

# DELAY INDUCED SUBCRITICAL HOPF BIFURCATION IN A DIFFUSIVE PREDATOR-PREY MODEL WITH HERD BEHAVIOR AND HYPERBOLIC MORTALITY\*

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**Abstract** In this paper, we consider the dynamics of a delayed diffusive predator-prey model with herd behavior and hyperbolic mortality under Neumann boundary conditions. Firstly, by analyzing the characteristic equations in detail and taking the delay as a bifurcation parameter, the stability of the positive equilibria and the existence of Hopf bifurcations induced by delay are investigated. Then, applying the normal form theory and the center manifold argument for partial functional differential equations, the formula determining the properties of the Hopf bifurcation are obtained. Finally, some numerical simulations are also carried out and we obtain the unstable spatial periodic solutions, which are induced by the subcritical Hopf bifurcation.

**Keywords** Predator-prey model with herd behavior, hyperbolic mortality, delay, diffusion, Hopf bifurcation, periodic solutions.

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## 1. Introduction

The study on dynamics of a biological model is one of the dominant subjects in mathematical biology due to its universal existence and importance. Numerous mathematical models have been proposed to study the relation between predator population and prey population. Thus, predator-prey model becomes one of the most important population dynamical models. We know that there are many factors which affect population dynamics in predator-prey models. One crucial component of predator-prey relationships is predator-prey interaction (also called functional response), which can be classified into many different types, such as Holling I-IV types, Hassell-Varley type, Crowley-Martin type, Beddington-DeAngelis type, and so on.

In natural ecosystems, many living beings live forming herds and all members of a group do not interact at a time. There are many reasons for this herd behaviour, such as searching for food resources, defending the predators, etc. Because of having more abundant and interesting dynamic characteristics, predator-prey systems with

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prey group defense ability attract attention of many scholars [1, 6, 11, 12, 19, 21, 37]. Of course, there are other ways of modelling group defence. More recently, More recently, in [2, 3], a new predator-prey interaction has been studied for a more elaborated social model, in which the individuals of one population in the large herbivores populating the savannas, gather together in huge herds, with generally the strongest individuals on the border and the weakest being concentrated in the middle of the bunch, while the other one shows a more individualistic behavior. This leads to the consequence that the capture of a prey by a successful predators attack occurs mainly on the boundary, involving therefore mostly the individuals that occupy the outermost positions in the herd. Based on the fact that predator-prey interactions occur mainly through the perimeter of the herd, the authors in [2, 3] have proposed a new predator-prey model with square root functional responses

$$\begin{cases} \frac{du}{dt} = u(1-u) - \frac{\sqrt{uv}}{1+\alpha\sqrt{u}}, \\ \frac{dv}{dt} = \beta v(-\gamma + \frac{\sqrt{u}}{1+\alpha\sqrt{u}}), \end{cases} \quad (1.1)$$

where  $u(t)$  and  $v(t)$  stand for the prey and predator densities, respectively, at time  $t$ .  $\beta\gamma$  is the death rate of the predator in the absence of prey,  $\gamma$  is the conversion or consumption rate of prey to predator.  $\alpha \geq 0$  is related to the search efficiency of  $v$  for  $u$  and average handling time, see [2, 3] in detail. This model is also known as the predator-prey model with herd behavior. When  $\alpha = 0$ , the authors in [2, 3] have obtained some meaningful results for system (1.1) by the Poincaré-Bendixson theorem, method of phase plane analysis and numerical calculation, respectively. Other types of models have been considered, see for instance [5, 7, 16, 18, 33, 34].

In the real world, the predator and the prey may move for many reasons, such as currents and turbulent diffusion. Thus, we should consider the spatial disperse. The spatiotemporal dynamics of the predator-prey models involving spatial diffusion have been increasingly studied by many researchers, see [23, 28, 29, 36, 44, 48]. Here, assuming the preys and the predators are in an isolate patch, we neglect the impact of migration, including immigration and emigration, and only consider the diffusion of the spatial domain. With the development of mathematical models for population dynamics, we know that the functional response and mortality rate of the predator are essential. Therefore, we further consider another crucial component of predator-prey relationships, that is, mortality rate including linear mortality and nonlinear mortality. Of course, linear mortality rate is intensively used by researchers, see [23, 24, 28–30, 36, 44, 48]. But nonlinear mortality rates, such as quadratic mortality, have also been used and can lead to richer dynamics, see [38, 42, 45, 49] and so on. Besides, in 1994, Cavani and Farkasin [8] introduced a new nonlinear mortality rates: hyperbolic mortality, with which predator-prey models have been studied by some scholars, see [8, 9, 42, 49]. So, in [31], we have proposed a spatial model with herd behavior and hyperbolic mortality as follows:

$$\begin{cases} \frac{\partial u}{\partial t} - d_1 \Delta u = u(1-u) - \frac{\sqrt{uv}}{1+\alpha\sqrt{u}}, & x \in (0, \pi), t > 0, \\ \frac{\partial v}{\partial t} - d_2 \Delta v = \beta v(-\frac{\gamma + \delta v}{1+v} + \frac{\sqrt{u}}{1+\alpha\sqrt{u}}), & x \in (0, \pi), t > 0, \\ u_x(0, t) = v_x(0, t) = u_x(\pi, t) = v_x(\pi, t) = 0, & t > 0, \\ u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, & x \in (0, \pi), \end{cases} \quad (1.2)$$

where  $d_1$  and  $d_2$  are the positive diffusion constants for the prey and predator, respectively. The hyperbolic mortality  $\frac{\gamma+\delta v}{1+v}$  of predators in absence of prey depends on the quantity of predator,  $\gamma$  is the mortality at low density, and  $\delta$  is the maximal mortality with the natural assumption  $0 < \gamma < \delta$ , see [8] in detail. In [31], we have investigated the Hopf bifurcation, steady state bifurcation and Turing instability of model (1.2) by Faria normal form and center manifold theory.

