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PERIODIC SOLUTIONS FOR A TYPE OF NEUTRAL SYSTEM WITH VARIABLE PARAMETERS*

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Abstract In this paper, we firstly analyze some properties of the linear difference operator $[Ax](t) = x(t) - C(t)x(t-\tau)$, where $C(t)$ is a $n \times n$ matrix function, and then using Mawhin's continuation theorem, a first-order neutral functional differential system is studied. Some new results on the existence and stability of periodic solutions are obtained. The results are related to the deviating arguments τ and μ. Meanwhile, the approaches to estimate a *prior* bounds of periodic solutions are different from the corresponding ones of the known literature.

Keywords Periodic solution, coincidence degree, neutral system.

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1. Introduction

Neutral functional differential equations (NFDEs) are not only an extension of functional differential equations but also provide good models in many fields including Biology, Mechanics and Economics. In particular, qualitative analysis such as periodicity and stability of solutions of neutral functional differential equations has been studied extensively by many authors. We refer to [5, 6, 9–12, 14, 15, 18, 21, 25–27, 31–33, 38] for some recent work on the subject of periodicity and stability of neutral equations.

In [20], J. Hale gave a definition for NFDE of $D-$operator as follows:

$$\frac{dDx_t}{dt} = f(t, x_t), \tag{1.1}$$

where $x_t(\theta) = x(t+\theta),\ \theta \in [-\tau, 0], \tau > 0$ is a constant, $D : C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is linear,continuous, and atomic at zero, and $f \in (C([-\tau, 0], \mathbb{R}^n), \mathbb{R}^n)$. The difficulty lies in the fact that the message of how properties reflect general properties of the solution is far from clear. For example, in the definition of a solution $u(t)$ of (1.1), it is only required that $D(u_t)$ is continuously differentiable in t, but, generally, $u(t)$ may not be differentiable in t, which is essentially different from the

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corresponding case of retarded functional differential equations. In the foundation of theory of NFDE, J. Hale gave an important definition named stable difference operator D: The linear difference operator $D : C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$, $D(\varphi) = \varphi(0) - \int_{-\tau}^{0} \varphi(\theta) d\mu(\theta)$ called stable, if the zero solution of the difference equation $dy_t = 0, y_0 = \varphi \in \{C([-\tau, 0], \mathbb{R}^n) : D\varphi = 0\}$ is uniformly asymptotically stable. Under the condition that the operator D is stable, many researchers studied the problem of existence of periodic solutions for Eq. (1.1) by means of some fixed point theorems and topology degree theory, see [22, 23, 37, 39].

Now, we consider a type special NFDE. If $r > 0$, B is an $n \times n$ constant matrix, $D(\phi) = \phi(0) - B\phi(r)$, and $f : \Omega \to \mathbb{R}^n$, the pair (D, f) defines a NFDE:

$$\frac{d}{dt}(x(t) - Bx(t - r)) = f(t, x_t).$$

When $||B|| < 1$, the operator D is stable. On the basis of the stability of $D-$operator, one can study NFDEs by using the similar methods to retarded equations, see [7,8,24,34,42]. For a long time, the treatment of NFDEs used the papers of Hale [20] and Henry [19]. But when $D-$ operator is not stable, how can we study existence and stability of solutions to NFDEs, which is very important subject for the theory and applications to NFDEs. In 1995, under the non-resonance condition, Zhang [43] studied a kind of neutral differential equation and relieved the above stability restriction. Zhang gave some properties for the difference operator A and obtained the following results: Define the operator A on C_T

$$A : C_T \to C_T, [Ax](t) = x(t) - cx(t - \tau), \forall t \in \mathbb{R}, \tag{1.2}$$

where $C_T = \{x : x \in C(\mathbb{R}, \mathbb{R}), x(t + T) \equiv x(t)\}$, c is a constant. When $|c| \neq 1$, then A has a unique continuous bounded inverse A^{-1} satisfying

$$[A^{-1}f](t) = \begin{cases} \sum\limits_{j \geq 0} c^j f(t - j\tau), & \text{if } |c| < 1, \quad \forall f \in C_T, \\ -\sum\limits_{j \geq 1} c^{-j} f(t + j\tau), & \text{if } |c| > 1, \quad \forall f \in C_T. \end{cases}$$

After that, using the properties of A^{-1} Lu et. al [28] gave some inequalities for A:

(1) $||A^{-1}|| \leq \frac{1}{|1-|c||}$;

(2) $\int_0^T |[A^{-1}f](t)| dt \leq \frac{1}{|1-|c||} \int_0^T |f(t)| dt, \forall f \in C_T$;

(3) $\int_0^T |[A^{-1}f](t)|^2 dt \leq \frac{1}{|1-|c||} \int_0^T |f(t)|^2 dt, \forall f \in C_T$.

But, when the constant c in (1.2) is a variable $c(t)$, there are no corresponding results for the neutral operator A. In 2009, when c is a variable $c(t)$, we obtained the properties of the neutral operator A in [9]:

Lemma 1.1. *If $c(t)$ is continuous $T-$periodic function with $|c(t)| \neq 1$ for $t \in \mathbb{R}$, then operator A has continuous inverse A^{-1} on C_T, satisfying*

(1)

$$[A^{-1}f](t) = \begin{cases} f(t) + \sum\limits_{j=1}^{\infty} \prod\limits_{i=1}^{j} c(t - (i-1)\tau) f(t - j\tau), \ c_0 < 1, \ \forall f \in C_T, \\ -\frac{f(t+\tau)}{c(t+\tau)} - \sum\limits_{j=1}^{\infty} \prod\limits_{i=1}^{j+1} \frac{1}{c(t+i\tau)} f(t + j\tau + \tau), \ \sigma > 1, \ \forall f \in C_T, \end{cases}$$

(2)

$$||A^{-1}|| \le \begin{cases} \frac{1}{1-c_0}, \ c_0 < 1, \ \forall f \in C_T, \\ \frac{1}{\sigma-1}, \ \sigma > 1, \ \forall f \in C_T, \end{cases}$$

(3)

$$\int_0^T |[A^{-1}f](t)|dt \le \begin{cases} \frac{1}{1-c_0} \int_0^T |f(t)|dt, \ c_0 < 1, \ \forall f \in C_T, \\ \frac{1}{\sigma-1} \int_0^T |f(t)|dt, \ \sigma > 1, \ \forall f \in C_T, \end{cases}$$

where

$$c_0 = \max_{t\in[0,T]} |c(t)|, \quad \sigma = \min_{t\in[0,T]} |c(t)|.$$

Remark 1.1. We can improve the result (3) in Lemma 1.1. In fact, if $c_0 < 1$, for $p > 1$,

$$\begin{aligned} \int_0^T |x(t)|^p dt &= \int_0^T |x(t)|^{p-1}|x(t)|dt \\ &= \int_0^T |x(t)|^{p-1}|f(t) + \sum_{j=1}^{\infty}\prod_{i=1}^{j} c(t-(i-1)\tau)f(t-j\tau)|dt \\ &\le \int_0^T |x(t)|^{p-1}||f(t)|dt + \frac{c_0}{1-c_0}\int_0^T |x(t)|^{p-1}||f(t-j\tau)|dt \\ &\le \left(\int_0^T |x(t)|^p dt\right)^{(p-1)/p}\left(\int_0^T |f(t)|^p dt\right)^{1/p} \\ &+ \frac{c_0}{1-c_0}\left(\int_0^T |x(t)|^p dt\right)^{(p-1)/p}\left(\int_0^T |f(t-j\tau)|^p dt\right)^{1/p} \\ &\le \left(\int_0^T |x(t)|^p dt\right)^{(p-1)/p}\left(\int_0^T |f(t)|^p dt\right)^{1/p} \\ &+ \frac{c_0}{1-c_0}\left(\int_0^T |x(t)|^p dt\right)^{(p-1)/p}\left(\int_0^T |f(t)|^p dt\right)^{1/p} \\ &= \frac{1}{1-c_0}\left(\int_0^T |x(t)|^p dt\right)^{(p-1)/p}\left(\int_0^T |f(t)|^p dt\right)^{1/p}, \end{aligned}$$

which implies

$$\int_0^T |x(t)|^p dt \le \frac{1}{(1-c_0)^p}\int_0^T |f(t)|^p dt.$$

If $\sigma > 1$, for $p > 1$, similar to the above proof, we have

$$\int_0^T |x(t)|^p dt \le \frac{1}{(\sigma-1)^p}\int_0^T |f(t)|^p dt.$$

Hence result (3) in Lemma 1.1 can be generalized the following form:

$$\int_0^T |[A^{-1}f](t)|^p dt \le \begin{cases} \frac{1}{(1-c_0)^p} \int_0^T |f(t)|^p dt, \ c_0 < 1, \ \forall f \in C_T, \ p \ge 1, \\ \frac{1}{(\sigma-1)^p} \int_0^T |f(t)|^p dt, \ \sigma > 1, \ \forall f \in C_T, \ p \ge 1. \end{cases}$$

Using the results of [9], we have obtained some existence results of periodic solutions for first-order, second-order and $p-$Laplacian neutral equations with variable parameter, see [12–14]. In very recent paper [40], motivated by the work of [9], when the neutral operator A has multiple variable parameters as follows

$$A : C_T \to C_T, \quad [Ax](t) = x(t) - \sum_{i=1}^{n} c_i(t)x(t - \tau_i).$$

Wang, Lu and Cao obtained the following results for the operator A:

If $\sum_{i=1}^{n} c_i^0 < 1$, then A has continuous inverse A^{-1} on C_T with the following properties:

(1) $[A^{-1}f](t) = f(t)+\sum_{m=1}^{\infty} \sum_{i_1}^{n} \sum_{i_2}^{n} \cdots \Sigma_{i_m}^{n} \Pi_{j=1}^{m} C_{i_j}(t-\Sigma_{k=j+1}^{m} r_{i_k})-f(t-\Sigma_{s=1}^{m} r_{i_s})$,

(2) $||A^{-1}|| \le \frac{1}{1-\sum_{i=1}^{n} c_i^0}$,

(3) $\int_0^T |[A^{-1}f](t)|^p dt \le \frac{1}{(1-\sum_{i=1}^{n} c_i^0)^p} \int_0^T |f(t)|^p dt, \quad f \in C_T, \ p > 1$,

(4)$[Af'](t) = [Af]'(t) + \sum_{i=1}^{n} c_i'(t) f(t - r_i), \quad f \in C_T^1$.

Using the above results and Mawhin's continuation theorem, the authors obtained the existence results of periodic solutions for a kind of $p-$Laplacian neutral functional differential equation with multiple variable parameters.

