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# SWITCHING SYNCHRONIZED CHAOTIC SYSTEMS APPLIED TO SECURE COMMUNICATION

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**Abstract** The purpose of this paper is to study the behavior of the solutions of two synchronized chaotic systems when the solutions switch from the first to the second system and vice-versa. The initial condition is chosen in the first system and the solutions travels for time $t \in [0, h]$, where $h > 0$. The value of the solution at time $h$ is then chosen as the initial condition for the solution of the second system and this solution travels for time $t \in [h, 2h]$. The value of the solution at time $2h$ is then chosen as the initial value for the solution of the first system and so on. The first system is composed of two subsystems, Master and Slave that are synchronized. We present applications using the Lorenz, Chua and Chen systems. Some simulations using Matlab are presented.

**Keywords** Synchronization, switching systems, chaotic systems.

**MSC(2010)** 34D06, 37D45.

## 1. Introduction

The subject of synchronization has been treated in several previous works. Chua, Matsumoto and Komuro [8] discovered an electronic circuit whose solutions have chaotic behaviors. In Afraimovich et all [1], the authors presented some techniques to prove synchronization of two coupled systems. In Afraimovich and Rodrigues [2] and in Rodrigues [23], using Liapunov like functions the authors presented some mathematical methods to prove synchronization of two chaotic systems. In Rodrigues, Alberto and Bretas [24, 26] synchronization of chaotic systems and applications to power systems ware discussed and an invariant principle for systems that depend on parameters was proved, extending the classical result proved by LaSalle [18]. In [4], Carvalho, Dlotko and Rodrigues presented a version for infinite dimensional systems with applications to partial differential equations. Labouriau

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and Rodrigues [15] studied synchronization of coupled equations of Hodgkin-Huxley type. Gameiro and Rodrigues [12] show how to use synchronization of chaotic systems for communication systems to codify and transmit signals. For discrete systems some new mathematical methods were presented in Rodrigues, Wu and Gabriel [27] and Rodrigues, Wu and Gameiro [28]. Some applications to communication systems were given in Rodrigues, Wu and Gameiro [28]. Switching systems and synchronization in switching systems have attracted considerable attention in recent years (see e.g., [5–7, 9, 11, 14, 19–21, 29]).

In this paper we introduce two new features. One feature allows us to send one part of the signal through one system and another part through the switched system. Another feature is to send two different signals using two different variables of these systems, like if we were using two different chanels. This is illustrated in Example 4.3 using Chen's system.

The outline of this paper is as follows: we assume that using Theorem 2.1, proved in [27], one can prove the global dissipativeness of two of our examples, namely, Chua's System and Lorenz System, as it is presented in [12]. For the global dissipativeness of Chen's system we rely in Barboza and Chen [3]. To prove synchronization in our three examples we use Theorem 2.2. The Liapunov functions that we use for this purpose are discussed when we consider each case. Theorem 3.1 is the main theoretical result of this paper. It shows that swiching between two different, but similar systems preserves the transmitted signal. In some sense this is a stability result. To prove stability of swiching systems many authors rely on the fact that a unique Liapunov function can be used. In our case we use different but similar, Liapunov functions which are Liapunov functions depending on parameters. In each case, they use different values of the parameters. However we prove that both have the same exponential decay. Essentially in our approach we use nonautonomous discontinuous systems and discontinuous Liapunov functions with the same estimate for exponential decay. In each example we obtain a Liapunov function with the corresponding estimate to obtain synchronization.

Finally we present simulation using Matlab of all these examples. These simulations are coherent with our mathematical results.

## 2. Dissipativeness and Synchronization

In this section, we state and prove some results, in a convenient form, that will be very useful on the applications of the next section. For details, see Rodrigues, Wu and Gameiro [28].

Let $E$ be a Banach space and $\Lambda \subset E$. Let $f \in C(\mathbb{R} \times \mathbb{R}^n \times \Lambda, \mathbb{R}^n)$ satisfy the following Lipschitz condition with respect to $x \in \mathbb{R}^n$: for each $\lambda \in \Lambda$ and each bounded set $B \subset \mathbb{R} \times \mathbb{R}^n$ there exists a constant $L \geq 0$ such that

$$\|f(t, x_1, \lambda) - f(t, x_2, \lambda)\| \leq L\|x_1 - x_2\|, \quad \forall (t, x_1), (t, x_2) \in B. \tag{2.1}$$

Consider the equation

$$\dot{x} = f(t, x, \lambda). \tag{2.2}$$

The next theorem gives conditions under which one can find a bounded set containing the attractor of (2.2) for all values of the parameter $\lambda \in \Lambda$. See Figure 1 for an illustration.