In fact, the reproduction of predator after consuming the prey is not instantaneous, but is mediated by some time lag required for gestation. Thus, the effects of delay on population dynamics have been widely investigated, and are believed to be one major reason accounting for the nonlinear scenarios in population dynamics [22]. Meanwhile, many researchers have also concentrated on the reaction diffusion equations with delay and have obtained some interesting results such as delay-induced Hopf bifurcation [20, 25–27, 39–41, 46, 47, 50–55], traveling wave solutions [4, 15, 32], stability and global attractivity [17, 43, 56]. Based on the above analysis, in this paper, we shall devote our attention to the following delayed diffusive predator-prey model with herd behavior and hyperbolic mortality

$$\begin{cases} \frac{\partial u}{\partial t} - d_1 \Delta u = u(1-u) - \frac{\sqrt{uv}}{1+\alpha\sqrt{u}}, \\ \frac{\partial v}{\partial t} - d_2 \Delta v = \beta v \left( -\frac{\gamma+\delta v}{1+v} + \frac{\sqrt{u_t}}{1+\alpha\sqrt{u_t}} \right), \\ u_x(0, t) = u_x(\pi, t) = v_x(0, t) = v_x(\pi, t) = 0, \quad t \geq 0, \\ u(x, t) = \phi(x, t) \geq 0, \quad v(x, t) = \psi(x, t) \geq 0, \quad (x, t) \in [0, \pi] \times [-\tau, 0], \end{cases} \quad (1.3)$$

where the delay item  $u_t = u(x, t - \tau)$  shows that the reproduction of predator after consuming the prey is not instantaneous, but mediated by some constant time lag  $\tau$  for gestation.

However, to the best of our knowledge, there are no results on the Hopf bifurcations of the above system (1.3). In this paper, we shall investigate the stability of the positive equilibrium, delay-induced Hopf bifurcation and the properties of Hopf bifurcation such as the direction of the bifurcation and stability of the bifurcating periodic solutions. The rest of this paper is organized as follows. In Section 2, the stability of the positive equilibrium and the existence of Hopf bifurcations are investigated by analyzing the characteristic equations. In Section 3, the results of determining the direction and stability of the bifurcating periodic solutions are obtained by Faria normal form and center manifold theory. In Section 4, we illustrate our results with numerical simulations, which support and extend the theoretical results. The paper ends with a conclusion.

## 2. Stability of positive equilibrium and existence of Hopf bifurcations

In this section, we consider the stability of the positive equilibrium for system (1.3) and existence of Hopf bifurcations by analyzing the distribution of eigenvalues in corresponding linear system of system (1.3), which can be induced by delay.

It is easy to check that system (1.3) has two boundary equilibria  $(0, 0)$  and  $(1, 0)$ . In [31], we have proved that system (1.3) has a unique positive equilibrium  $(u^*, v^*)$ .

For convenience of readers, we list the result of a unique positive equilibrium as follows:

**Lemma 2.1.** *Assume that  $0 < \gamma < \delta < 1$  and  $\frac{1}{\delta} - 1 < \alpha < \min\{\frac{1}{\delta} - \frac{1}{3}, \frac{1}{\gamma} - 1\}$  hold. Then system (1.3) has a unique positive equilibrium  $(u^*, v^*)$ .*

In the following, we only study the effect of delay on system (1.3). For simplification of notations, we always use  $u(t)$  for  $u(x, t)$ ,  $v(t)$  for  $v(x, t)$ ,  $u(t - \tau)$  for  $u(x, t - \tau)$ , and  $v(t - \tau)$  for  $v(x, t - \tau)$ . Let

$$f(u, v) = u(1 - u) - \frac{\sqrt{uv}}{1 + \alpha\sqrt{u}}, \quad g(v, w) = \beta v \left( -\frac{\gamma + \delta v}{1 + v} + \frac{\sqrt{w}}{1 + \alpha\sqrt{w}} \right). \quad (2.1)$$

Linearizing system (1.3) at positive equilibrium  $(u^*, v^*)$ , then we have that

$$\begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix} = D\Delta \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + A_0 \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + A_1 \begin{pmatrix} u(t - \tau) \\ v(t - \tau) \end{pmatrix}, \quad (2.2)$$

with

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad A_0 = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22}\beta \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ a_{21}\beta & 0 \end{pmatrix},$$

where

$$\begin{aligned} a_{11} &= \frac{\partial f^{(1)}}{\partial u}(u^*, v^*) = 1 - 2u^* - \frac{v^*}{2\sqrt{u^*}(1 + \alpha\sqrt{u^*})^2}, \\ a_{12} &= \frac{\partial f^{(1)}}{\partial v}(u^*, v^*) = -\frac{\sqrt{u^*}}{1 + \alpha\sqrt{u^*}}, \quad a_{21} = \frac{\partial f^{(2)}}{\partial u}(u^*, v^*) = \frac{v^*}{2\sqrt{u^*}(1 + \alpha\sqrt{u^*})^2}, \\ a_{22} &= \frac{\partial f^{(2)}}{\partial v}(u^*, v^*) = -\frac{v^*(\delta - \gamma)}{(1 + v^*)^2}. \end{aligned} \quad (2.3)$$

Thus, we can write out the characteristic equation of (2.2) as follows:

$$\det(\lambda I - M_k - A_0 - A_1 e^{-\lambda\tau}) = 0, \quad (2.4)$$

where  $I$  is the  $2 \times 2$  identity matrix and  $M_k = -k^2 \text{diag}\{d_1, d_2\}$ ,  $k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ . It follows from (2.4) that the characteristic equations for the positive constant equilibrium  $(u^*, v^*)$  are the following sequence of quadratic transcendental equations

$$\begin{aligned} \lambda^2 - [a_{11} + a_{22}\beta - (d_1 + d_2)k^2]\lambda + d_1 d_2 k^4 \\ - (a_{11}d_2 + a_{22}\beta d_1)k^2 + (a_{11}a_{22} - a_{12}a_{21}e^{-\lambda\tau})\beta = 0. \end{aligned} \quad (2.5)$$

When  $\tau = 0$ , the characteristic equation (2.5) becomes the following sequence of quadratic polynomial equations

$$\lambda^2 - T_k \lambda + D_k = 0, \quad (2.6)$$

where

$$\begin{aligned} T_k &= a_{11} + a_{22}\beta - (d_1 + d_2)k^2, \\ D_k &= d_1d_2k^4 - (a_{11}d_2 + a_{22}\beta d_1)k^2 + (a_{11}a_{22} - a_{12}a_{21})\beta. \end{aligned} \quad (2.7)$$