On the other hand, neutral differential system is an important subject for NFDEs. In 2008, when the constant c of (1.2) is a $n \times n$ real symmetrical matrix C, Lu [31] gave the following results:

Lemma 1.2. *Suppose that $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of real symmetrical matrix C. If $|\lambda_i| \ne 1, \ i = 1, 2, \cdots, n$, then A has continuous bounded inverse with the following relationships:*
(1)

$$\int_0^T [A^{-1}f](t)dt = (E - C)^{-1} \int_0^T f(t)dt, \quad \forall f \in Z,$$
$$||A^{-1}f||_Z \le \left(\sum_{i=1}^{n} \frac{1}{|1 - |\lambda_i||} \right) ||f||_Z,$$

where $E = \mathrm{diag}(1, 1, \cdots, 1)$;
(2)

$$\int_0^T |[A^{-1}f](t)|^p dt \le \sigma \int_0^T |f(t)|^p dt, \quad \forall f \in Z, \ p \ge 1,$$
$$\sigma = \begin{cases} \max_{i \in I_n} \frac{1}{|1-|\lambda_i||^2}, & p = 2, \\ \left(\frac{1}{(1-|\lambda_i|)^{2p/2-p}} \right)^{(2-p)/p}, & p \in [1, 2), \\ \left(\frac{1}{(1-|\lambda_i|)^q} \right)^{p/q}, & p \in (2, +\infty), \end{cases}$$

where q is a constant $1/p + 1/q = 1$;
(3)

$$Ax' = (Ax)', \quad \forall x \in X.$$

In 2011, when the constant c of (1.2) is a $n \times n$ real matrix C, Lu, Xu and Xia [33] gave the following results:

Lemma 1.3. *Suppose that the matrix U and the operator A are defined by (2.5) and (2.2), respectively, and for all $i = 1, 2, \cdots, l$, $|\lambda_i| \neq 1$, where (2.5) and (2.2) are defined in [33]. Then A has its inverse $A^{-1} : C_T \to C_T$ with the following properties:*
(1) $||A^{-1}|| \leq |U^{-1}||U|\sigma_0$, $\sigma_0 = \Sigma_{i=1}^{l}\Sigma_{j=1}^{n_i}\Sigma_{k=1}^{j}\frac{1}{|1-\lambda_i|^k}$;
(2) For all $f \in C_T$, $\int_0^T |[A^{-1}f](s)|^p ds \leq |U^{-1}|^p|U|^p\sigma_1 \int_0^T |f(s)|^p ds$, $p \in [1, +\infty)$, where

$$\sigma_1 = \begin{cases} \Sigma_{i=1}^{l}\Sigma_{j=1}^{n_i}\left(\Sigma_{k=1}^{j}\frac{1}{|1-\lambda_i|^k}\right)^2, \ p = 2, \\ n^{\frac{2-p}{2}}\left[\Sigma_{i=1}^{l}\Sigma_{j=1}^{n_i}\left(\Sigma_{k=1}^{j}\frac{1}{|1-\lambda_i|^k}\right)^q\right]^{\frac{p}{q}}, \ p \in [1, 2) \\ \left[\Sigma_{i=1}^{l}\Sigma_{j=1}^{n_i}\left(\Sigma_{k=1}^{j}\frac{1}{|1-\lambda_i|^k}\right)^q\right]^{\frac{p}{q}}, \ p \in [2, +\infty) \end{cases}$$

and $q > 0$ is a constant with $1/p + 1/q = 1$;
(3)$A^{-1}f \in C_T^1$, $[A^{-1}f]'(t) = [A^{-1}f'](t)$, for all $f \in C_\omega^1$, $t \in \mathbb{R}$.

In this paper, we also need the following lemmas:

Lemma 1.4 ([29]). *Let $p \in (1, +\infty)$ be a constant, $s \in C(\mathbb{R}, \mathbb{R})$ such that $s(t+T) \equiv s(t)$, $u \in C_T^1$. Then*

$$\int_0^T |u(t) - u(t - s(t))|^p dt \leq 2\big(\max_{t\in[0,T]} |s(t)|\big)^p \int_0^T |u'(t)|^p dt.$$

Lemma 1.5 ([30]). *Let $s, \sigma \in C(\mathbb{R}, \mathbb{R})$ with $s(t + T) \equiv s(t)$ and $\sigma(t + T) \equiv \sigma(t)$. Suppose that the function $t - \sigma(t)$ has a unique inverse $\mu(t)$, $t \in \mathbb{R}$. Then $s(\mu(t + T)) \equiv s(\mu(t))$.*

Lemma 1.6 ([16]). *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$, is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \bar{\Omega} \to Y$ is L-compact on $\bar{\Omega}$. if all the following conditions hold:*

(1) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \forall\lambda \in (0, 1)$,

(2) $Nx \notin ImL, \forall x \in \partial\Omega \cap KerL$,

(3) $deg\{JQN, \Omega \cap KerL, 0\} \neq 0$,

where $J : ImQ \to KerL$ is an isomorphism. Then equation $Lx = Nx$ has a solution on $\bar{\Omega} \cap D(L)$.

As a continuity of the previous study [31, 33], it is very natural and interesting to investigate the case of which the constant c of (1.2) is a $n \times n$ real matrix function $C(t)$. This will be the main purpose of the present paper.

In this work, we consider the following nonlinear neutral functional differential system as follows:

$$(Ax)'(t) + f(x(t)) + g(x(t - \mu(t))) = e(t), \tag{1.3}$$

where

$$(Ax)(t) = x(t) - C(t)x(t - \tau) = ((A_1x_1)(t), (A_2x_2)(t), \cdots, (A_nx_n)(t))^\top,$$

$x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^\top$, $f, g \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ with

$$f(x) = (f_1(x_1), f_2(x_2), \cdots, f_n(x_n))^\top,\ g(x) = (g_1(x_1), g_2(x_2), \cdots, g_n(x_n))^\top,$$

$\tau > 0$ is a constant, $\mu \in C^1(\mathbb{R}, \mathbb{R})$ with $\mu(t) > 0$, $\mu(t+T) = \mu(t)$ and $\mu'(t) < 1$, $e \in C(\mathbb{R}, \mathbb{R}^n)$ with $e(t+T) = e(t)$, $C(t) = (c_{ij}(t))_{n\times n}$ is a continuous matrix function.

Throughout this paper, let

$$I_n = \{1, 2, \cdots, n\},\quad |a| = \left(\sum_{i=1}^n |a_i|^2\right)^{1/2}, \forall a = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n,$$
$$|A| = \max\{|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|\},$$

where $\lambda_i (i \in I_n)$ is eigenvalue of matrix $A = (a_{ij})_{n\times n}$,

$$|u|_0 = \max_{t\in[0,T]} |u(t)|,$$

where u is a $T-$periodic continuous function,

$$X = \{x : x \in C(\mathbb{R}, \mathbb{R}^n),\ x(t+T) = x(t)\}$$

with the norm $||x||_X = \max_{t\in\mathbb{R}} |x(t)|$,

$$Y = \{x : x \in C^1(\mathbb{R}, \mathbb{R}^n), x(t+T) = x(t)\}$$

with the norm
$$\|\varphi\|_Y = \max_{t\in[0,T]}\{||\varphi||_X, ||\varphi'||_X\},\ \forall \varphi \in Y.$$

Clearly, X and Y are Banach spaces.

The article is organized as follows. In Section 2, when $C(t)$ is a diagonal matrix function, we obtain some existence results of periodic solutions to system (1.3). In Section 3, when $C(t)$ is a symmetrical matrix function, we obtain some existence results of periodic solutions to system (1.3). Section 4 contains some stability results. Section 5 provides an illustrative examples. Section 6 concludes this article with a summary of our results.

2. C(t) is a diagonal matrix function

Let
$$A :\ X \to X, [Ax](t) = x(t) - C(t)x(t-\tau),$$

where $C(t) = \mathrm{diag}(c_1(t), c_2(t), \cdots, c_n(t))$, where $c_i(t) (i \in I_N)$ is continuously differentiable $T-$periodic function. We first give the following theorem:

Theorem 2.1. *Let $\int_0^T e(s)ds = \mathbf{0}$ and $\int_0^T \varphi^\top(t)\varphi(t)dt \neq 0$, where $\varphi(t)$ is defined by (2.4). Suppose that the following conditions hold:*
(H_1) there is a constant $M > 0$ such that for all $u = (u_1, u_2, \cdots, u_n)^\top \in X$ with $\min_{t\in[0,T], j\in I_n} |u_j(t)| > M$,

$$\int_0^T [f(u(t)) + \frac{1}{1-\mu'(\gamma(t))} g(u(t))]dt \neq \mathbf{0};$$

(H_2) there is a constant $r > 0$ such that

$$\lim_{|x_i|\to+\infty} \frac{|g_i(x_i)|}{|x_i|} \le r, \quad i \in I_n;$$

(H_3) there is a constant $K > 0$ such that

$$|f_i(x_i)| \le K, \quad i \in I_n;$$

(H_4) there is a constant $D > 0$ such that

$$x^\top[f(x) + \frac{1}{1-\mu'(\gamma(t))} g(x)] > 0 \quad for\ |x| > D.$$

Then system (1.3) has at least one $T-$periodic solution, if following condition holds:

$$\max\left\{\frac{c_{1,i}T}{1-c_{0,i}}, \frac{Tr}{(1-\mu_0)(1-c_{0,i}-c_{1,i}T)}\right\} < 1 \text{ for } c_{0,i} < \frac{1}{2}$$

or

$$\max\left\{\frac{c_{1,i}T}{\sigma_i - 1}, \frac{Tr}{(1-\mu_0)(\sigma_i - 1 - c_{1,i}T)}\right\} < 1 \text{ for } \sigma_i > 1,$$

where $c_{1,i} = \max\limits_{t\in[0,T]} |c_i'(t)|, \ i \in I_n$.

Proof. Define

$$N : X \to X, (Nx)(t) = -f(x(t)) - g(x(t-\mu(t))) + e(t),$$
$$L : D(L) \subset X \to Y, \quad Lx = (Ax)',$$

where $D(L) = \{x : x \in X, (Ax)' \in Y\}$. Then system (1.3) is the operator equation $Lx = Nx$. Since for all $x \in KerL$, $(x(t) - C(t)x(t-\tau))' = \mathbf{0}$, we have

$$x(t) - C(t)x(t-\tau) = \mathbf{1}, \tag{2.4}$$

where $\mathbf{1} = (1, 1, \cdots, 1)^\top$. Let $\varphi(t)$ be the unique $T-$periodic solution of (2.4), then for all $t \in \mathbb{R}$, $\varphi(t) \ne \mathbf{0}$ and

$$KerL = \{a\varphi(t), a \in \mathbb{R}\},$$

where $\varphi(t) = (\varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t))^\top$. Obviously, ImL is a closed in X and $\dim KerL = \operatorname{condim} ImL = n$. So L is a Fredholm operator with index zero. Define continuous projectors P, Q

$$P : X \to KerL, (Px)(t) = \frac{\int_0^T x^\top(t)\varphi(t)dt}{\int_0^T \varphi^\top\varphi dt}\varphi(t)$$

and

$$Q : X \to X/ImL, Qy = \frac{1}{T}\int_0^T y(s)ds.$$

Let

$$L_P = L|_{D(L)\cap KerP} : D(L) \cap KerP \to ImL,$$

then

$$L_P^{-1} = K_P : ImL \to D(L) \cap KerP.$$

Since $ImL \subset X$ and $D(L) \cap KerP \subset Y$, so K_P is an embedding operator. Hence K_P is a completely continuous operator in ImL. By the definitions of Q and N, it knows that $QN(\overline{\Omega})$ is bounded on $\overline{\Omega}$. Hence nonlinear operator N is L-compact on $\overline{\Omega}$. We complete the proof by three steps.