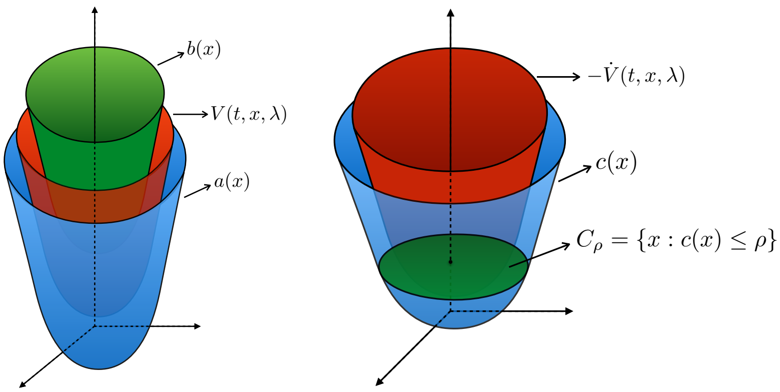


**Figure 1.** Figures illustrating the bounds in Theorem 2.1.

**Theorem 2.1.** *Let $V \in C^1(\mathbb{R} \times \mathbb{R}^n \times \Lambda, \mathbb{R})$. Suppose there are $a, b, c \in C(\mathbb{R}^n, \mathbb{R})$ such that $a(x) \to +\infty$ as $\|x\| \to +\infty$ and*

$$a(x) \leq V(t,x,\lambda) \leq b(x), \quad -\dot{V}(t,x,\lambda) \geq c(x)$$

*for every $(t,x,\lambda) \in \mathbb{R} \times \mathbb{R}^n \times \Lambda$, where*

$$\dot{V}(t,x,\lambda) := \frac{\partial}{\partial t} V(t,x,\lambda) + \left[\frac{\partial}{\partial x} V(t,x,\lambda)\right] f(t,x,\lambda).$$

*We also assume that there exists $\rho > 0$, such that the set $C_\rho := \{x \in \mathbb{R}^n : c(x) \leq \rho\}$ is nonempty and bounded. Let $r > 0$ be sufficiently large in such a way that $r > \sup_{x \in C_\rho} b(x)$. Let $\mathcal{A}_r := \{x \in \mathbb{R}^n : a(x) \leq r\}$. Then the following holds for Equation (2.2):*

*(i) Given $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ and $\lambda \in \Lambda$, the solution $x(t, t_0, x_0, \lambda)$ of (2.2) is defined in $[t_0, +\infty)$ and there exists $t_1 \geq t_0$ such that $x(t, t_0, x_0, \lambda) \in \mathcal{A}_r$ for every $t \geq t_1$.*

*(ii) If $x(t)$ is a solution of (2.2) defined for every $t \in \mathbb{R}$, with $\sup_{t\in\mathbb{R}} \|x(t)\| < +\infty$, then $x(t) \in \mathcal{A}_r$ for every $t \in \mathbb{R}$.*

In the next theorem, we give sufficient conditions to obtain synchronization of two systems of ordinary differential equations.

Let $f, g \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \Lambda, \mathbb{R}^n)$ satisfy the Lipschitz condition on bounded sets (2.1), with respect to $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. For $\lambda_1, \lambda_2 \in \Lambda$, consider the system

$$\begin{cases} \dot{x} = f(t,x,y,\lambda_1), \\ \dot{y} = g(t,x,y,\lambda_2). \end{cases} \tag{2.3}$$

**Theorem 2.2.** *We assume the following hypotheses:*

*(i) There exists a bounded set $\mathcal{B} \subset \mathbb{R}^n \times \mathbb{R}^n$ such that, for each $(t_0, x_0, y_0) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda_1, \lambda_2 \in \Lambda$, the solution $x(t) := x(t, t_0, x_0, y_0, \lambda_1, \lambda_2)$, $y(t) := y(t, t_0, x_0, y_0, \lambda_1, \lambda_2)$ of (2.3) is defined for $t \in [t_0, +\infty)$ and there exists $t_1 \geq t_0$ such that*

$$\big(x(t), y(t)\big) \in \mathcal{B}, \quad \textit{for every } t \geq t_1.$$

*(ii)* *There exist $V \in C^1(\mathbb{R} \times \mathbb{R}^n \times \Lambda, \mathbb{R})$ and positive constants $k_1$, $\rho$, $\alpha_1$, $\alpha_2$ and $\beta \geq 1$, such that*

$$\frac{\partial}{\partial t}V(t, y-x, \lambda)+\Big\langle \nabla V(t, y-x, \lambda), g(t,x,y,\lambda)-f(t,x,y,\lambda)\Big\rangle \leq -\rho V(t,y-x,\lambda),$$
$$\alpha_1\|y-x\|^\beta \leq V(t,y-x,\lambda) \leq \alpha_2\|y-x\|^\beta \quad \text{and} \quad \big\|\nabla V\big(t,y-x,\lambda\big)\big\| \leq k_1 \tag{2.4}$$

*for every $(x,y) \in \mathcal{B}$, $t \in \mathbb{R}$ and $\lambda \in \Lambda$, where*

$$\nabla V(t,x,\lambda) = \left(\frac{\partial V}{\partial x_1}(t,x,\lambda), \ldots, \frac{\partial V}{\partial x_n}(t,x,\lambda)\right),$$

*and $\nabla V(t,y-x,\lambda) = \nabla V(t,z,\lambda)\big|_{z=y-x}$.*

*(iii)* *There exists a function $H_1 : \mathbb{R}_+ \to \mathbb{R}_+$, with $H_1(0) = 0$, $H_1$ being continuous at $0 \in \mathbb{R}_+$, such that*

$$\|f(t,x,y,\lambda_2) - f(t,x,y,\lambda_1)\| \leq H_1(\|\lambda_2 - \lambda_1\|)$$

*for every $t \in \mathbb{R}$, $(x,y) \in \mathcal{B}$ and $\lambda_1, \lambda_2 \in \Lambda$.*