Now, we begin to discuss that the Hopf bifurcation is induced by delay when  $\tau \neq 0$ . For this aim, we always assume that positive equilibrium  $(u^*, v^*)$  of system (1.3) is locally asymptotically stable when  $\tau = 0$ , which implies  $T_k < 0$ ,  $D_k > 0$  for any  $k \in \mathbb{N}_0$ . Assume that  $i\omega$  ( $\omega > 0$ ) be a root of the characteristic equation (2.5), then  $\omega$  satisfies the following equation:

$$\begin{aligned} -\omega^2 - [a_{11} + a_{22}\beta - (d_1 + d_2)k^2]i\omega + d_1d_2k^4 \\ - (a_{11}d_2 + a_{22}\beta d_1)k^2 + (a_{11}a_{22} - a_{12}a_{21}e^{-i\omega\tau})\beta = 0. \end{aligned} \quad (2.8)$$

Separating the real and imaginary parts of Eq. (2.8) leads to

$$\begin{cases} -\omega^2 + d_1d_2k^4 - (a_{11}d_2 + a_{22}\beta d_1)k^2 + a_{11}a_{22}\beta - a_{12}a_{21}\beta \cos \omega\tau = 0, \\ (a_{11} + a_{22}\beta - (d_1 + d_2)k^2)\omega - a_{12}a_{21}\beta \sin \omega\tau = 0, \end{cases} \quad (2.9)$$

which implies that

$$\omega^4 + P_k\omega^2 + Q_k = 0, \quad (2.10)$$

where

$$\begin{aligned} P_k &= [d_1k^2 - a_{11}]^2 + [d_2k^2 - a_{22}\beta]^2, \\ Q_k &= D_k[d_1d_2k^4 - (a_{11}d_2 + a_{22}\beta d_1)k^2 + (a_{11}a_{22} + a_{12}a_{21})\beta]. \end{aligned} \quad (2.11)$$

Setting

$$R_k = d_1d_2k^4 - (a_{11}d_2 + a_{22}\beta d_1)k^2 + (a_{11}a_{22} + a_{12}a_{21})\beta, \quad (2.12)$$

then the sign of  $Q_k$  coincides with that of  $R_k$  since  $D_k > 0$ .

To obtain the values of  $k$ , we need the following condition.

(H)  $a_{11}a_{22} + a_{12}a_{21} < 0$ .

Notice that  $R_k$  is a quadratic polynomial with respect to  $k^2$  and  $R_0 < 0$ . Thus, from (2.12), we can conclude that there exists  $N_1 \in \mathbb{N}_0$  such that

$$R_k < 0, \text{ for } 0 \leq k \leq N_1, \text{ and } R_k > 0 \text{ for } k \geq N_1 + 1, k \in \mathbb{N}_0. \quad (2.13)$$

From  $P_k > 0$  and (2.13), we can conclude that for each  $k \in \{0, 1, 2, \dots, N_1\}$ , Eq. (2.10) has only one positive real root  $\omega_k$ , where

$$\omega_k = \sqrt{\frac{-P_k + \sqrt{P_k^2 - 4Q_k}}{2}}, \quad (2.14)$$

but has no positive real roots for  $k \geq N_1 + 1$ ,  $k \in \mathbb{N}_0$ .

According to the above analysis, we have the following results:

**Lemma 2.2.** *Assume that the condition of lemma 2.1 and (H) hold.  $J_k < 0$ ,  $D_k > 0$  for any  $k \in \mathbb{N}_0$ . Then (2.5) has a pair of purely imaginary roots  $i\omega_k$  for each  $k \in \{0, 1, 2, \dots, N_1\}$  and (2.5) has no purely imaginary roots for  $k \geq N_1 + 1$ .*

By (2.9), we can obtain

$$\tau_{kj} = \tau_{k0} + \frac{2\pi j}{\omega_k} \quad \text{and} \quad \tau_{k0} = \frac{1}{\omega_k} \arccos \frac{\omega_k^2 - D_k - a_{12}a_{21}\beta}{-a_{12}a_{21}\beta} \quad (2.15)$$

for  $k \in \{0, 1, 2, \dots, N_1\}$ .

Clearly,

$$\tau_{k0} = \min_{j \in \mathbb{N}_0} \{\tau_{kj}\}, \quad k \in \{0, 1, 2, \dots, N_1\}. \quad (2.16)$$

Let  $\lambda(\tau) = \alpha(\tau) + i\beta(\tau)$  be the roots of Eq. (2.5) near  $\tau = \tau_{kj}$  satisfying  $\alpha(\tau_{kj}) = 0$ ,  $\beta(\tau_{kj}) = \omega_k$ . Then we have the following transversality condition.

**Lemma 2.3.** *For  $k \in \{0, 1, 2, \dots, N_1\}$ , and  $j \in \mathbb{N}_0$ ,  $\frac{d\operatorname{Re}(\lambda)}{d\tau}|_{\tau=\tau_{kj}} > 0$ .*

**Proof.** Differentiating two sides of Eq. (2.5) on  $\tau$ , we get

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(2\lambda - T_k)e^{\lambda\tau}}{-a_{12}a_{21}\beta\lambda} - \frac{\tau}{\lambda}.$$

Thus, by (2.9) and (2.11), we have

$$\begin{aligned} \operatorname{Re}\left(\left(\frac{d\lambda}{d\tau}\right)^{-1}\right)|_{\tau=\tau_{kj}} &= \operatorname{Re}\left(\frac{(2\lambda - T_k)e^{\lambda\tau}}{-a_{12}a_{21}\beta\lambda} - \frac{\tau}{\lambda}\right)|_{\tau=\tau_{kj}} \\ &= \operatorname{Re}\left(\frac{(2i\omega_k - T_k)e^{i\omega_k\tau_{kj}}}{-a_{12}a_{21}\beta i\omega_k} - \frac{\tau_{kj}}{i\omega_k}\right) \\ &= \frac{T_k a_{12} a_{21} \beta \sin \omega_k \tau_{kj} - 2\omega_k a_{12} a_{21} \beta \cos \omega_k \tau_{kj}}{a_{12}^2 a_{21}^2 \beta^2 \omega_k} \\ &= \frac{2\omega_k^2 + P_k}{a_{12}^2 a_{21}^2 \beta^2} > 0. \end{aligned}$$

This completes the proof.  $\square$

According to Lemmas 2.2 and 2.3 and the qualitative theory of partial functional differential equations [35], we can obtain the following results on the stability and Hopf bifurcation.