Step 1. Let $\Omega_1 = \{x \in D(L) \subset Y : Lx = \lambda Nx, \lambda \in (0,1)\}$. We show that Ω_1 is a bounded set. If $\forall x \in \Omega_1$, then $Lx = \lambda Nx$, i.e.,

$$(x(t) - C(t)x(t-\tau))' = -\lambda f(x(t)) - \lambda g(x(t-\mu(t))) + \lambda e(t). \tag{2.5}$$

Integrating both sides of (2.5) over $[0,T]$, we have

$$\int_0^T [f(x(t)) + g(x(t-\mu(t)))]dt = \mathbf{0}. \tag{2.6}$$

Let $t - \mu(t) = u$, since $\mu'(t) < 1$, so $t - \mu(t)$ exists a inverse function γ, then by (2.6) and Lemma 1.5 we have $t = \gamma(u)$ and

$$\int_0^T [f(x(t)) + \frac{g(x(t))}{1-\mu'(\gamma(t))}]dt = \mathbf{0},$$

i.e., $\forall i \in I_n$,

$$\int_0^T [f_i(x_i(t)) + \frac{g_i(x_i(t))}{1-\mu'(\gamma(t))}]dt = 0. \tag{2.7}$$

We claim that there exists a point $t_1 \in [0,T]$ such that

$$|x_i(t_1)| \le M. \tag{2.8}$$

In fact, if (2.8) does not hold, then by assumption (H_1)

$$f_i(x_i(t_1)) + \frac{g_i(x_i(t_1))}{1-\tau'(\gamma(t_1))} \ne 0,$$

which is contradiction to (2.7). So (2.8) holds. Hence we get

$$\begin{aligned} |x_i|_0 = \max_{t\in[0,T]} \left| x_i(t_1) + \int_{t_1}^t x_i'(s)ds \right| &\le |x_i(t_1)| + \int_0^T |x_i'(s)|ds \\ &\le M + \int_0^T |x_i'(s)|ds, \quad i \in I_n. \end{aligned} \tag{2.9}$$

From $[A_i x_i](t) = x_i(t) - c_i(t)x_i(t-\tau)$, we have

$$[A_i x_i'](t) = (A_i x_i)'(t) + c_i'(t)x_i(t-\tau),$$

then from Lemma 1.1, if $c_{0,i} < 1/2$ $(i \in I_n)$ we have

$$\begin{aligned} \int_0^T |x_i'(t)|dt = \int_0^T |(A_i^{-1}A_i x_i')(t)|dt &\le \int_0^T \frac{|(A_i x_i')(t)|}{1-c_{0,i}}dt \\ &= \int_0^T \frac{|(A_i x_i)'(t) + c_i'(t)x_i(t-\tau)|}{1-c_{0,i}}dt \\ &\le \int_0^T \frac{|(A_i x_i)'(t)|}{1-c_{0,i}}dt + \frac{c_{1,i}T}{1-c_{0,i}}\left(M + \int_0^T |x_i'(t)|dt\right). \end{aligned}$$

In view of $c_{1,i}T/(1-c_{0,i})<1$, we have

$$\int_0^T |x_i'(t)|dt \le \int_0^T \frac{|(A_ix_i)'(t)|}{1-c_{0,i}-c_{1,i}T}dt + \frac{c_{1,i}TM}{1-c_{0,i}-c_{1,i}T}, \quad i\in I_n. \tag{2.10}$$

From (2.5) and (H_3), for $i\in I_n$ we have

$$\begin{aligned}\int_0^T |(A_ix_i)'(t)|dt &\le \int_0^T |f_i(x_i(t))|dt + \int_0^T |g_i(x_i(t-\mu(t)))|dt + \int_0^T |e_i(t)|dt\\ &\le KT + \frac{1}{1-\mu_0}\int_0^T |g_i(x_i(t))|dt + T|e_i|_0.\end{aligned} \tag{2.11}$$

In view of $\frac{Tr}{(1-\mu_0)(1-c_{0,i}-c_{1,i}T)}<1$, there exists a constant $\varepsilon>0$ such that

$$\frac{T(r+\varepsilon)}{(1-\mu_0)(1-c_{0,i}-c_{1,i}T)}<1.$$

For such a positive constant ε, in view of (H_2), we obtain that there exists a constant $\rho>0$ such that

$$|g_i(x_i)| \le (r+\varepsilon)|x_i|, \text{ for } |x_i|>\rho, \quad i\in I_n. \tag{2.12}$$

Let

$$E_1=\{t\in[0,T]:|x_i(t)|>\rho, \quad i\in I_n\},\ E_2=\{t\in[0,T]:|x_i(t)|\le\rho, \quad i\in I_n\}.$$

By (2.12), for $i\in I_n$, we have

$$\begin{aligned}\int_0^T |g_i(x_i(t))|dt &= \int_{E_1}|g_i(x_i(t))|dt + \int_{E_2}|g_i(x_i(t))|dt\\ &\le T(r+\varepsilon)|x_i|_0 + Tg_\rho,\end{aligned} \tag{2.13}$$

where $g_\rho=\max_{|x_i|\le\rho}|g(x_i)|$. From (2.11) and (2.13), for $i\in I_n$, we have

$$\begin{aligned}\int_0^T |(A_ix_i)'(t)|dt &\le \int_0^T |f_i(x_i(t))|dt + \int_0^T |g_i(x_i(t-\mu(t)))|dt + \int_0^T |e_i(t)|dt\\ &\le KT + \frac{T(r+\varepsilon)}{1-\mu_0}|x_i|_0 + \frac{Tg_\rho}{1-\mu_0} + T|e_i|_0.\end{aligned} \tag{2.14}$$

From (2.10) and (2.14), for $i\in I_n$, we have

$$\begin{aligned}\int_0^T |x_i'(t)|dt \le &\frac{KT}{1-c_{0,i}-c_{1,i}T} + \frac{T(r+\varepsilon)}{(1-\mu_0)(1-c_{0,i}-c_{1,i}T)}|x_i|_0\\ &+\frac{Tg_\rho}{(1-\mu_0)(1-c_{0,i}-c_{1,i}T)} + \frac{T|e_i|_0}{1-c_{0,i}-c_{1,i}T} + \frac{c_{1,i}TM}{1-c_{0,i}-c_{1,i}T}.\end{aligned} \tag{2.15}$$

From (2.9) and (2.15), for $i\in I_n$, we have

$$\begin{aligned}|x_i|_0 \le &M + \int_0^T |x_i'(s)|ds\\ \le &M + \frac{KT}{1-c_{0,i}-c_{1,i}T} + \frac{T(r+\varepsilon)}{(1-\mu_0)(1-c_{0,i}-c_{1,i}T)}|x_i|_0\\ &+\frac{Tg_\rho}{(1-\mu_0)(1-c_{0,i}-c_{1,i}T)} + \frac{T|e_i|_0}{1-c_{0,i}-c_{1,i}T} + \frac{c_{1,i}TM}{1-c_{0,i}-c_{1,i}T}.\end{aligned} \tag{2.16}$$

By $\frac{T(r+\varepsilon)}{(1-\mu_0)(1-c_{0,i}-c_{1,i}T)} < 1$, there exists a constant $M_1 > 0$ such that

$$|x_i|_0 \le M_1, \ ||x||_X \le \sqrt{n}M_1, \quad i \in I_n.$$

If $\sigma_i > 1$ $(i \in I_N)$, based on the conditions of Theorem 2.1, similar to the above proof, there exist constants M_1' and M_2' such that

$$||x||_X \le M_1', \quad ||x'||_X \le M_2'.$$

Then we have

$$||x||_X \le \max\{\sqrt{n}M_1, M_1'\} + 1 := \widetilde{M}.$$

Step 2. Let $\Omega_2 = \{x \in KerL : QNx = \mathbf{0}\}$, we shall prove that Ω_2 is a bounded set. $\forall x \in \Omega_2$, then $x(t) = a_0\varphi(t), a_0 \in \mathbb{R}$ satisfying

$$\int_0^T [f_i(a_0\varphi_i(t)) + \frac{1}{1-\mu'(\gamma(t))} g_i(a_0\varphi_i(t)]dt = 0, \quad i \in I_n. \tag{2.17}$$

When $c_{0,i} < \frac{1}{2}, i \in I_n$, we have

$$\begin{aligned}\varphi_i(t) = A^{-1}(1) &= 1 + \sum_{j=1}^{\infty}\prod_{k=1}^{j} c_i(t-(k-1)\tau) \\ &\ge 1 - \sum_{j=1}^{\infty}\prod_{k=1}^{j} c_{0,i} = 1 - \frac{c_{0,i}}{1-c_{0,i}} \\ &= \frac{1-2c_{0,i}}{1-c_{0,i}} := \delta > 0.\end{aligned}$$

Thus

$$a_0 \le \frac{M}{\delta}.$$

Otherwise, $\forall t \in [0,T], |a_0\varphi_i(t)| > M$, from assumption (H_1), we have

$$\int_0^T [f(a_0\varphi_i(t)) + \frac{1}{1-\mu'(\gamma(t))} g(a_0\varphi_i(t)]dt > 0 \ (or < 0), \ i \in I_n,$$

which is contradiction to (2.17). When $\sigma_i > 1, i \in I_n$, we have

$$\begin{aligned}\varphi_i(t) = A^{-1}(1) &= -\frac{1}{c_i(t+\tau)} - \sum_{j=1}^{\infty}\prod_{k=1}^{j} \frac{1}{c_i(t+k\tau)} \\ &\le -\frac{1}{\sigma_i} - \sum_{j=1}^{\infty}\prod_{k=1}^{j+1} \frac{1}{\sigma_i} \\ &= -\frac{1}{\sigma_i - 1} := \gamma < 0.\end{aligned}$$

Thus

$$a_0 \le \frac{M}{|\gamma|}.$$

Otherwise, $\forall t \in [0,T],\ |a_0\varphi_i(t)| > M$, from assumption (H_1), we have

$$\int_0^T [f_i(a_0\varphi_i(t)) + \frac{1}{1-\mu'(\gamma(t))} g_i(a_0\varphi_i(t)]dt > 0 \ (or < 0),\ i \in I_n,$$

which is contradiction to (2.17). Hence Ω_2 is a bounded set.