*Then we have*

$$\|y(t) - x(t)\| \leq M\|y(t_1) - x(t_1)\|e^{-\alpha(t-t_1)} + kH(\|\lambda_2 - \lambda_1\|), \quad \forall t \geq t_1,$$

*where $\alpha := \rho/\beta$, $k := (\frac{k_1}{\rho\alpha_1})^{1/\beta}$, $M := (\alpha_2/\alpha_1)^{1/\beta}$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$ is given by $H(r) := \big(H_1(r)\big)^{1/\beta}$.*

## 3. Switching Systems and Synchronization

Let $\Lambda \subset \mathbb{R}^m$ be a set of parameters. Let $f, g, F, G : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\times\Lambda \to \mathbb{R}^n$. Consider the two following systems of ordinary differential equations

$$\begin{cases} \dot{x} = f(t,x,y,\lambda_1), \\ \dot{y} = g(t,x,y,\lambda_2); \end{cases} \tag{3.1}$$

$$\begin{cases} \dot{u} = F(t,u,v,\mu_1), \\ \dot{v} = G(t,u,v,\mu_2). \end{cases} \tag{3.2}$$

Given $h > 0$ we consider the following switch system

$$\begin{cases} \dot{x} = f_h(t,x,y,\lambda_1), \\ \dot{y} = g_h(t,x,y,\lambda_2), \end{cases} \tag{3.3}$$

where

$$f_h(t,x,y,\lambda_1) = \begin{cases} f(t,x,y,\lambda_1), & \text{if } t \in [2kh,(2k+1)h], \quad k = 0,1,2,\ldots \\ F(t,x,y,\lambda_1), & \text{if } t \in [(2k+1)h, 2kh], \quad k = 0,1,2,\ldots \end{cases} \tag{3.4}$$

and

$$g_h(t,x,y,\lambda_1) = \begin{cases} g(t,x,y,\lambda_1), & \text{if } t \in [2kh,(2k+1)h], \quad k = 0,1,2,\ldots \\ G(t,x,y,\lambda_1), & \text{if } t \in [(2k+1)h, 2kh], \quad k = 0,1,2,\ldots \end{cases} \tag{3.5}$$

**Theorem 3.1.** *Assume that both system* (3.1) *and system* (3.2) *satisfiy the hypotheses of Teorem 2.2. Then there exists* $h > 0$ *such that the switch system* (3.3) *synchronizes.*

**Proof.** Let $(x(t), y(t))$ be a solution of (3.3). Then for $t \in [0, h]$ we have that $(x(t), y(t))$ is a solution of (3.1) and, since (3.1) satisfies the hypotheses of Theorem 2.2, we have that

$$|y(t) - x(t)| \leq Me^{-\alpha(t-0)}|y(0) - x(0)|$$

which implies that

$$|y(h) - x(h)| \leq Me^{-\alpha h}|y(0) - x(0)|.$$

Analogously, if $t \in [h, 2h]$ we have that $(x(t), y(t))$ is a solution of (3.2), and we have that

$$|y(t) - x(t)| \leq Me^{-\alpha(t-h)}|y(h) - x(h)| \leq Me^{-\alpha(t-h)}Me^{-\alpha h}|y(0) - x(0)|$$

which implies that

$$|y(2h) - x(2h)| \leq (Me^{-\alpha h})^2|y(0) - x(0)|.$$

Continuing this way, for $t \in [nh, (n+1)h]$ we have that

$$|y(t) - x(t)| \leq Me^{-\alpha(t-nh)}(Me^{-\alpha h})^n|y(0) - x(0)|.$$

Taking $h$ sufficiently large such that $Me^{-\alpha h} < 1$, given $\varepsilon > 0$ take $n$ such that $M(Me^{-\alpha h})^n|y(0) - x(0)| < \varepsilon$. For $t \in [(n+1)h, (n+2)h]$, we have

$$\begin{aligned}
|y(t) - x(t)| &\leq Me^{-\alpha(t-(n+1)h)}(Me^{-\alpha h})^{n+1}|y(0) - x(0)| \\
&\leq M(Me^{-\alpha h})^n Me^{-\alpha h}|y(0) - x(0)| \\
&\leq M(Me^{-\alpha h})^n|y(0) - x(0)| < \varepsilon.
\end{aligned}$$

□

# 4. Applications

In this section we present some examples to illustrate the results in the paper. The examples and the simulations in the examples demonstrate the applicability of the results to known systems of ODEs, hence showing the possibility for practical applications of these results. The code to perform the simulations are available at [25].

**Example 4.1** (Lorenz System)**.** We start with the classical Lorenz System:

$$\begin{cases}
\dot{x} = -\sigma x + \sigma y, \\
\dot{y} = rx - y - xz, \\
\dot{z} = -bz + xy,
\end{cases}$$

with the following reference values for the parameters: $\sigma = 10,\ r = 28,\ b = \frac{8}{3}$.