**Theorem 2.1.** *Assume that the condition of lemma 2.1 and (H) hold.  $J_k < 0$ ,  $D_k > 0$  for any  $k \in \mathbb{N}_0$ ,  $\omega_k$  and  $\tau_{kj}$  is defined by (2.14) and (2.15), respectively. Denote the minimum of the critical values of delay by  $\tau^* = \min_{k \in \{0, 1, \dots, N_1\}} \tau_{k0}$ .*

(i) *The positive equilibrium  $(u^*, v^*)$  of system (1.3) is asymptotically stable for  $\tau \in [0, \tau^*)$  and unstable for  $(\tau^*, +\infty)$ ;*

(ii) *System (1.3) undergoes Hopf bifurcations near the positive equilibrium  $(u^*, v^*)$  at  $\tau_{kj}$  for  $k \in \{0, 1, 2, \dots, N_1\}$  and  $j \in \mathbb{N}_0$ . If  $k = 0$ , the bifurcating periodic solutions are all spatially homogeneous. Otherwise, these bifurcating periodic solutions are spatially inhomogeneous.*

### 3. Direction and stability of Hopf bifurcation

From Theorem 2.1, we can know that system (1.3) undergoes Hopf bifurcations near the equilibrium  $(u^*, v^*)$  at  $\tau_{kj}$ ,  $k \in \{0, 1, 2, \dots, N_1\}$  and  $j \in \mathbb{N}_0$ . In this section, we continue to investigate the stability, the direction and the period of bifurcating periodic solutions by using the normal form theory and the center manifold theorem of partial functional differential equation presented in [13, 14]. Since the methods used are standard, we omit the detailed process and only give the main results. The readers can see [13, 14, 27] for more details on the derivation process. Without loss of generality, denote any one of these critical values  $\tau_{kj}$ ,  $k \in \{0, 1, 2, \dots, N_1\}$  and  $j \in \mathbb{N}_0$  by  $\tau^*$  at which Eq. (2.5) has a pair of simply purely imaginary roots  $\pm i\omega_k$  denoted by  $\pm i\omega^*$ .

Let  $\tilde{u}(\cdot, t) = u(\cdot, \tau t) - u^*$ ,  $\tilde{v}(\cdot, t) = v(\cdot, \tau t) - v^*$  and  $\tilde{U}(t) = (\tilde{u}(\cdot, t), \tilde{v}(\cdot, t))$ , then dropping the tildes for simplification of notation, system (1.3) can be written as the equation in the space  $\mathcal{C} = C([-1, 0], X)$ ,  $X = \{(u, v) \in W^{2,2}(0, \pi) \mid \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \text{ at } x = 0, \pi\}$ ,

$$\frac{dU(t)}{dt} = \tau D\Delta U(t) + L(\tau)(U_t) + f(U_t, \tau), \text{ for } \varphi = (\varphi_1, \varphi_2)^T \in \mathcal{C}, \quad (3.1)$$

where  $L(\tau)(\cdot) : \mathcal{C} \rightarrow X$ , and  $f : \mathcal{C} \times \mathbb{R} \rightarrow X$  are given, respectively, by

$$L(\tau)(\varphi) = \tau \begin{pmatrix} a_{11}\varphi_1(0) + a_{12}\varphi_2(0) \\ a_{21}\varphi_1(-1) + a_{22}\varphi_2(0) \end{pmatrix}, \quad (3.2)$$

$$f(\varphi, \tau) = \tau \begin{pmatrix} f^{(1)}(\tau) \\ f^{(2)}(\tau) \end{pmatrix} = \tau \begin{pmatrix} \sum_{i+j \geq 2} \frac{1}{i!j!} f_{ij}^{(1)} \varphi_1^i(0) \varphi_2^j(0) \\ \sum_{i+j \geq 2} \frac{1}{i!j!} f_{ij}^{(2)} \varphi_1^i(-1) \varphi_2^j(0) \end{pmatrix},$$

where

$$f^{(1)}(u, v) = (u + u^*)(1 - u - u^*) - \frac{\sqrt{u + u^*}(v + v^*)}{1 + \alpha\sqrt{u + u^*}},$$

$$f^{(2)}(w, v) = \beta(v + v^*) \left( -\frac{\gamma + \delta(v + v^*)}{1 + v + v^*} + \frac{\sqrt{w + u^*}}{1 + \alpha\sqrt{w + u^*}} \right),$$

$$f_{ij}^{(1)} = \frac{\partial^{i+j} f^{(1)}}{\partial u^i \partial v^j}(u^*, v^*), \quad f_{ij}^{(2)} = \frac{\partial^{i+j} f^{(2)}}{\partial w^i \partial v^j}(u^*, v^*).$$

Setting  $\tau = \tau^* + \alpha$ ,  $\alpha \in \mathbb{R}$ , and consider only the case  $\Lambda_0 = \{-i\tau^*\omega^*, i\tau^*\omega^*\}$  is the set of eigenvalues on the imaginary axis of the infinitesimal generator associated with the flow of

$$\frac{dU(t)}{dt} = \tau^* D\Delta U(t) + L(\tau^*)(U_t), \quad (3.3)$$

Thus, Eq. (3.1) can be written as

$$\frac{dU(t)}{dt} = \tau D\Delta U(t) + L(\tau)(U_t) + F(U_t, \alpha), \quad (3.4)$$

where  $F(\varphi, \alpha) = \alpha D\Delta\varphi(0) + L(\alpha)(\varphi) + f(\varphi, \tau^* + \alpha)$  for  $\varphi \in \mathcal{C}$ . The eigenvalues of  $\tau^* D\Delta$  on  $X$  are  $\mu_k^i = -d_i \tau^* k^2$ ,  $i = 1, 2$ ,  $k \in \mathbb{N}_0$ , with corresponding normalized eigenfunctions  $\beta_k^j$ , where

$$\beta_k^1 = \begin{pmatrix} \gamma_k(x) \\ 0 \end{pmatrix}, \quad \beta_k^2 = \begin{pmatrix} 0 \\ \gamma_k(x) \end{pmatrix}, \quad \gamma_k(x) = \frac{\cos kx}{\|\cos kx\|_{2,2}}, \quad k \in \mathbb{N}_0. \quad (3.5)$$

Let  $\mathcal{B}_k = \text{span}\{\langle v(\cdot), \beta_k^i \rangle \mid v \in \mathcal{C}, i = 1, 2\}$ . Assume that  $z_t(\theta) \in C = C([-1, 0], \mathbb{R}^2)$  and

$$z_t^T(\theta) \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix} \in \mathcal{B}_k, \quad (3.6)$$

then linear PDE (3.3) restricted on  $\mathcal{B}_k$  is equivalent to the FDE on  $C = C([-1, 0], \mathbb{R}^2)$

$$\dot{z}(t) = \begin{pmatrix} \mu_k^1 & 0 \\ 0 & \mu_k^2 \end{pmatrix} z(t) + L(\tau^*)(z_t) \quad (3.7)$$

with the characteristic equation given by (2.5).