Step 3. Let $\Omega = \{x \in X : ||x||_X < \widetilde{M}\}$, then $\Omega_1 \cup \Omega_2 \subset \Omega$, $\forall (x,\lambda) \in \partial\Omega \times (0,1)$, from the above proof, $Lx \neq \lambda Nx$ is satisfied. Obviously, condition (2) of Lemma 1.6 is also satisfied. Now we prove that condition (3) of Lemma 1.6 is satisfied. $\forall x^0 \in \partial\Omega \cap KerL$, we have $|x^0| = ||a_0\varphi||_X,\ a_0 \in \mathbb{R}$. Then $|x^0| = \widetilde{M} > D$. Take the homotopy

$$H(x,\mu) = \mu x + (1-\mu)QNx,\ x \in \overline{\Omega} \cap KerL,\ \mu \in [0,1].$$

Then, by using assumption (H_4), we have $H(x,\mu) \neq 0$. And then by the degree theory,

$$\begin{aligned} deg\{QN, \Omega \cap KerL, 0\} &= deg\{H(\cdot,0), \Omega \cap kerL, 0\} \\ &= deg\{H(\cdot,1), \Omega \cap kerL, 0\} \\ &= deg\{I, \Omega \cap kerL, 0\} \neq 0. \end{aligned}$$

Applying Lemma 1.6, we reach the conclusion.

□

3. C(t) is a symmetrical matrix function

Let $C(t) = c(t)B$ in (1.3), where $c(t)$ is a $T-$periodic continuous function, B is a $n \times n$ real symmetrical matrix. Then (1.3) is changed into the following form:

$$(x(t) - c(t)Bx(t-\tau))' + f(x(t)) + g(x(t-\mu(t))) = e(t). \tag{3.1}$$

Denote the operator by

$$\tilde{A} : X \to X,\quad [\tilde{A}x](t) = x(t) - c(t)Bx(t-\tau). \tag{3.2}$$

We first give the following lemma.

Lemma 3.1. *Let $p \geq 1$ be a constant. Suppose that $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of matrix B. $\forall i \in I_n$, operator $\tilde{A}$ has continuous inverse $\tilde{A}^{-1}$ on X, satisfying*

(1) If $c_0\lambda_i < 1$, then

$$||\tilde{A}^{-1}f||_X \leq \sum_{i=1}^n \frac{1}{1-c_0\lambda_i}||f||_X.$$

If $\sigma\lambda_i > 1$, then

$$||\tilde{A}^{-1}f||_X \leq \sum_{i=1}^n \frac{1}{\sigma\lambda_i - 1}||f||_X.$$

(2) If $c_0\lambda_i < 1$, then

$$\int_0^T |\tilde{A}^{-1}f(t)|^p dt \leq \hbar_1 \int_0^T |f(t)|^p dt,$$

where

$$\hbar_1 = \begin{cases} \left(\sum_{i=1}^n \frac{1}{(1-c_0\lambda_i)^{2p/(2-p)}} \right)^{(2-p)/2}, & p \in [1,2), \\ \sum_{i=1}^n \frac{1}{(1-c_0\lambda_i)^2}, & p = 2, \\ \left(\sum_{i=1}^n \frac{1}{(1-c_0\lambda_i)^q} \right)^{p/q}, & p > 2. \end{cases}$$

If $\sigma\lambda_i > 1$, *then*

$$\int_0^T |\tilde{A}^{-1} f(t)|^p dt \le \hbar_2 \int_0^T |f(t)|^p dt,$$

where

$$\hbar_2 = \begin{cases} \left(\sum_{i=1}^n \frac{1}{(\sigma\lambda_i-1)^{2p/(2-p)}} \right)^{(2-p)/2}, & p \in [1,2), \\ \sum_{i=1}^n \frac{1}{(\sigma\lambda_i-1)^2}, & p = 2, \\ \left(\sum_{i=1}^n \frac{1}{(\sigma\lambda_i-1)^q} \right)^{p/q}, & p > 2. \end{cases}$$

Proof. Let

$$[\tilde{A}x](t) = x(t) - c(t)Bx(t-\tau) = f(t), \quad \forall f \in X. \tag{3.3}$$

Since B is a symmetric matrix, we find that there is an orthogonal matrix U such that

$$UBU^\top = E_\lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}.$$

Setting $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))^\top = Ux(t)$, by (3.3) we have

$$y(t) - c(t)E_\lambda y(t-\tau) = \tilde{f}(t),$$

i.e.,

$$y_i(t) - c(t)\lambda_i y_i(t-\tau) = \tilde{f}_i(t), \ i \in I_N,$$

where $\tilde{f}(t) = (\tilde{f}_1(t), \tilde{f}_2(t), \cdots, \tilde{f}_n(t))^\top = Uf(t)$. From Lemma 1.1, $\tilde{A}$ has continuous inverse $\tilde{A}^{-1}$ with

$$\tilde{A}^{-1} : X \to X, \ (\tilde{A}^{-1}f)(t) = U^\top y(t).$$

(1) From Lemma 1.1, if $c_0\lambda_i < 1$, we have

$$\begin{aligned} ||\tilde{A}^{-1}f||_X = ||y||_X = \max_{t\in[0,T]} |y(t)| &= \max_{t\in[0,T]} \sqrt{\sum_{i=1}^n |y_i(t)|^2} \\ &\le \sum_{i=1}^n \max_{t\in[0,T]} |y_i(t)| \\ &\le \sum_{i=1}^n \frac{1}{1-c_0\lambda_i} \max_{t\in[0,T]} |\tilde{f}_i(t)| \\ &\le \sum_{i=1}^n \frac{1}{1-c_0\lambda_i} ||\tilde{f}||_X \\ &= \sum_{i=1}^n \frac{1}{1-c_0\lambda_i} ||f||_X. \end{aligned}$$

Similar to the above proof, for $\sigma\lambda_i > 1$, we have

$$||\tilde{A}^{-1}f||_X \leq \sum_{i=1}^{n} \frac{1}{\sigma\lambda_i - 1} ||f||_X.$$

(2) Case 1: $p \in [1, 2)$. By Lemma 1.1, for $c_0\lambda_i < 1$, we have

$$\begin{aligned}
\int_0^T |y(t)|^p dt &= \int_0^T \big[\sum_{i=1}^{n} y_i^2(t)\big]^{p/2} dt \leq \sum_{i=1}^{n} \int_0^T |y_i(t)|^p dt \\
&\leq \sum_{i=1}^{n} \frac{1}{(1 - c_0\lambda_i)^p} \int_0^T |\tilde{f}_i(t)|^p dt \\
&\leq \left(\sum_{i=1}^{n} \frac{1}{(1 - c_0\lambda_i)^{2p/(2-p)}} \right)^{(2-p)/2} \int_0^T \left(\sum_{i=1}^{n} |\tilde{f}_i(t)|^2 \right)^{p/2} dt \\
&= \left(\sum_{i=1}^{n} \frac{1}{(1 - c_0\lambda_i)^{2p/(2-p)}} \right)^{(2-p)/2} \int_0^T |\tilde{f}(t)|^p dt \\
&= \left(\sum_{i=1}^{n} \frac{1}{(1 - c_0\lambda_i)^{2p/(2-p)}} \right)^{(2-p)/2} \int_0^T |f(t)|^p dt.
\end{aligned}$$

Case 2: $p = 2$. For $c_0\lambda_i < 1$, by Lemma 1.1 we have

$$\begin{aligned}
\int_0^T |y(t)|^2 dt &= \int_0^T \sum_{i=1}^{n} y_i^2(t) dt = \sum_{i=1}^{n} \int_0^T y_i^2(t) dt \\
&\leq \sum_{i=1}^{n} \frac{1}{(1 - c_0\lambda_i)^2} \int_0^T \tilde{f}_i^2(t) dt \\
&\leq \sum_{i=1}^{n} \frac{1}{(1 - c_0\lambda_i)^2} \int_0^T |\tilde{f}(t)|^2 dt \\
&= \sum_{i=1}^{n} \frac{1}{(1 - c_0\lambda_i)^2} \int_0^T |f(t)|^2 dt.
\end{aligned}$$

Case 3: $p > 2$. For $c_0\lambda_i < 1$, by Lemma 1.1 we have

$$\begin{aligned}
\left(\int_0^T |y(t)|^p dt \right)^{1/p} &= \left(\int_0^T \big[\sum_{i=1}^{n} y_i^2(t)\big]^{p/2} dt \right)^{1/p} \leq \left(\int_0^T [\sum_{i=1}^{n} |y_i(t)|]^p dt \right)^{1/p} \\
&\leq \sum_{i=1}^{n} \left(\int_0^T |y_i(t)|^p dt \right)^{1/p} \\
&\leq \sum_{i=1}^{n} \frac{1}{(1 - c_0\lambda_i)} \left(\int_0^T |\tilde{f}_i(t)|^p dt \right)^{1/p} \\
&\leq \left(\sum_{i=1}^{n} \frac{1}{(1 - c_0\lambda_i)^q} \right)^{1/q} \left(\int_0^T \sum_{i=1}^{n} |\tilde{f}_i(t)|^p dt \right)^{1/p} \\
&= \left(\sum_{i=1}^{n} \frac{1}{(1 - c_0\lambda_i)^q} \right)^{1/q} \left(\int_0^T \sum_{i=1}^{n} |\tilde{f}_i(t)|^p dt \right)^{1/p}.
\end{aligned}$$

thus

$$\begin{aligned}\int_0^T |y(t)|^p d &\le \left(\sum_{i=1}^n \frac{1}{(1-c_0\lambda_i)^q}\right)^{p/q} \int_0^T \sum_{i=1}^n |\tilde{f}_i(t)|^p dt\\ &= \left(\sum_{i=1}^n \frac{1}{(1-c_0\lambda_i)^q}\right)^{p/q} \int_0^T \sum_{i=1}^n (|\tilde{f}_i(t)|^2)^{p/2} dt\\ &\le \left(\sum_{i=1}^n \frac{1}{(1-c_0\lambda_i)^q}\right)^{p/q} \int_0^T \left(\sum_{i=1}^n (|\tilde{f}_i(t)|^2\right)^{p/2} dt\\ &= \left(\sum_{i=1}^n \frac{1}{(1-c_0\lambda_i)^q}\right)^{p/q} \int_0^T |\tilde{f}(t)|^p dt\\ &= \left(\sum_{i=1}^n \frac{1}{(1-c_0\lambda_i)^q}\right)^{p/q} \int_0^T |f(t)|^p dt.\end{aligned}$$

Similar to the above proof, for $\sigma\lambda_i > 1$, we have

$$\int_0^T |\tilde{A}^{-1} f(t)|^p dt \le \hbar_2 \int_0^T |f(t)|^p dt.$$

□

Next, we will study the existence of periodic solution for system (3.1).