Now we consider the system:

$$\begin{cases} \dot{x} = -\sigma x + \sigma y, \\ \dot{y} = r\,(x + m(t)) - y - (x + m(t))\,z, \\ \dot{z} = -bz + (x + m(t))\,y, \\ \dot{u} = -\sigma u + \sigma v, \\ \dot{v} = r\,(x + m(t)) - v - (x + m(t))\,w, \\ \dot{w} = -bw + (x + m(t))\,v, \end{cases} \tag{4.1}$$

where $m(t)$ represents the signal to be codified and transmitted. If we let $X = x - u,\quad Y = y - v,\quad Z = z - w$ we obtain the following system:

$$\begin{cases} \dot{X} = -\sigma X + \sigma Y, \\ \dot{Y} = -Y - (x + m(t))\,Z, \\ \dot{Z} = -b\;Z + (x + m(t))\,Y. \end{cases} \tag{4.2}$$

We have

$$X\dot{X} = -\sigma X^2 + \sigma XY,\;\; Y\dot{Y} = -Y^2 - (x + m(t))\,YZ,\;\; Z\dot{Z} = -b\;Z^2 + (x + m(t))\,YZ.$$

If we consider the following Liapunov Function $\Phi = \frac{1}{2}(X^2 + \sigma Y^2 + \sigma Z^2)$ we have:

$$\begin{aligned} &\dot{\Phi} = -\sigma X^2 + \sigma XY + \sigma(-Y^2 - (x + m(t))\,YZ) + \sigma(-b\;Z^2 + (x + m(t))\,YZ), \\ &-\dot{\Phi} = \sigma X^2 - \sigma XY + \sigma Y^2 + \sigma b Z^2, \\ &-\dot{\Phi} - \rho\Phi = (\sigma - \frac{\rho}{2})X^2 - \sigma XY + \sigma(1 - \frac{\rho}{2})Y^2 + \sigma(b - \frac{\rho}{2})Z^2. \end{aligned}$$

The above quadratic form will be definitely positive if $\rho < 2b$ and

$$\begin{aligned} \begin{vmatrix} \sigma - \frac{\rho}{2} & \frac{\sigma}{2} \\ \frac{\sigma}{2} & \sigma(1 - \frac{\rho}{2}) \end{vmatrix} &= \sigma(\sigma - \frac{\rho}{2})(1 - \frac{\rho}{2}) - \frac{\sigma^2}{4} = \sigma\left[\sigma - \frac{(1+\sigma)\rho}{2} + \frac{\rho^2}{4}\right] - \frac{\sigma^2}{4} \\ &= \sigma\left[\sigma - \frac{(1+\sigma)\rho}{2} + \frac{\rho^2}{4} - \frac{\sigma}{4}\right] = \sigma\left[\frac{3\sigma}{4} - \frac{(1+\sigma)\rho}{2} + \frac{\rho^2}{4}\right] \\ &= \frac{\sigma}{4}\left[3\sigma - 2(1+\sigma)\rho + \rho^2\right] > 0. \end{aligned}$$

Now we will analyze the roots of:

$$\rho^2 - 2(1+\sigma)\rho + 3\sigma = 0.$$

If we let $\Delta = 4(1+\sigma)^2 - 12\sigma$ we obtain

$$\Delta = 4(\sigma^2 + 2\sigma + 1) - 12\sigma = 4(\sigma^2 + 2\sigma + 1 - 3\sigma) = 4(\sigma^2 - \sigma + 1) > 0.$$

Both roots are positive and they are given by:

$$\begin{aligned} \rho_1 &= \frac{2(1+\sigma) - \sqrt{4(\sigma^2 - \sigma + 1)}}{2} = (1+\sigma) - \sqrt{(\sigma^2 - \sigma + 1)}, \\ \rho_2 &= (1+\sigma) + \sqrt{(\sigma^2 - \sigma + 1)}. \end{aligned}$$

Therefore, we should choose:

$$0 < \rho \leq \inf\{2b,\ \rho_1\}.$$

This implies that

$$\dot{\Phi} \leq -\rho\ \Phi.$$

Consider the systems of ODEs

$$\begin{cases} \dot{x} = -\sigma_1 x + \sigma_1 y, \\ \dot{y} = r_1\left(x + m(t)\right) - y - \left(x + m(t)\right) z, \\ \dot{z} = -b_1 z + \left(x + m(t)\right) y, \\ \dot{u} = -\sigma_1 u + \sigma_1 v, \\ \dot{v} = r_1\left(x + m(t)\right) - v - \left(x + m(t)\right) w, \\ \dot{w} = -b_1 w + \left(x + m(t)\right) v; \end{cases} \tag{4.3}$$

$$\begin{cases} \dot{X} = -\sigma_2 X + \sigma_2 Y, \\ \dot{Y} = r_2\left(X + m(t)\right) - Y - \left(X + m(t)\right) Z, \\ \dot{Z} = -b_2 Z + \left(X + m(t)\right) Y, \\ \dot{U} = -\sigma_2 U + \sigma_2 V, \\ \dot{V} = r_2\left(X + m(t)\right) - V - \left(X + m(t)\right) W, \\ \dot{W} = -b_2 W + \left(X + m(t)\right) V. \end{cases} \tag{4.4}$$

The first system is used to transmit the message $m(t)$ in the interval $[0, h]$ the second to transmit the message $m(t)$ in the interval $[h, 2h]$, then the first is used to transmit in the interval $[2h, 3h]$, and so on.

For the simulations presented in Figure 2 we used the following parameter values: $\sigma_1 = 10$, $r_1 = 28$, $b_1 = 8/3$, $\sigma_2 = 13$, $r_2 = 32$, $b_2 = 5$, $h = 10$, and $m(t) = 3\cos(5t)$.