Suppose that there exists a  $k \in \mathbb{N}_0$  such that (2.5), for fixed  $k$ , has a pair of purely imaginary roots  $\pm i\omega^*$  and all other roots of (2.5) have negative real parts when  $\tau = \tau^*$ . Define  $\eta(\theta) \in \text{BV}([-1, 0], \mathbb{R})$  such that

$$\mu_k \psi(0) + L(\tau^*)\psi = \int_{-1}^0 d\eta(\theta)\psi(\theta) \quad (3.8)$$

and the adjoint bilinear form on  $C^* \times C$ ,  $C^* = C([0, 1], \mathbb{R}^{2*})$ , as follows

$$\langle \psi(s), \phi(\theta) \rangle = \psi(0)\phi(0) - \int_{-1}^0 \int_0^\theta \psi(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi \quad \text{for } \psi \in C^*, \phi \in C.$$

Then, for Eq. (3.7) with fixed  $k$ , the dual bases  $\Phi_k$  and  $\Psi_k$  for its eigenspace  $P$  and its dual space  $P^*$  are, respectively, given by

$$\Phi_k = (pe^{i\omega^* \tau^* \theta}, \bar{p}e^{-i\omega^* \tau^* \theta}), \quad \text{and} \quad \Psi_k = \text{col}(q^T e^{-i\omega^* \tau^* s}, \bar{q}^T e^{i\omega^* \tau^* s})$$

such that  $\langle \Phi_k, \Psi_k \rangle = I_2$ , where  $I_2$  is a  $2 \times 2$  matrix and

$$p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{i\omega^* + d_1 k^2 - a_{11}}{a_{12}} \end{pmatrix}, \quad q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \rho \begin{pmatrix} 1 \\ \frac{i\omega^* + d_1 k^2 - a_{11}}{a_{21}} e^{i\omega^* \tau^*} \end{pmatrix}, \quad (3.9)$$

with

$$\rho = \left( 1 + \tau^* (i\omega^* + d_1 k^2 - a_{11}) + \frac{(\tau^* a_{22} + e^{i\omega^* \tau^*})(i\omega^* + d_1 k^2 - a_{11})^2}{a_{12} a_{21}} \right)^{-1}.$$

Following the standard procedure in [13], especially [27], we can obtain the following normal form on the center manifold

$$\dot{z} = Bz + \begin{pmatrix} A_{k1} z_1 \alpha \\ \bar{A}_{k1} z_2 \alpha \end{pmatrix} + \begin{pmatrix} A_{k2} z_1^2 z_2 \\ \bar{A}_{k2} z_1 z_2^2 \end{pmatrix} + O(|z|\alpha^2 + |z|^4), \quad (3.10)$$

where

$$A_{k1} = -k^2(d_1 q_1 p_1 + d_2 q_2 p_2) + i\omega^* q^T p, \quad (3.11)$$

and

$$A_{k2} = \frac{i}{2\omega^* \tau^*} \left( B_{k20} B_{k11} - 2|B_{k11}|^2 - \frac{1}{3}|B_{k02}|^2 \right) + \frac{1}{2}(B_{k21} + D_{k21}), \quad (3.12)$$

with

$$B_{k20} = \begin{cases} \frac{\tau^*}{\sqrt{\pi}}(C_1 q_1 + C_2 q_2), & k = 0, \\ 0, & k \neq 0, \end{cases} \quad B_{k11} = \begin{cases} \frac{\tau^*}{\sqrt{\pi}}(C_3 q_1 + C_4 q_2), & k = 0, \\ 0, & k \neq 0, \end{cases}$$

$$B_{k02} = \begin{cases} \frac{\tau^*}{\sqrt{\pi}}(\bar{C}_1 q_1 + \bar{C}_2 q_2), & k = 0, \\ 0, & k \neq 0, \end{cases} \quad B_{k21} = \begin{cases} \frac{\tau^*}{\pi} C_5, & k = 0, \\ \frac{3\tau^*}{2\pi} C_5, & k \neq 0, \end{cases}$$

where

$$C_1 = f_{20}^{(1)} p_1^2 + 2f_{11}^{(1)} p_1 p_2 + f_{02}^{(1)} p_2^2, \quad C_3 = f_{20}^{(1)} |p_1|^2 + 2f_{11}^{(1)} \operatorname{Re}\{p_1 \bar{p}_2\} + f_{02}^{(1)} |p_2|^2,$$

$$C_2 = f_{20}^{(2)} p_1^2 e^{-2i\omega^* \tau^*} + 2f_{11}^{(2)} p_1 p_2 e^{-i\omega^* \tau^*}, \quad C_4 = f_{20}^{(2)} |p_1|^2 + 2f_{11}^{(2)} \operatorname{Re}\{p_1 \bar{p}_2 e^{-i\omega^* \tau^*}\},$$

$$C_5 = q_1 (f_{30}^{(1)} p_1 |p_1|^2 + f_{03}^{(1)} p_2 |p_2|^2 + f_{21}^{(1)} (p_1^2 |\bar{p}_2| + 2|p_1|^2 p_2) + f_{12}^{(1)} (|\bar{p}_1| p_2^2 + 2p_1 |p_2|^2)) + q_2 (f_{30}^{(2)} p_1 |p_1|^2 e^{-i\omega^* \tau^*} + f_{21}^{(2)} (p_1^2 \bar{p}_2 e^{-2i\omega^* \tau^*} + 2|p_1|^2 p_2)),$$

