$, $d = 1 - \sum\limits_{k=1}^{\infty} a_k \xi_k^{\alpha-\gamma-1} = 1 - \frac{\pi^2}{12}$. Clearly, the conditions $(H_1), (H_2), (H_3)$ and (H_4) hold. From Theorem 3.1 we conclude that the singular problem (5.1) has a nontrivial solution.

Example 5.2. Consider nonlocal boundary problems of nonlinear higher-order singular fractional differential equations:

$$\begin{cases} D_{0+}^{\frac{7}{2}}u(t)+\dfrac{f(t,u(t))}{1-t}=0,\ t\in(0,1),\\ u(0)=u'(0)=u''(0)=0, u'(1)=\sum\limits_{k=1}^{\infty}\dfrac{1}{2\sqrt{k}}u'\Big(\dfrac{1}{k}\Big), \end{cases} \tag{5.2}$$

where $h(t)=(1-t)^{-1}, f(t,u)=u^{\frac{2}{3}}+4\sin u^2$ is usually sign-changing. It is easy to verify that the conditions $(H_1),(H_5),(H_6)$ and (H_7) hold. From Theorem 3.2 we conclude that the singular problem (5.2) has a nontrivial solution.

Example 5.3. Consider nonlocal boundary problems of nonlinear higher-order singular fractional differential equations:

$$\begin{cases} D_{0+}^{\frac{7}{2}}u(t)+\dfrac{f(t,u(t))}{1-t}=0,\ t\in(0,1),\\ u(0)=u'(0)=u''(0)=0, u'(1)=\sum\limits_{k=1}^{\infty}\dfrac{1}{2\sqrt{k}}u'\Big(\dfrac{1}{k}\Big), \end{cases} \tag{5.3}$$

where $h(t)=(1-t)^{-1}$,

$$f(t,u)=k_1^{-1}\begin{cases} u^3,\ t\in[0,1],-1\le u\le 0,\\ u,\quad t\in[0,1],0\le u\le 1,\\ u^{\frac{1}{3}}, t\in[0,1],u\ge 1. \end{cases}$$

Take $r=1$, it is easy to see that $f(t,u)\ge k_1^{-1}u, |u|\le 1, t\in[0,1]$ and the condition (H_7) holds. From Theorem 3.3 we conclude that the singular problem (5.3) has a nontrivial solution.

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