**Example 4.2** (Chua Like System.)**.** We first consider the Chua Like System:

$$\begin{cases} \dot{x} = -\alpha x + \alpha y - \alpha\ h\left(x, a, b\right), \\ \dot{y} = -x - y + z, \\ \dot{z} = -\beta y - \sigma z, \end{cases} \tag{4.5}$$

where $h\left(x, a, b\right) = bx + \frac{(a-b)}{2}\left(|x+1| - |x-1|\right)$. $a > b$ and $b < 1$. Here we use the following reference values $\alpha = 7, \beta = 100,\ a = \frac{8}{7},\ \sigma = \frac{1}{2},\ b = \frac{5}{7}$.

Now we consider the coupled master-slave system:

$$\begin{cases} \dot{x} = -\alpha x + \alpha y - \alpha\ h\left(x + m(t), a, b\right), \\ \dot{y} = -\left(x + m(t)\right) - y + z, \\ \dot{z} = -\beta y - \sigma z, \\ \dot{u} = -\alpha u + \alpha v - \alpha\ h\left(x + m(t), a, b\right), \\ \dot{v} = -\left(x + m(t)\right) - v + w, \\ \dot{w} = -\beta v - \sigma w. \end{cases} \tag{4.6}$$

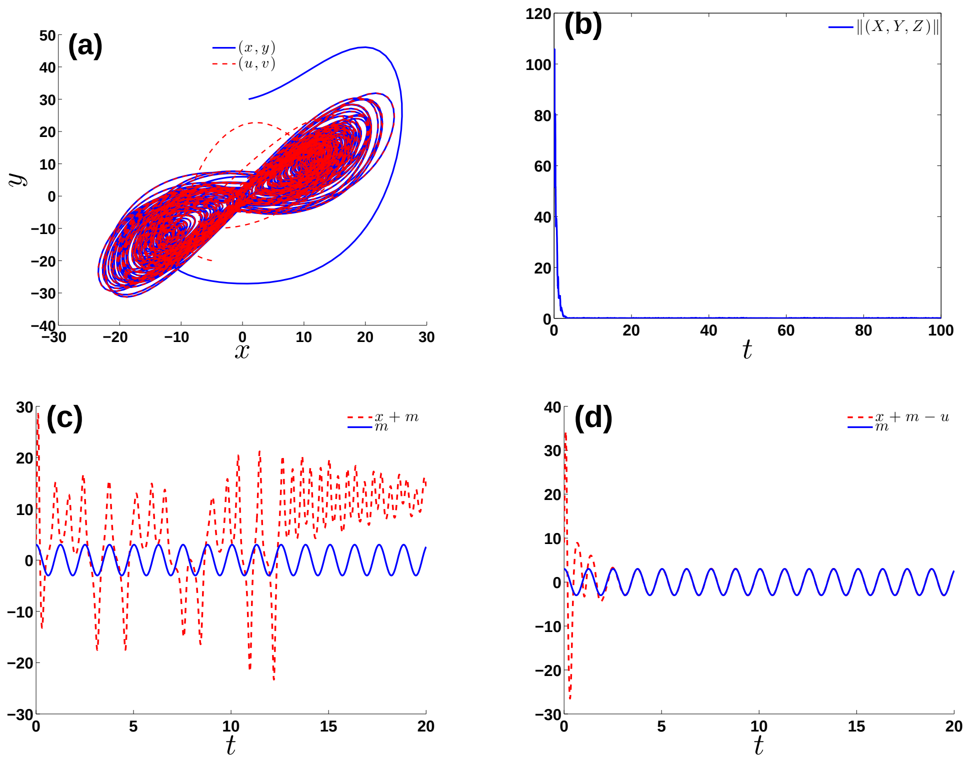


**Figure 2.** Simulations for the Lorenz system. In **(a)** we plot the solutions $(x(t), y(t))$ and $(u(t), v(t))$; in **(b)** we show $|x(t) - u(t)| + |y(t) - v(t)| + |z(t) - w(t)|$; in **(c)** we plot the original message $m(t)$ and the coded message $x(t) + m(t)$; and in **(d)** we plot the original message $m(t)$ and the decoded message $x(t) + m(t) - u(t)$.

If we let $X = x - u,\ Y = y - v,\ Z = z - w$ and take the difference between the above two systems we obtain the system:

$$\begin{cases} \dot{X} = -\alpha X + \alpha Y, \\ \dot{Y} = -Y + Z, \\ \dot{Z} = -\beta Y - \sigma Z. \end{cases} \tag{4.7}$$

We have:

$$X\dot{X} = -\alpha X^2 + \alpha XY,\ Y\dot{Y} = -Y^2 + YZ,\ Z\dot{Z} = -\beta YZ - \sigma Z^2.$$

We consider the following Liapunov Function:

$$\begin{aligned} &\Phi = \frac{1}{2}(X^2 + \beta Y^2 + Z^2), \\ &\dot{\Phi} = X\dot{X} + \beta Y\dot{Y} + Z\dot{Z} = -\alpha X^2 + \alpha XY - \beta Y^2 + \beta YZ + -\beta YZ - \sigma Z^2, \\ &-\dot{\Phi} = \alpha X^2 - \alpha XY + \beta Y^2 + \sigma Z^2, \\ &-\dot{\Phi} - \rho\Phi = (\alpha - \frac{\rho}{2})X^2 - \alpha XY + \beta(1 - \frac{\rho}{2})Y^2 + (\sigma - \frac{\rho}{2})Z^2. \end{aligned}$$