Theorem 3.1. *Let $\lambda_m = \max\{|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|\}$, where $\lambda_i (i \in I_n\ )$are eigenvalues of matrix B, $\int_0^T e(s)ds = \mathbf{0}$ and $\int_0^T \varphi^\top(t)\varphi(t)dt \neq 0$, where $\varphi(t)$ is defined by (3.6). Suppose that the following conditions hold:*
(H_5) there is a constant $d > 0$ such that for all $u = (u_1, u_2, \cdots, u_n)^\top \in X$ with $\min_{t\in[0,T], j\in I_n} |u_j(t)| > d$,

$$\int_0^T [f(u(t)) + \frac{1}{1-\mu'(\gamma(t))} g(u(t))]dt \neq \mathbf{0};$$

(H_6) there is a constant $r > 0$ such that

$$\lim_{|x|\to+\infty} \frac{|g(x)|}{|x|} \le r;$$

(H_7) there is a constant $K > 0$ such that

$$|f(x)| \le K, \quad \forall x \in X;$$

(H_8) there is a constant $D > 0$ such that

$$x^\top [f(x) + \frac{1}{1-\mu'(\gamma(t))} g(x)] > 0, \quad \forall x \in X \text{ with } |x| > D > d.$$

Then system (3.1) has at least one $T-$periodic solution, if the following condition holds:

For $c_0\lambda_i < \frac{1}{2},\ i \in I_n,$

$$\begin{aligned}
&\hbar_1 \max_{t\in[0,T]} \left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2} r(1+c_0\lambda_m)\frac{n^2T^2}{\pi^2} \\
&+ \sqrt{2}r\tau \max_{t\in[0,T]} \left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2} \frac{nT}{\pi} + \hbar_1 c_1^2\lambda_m^2 \frac{n^2T^2}{\pi^2} \\
&+ 2\hbar_1 c_1\lambda_m \Bigg(\max_{t\in[0,T]} \left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2} r(1+c_0\lambda_m)\frac{n^2T^2}{\pi^2} \\
&+ \sqrt{2}r\tau \max_{t\in[0,T]} \left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2} \frac{nT}{\pi}\Bigg)^{1/2} \frac{nT}{\pi} < 1,
\end{aligned}$$

or for $\sigma\lambda_i > 1,\ i \in I_n,$

$$\begin{aligned}
&\hbar_2 \max_{t\in[0,T]} \left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2} r(1+c_0\lambda_m)\frac{n^2T^2}{\pi^2} + \sqrt{2}r\tau \max_{t\in[0,T]} \left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2} \frac{nT}{\pi} \\
&+ \hbar_2 c_1^2\lambda_m^2 \frac{n^2T^2}{\pi^2} + 2\hbar_2 c_1\lambda_m \Bigg(\max_{t\in[0,T]} \left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2} r(1+c_0\lambda_m)\frac{n^2T^2}{\pi^2} \\
&+ \sqrt{2}r\tau \max_{t\in[0,T]} \left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2} \frac{nT}{\pi}\Bigg)^{1/2} \frac{nT}{\pi} < 1,
\end{aligned}$$

where $c_1 = \max_{t\in[0,T]} |c'(t)|$, $\hbar_1$ *and* $\hbar_2$ *are defined by Lemma 3.1.*

Proof. In order to use Lemma 1.6 to study the existence of periodic solutions for system (3.1), we set

$$L : D(L) \subset X \to Y, \quad Lx = (\tilde{A}x)'(t), \tag{3.4}$$

where $D(L) = \{x : x \in X, \tilde{A}x \in Y\}$,

$$N : X \to X, \quad (Nx)(t) = -f(x(t)) - g(x(t-\mu(t))) + e(t). \tag{3.5}$$

Since for all $x \in KerL,\ (x(t) - c(t)Bx(t-\tau))' = \mathbf{0}$, we have

$$x(t) - c(t)Bx(t-\tau) = \mathbf{1}, \tag{3.6}$$

where $\mathbf{1} = (1, 1, \cdots, 1)^\top$. Let $\varphi(t)$ be the unique $T-$periodic solution of (3.6), then $\forall t \in [0,T],\ \varphi(t) \neq \mathbf{0}$ and

$$KerL = \{a\varphi(t), a \in \mathbb{R}\},$$

where $\varphi(t) = (\varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t))^\top$. Obviously, ImL is a closed in X and $dimKerL = condimImL = n$. So L is a Fredholm operator with index zero. Similar to the proof of Theorem 2.1, we can define the operators P, Q, L_P, K_P and prove that K_P is a completely continuous operator in ImL. Then We complete the proof by three steps.

Step 1. Let $\Omega_1 = \{x \in D(L) \subset X : Lx = \lambda Nx, \lambda \in (0,1)\}$. We show that Ω_1 is a bounded set. If $\forall x \in \Omega_1$, then $Lx = \lambda Nx$, i.e.,

$$(\tilde{A}x)'(t) = -\lambda f(x(t)) - \lambda g(x(t-\tau)) + \lambda e(t). \tag{3.7}$$

Similar to the proof of Theorem 2.1, assumption (H_5) leads to the fact that there is a $\xi_i \in [0,T]$ such that

$$|x_i(\xi_i)| \le d, \quad \forall i \in I_n. \tag{3.8}$$

By (3.8) we have

$$|x| \le \sqrt{n}d + \int_0^T |x'(t)|dt \le \sqrt{n}d + T^{1/2}\bigg(\int_0^T |x'(t)|^2 dt\bigg)^{1/2}. \tag{3.9}$$

For all $i \in I_n$, let $\omega_i = x_i(t+\xi_i)$, and hence $\omega_i(0) = \omega_i(T) = 0$. From [17], we have

$$\int_0^T |\omega_i(t)|^2 dt \le \frac{T^2}{\pi^2}\int_0^T |\omega_i'(t)|^2 dt = \frac{T^2}{\pi^2}\int_0^T |x_i'(t)|^2 dt,\ i \in I_n. \tag{3.10}$$

For all $i \in I_n$, (3.8) and Minkowski's inequality yields

$$\begin{aligned}\bigg(\int_0^T |x_i(t)|^2 dt\bigg)^{1/2} &= \bigg(\int_0^T |\omega_i(t) + x_i(t)|^2 dt\bigg)^{1/2} \\ &\le \bigg(\int_0^T |\omega_i(t)|^2 dt\bigg)^{1/2} + \bigg(\int_0^T |x_i(t)|^2 dt\bigg)^{1/2} \\ &\le \frac{T}{\pi}\bigg(\int_0^T |x_i'(t)|^2 dt\bigg)^{1/2} + dT^{1/2} \\ &\le \frac{T}{\pi}\bigg(\int_0^T |x'(t)|^2 dt\bigg)^{1/2} + dT^{1/2}.\end{aligned}$$

Thus

$$\bigg(\int_0^T |x(t)|^2 dt\bigg)^{1/2} \le \frac{nT}{\pi}\bigg(\int_0^T |x'(t)|^2 dt\bigg)^{1/2} + 2ndT^{1/2} \tag{3.11}$$

and

$$\int_0^T |x(t)|^2 dt \le \frac{n^2T^2}{\pi^2}\int_0^T |x'(t)|^2 dt + \frac{2n^2dT^{3/2}}{\pi}\bigg(\int_0^T |x'(t)|^2 dt\bigg)^{1/2} + 4n^2d^2T. \tag{3.12}$$

Multiplying the two sides of (3.7) by $((\tilde{A}x)(t))^\top$ and integrating them on $[0,T]$, we have

$$\begin{aligned}\int_0^T |(\tilde{A}x)'(t)|^2 dt =&\lambda\int_0^T (x^\top(t) - c(t)x^\top(t-\tau)B)f(x(t))dt \\ &+ \lambda\int_0^T (x^\top(t) - c(t)x^\top(t-\tau)B)g(x(t-\mu(t)))dt \\ &- \lambda\int_0^T (x^\top(t) - c(t)x^\top(t-\tau)B)e(t)dt \\ =&\lambda\int_0^T (x^\top(t) - c(t)x^\top(t-\tau)B)f(x(t))dt \\ &+ \lambda\int_0^T x^\top(t-\tau)(I - c(t)B)g(x(t-\mu(t)))dt \\ &+ \lambda\int_0^T [x^\top(t) - x^\top(t-\tau)]g(x(t-\mu(t)))dt \\ &- \lambda\int_0^T (x^\top(t) - c(t)x^\top(t-\tau)B)e(t)dt.\end{aligned} \tag{3.13}$$

In view of

$$
\begin{aligned}
&\hbar_1 \max_{t\in[0,T]}\left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2} r(1+c_0\lambda_m)\frac{n^2T^2}{\pi^2}+\sqrt{2}r\tau \max_{t\in[0,T]}\left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2}\frac{nT}{\pi}\\
&+\hbar_1 c_1^2\lambda_m^2\frac{n^2T^2}{\pi^2}+2\hbar_1 c_1\lambda_m\left(\max_{t\in[0,T]}\left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2} r(1+c_0\lambda_m)\frac{n^2T^2}{\pi^2}\right.\\
&\left.+\sqrt{2}r\tau \max_{t\in[0,T]}\left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2}\frac{nT}{\pi}\right)^{1/2}\frac{nT}{\pi}<1,
\end{aligned}
$$

there must be a sufficiently small constant $\varepsilon > 0$ such that

$$
\hbar_1\beta_1+\hbar_1 c_1^2\lambda_m^2\frac{n^2T^2}{\pi^2}+2\hbar_1 c_1\lambda_m\beta_1^{1/2}\frac{nT}{\pi}<1, \tag{3.14}
$$

where

$$
\begin{aligned}
\beta_1 = &\max_{t\in[0,T]}\left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2}(r+\varepsilon)(1+c_0\lambda_m)\frac{n^2T^2}{\pi^2}\\
&+\sqrt{2}(r+\varepsilon)\tau\max_{t\in[0,T]}\left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2}\frac{nT}{\pi}.
\end{aligned} \tag{3.15}
$$

For such an $\varepsilon > 0$, in view of assumption (H_6), there is a constant $\rho > d$, where d is defined by assumption (H_5), such that

$$
|g(x)| < (r+\varepsilon)|x|, \quad \forall x\in\mathbb{R}^n \text{ with } |x|>\rho. \tag{3.16}
$$

Let

$$
E_1=\{t: t\in[0,T], |x(t-\mu(t))|\le\rho\},\quad E_2=\{t: t\in[0,T], |x(t-\mu(t))|>\rho\}.
$$