and

$$D_{k21} = \begin{cases} E_0, & k = 0, \\ E_0 + \frac{\sqrt{2}}{2} E_{2k}, & k \neq 0, \end{cases}$$

where for  $j = 0, 2k$ ,

$$E_j = \frac{2\tau^*}{\sqrt{\pi}} q^T \begin{pmatrix} F_1 h_{j11}^{(1)}(0) + F_2 h_{j11}^{(2)}(0) + \bar{F}_1 h_{j20}^{(1)}(0) + \bar{F}_2 h_{j20}^{(2)}(0) \\ F_3 h_{j11}^{(1)}(-1) + \bar{F}_3 h_{j20}^{(1)}(-1) + F_4 h_{j11}^{(2)}(0) + \bar{F}_4 h_{j20}^{(2)}(0) \end{pmatrix},$$

with

$$F_1 = f_{20}^{(1)} p_1 + f_{11}^{(1)} p_2, \quad F_2 = f_{11}^{(1)} p_1 + f_{02}^{(1)} p_2,$$

$$F_3 = f_{20}^{(2)} p_1 e^{-i\omega^* \tau^*} + f_{11}^{(2)} p_2, \quad F_4 = f_{11}^{(2)} p_1 e^{-i\omega^* \tau^*},$$

and

$$h_{k20}(\theta) = -\frac{1}{i\omega^*\tau^*} \left( B_{k20}e^{i\omega^*\tau^*\theta} p + \frac{1}{3}\bar{B}_{k02}e^{-i\omega^*\tau^*\theta}\bar{p} \right) + e^{2i\omega^*\tau^*\theta} G_{k1},$$

$$h_{k11}(\theta) = \frac{2}{i\omega^*\tau^*} \left( B_{k11}e^{i\omega^*\tau^*\theta} p - \bar{B}_{k11}e^{-i\omega^*\tau^*\theta}\bar{p} \right) + G_{k2},$$

where

$$G_{k1} = \left( \begin{array}{c} \frac{\sigma_{kj}(C_1(2i\omega^* - a_{22}e^{-2i\omega^*\tau^*}) + C_2a_{12})}{(2i\omega^* - a_{11})(2i\omega^* - a_{22}e^{-2i\omega^*\tau^*}) - a_{12}a_{21}e^{-2i\omega^*\tau^*}} \\ \frac{\sigma_{kj}(C_1a_{21}e^{-2i\omega^*\tau^*} + C_2(2i\omega^* - a_{11}))}{(2i\omega^* - a_{11})(2i\omega^* - a_{22}e^{-2i\omega^*\tau^*}) - a_{12}a_{21}e^{-2i\omega^*\tau^*}} \end{array} \right),$$

$$G_{k2} = \left( \begin{array}{c} \frac{2\sigma_{kj}(C_4a_{12} - C_3a_{22})}{a_{11}a_{22} - a_{12}a_{21}} \\ \frac{2\sigma_{kj}(C_3a_{21} - C_4a_{11})}{a_{11}a_{22} - a_{12}a_{21}} \end{array} \right),$$

and

$$\sigma_{kj} = \begin{cases} \frac{1}{\sqrt{\pi}}, & j = k = 0, \\ \frac{1}{\sqrt{\pi}}, & j = 0, k \neq 0, \\ \frac{1}{\sqrt{2\pi}}, & j = 2k \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Through the change of variables  $z_1 = w_1 - iw_2$ ,  $z_2 = w_1 + iw_2$  and  $w_1 = \rho \cos \phi$ ,  $w_2 = \rho \sin \phi$ , then the normal form (3.10) becomes the following polar coordinate system

$$\begin{cases} \dot{\rho} = \kappa_{k1}\alpha\rho + \kappa_{k2}\rho^3 + O(\alpha^2\rho + |(\rho, \alpha)|^4), \\ \dot{\phi} = -\omega^*\tau^* + O(|(\rho, \alpha)|), \end{cases}$$

where  $\kappa_{k1} = \text{Re}A_{k1}$ ,  $\kappa_{k2} = \text{Re}A_{k2}$ . Thus, from [10], we can know that the sign of  $\kappa_{k1}\kappa_{k2}$  determines the direction of the bifurcation and the sign of  $\kappa_{k2}$  determines the stability of the nontrivial periodic orbits and have the following results.

(i) When  $\kappa_{k1}\kappa_{k2} < 0$ , the Hopf bifurcation that system (1.3) undergoes at the critical value  $\tau = \tau^*$  is a supercritical bifurcation. Moreover, if  $\kappa_{k2} < 0$ , the bifurcating periodic solution is stable; if  $\kappa_{k2} > 0$ , the bifurcating periodic solution is unstable.

(ii) When  $\kappa_{k1}\kappa_{k2} > 0$ , the Hopf bifurcation that system (1.3) undergoes at the critical value  $\tau = \tau^*$  is a subcritical bifurcation. Moreover, if  $\kappa_{k2} < 0$ , the bifurcating periodic solution is stable; if  $\kappa_{k2} > 0$ , the bifurcating periodic solution is unstable.

## 4. Numerical calculations and simulations

In this section, we present some numerical simulations that support and supplement the analytic results given in the previous sections. Taking  $\alpha = 1.5$ ,  $\gamma = 0.2$ ,  $\delta = 0.5$

and  $d_1 = 0.01$ , together with Lemma 2.1, we can confirm that (1.3) has a uniqueness positive equilibrium  $(u^*, v^*) = (0.4153, 0.7410)$ . Further, we have that  $a_{11} = 0.0207$ ,  $a_{12} = -0.3277$ ,  $a_{21} = 0.1487$  and  $a_{22} = -0.0733$  at  $(u^*, v^*) = (0.4153, 0.7410)$ .