The above quadratic function will be definitely positive if we impose that $\rho < \inf\{2\sigma, 2\alpha\}$ and

$$\begin{vmatrix} \alpha - \frac{\rho}{2} & -\frac{\alpha}{2} \\ -\frac{\alpha}{2} & \beta(1-\frac{\rho}{2}) \end{vmatrix} = \beta(\alpha - \frac{\rho}{2})(1-\frac{\rho}{2}) - \frac{\alpha^2}{4} = \beta\left[\alpha - \frac{(1+\alpha)\rho}{2} + \frac{\rho^2}{4}\right] - \frac{\alpha^2}{4} > 0,$$

$$4\left[\alpha - \frac{(1+\alpha)\rho}{2} + \frac{\rho^2}{4} - \frac{\alpha^2}{4\beta}\right] = \rho^2 - 2(1+\alpha)\rho + 4\alpha - \frac{\alpha^2}{\beta} > 0.$$

Let

$$\Delta = 4(1+\alpha)^2 - 4(4\alpha - \frac{\alpha^2}{\beta}) = 4(\alpha^2 + 2\alpha + 1) - 16\alpha + \frac{4\alpha^2}{\beta} = 4(\alpha-1)^2 + \frac{4\alpha^2}{\beta} > 0.$$

The roots of the equation $\rho^2 - 2(1+\alpha)\rho + 4\alpha - \frac{\alpha^2}{\beta} = 0$ are both positive and are given by:

$$\rho_1 = (1+\alpha) - \sqrt{(\alpha-1)^2 + \frac{\alpha^2}{\beta}}, \quad \rho_2 = (1+\alpha) + \sqrt{(\alpha-1)^2 + \frac{\alpha^2}{\beta}}.$$

Therefore we could take $\rho < \inf\{2\alpha, 2\sigma, \rho_1\}$. This implies that

$$\dot{\Phi} \leq -\rho\,\Phi.$$

Consider the systems of ODEs:

$$\begin{cases} \dot{x} = -\alpha_1 x + \alpha_1 y - \alpha_1 h\left(x + m(t), a_1, b_1\right), \\ \dot{y} = -\left(x + m(t)\right) - y + z, \\ \dot{z} = -\beta_1 y - \sigma_1 z, \\ \dot{u} = -\alpha_1 u + \alpha_1 v - \alpha_1 h\left(x + m(t), a_1, b_1\right), \\ \dot{v} = -\left(x + m(t)\right) - v + w, \\ \dot{w} = -\beta_1 v - \sigma_1 w; \end{cases} \tag{4.8}$$

$$\begin{cases} \dot{X} = -\alpha_2 X + \alpha_2 Y - \alpha_2 h\left(X + m(t), a_2, b_2\right), \\ \dot{Y} = -\left(X + m(t)\right) - Y + Z, \\ \dot{Z} = -\beta_2 Y - \sigma_2 Z, \\ \dot{U} = -\alpha_2 U + \alpha_2 V - \alpha_2 h\left(X + m(t), a_2, b_2\right), \\ \dot{V} = -\left(X + m(t)\right) - V + W, \\ \dot{W} = -\beta_2 V - \sigma_2 W, \end{cases} \tag{4.9}$$

where

$$h(x, a, b) = bx + \frac{(a-b)}{2}\left(|x+1| - |x-1|\right).$$

As in the previous example these systems will play the role of our switching systems. In Figure 3 we show some simulations for these systems, where the following parameter values are used: $\alpha_1 = 7$, $\beta_1 = 100$, $\mu_1 = 1/2$, $a_1 = 8/7$, $b_1 = 5/7$, $\alpha_2 = 12$, $\beta_2 = 80$, $\mu_2 = 1/3$, $a_2 = 6/7$, $b_2 = 4/7$, $h = 10$, and $m(t) = 3\cos(5t)$.