From (3.16), Lemma 1.4 and 1.5, we get

$$
\begin{aligned}
&\int_0^T x^\top(t-\tau)(I-c(t)B)g(x(t-\mu(t)))dt\\
\le&(1+c_0\lambda_m)\int_0^T |x^\top(t-\tau)||g(x(t-\mu(t)))|dt\\
\le&(1+c_0\lambda_m)\left(\int_0^T|x|^2dt\right)^{1/2}\left(\alpha_0^2T+\int_{E_2}|g(x(t-\mu(t)))|^2dt\right)^{1/2}\\
\le&(1+c_0\lambda_m)\left(\int_0^T|x|^2dt\right)^{1/2}\left(\alpha_0^2T+\max_{t\in[0,T]}\frac{1}{1-\mu'(\gamma(t))}(r+\varepsilon)^2\int_0^T|x(t)|^2dt\right)^{1/2}\\
\le&\max_{t\in[0,T]}\left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2}(r+\varepsilon)(1+c_0\lambda_m)\int_0^T|x(t)|^2dt\\
&+(1+c_0\lambda_m)\tau_0T^{1/2}\left(\int_0^T|x|^2dt\right)^{1/2}\\
\le&\max_{t\in[0,T]}\left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2}(r+\varepsilon)(1+c_0\lambda_m)\frac{n^2T^2}{\pi^2}\int_0^T|x'(t)|^2dt\\
&+\max_{t\in[0,T]}\left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2}(r+\varepsilon)(1+c_0\lambda_m)\frac{2n^2dT^{2/3}}{\pi}\left(\int_0^T|x'(t)|^2dt\right)^{1/2}
\end{aligned}
$$

$$+ 4 \max_{t\in[0,T]} \left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2} (r+\varepsilon)(1+c_0\lambda_m)n^2d^2T$$
$$+ (1+c_0\lambda_m)\alpha_0 T^{1/2}\frac{nT}{\pi}\left(\int_0^T |x'(t)|^2dt\right)^{1/2} + 2nd(1+c_0\lambda_m)\alpha_0 T \tag{3.17}$$

where $\alpha_0 = \max_{|x|\leq\rho}|g(x)|$, and

$$\begin{aligned}
&\int_0^T [x^\top(t) - x^\top(t-\tau)]g(x(t-\mu(t)))dt \\
\leq&\left(\int_0^T |x(t)-x(t-\tau)|^2dt\right)^{1/2}\left(\int_0^T |g(x(t-\mu(t)))|^2dt\right)^{1/2} \\
=&\left(\int_0^T |x(t)-x(t-\tau)|^2dt\right)^{1/2}\left(\int_{E_1} |g(x(t-\mu(t)))|^2dt + \int_{E_2} |g(x(t-\mu(t)))|^2dt\right)^{1/2} \\
\leq&\left(\int_0^T |x(t)-x(t-\tau)|^2dt\right)^{1/2}\left(\alpha_0^2T + \int_{E_2} |g(x(t-\mu(t)))|^2dt\right)^{1/2} \\
\leq&\sqrt{2}\tau\left(\int_0^T |x'(t)|^2dt\right)^{1/2}\left(\alpha_0^2T + \max_{t\in[0,T]}\frac{1}{1-\mu'(\gamma(t))}(r+\varepsilon)^2\int_0^T |x(t)|^2dt\right)^{1/2} \\
\leq&\sqrt{2T}\alpha_0\tau\left(\int_0^T |x'(t)|^2dt\right)^{1/2} \\
&+ \sqrt{2}(r+\varepsilon)\tau\max_{t\in[0,T]}\left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2}\left(\int_0^T |x'(t)|^2dt\right)^{1/2}\left(\int_0^T |x(t)|^2dt\right)^{1/2} \\
\leq&\sqrt{2T}\alpha_0\tau\left(\int_0^T |x'(t)|^2dt\right)^{1/2} + \sqrt{2}(r+\varepsilon)\tau\max_{t\in[0,T]}\left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2}\frac{nT}{\pi}\int_0^T |x'(t)|^2dt \\
&+ 2\sqrt{2}(r+\varepsilon)\tau\max_{t\in[0,T]}\left(\frac{1}{1-\mu'(\gamma(t))}\right)^{1/2} ndT^{1/2}\left(\int_0^T |x'(t)|^2dt\right)^{1/2}
\end{aligned} \tag{3.18}$$

Furthermore, by (H_7) we have

$$\begin{aligned}
&\lambda\int_0^T (x^\top(t) - c(t)x^\top(t-\tau)B)f((x(t)))dt \\
\leq&K\int_0^T |(I-c(t)B)x(t)|dt \\
\leq&(1+c_0\lambda_m)K\int_0^T |x(t)|dt \\
\leq&(1+c_0\lambda_m)KT^{1/2}\left(\int_0^T |x(t)|^2dt\right)^{1/2} \\
\leq&(1+c_0\lambda_m)KT^{3/2}\frac{n}{\pi}\left(\int_0^T |x'(t)|^2dt\right)^{1/2} + 2nd(1+c_0\lambda_m)KT.
\end{aligned} \tag{3.19}$$

Obviously,

$$-\lambda\int_0^T (x^\top(t) - c(t)x^\top(t-\tau)B)e(t)dt \leq ||e||_X\int_0^T |(I-c(t)B)x(t)|dt$$

$$\leq (1+c_0\lambda_m)||e||_X T^{1/2}\bigg(\int_0^T |x(t)|dt\bigg)^{1/2}$$
$$\leq (1+c_0\lambda_m)||e||_X T^{3/2}\frac{n}{\pi}\bigg(\int_0^T |x'(t)|dt\bigg)^{1/2} + 2nd(1+c_0\lambda_m)||e||_X T. \tag{3.20}$$

Substituting (3.17)-(3.20) into (3.13), we have

$$\int_0^T |(\tilde{A}x)'(t)|^2 dt \leq \beta_1 \int_0^T |x'(t)|^2 dt + \beta_2\bigg(\int_0^T |x'(t)|^2 dt\bigg)^{1/2} + \beta_3, \tag{3.21}$$

where β_1 is defined by (3.15),

$$\begin{aligned}\beta_2 = {} & \max_{t\in[0,T]}\bigg(\frac{1}{1-\mu'(\gamma(t))}\bigg)^{1/2}(r+\varepsilon)(1+c_0\lambda_m)\frac{2n^2dT^{2/3}}{\pi}\\ & +(1+c_0\lambda_m)\alpha_0 T^{1/2}\frac{nT}{\pi} + \sqrt{2T}\alpha_0\delta\\ & +\sqrt{2}(r+\varepsilon)\delta \max_{t\in[0,T]}\bigg(\frac{1}{1-\mu'(\gamma(t))}\bigg)^{1/2}\\ & +(1+c_0\lambda_m)KT^{3/2}\frac{n}{\pi} + (1+c_0\lambda_m)||e||_X T^{3/2}\frac{n}{\pi},\\ \beta_3 = {} & 4\max_{t\in[0,T]}\bigg(\frac{1}{1-\mu'(\gamma(t))}\bigg)^{1/2}(r+\varepsilon)(1+c_0\lambda_m)n^2d^2T + 2nd(1+c_0\lambda_m)\alpha_0 T\\ & +2\sqrt{2}(r+\varepsilon)\delta\max_{t\in[0,T]}\bigg(\frac{1}{1-\mu'(\gamma(t))}\bigg)^{1/2} ndT^{1/2}\bigg(\int_0^T |x'(t)|^2 dt\bigg)^{1/2}\\ & +2nd(1+c_0\lambda_m)KT + 2nd(1+c_0\lambda_m)||e||_X T.\end{aligned}$$

For $x \in X$, from $(\tilde{A}x)(t) = x(t) - c(t)Bx(t-\tau)$, we have

$$(\tilde{A}x')(t) = (\tilde{A}x)'(t) + c'(t)Bx(t-\tau). \tag{3.23}$$

Then from (3.11),(3.12),(3.21),(3.23) and Lemma 3.1, if $c_0\lambda_i < 1/2$ ($i \in I_n$) we have

$$\begin{aligned}\int_0^T |x'(t)|^2 dt &= \int_0^T |(\tilde{A}^{-1}\tilde{A}x'(t)|^2 dt\\ &\leq \hbar_1 \int_0^T |(\tilde{A}x'(t)|^2 dt\\ &= \hbar_1 \int_0^T |(\tilde{A}x)'(t) + c'(t)Bx(t-\tau)|^2 dt\\ &\leq \hbar_1 \int_0^T |(\tilde{A}x)'(t)|^2 dt + \hbar_1 c_1^2\lambda_m^2 \int_0^T |x(t)|^2 dt\\ &\quad + 2\hbar_1 c_1\lambda_m \int_0^T |(\tilde{A}x)'(t)||x(t-\tau)|dt\\ &\leq \hbar_1\beta_1 \int_0^T |x'(t)|^2 dt + \hbar_1\beta_2\bigg(\int_0^T |x'(t)|^2 dt\bigg)^{1/2} + \hbar_1\beta_3\\ &\quad + \hbar_1 c_1^2\lambda_m^2\int_0^T |x(t)|^2 dt + 2\hbar_1 c_1\lambda_m\bigg(\int_0^T |(\tilde{A}x)'(t)|^2 dt\bigg)^{1/2}\bigg(\int_0^T |x(t)|^2 dt\bigg)^{1/2}\end{aligned}$$

$$
\begin{aligned}
&\le \hbar_1\beta_1 \int_0^T |x'(t)|^2dt + \hbar_1\beta_2\left(\int_0^T |x'(t)|^2dt\right)^{1/2} + \hbar_1\beta_3 \\
&\quad + \hbar_1 c_1^2\lambda_m^2 \frac{n^2T^2}{\pi^2}\int_0^T |x'(t)|^2dt \\
&\quad + \hbar_1 c_1^2\lambda_m^2 \frac{2n^2dT^{3/2}}{\pi}\left(\int_0^T |x'(t)|^2dt\right)^{1/2} + 4n^2d^2T\hbar_1c_1^2\lambda_m^2 \\
&\quad + 2\hbar_1c_1\lambda_m\left(\beta_1\int_0^T |x'(t)|^2dt + \beta_2\left(\int_0^T |x'(t)|^2dt\right)^{1/2} + \beta_3\right)^{1/2} \\
&\times\left[\frac{nT}{\pi}\left(\int_0^T |x'(t)|^2dt\right)^{1/2} + 2ndT^{1/2}\right]. \qquad (3.24)
\end{aligned}
$$

From (3.24) and (3.14), there is a constant $D_1 > 0$ such that

$$\int_0^T |x'(t)|^2dt \le D_1$$

which together with (3.9) gives

$$||x||_X \le \sqrt{n}d + T^{1/2}D_1^{1/2} := D_2. \qquad (3.25)$$

If $\sigma\lambda_i > 1$ ($i \in I_n$), similar to the above proof, we obtain that there exists a constant $D_3, D_3' > 0$ such that

$$||x||_X \le D_3, \quad ||x'||_X \le D_3'. \qquad (3.27)$$

From (3.25)-(3.27), we have

$$||x||_X < \max\{D_2, D_3\} + 1 := \widetilde{M}.$$

The proof of Step 2 and Step 3 is similar to the Theorem 2.1, we omit it here.

□

Remark 3.1. When $C(t)$ is a symmetrical matrix function $V(t)$ in (1.3), it is very difficult to obtain the existence results of periodic solutions. Define

$$\mathbb{A}_1 : X \to X,\ Ax(t) = x(t) - V(t)x(t-\tau),$$

where $V(t)$ is a symmetrical matrix function. Although Lemma 1.2 gives some properties for the case of $C(t)$ is a symmetrical constant matrix C, and we can obtain that $\mathbb{A}_1$ exists inverse operator $\mathbb{A}_1^{-1}$ and some related properties for $\mathbb{A}_1$, but in this case, the prior bound of solutions to (1.3) can not been obtained, we hope that some results for (1.3) will be obtained in the case of $C(t)$ is a symmetrical matrix function.