Choosing  $\beta = 0.35$ ,  $d_2 = 1$  together with a direct computation, we have  $T_k < 0$ ,  $D_k > 0$  for any  $k \in \mathbb{N}_0$ , which implies that the positive equilibrium  $(u^*, v^*)$  of model (1.3) without delay is asymptotically stable. Further, from Theorem 2.1, we can know that the positive equilibrium  $(u^*, v^*)$  of model (1.3) is asymptotically stable for  $\tau \in (0, \tau^*)$  and unstable for  $\tau \in (\tau^*, +\infty)$ , where  $\tau^*$  is a critical value of delay  $\tau$ . By (2.11), we have that

$$P_k = 1.0001k^4 + 0.0509k^2 + 0.0011 > 0, \quad \text{for all } k \in \mathbb{N}_0$$

and

$$Q_k = (0.01k^4 - 0.0204k^2 + 0.0165)(0.01k^4 - 0.0204k^2 - 0.0176) \begin{cases} < 0, & k = 0, 1, \\ > 0, & k \geq 2. \end{cases}$$

From this, we can know that Hopf bifurcations induced by delay occur for  $k = 0$  and  $k = 1$ . By (2.15), we can obtain the critical values of delay as follows

$$\tau_{00} \approx 0.2906, \quad \tau_{01} \approx 49.1871, \dots; \quad \tau_{10} \approx 67.5179, \quad \tau_{11} \approx 561.2084, \dots$$

When  $k = 0$ , from Theorem 2.1, we can conclude that the positive equilibrium  $(u^*, v^*)$  is asymptotically stable for  $\tau < \tau_{00}$  and unstable for  $\tau > \tau_{00}$ . So, system (1.3) undergoes Hopf bifurcation near the positive equilibrium  $(u^*, v^*)$  when the delay  $\tau$  increasingly crosses through the critical value  $\tau_{00}$ . By a direct computation, together with (2.14), (3.2) and (3.9), we have that  $\omega^* = 0.1285$ ,  $p_1 = 1$ ,  $p_2 = 0.0632 + 0.3921i$ ,  $q_1 = 0.7511 - 0.0895i$ ,  $q_2 = -0.0519 + 0.6601i$  and

$$f_{20} = \begin{pmatrix} -1.6451 \\ -0.1242 \end{pmatrix}, \quad f_{11} = \begin{pmatrix} -0.2006 \\ 0.0702 \end{pmatrix}, \quad f_{02} = \begin{pmatrix} 0 \\ -0.0398 \end{pmatrix},$$

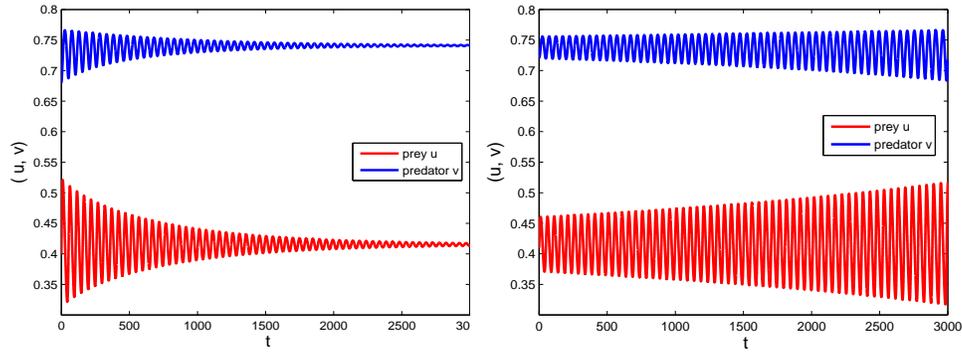
$$f_{30} = \begin{pmatrix} -1.5941 \\ 0.5579 \end{pmatrix}, \quad f_{12} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f_{21} = \begin{pmatrix} 0.4789 \\ -0.1676 \end{pmatrix}, \quad f_{03} = \begin{pmatrix} 0 \\ 0.0686 \end{pmatrix}.$$

Thus, from (3.10)-(3.12), we can get the following normal form truncated to the third-order term:

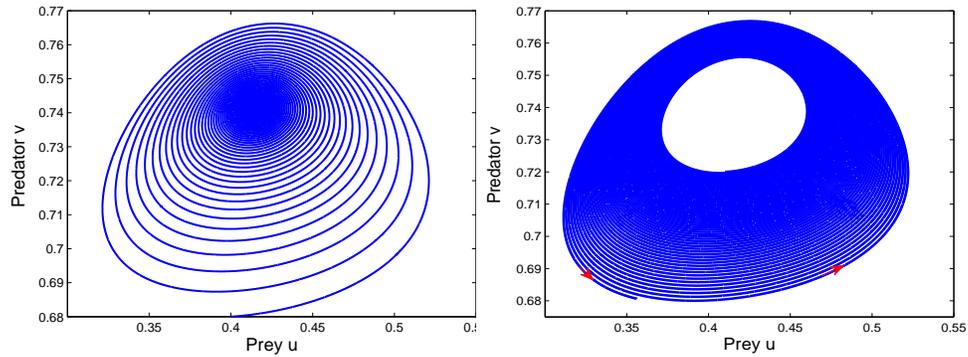
$$\dot{\rho} = 0.0035\alpha\rho + 0.0475\rho^3,$$

which implies  $\kappa_{01} = 0.0035 > 0$  and  $\kappa_{02} = 0.0475 > 0$ . Then, from the results in Section 3, we can conclude that Hopf bifurcation that system (1.3) undergoes near the positive equilibrium  $(u^*, v^*)$  when the delay  $\tau$  increasingly crosses through the critical value  $\tau_{00}$  is subcritical and the bifurcating periodic solution is unstable. That is, there exists a unstable spatially homogenous periodic orbit bifurcating from the positive equilibrium  $(u^*, v^*)$ . These facts for system (1.3) without diffusion are shown in Figs. 1 and 2, and for system (1.3) are shown in Figs. 3 and 4.

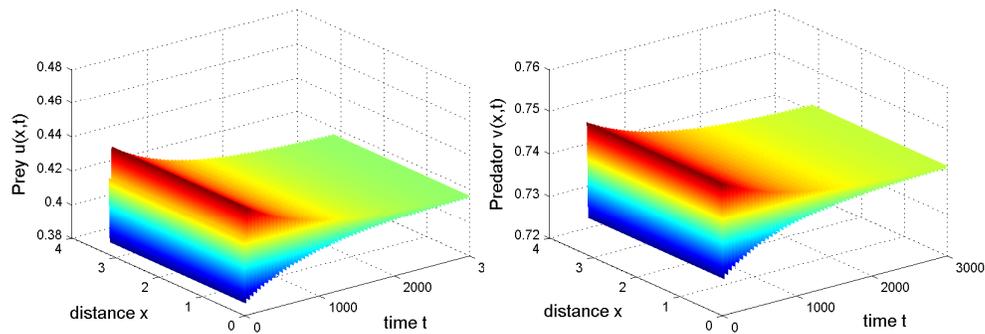
When  $k = 1$ , from Theorem 2.1, we can conclude that system (1.3) undergoes Hopf bifurcation near the positive equilibrium  $(u^*, v^*)$  when the delay  $\tau$  increasingly crosses through the critical value  $\tau_{10}$  and exits an unstable spatially inhomogenous periodic solution bifurcating from the positive equilibrium  $(u^*, v^*)$ .