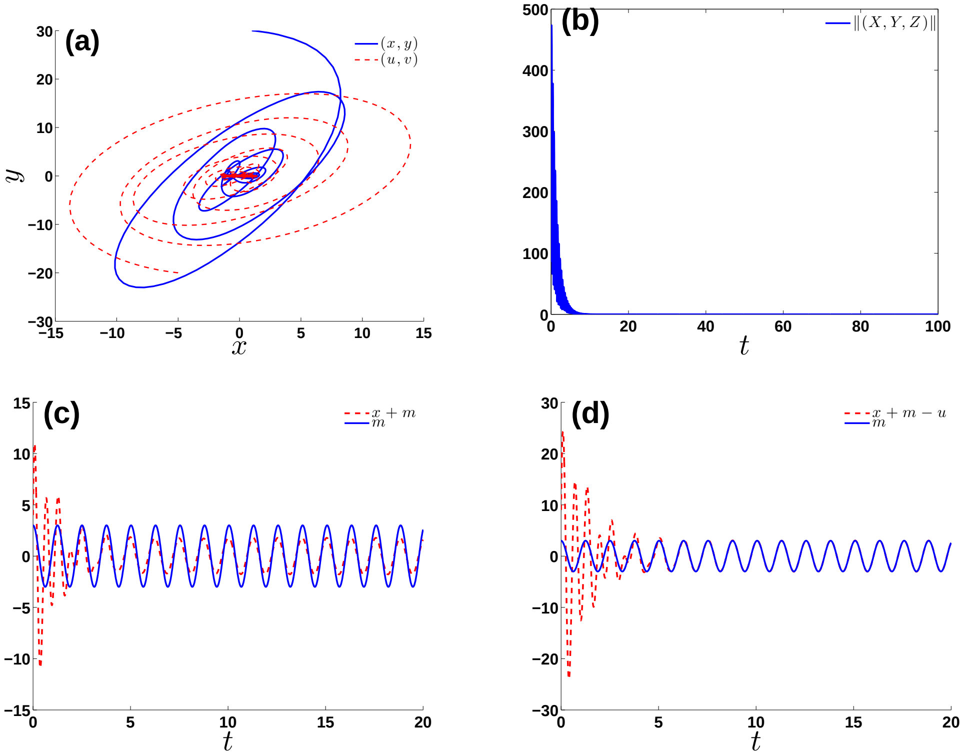


**Figure 3.** Simulations for Chua's system. In **(a)** we plot the solutions $(x(t), y(t))$ and $(u(t), v(t))$; in **(b)** we show $|x(t) - u(t)| + |y(t) - v(t)| + |z(t) - w(t)|$; in **(c)** we plot the original message $m(t)$ and the coded message $x(t) + m(t)$; and in **(d)** we plot the original message $m(t)$ and the decoded message $x(t) + m(t) - u(t)$.

**Example 4.3** (Chen's System)**.** Consider the Chen's System:

$$\begin{cases} \dot{x} = -ax + ay, \\ \dot{y} = -ax + cx + cy - xz, \\ \dot{z} = -bz + xy, \end{cases} \tag{4.10}$$

with the reference values $a = 35,\ b = 3,\ c = 28$.

Consider the coupled master-slave system of ODEs

$$\begin{cases} \dot{x} = -ax + ay, \\ \dot{y} = (c-a)x + (c-a)y - (x + m(t))\, z + a(y + \ell(t)), \\ \dot{z} = -bz + (x + m(t))\, y, \\ \dot{u} = -au + av, \\ \dot{v} = (c-a)u + (c-a)v - (x + m(t))\, w + a(y + \ell(t)), \\ \dot{w} = -bw + (x + m(t))\, v. \end{cases} \tag{4.11}$$

Consider the difference between the two above systems considering $X = x -$

$u,\ Y = y - v,\ Z = z - w$:

$$\begin{cases} \dot{X} = -aX + aY, \\ \dot{y} = (c-a)X + (c-a)Y - (x + m(t))\, Z, \\ \dot{z} = -bZ + (x + m(t))\, Y. \end{cases} \tag{4.12}$$

We take the Liapunov function:

$$\Phi = \frac{1}{2}\left[\frac{a-c}{a}X^2 + Y^2 + Z^2\right],$$
$$X\dot{X} = aYX - aX^2,$$
$$Y\dot{Y} = (c-a)XY - (x+m(t))\, ZY + (c-a)Y^2,$$
$$Z\dot{Z} = -bZ^2 + (x+m(t))\, YZ.$$

Therefore,

$$\begin{aligned} \dot{\Phi} &= \frac{a-c}{a}[aYX - aX^2] + c-a)XY - (x+m(t))\, ZY + (c-a)Y^2 - bZ^2 \\ &\quad + (x+m(t))\, YZ \\ &= -(a-c)X^2 - (a-c)Y^2 - bZ^2, \\ -\dot{\Phi} &= (a-c)X^2 + (a-c)Y^2 + bZ^2, \\ -\dot{\Phi} - \rho\Phi &= (a-c)X^2 + (a-c)Y^2 + bZ^2 - \frac{\rho}{2}\left[\frac{a-c}{a}X^2 + Y^2 + Z^2\right] \\ &= [(a-c) - \frac{\rho}{2}\frac{(a-c)}{a}]X^2 + [(a-c) - \frac{\rho}{2}]Y^2 + (b - \frac{\rho}{2})Z^2. \end{aligned}$$

This implies that we should chose $\rho > 0$ satisfying:

$$\rho < 2a,\ \rho < 2(a-c),\ \rho < 2b.$$

Due to the initial established parameters we choose $\rho < 2b$.
This implies that

$$\dot{\Phi} \le -\rho\ \Phi.$$

The two systems below will play the role of our switching systems.

$$\begin{cases} \dot{x} = -a_1x + a_1y, \\ \dot{y} = -a_1x + c_1\,(x+m(t)) + (c_1-a_1)y - (x+m(t))\, z + a_1(y+\ell(t)), \\ \dot{z} = -b_1z + (x+m(t))\, y, \\ \dot{u} = -a_1u + a_1v, \\ \dot{v} = -a_1u + c_1\,(x+m(t)) + (c_1-a_1)v - (x+m(t))\, w + a_1(y+\ell(t)), \\ \dot{w} = -b_1w + (x+m(t))\, v; \end{cases} \tag{4.13}$$

$$\begin{cases} \dot{X} = -a_2X + a_2Y, \\ \dot{Y} = -a_2X + c_2\,(x+m(t)) + (c_2-a_2)Y - (x+m(t))\, Z + a_2(y+\ell(t)), \\ \dot{Z} = -b_2Z + (x+m(t))\, Y, \\ \dot{U} = -a_2U + a_2V, \\ \dot{V} = -a_2U + c_2\,(x+m(t)) + (c_2-a_2)V - (x+m(t))\, W + a_2(y+\ell(t)), \\ \dot{W} = -b_2W + (x+m(t))\, V. \end{cases} \tag{4.14}$$