Remark 3.2. When $C(t)$ is a real matrix function in (1.3), it is very difficult to obtain the existence results of periodic solutions. Define

$$\mathbb{A}_2 : X \to X,\ Ax(t) = x(t) - C(t)x(t-\tau),$$

where $C(t)$ is a real matrix function. Although Lemma 1.3 gives some properties for the case of $C(t)$ is a real constant matrix C, but we can not obtain that $\mathbb{A}_2$ exists inverse operator $\mathbb{A}^{-1}$ and some related properties for $\mathbb{A}_2$, so when $C(t)$ is a real matrix function in (1.3), we hope that some results for (1.3) will be obtained.

4. Asymptotic behaviors of periodic solution

In this section, we will study asymptotic behaviors of periodic solution to system (1.3) for the cases of $C(t)$ is a diagonal and symmetrical matrix function.

Definition 4.1. If $x^*(t) = (x_1^*(t), x_2^*(t), \cdots, x_n^*(t))^\top$ is an periodic solution of (1.3) and $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^\top$ is any solution of (1.3) satisfying

$$\lim_{t\to+\infty} \sum_{i=1}^{n} |x_i(t) - x_i^*(t)| = 0.$$

We call $x^*(t)$ is globally asymptotic stable.

Definition 4.2. If $x^*(t) = (x_1^*(t), x_2^*(t), \cdots, x_n^*(t))^\top$ is an periodic solution of (1.3) with initial value $\phi^*(t) = (\phi_1^*(t), \phi_2^*(t), \cdots, \phi_n^*(t))^\top$. If there exist constants $\lambda > 0,\ M_\phi > 1$ such that, for any solution $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^\top$ of (1.3) with initial value $\phi(t) = (\phi_1(t), \phi_2(t), \cdots, \phi_n(t))^\top$,

$$|x_i(t) - x_i^*(t)| \le M_\phi ||\phi - \phi^*|| e^{-\lambda t} \quad \forall t > 0,\ i \in I_n.$$

We call $x^*(t)$ is globally exponential stable.

For convenience of obtaining globally asymptotic stability, let $C(t) = \mathrm{diag}\{c_1(t), c_2(t), \cdots, c_n(t)\},\ e_i(t) = 0,\ f_i(0) = g_i(0) = 0,\ i \in I_N$. Then (1.3) is changed into

$$(A_i x_i)'(t) = (x_i(t) - c_i(t)x_i(t-\tau))' = -f_i(x_i(t)) - g_i(x_i(t-\mu(t))), i \in I_N,\ t > 0 \tag{4.1}$$

with initial condition

$$x_i(t) = \phi_i(t),\ t \in [-r, 0],$$

where $r = \max_{t\in[0,T]}\{\tau, \mu(t)\},\ i \in I_N$. Clearly, $x = \mathbf{0}$ is the equilibrium point of (4.1). Now, we give the following theorem:

Theorem 4.1. *Under conditions of Theorem 2.1, assume further that*
(i) there exist $L_{1i} > 0,\ L_{2i} > 0$ such that

$$|f_i(x) - f_i(y)| \le L_{1i}|x - y|, \quad |g_i(x) - g_i(y)| \le L_{2i}|x - y|,\ \forall x, y \in \mathbb{R}, \quad i \in I_n.$$

Then (4.1) has unique $T-$periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \cdots, x_n^*(t))^\top = \mathbf{0}$ which is globally asymptotic stable if

$$2\delta - L_{2i} - c_{0,i}L_{2i} - (c_{0,i}L_{2i} + L_{2i}) \max_{t\in[0,T]} \frac{1}{1 - \mu'(\gamma(t))} > 0,$$

where $i \in I_N,\ t > 0,\ \gamma$ is a inverse function of $t - \mu(t)$.

Proof. Assumptions of Theorem 2.1 imply that (4.1) has unique $T-$periodic solution $x^*(t) = \mathbf{0}$. Suppose $x(t)$ be any solution of (4.1). Let

$$V_i(t) = (A_i x_i)^2, \quad i \in I_N,\ t > 0.$$

Derivation of it along the solution of (4.1) gives

$$
\begin{aligned}
V_i'(t) &= -2x_i(t)f_i(x_i(t)) - 2x_i(t)g_i(x_i(t-\mu(t))) \\
&\quad + 2c_i(t)x_i(t-\tau)f_i(x_i(t)) + 2c_ix_i(t-\tau)g_i(x_i(t-\mu(t))) \\
&\le -2\delta x_i^2(t) + 2|x_i(t)|L_{2i}|x_i(t-\mu(t))| \\
&\quad + 2c_{0,i}|x_i(t-\tau)|L_{1i}|x_i(t)| + 2c_{0,i}|x_i(t-\tau)|L_{2i}|x_i(t-\mu(t))| \\
&\le -2\delta x_i^2(t) + L_{2i}x_i^2(t) + L_{2i}x_i^2(t-\mu(t)) \\
&\quad + c_{0,i}L_{1i}x_i^2(t) + c_{0,i}L_{1i}x_i^2(t-\tau) + c_{0,i}L_{2i}x_i^2(t-\tau) + c_{0,i}L_{2i}x_i^2(t-\mu(t)) \\
&= -(2\delta - L_{2i} - c_{0,i}L_{1i})x_i^2(t) + (c_{0,i}L_{2i} + c_{0,i}L_{1i})x_i^2(t-\tau) \\
&\quad + (c_{0,i}L_{2i} + L_{2i})x_i^2(t-\mu(t)).
\end{aligned}
$$

For $t > 0$, define further that

$$
\begin{aligned}
V_{\tau_i}(t) &= (c_{0,i}L_{2i} + c_{0,i}L_{1i})\int_{t-\tau}^{t} x_i^2(s)ds, \quad V_{\mu i}(t) \\
&= (c_{0,i}L_{2i} + L_{2i})\int_{t-\mu(t)}^{t} \frac{1}{1-\mu'(\gamma(s))}x_i^2(s)ds.
\end{aligned}
$$

Then we have

$$
V_{\tau_i}'(t) = (c_{0,i}L_{2i} + c_{0,i}L_{1i})[x_i^2(t) - x_i^2(t-\tau)]
$$

and

$$
V_{\mu i}'(t) = (c_{0,i}L_{2i} + L_{2i})[\frac{1}{1-\mu'(\gamma(t))}x_i^2(t) - x_i^2(t-\mu(t))].
$$

Choose the Lyapunov functional for (4.1) in the following form:

$$
V(t) = \sum_{i=1}^{n}[V_i(t) + V_{\tau_i}(t) + V_{\mu i}(t)], \ t > 0.
$$

Derivating it along the solution of (4.1) gives

$$
\begin{aligned}
V'(t) &\le \sum_{i=1}^{n}\Big[-(2\delta - L_{2i} - c_{0,i}L_{1i})x_i^2(t) + (c_{0,i}L_{2i} + c_{0,i}L_{1i})x_i^2(t-\tau) \\
&\quad + (c_{0,i}L_{2i} + L_{2i})x_i^2(t-\mu(t)) + (c_{0,i}L_{2i} + c_{0,i}L_{1i})[x_i^2(t) - x_i^2(t-\tau)] \\
&\quad + (c_{0,i}L_{2i} + L_{2i})[\frac{1}{1-\mu'(\gamma(t))}x_i^2(t) - x_i^2(t-\mu(t))]\Big] \\
&= -\sum_{i=1}^{n}\left[2\delta - L_{2i} - c_{0,i}L_{1i} - (c_{0,i}L_{2i} + c_{0,i}L_{1i}) - \frac{c_{0,i}L_{2i} + L_{2i}}{1-\mu'(\gamma(t))}\right]x_i^2(t).
\end{aligned}
$$

From $2\delta - L_{2i} - c_{0,i}L_{2i} - (c_{0,i}L_{2i} + L_{2i})\max_{t\in\mathbb{R}} \frac{1}{1-\mu'(\gamma(t))} > 0$, we have

$$
V'(t) < 0.
$$

From Barbalat's Lemma [4], we have

$$
\lim_{t\to+\infty}\sum_{i=1}^{n}|x_i(t)| = 0.
$$

The proof of Theorem 4.1 is now finished. □

Let $C(t) = c(t)B,\ e_i(t) = 0,\ f_i(0) = g_i(0) = 0$. Then (1.3) is changed into

$$(A_i x_i)'(t) = (x_i(t) - c(t)\sum_{j=1}^{n} b_{ij} x_i(t-\tau))' = -f_i(x_i(t)) - g_i(x_i(t-\mu(t))),\ t > 0 \quad (4.2)$$

with initial condition

$$x_i(t) = \tilde{\phi}_i(t),\ t \in [-r, 0],$$

where $r = \max_{t\in[0,T]}\{\tau, \mu(t)\},\ i \in I_N$, $(A_i x_i)(t) = x_i(t) - c(t)\sum_{j=1}^{n} b_{ij} x_i(t-\tau),\ B = (b_{ij})_{n\times n}$ is $n\times n$ real symmetrical matrix, $c(t)$ is $T-$periodic function. Clearly, $x = \mathbf{0}$ is the equilibriun point of (4.2), Similar to the proof of Theorem 4.1, we have the following theorem:

Theorem 4.2. *Under conditions of Theorem 3.1, assume further that condition (i) of Theorem 4.1 holds. Then (4.2) has unique $T-$periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \cdots, x_n^*(t))^\top = \mathbf{0}$ which is globally asymptotic stable if*

$$2\delta - L_{2i} - c_0\sum_{j=1}^{n}|b_{ij}|L_{2i} - (c_0\sum_{j=1}^{n}|b_{ij}|L_{2i} + L_{2i})\max_{t\in\mathbb{R}}\frac{1}{1-\mu'(\gamma(t))} > 0,$$

where $i \in I_N,\ t > 0,\ \gamma$ is a inverse function of $t - \mu(t)$.

For convenience of obtaining globally exponential stability, for $i \in I_N$, let $f_i(t) = a_i(t)x_i(t) + \tilde{f}(x_i(t))$ with $a_i(t) > 0,\ e_i(t) = 0,\ f_i(0) = g_i(0) = 0$.