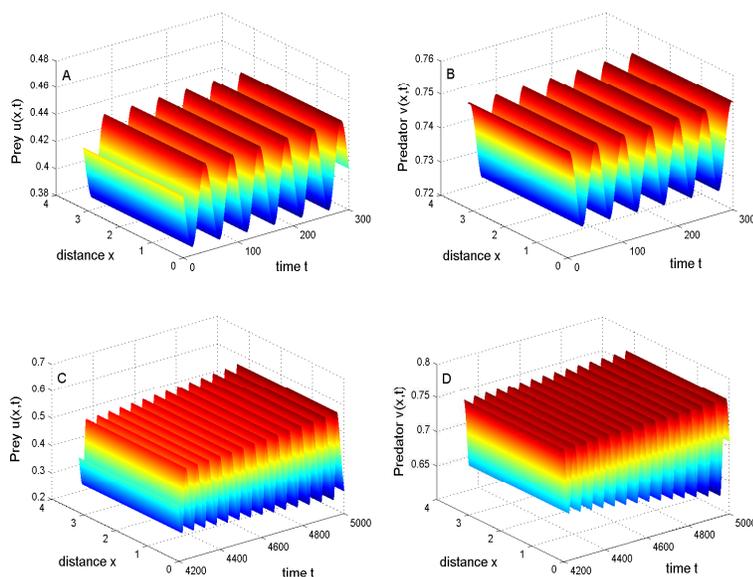
**Figure 1.** The trajectory portraits of system (1.3) without diffusion. Left:  $(u^*, v^*)$  is locally asymptotically stable when  $\beta = 0.15 < \tau_{00} = 0.2906$ , and the initial value is  $u_0 = 0.40, v_0 = 0.68$ ; right: the unstable periodic orbit bifurcates from the positive equilibrium  $(u^*, v^*)$  when  $\tau = 0.32 > \tau_{00}$ , and the initial value is  $u_0 = 0.41, v_0 = 0.72$ .



**Figure 2.** The phase portraits of system (1.3) without diffusion. Left: the positive equilibrium  $(u^*, v^*)$  is asymptotically stable when  $\tau = 0.15 < \tau_{00}$ , and the initial value is  $u_0 = 0.40, v_0 = 0.68$ ; right: there exists an unstable limit cycle bifurcating from the positive equilibrium  $(u^*, v^*)$  when  $\tau = 0.32 > \tau_{00}$ , and the initial value is  $u_0 = 0.41, v_0 = 0.72$ .



**Figure 3.** The positive equilibrium  $(u^*, v^*) = (0.4153, 0.7410)$  of system (1.3) is asymptotically stable when  $\tau = 0.15 < \tau_{00}$ . Here, we set parameter values as  $d_1 = 0.01, d_2 = 1, \alpha = 1.5, \gamma = 0.2, \delta = 0.5, \beta = 0.35$  and the initial conditions  $u(x, 0) = u^* + 0.01, v(x, 0) = v^* + 0.01$ .



**Figure 4.** The unstable spatial periodic solution bifurcating from the positive constant equilibrium when  $\tau = 0.32 > \tau_{00}$ . Here, we set parameter values as  $d_1 = 0.01$ ,  $d_2 = 1$ ,  $\alpha = 1.5$ ,  $\gamma = 0.2$ ,  $\delta = 0.5$ ,  $\beta = 0.35$ , and the initial value is  $u(x, 0) = u^* + 0.01$ ,  $v(x, 0) = v^* + 0.01$ . (A) and (B) are transient behaviours for  $u$  and  $v$ , respectively. (C) and (D) are long-term behaviours for  $u$  and  $v$ , respectively.

## 5. Conclusions and discussion

In this paper, we consider the effects of delay on dynamics of a diffusive predator-prey model with herd behavior and hyperbolic mortality under Neumann boundary conditions. Firstly, the stability of the positive equilibria and the existence of Hopf bifurcations induced by delay are investigated by analyzing the characteristic equations. Then, by the normal forms on the center manifold, the results determining the direction and stability of Hopf bifurcations are derived. Finally, choosing the parameter values:  $d_1 = 0.01$ ,  $d_2 = 1$ ,  $\alpha = 1.5$ ,  $\beta = 0.35$ ,  $\gamma = 0.2$  and  $\delta = 0.5$ , together with a direct computation, we obtain the critical value of delay for  $k = 0$  and  $k = 1$ , that is,  $\tau_{00} = 0.2906 (k = 0)$  and  $\tau_{10} = 67.5179 (k = 1)$ . When  $k = 0$ , from Theorem 2.1, we can conclude that the positive equilibrium  $(u^*, v^*)$  is asymptotically stable for  $\tau < \tau_{00}$  and unstable for  $\tau > \tau_{00}$ . So, system (1.3) undergoes Hopf bifurcation near the positive equilibrium  $(u^*, v^*)$  when the delay  $\tau$  increasingly crosses through the critical value  $\tau_{00}$ . By  $\kappa_{01} = 0.0035 > 0$  and  $\kappa_{02} = 0.0475 > 0$ , combining with the results in Section 3, we can know that the direction of Hopf bifurcation is subcritical and the bifurcating periodic solution is unstable. These facts are shown in Figs. 1-4. Here, we have to point out a fact, that is, our results in this paper are different from the ones in the known literatures [20, 25–28, 39–41, 46, 47, 50–55], and so on, in which the direction of Hopf bifurcation is supercritical and the bifurcating periodic solution is stable. Therefore, we think that our results in this paper are new. Meanwhile, when  $k = 1$ , we obtain the unstable spatially inhomogeneous periodic solution. Of course, we hope that our work could be instructive to study the population.

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