We presente some simulations for these systems in Figure 4. In these simulations we used the following parameter values: $a_1 = 35$, $b_1 = 3$, $c_1 = 28$, $a_2 = 37$, $b_2 = 5$, $c_2 = 23$, $h = 10$, $m(t) = 3\cos(5t)$, and $\ell(t) = 2\sin(3t)$.

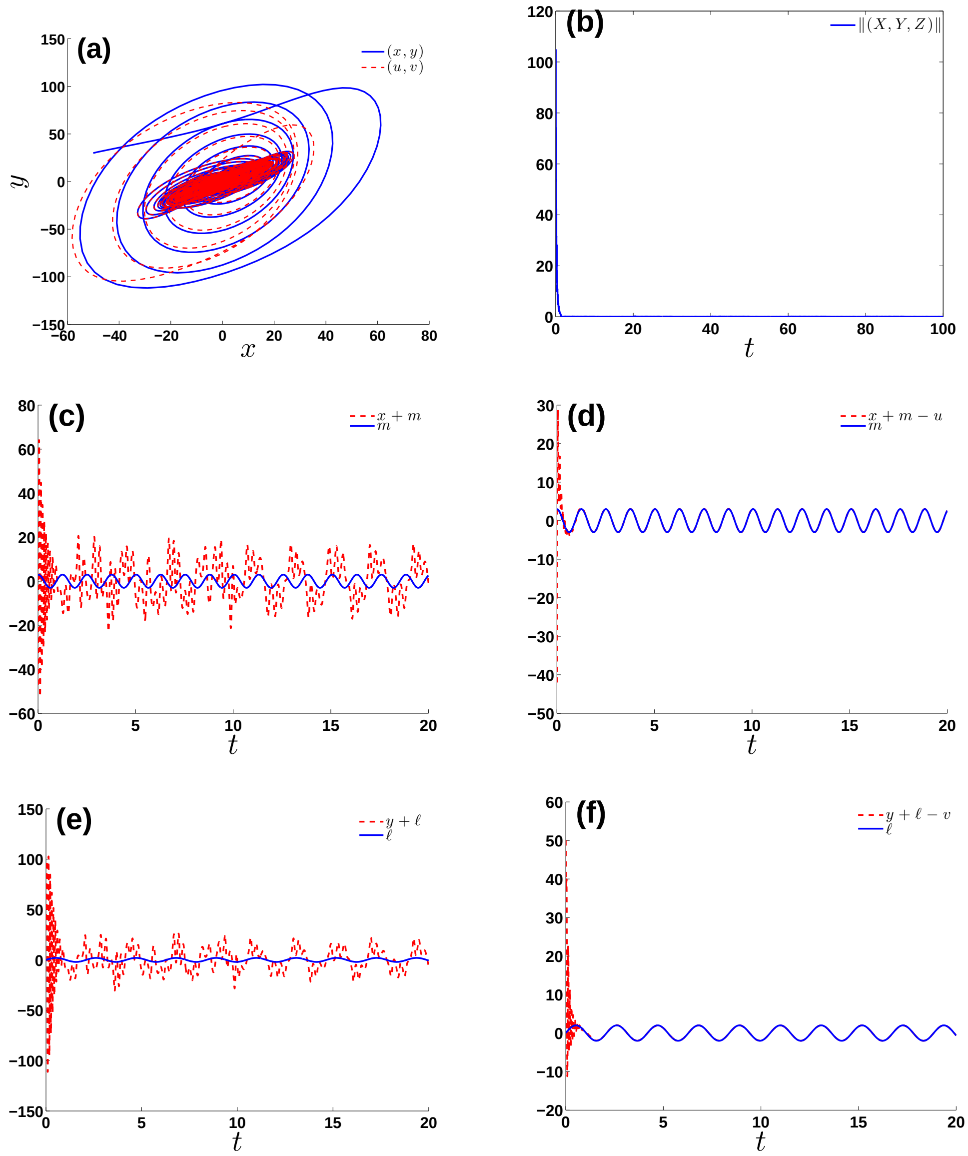


**Figure 4.** Simulations for Chen's system. In **(a)** we plot the solutions $(x(t), y(t))$ and $(u(t), v(t))$; in **(b)** we show $|x(t) - u(t)| + |y(t) - v(t)| + |z(t) - w(t)|$; in **(c)** we plot the original message $m(t)$ and the coded message $x(t) + m(t)$; and in **(d)** we plot the original message $m(t)$ and the decoded message $x(t) + m(t) - u(t)$; in **(e)** we plot the original message $\ell(t)$ and the coded message $y(t) + \ell(t)$; and in **(f)** we plot the original message $\ell(t)$ and the decoded message $y(t) + \ell(t) - v(t)$.

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