Theorem 4.3. *Under conditions of Theorem 2.1, assume further that*
(i) there exist $L_{1i} > 0,\ L_{2i} > 0$ such that

$$|\tilde{f}_i(x) - \tilde{f}_i(y)| \le L_{1i}|x-y|, \quad |g_i(x) - g_i(y)| \le L_{2i}|x-y|,\ \forall x, y \in \mathbb{R}, \quad i \in I_n;$$

(ii) there exists a constant vector $\xi = (\xi_1, \xi_2, \cdots, \xi_n)^\top > 0$ such that

$$(-a_i^- + c_0 a_{0,i}\kappa_i + L_{1i}\kappa_i + L_{2i}\kappa_i)\xi_i < 0,$$

where $a_i^- = \min_{t\in[0,T]} a_i(t),\ a_{0,i} = \max_{t\in[0,T]} a_i(t),\ \kappa_i = \max\{\frac{1}{1-c_{0,i}}, \frac{1}{\sigma_{0,i}-1}\},\ i \in I_N$. Then (4.1) has unique $T-$periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \cdots, x_n^*(t))^\top = \mathbf{0}$ satisfying initial condition $x_i^*(t) = \phi_i^*(t),\ t \in [-r, 0]$, which is globally exponential stable.

Proof. For $i \in I_n$, define function

$$\Xi_i(t) = (t - a_i^- + c_0 a_{0,i}\kappa_i e^{t\tau} + L_{1i}\kappa_i + L_{2i}\kappa_i e^{t\mu_0})\xi_i.$$

In view of condition (ii), $\Xi_i(t)$ is continuous on interval $[0, \lambda_0]$ with

$$\Xi_i(0) = (-a_i^- + c_0 a_{0,i}\kappa_i + L_{1i}\kappa_i + L_{2i}\kappa_i)\xi_i < 0,\ i \in I_n.$$

Thus there must be a positive constant $\lambda \in [0, \lambda_0]$ such that

$$\Xi_i(\lambda) = (t - a_i^- + c_0 a_{0,i}\kappa_i e^{\lambda\tau} + L_{1i}\kappa_i + L_{2i}\kappa_i e^{\lambda\mu_0})\xi_i < 0,\ i \in I_n. \quad (4.3)$$

For above λ, we choose the following Lyapunov functional

$$V_i(t) = |(A_i x_i)(t)|e^{\lambda t}, \quad i \in I_n, \forall t > 0.$$

It is easy to check that

$$V_i(t) = |(A_i x_i)(t)|e^{\lambda t} < \xi_i, \quad i \in I_n, \ \forall t > 0. \tag{4.4}$$

Otherwise, there must be an $i \in I_n$ and $t_i > 0$ such that

$$V_i(t_i) = \xi_i \ \text{ and } \ V_i(t) < \xi_i, \ i \in I_n, \ \text{for } 0 < t < t_i.$$

From (4.1), condition (i) and Lemma 1.1, for $i \in I_n$, we have

$$\begin{aligned}
0 &\le D^+V_i(t_i - \xi_i) = D^+V_i(t_i) \\
&= sgn\{(A_i x_i)(t_i)\}(A_i x_i)'(t_i)e^{\lambda t_i} + \lambda |(A_i x_i)(t_i)|e^{\lambda t_i} \\
&= sgn\{(A_i x_i)(t_i)\}\Big(- a_i x_i(t_i) - \tilde{f}_i(x_i(t)) - g_i(x_i(t_i - \mu(t_i)))\Big)e^{\lambda t_i} + \lambda |(A_i x_i)(t_i)|e^{\lambda t_i} \\
&\le (\lambda - a_i^-)|(A_i x_i)(t_i)|e^{\lambda t_i} + c_0 a_{0,i}|x_i(t_i - \tau)|e^{\lambda(t_i-\tau)}e^{\lambda\tau} \\
&\quad + L_{1i}|x_i(t_i)|e^{\lambda t_i} + L_{2i}|x_i(t_i - \mu(t_i))|e^{\lambda(t_i-\mu(t_i))}e^{\lambda\mu(t_i)} \\
&\le (\lambda - a_i^-)|(A_i x_i)(t_i)|e^{\lambda t_i} + c_0 a_{0,i}|(A_i^{-1}A_i x_i)(t_i - \tau)|e^{\lambda(t_i-\tau)}e^{\lambda\tau} \\
&\quad + L_{1i}|(A_i^{-1}A_i x_i)(t_i)|e^{\lambda t_i} + L_{2i}|(A_i^{-1}A_i x_i)(t_i - \mu(t_i))|e^{\lambda(t_i-\mu(t_i))}e^{\lambda\mu(t_i)} \\
&\le (\lambda - a_i^-)|(A_i x_i)(t_i)|e^{\lambda t_i} + c_0 a_{0,i}\kappa_i|(A_i x_i)(t_i - \tau)|e^{\lambda(t_i-\tau)}e^{\lambda\tau} \\
&\quad + L_{1i}\kappa_i|(A_i x_i)(t_i)|e^{\lambda t_i} + L_{2i}\kappa_i|(A_i x_i)(t_i - \mu(t_i))|e^{\lambda(t_i-\mu(t_i))}e^{\lambda\mu(t_i)} \\
&\le (\lambda - a_i^+ + c_0 a_{0,i}\kappa_i e^{\lambda\tau} + L_{1i}\kappa_i + L_{2i}\kappa_i e^{\lambda\mu_0})\xi_i,
\end{aligned}$$

which is a contravention to (4.3). Thus, (4.4) holds. It follows from (4.4) and Lemma 1.1 that, $\forall i \in I_n, \ t > 0$

$$|x_i(t)| = |(A_i^{-1}A_i x_i)(t)| \le \kappa_i|(A_i x_i)(t)| < \kappa_i e^{-\lambda t}\xi_i \le M_\phi||\phi - \phi^*||e^{-\lambda t},$$

where $M_\phi > 1$ is a constant such that

$$M_\phi||\phi - \phi^*|| \ge \kappa_i^{-1}\xi_i, \ i \in I_n.$$

The proof is completed. □

For system (4.2), similar to the proof of Theorem 4.3, we have the following theorem:

Theorem 4.4. *Under conditions of Theorem 3.1, assume further that*
(i) there exist $L_{1i} > 0, \ L_{2i} > 0$ such that

$$|\tilde{f}_i(x) - \tilde{f}_i(y)| \le L_{1i}|x - y|, \quad |g_i(x) - g_i(y)| \le L_{2i}|x - y|, \ \forall x, y \in \mathbb{R}, \quad i \in I_n;$$

(ii) there exists a constant vector $\xi = (\xi_1, \xi_2, \cdots, \xi_n)^\top > 0$ such that

$$(-a_i^- + c_0 a_{0,i}\chi_i + L_{1i}\chi_i + L_{2i}\chi_i)\xi_i > 0,$$

where $a_i^- = \min_{t\in[0,T]} a_i(t), a_{0,i} = \max_{t\in[0,T]} a_i(t), \chi_i = \max\{\frac{1}{1-c_0\sum_{j=1}^n |b_{ij}|}, \frac{1}{\sigma_0 \sum_{j=1}^n |b_{ij}|-1}\}$ with $c_0 \sum_{j=1}^n |b_{ij}| < 1$ and $\sigma_0 \sum_{j=1}^n |b_{ij}| > 1, c_0 = \max_{t\in[0,T]}|c(t)|, \sigma_0 = \min_{t\in[0,T]}|c(t)|$. Then (4.2) has unique $T-$periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \cdots, x_n^*(t))^\top = \mathbf{0}$ satisfying initial condition $x_i^*(t) = \tilde{\phi}_i^*(t), \ t \in [-r, 0]$, which is globally exponential stable.

Remark 4.1. From Definitions 4.1 to 4.2, one knows that the global exponential stability of the periodic solutions of (4.1) and (4.2) implies its global asymptotic stability. In fact, for $M_\phi > 1$ and $\lambda > 0$, we have

$$|x_i(t) - x_i^*(t)| \leq M_\phi e^{\lambda t} \to 0,\ t \to +\infty,\ i \in I_n.$$

However, by comparing Theorem 4.1 (or 4.2) and 4.3 (or 4.4), we find that the conditions presented in Theorem 4.3 (or 4.4) are weaker than the corresponding ones in 4.1 (or 4.2). Obviously, in Theorem 4.1 (or 4.2) we need $\mu'(t) < 1,\ t \in \mathbb{R}$, which are not needed in Theorem 4.3 (or 4.4).

5. Example

In order to verify the feasibility of our results, consider the following two neutral type systems:

Example 5.1.

$$\begin{cases} (A_1x_1)'(t) + a_1(t)x_1(t) + f_1(x_1(t)) + g_1(x_1(t-\mu(t))) = 0, \\ (A_2x_2)'(t) + a_2(t)x_2(t) + f_2(x_2(t)) + g_2(x_2(t-\mu(t))) = 0, \end{cases} \tag{5.1}$$

where

$$\begin{aligned} &(A_1x_1)(t) = x_1(t) - c_1(t)x_1(t-\tau),\quad (A_2x_2)(t) = x_2(t) - c_2(t)x_2(t-\tau). \\ &T = 2\pi,\ \tau = \pi,\ a_1(t) = a_2(t) = 2,\ c_1(t) = c_2(t) = 0.01\cos t, \\ &\mu(t) = \frac{1}{2\pi}\sin t,\ f_i(x_i) = g_i(x_i) = 0.2\sin x_i,\ i = 1, 2. \end{aligned}$$

Obviously, based on the above parameters, all the conditions of Theorem 2.1 and 4.1 hold, hence, system (5.1) has a unique periodic solution $x(t) = (x_1(t), x_2(t))^\top$ which is globally asymptotic stable.

Example 5.2.

$$\begin{cases} (\mathbb{A}_1x_1)'(t) + a_1(t)x_1(t) + f_1(x_1(t)) + g_1(x_1(t-\mu(t))) = 0, \\ (\mathbb{A}_2x_2)'(t) + a_2(t)x_2(t) + f_2(x_2(t)) + g_2(x_2(t-\mu(t))) = 0, \end{cases} \tag{5.2}$$

where

$$\begin{aligned} &(\mathbb{A}_1x_1)(t) = x_1(t) - c(t)\sum_{j=1}^{2} b_{1j}x_1(t-\tau),\quad (\mathbb{A}_2x_2)(t) = x_2(t) - c(t)\sum_{j=1}^{2} b_{2j}x_2(t-\tau). \\ &T = 2\pi,\ \tau = \pi,\ a_1(t) = a_2(t) = 2,\ c(t) = 0.01\cos t, b_{11} = \ b_{22} = 1, b_{12} = b_{21} = -1, \\ &\mu(t) = \frac{1}{2\pi}\sin t,\ f_i(x_i) = g_i(x_i) = 0.2\sin x_i,\ i = 1, 2. \end{aligned}$$

Obviously, based on the above parameters, all the conditions of Theorem 3.1 and 4.3 hold, hence, system (5.2) has a unique periodic solution $x(t) = (x_1(t), x_2(t))^\top$ which is globally exponential stable.

6. Conclusions

In this article, we have investigated periodic problems for a class of neutral type system with variable parameters. The methods under study are Mawhin's continuation theorem and some considerate analysis techniques. For asymptotic behaviors of periodic solution, we developed a Lyapunov based framework and derived the theoretical results that the time-varying delays. Examples further illustrate our theoretical approach. It is possible to extend the main results to the more complicated cases such as the neutral systems with the finite and infinite distributed time delays, or with impulse terms, which are the future research topics.

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