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ON THE STABILITY OF A POPULATION MODEL WITH NONLOCAL DISPERSAL*

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Abstract This paper is concerned with a nonlocal dispersal population model with spatial competition and aggregation. We establish the existence and uniqueness of positive solutions by the method of coupled upper-lower solutions. We obtain the global stability of the stationary solutions.

Keywords Nonlocal dispersal, existence and uniqueness, stability.

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1. Introduction

Let $J : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a non-negative function. It is known from [1,13] that the nonlocal dispersal operator of form

$$\mathscr{D}[u](x,t) = \int_{\mathbb{R}^N} J(x,y)u(y,t)dy - u(x,t)$$

and its variations have been widely used to model different dispersal phenomena in material science and ecology. The nonlocal dispersal operator $\mathscr{D}[u]$ also characterizes the diffusion of species which may occur between nonadjacent locations [5, 6, 10, 20–22, 24]. There is quite an extensive literature on the study of nonlocal dispersal problems, for example, the papers [4, 9, 14, 15, 18, 19, 21, 23]. In this paper, we consider the nonlocal dispersal equation

$$\begin{cases} u_t(x,t) = \mathscr{D}[u](x,t) + u(x,t)(1 + \alpha u(x,t) - \beta u^2(x,t) - \delta G * u), \\ u(x,0) = u_0(x) \end{cases} \tag{1.1}$$

for $(x,t) \in \mathbb{R}^N \times (0,\infty)$, where α, β are positive constants such that $\delta = 1+\alpha-\beta > 0$ and $G * u$ is given by

$$G * u(x,t) = \int_{\mathbb{R}^N} G(x-y)u(y,t)dy.$$

Throughout this paper, we make the following assumptions.

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($A1$) $J \in C(\mathbb{R}^N)$ is bounded, non-negative and $\int_{\mathbb{R}^N} J(x,y)dy = 1$ for all $x \in \mathbb{R}^N$. There exist $\alpha > 0$ and $l > 0$ such that $J(x,y) > \alpha$ if $x, y \in \bar{\Omega}$ and $|x-y| < l$.

($A2$) $G \in C(\mathbb{R}^N)$ is non-negative and $G * 1 = 1$.

($A3$) $u_0 \in [\epsilon, 1/\epsilon]$ for some positive constant ϵ.

Nonlocal dispersal equation (1.1) is used to model the diffusion of a single species in $\mathbb{R}^N$ whose density is given by u. In (1.1), the term αu represents an advantage to the population in local aggregation or grouping, by making available different food success or protecting measure against predation [2]. The term $-\beta u^2$ represents competition for space. The integral term $\delta G * u$ represents intraspecific competition for food resources with non-negative weight function G. The reader is referred to [3, 7, 11, 12] for the study of diffusion population model with aggregation and nonlocal competition effects.

Among this paper, we mainly focus on the stability analysis of the stationary solution to (1.1), that is the solution of (1.1) which is independent of time t. It follows from [1] that (1.1) is a nonlocal version of the reaction-diffusion equation

$$u_t(x,t) = \Delta u + u(x,t)(1 + \alpha u(x,t) - \beta u^2(x,t) - \delta G * u),$$

which is well investigated [3]. For the reaction-diffusion equation with aggregations and nonlocal competitions as considered in [2], it could be transformed into a system by using a special form of function G. Then the nonlocal term which contains a spatial average is transformed into local term. So the linear stability of uniform state and some bifurcation phenomena of the local problem are well studied. It is not the case for nonlocal problems, as the dispersal operator $\mathscr{D}$ is nonlocal and there is a deficiency of regularization [4]. In order to overcome these differences, inspired by the recent work of Deng and Wu [8], we define the coupled upper-lower solutions and obtain the existence and uniqueness of global solution to (1.1). We then prove the stability of the stationary solution by approximation method.

The paper is organized as follows. In Section 2, we establish the existence and uniqueness of non-negative solutions to (1.1). We discuss the stability of stationary solution in Section 3.

2. Existence and uniqueness of solution

It is well-known from [16, 17] that the monotone iteration method is effective in the study of existence and uniqueness of solutions to classical reaction-diffusion equations. In this paper, we consider the nonlocal dispersal equation (1.1). Since the comparison principle is not valid for (1.1), we cannot use the classical nonlocal upper-lower solutions method [1]. By ($A3$), we know that $u_0 \notin L^1(\mathbb{R}^N)$, the argument of [7, 18] do not apply. In order to obtain the existence and uniqueness of the solutions to (1.1), we need to define new type upper-lower solutions. To do this, we consider the nonlocal dispersal equation

$$\begin{cases} v_t(x,t) = \mathscr{D}[v](x,t) + v(x,t)(f(v) - \delta G * v), \\ v(x,0) = v_0(x) \end{cases} \tag{2.1}$$

for $(x,t) \in \mathbb{R}^N \times (0,\infty)$. We impose the following assumptions.

($H1$) f is a continuously differentiable function on $[0,\infty)$.

$(H2)$ $v_0 \in L^\infty(\mathbb{R}^N)$ is positive.

Let us give the definition of upper-lower solutions.

Definition 2.1. A pair of functions $\omega(x,t)$ and $\phi(x,t)$ are called upper-lower solutions of (2.1) on $\mathbb{R}^N \times (0,T)$ if they satisfy the following conditions.

(i) ω and ϕ are bounded.

(ii) $\omega(\cdot,x), \phi(\cdot,x) \in C^1(0,T) \bigcap C[0,T)$ and $\omega(x,0) \geq v_0(x) \geq \phi(x,0)$ in $\mathbb{R}^N$.

(iii) For any $(x,t) \in \mathbb{R}^N \times (0,T)$,

$$\omega_t(x,t) \geq \mathscr{D}[\omega](x,t) + \omega(x,t)(f(\omega) - \delta G * \phi), \tag{2.2}$$

$$\phi_t(x,t) \leq \mathscr{D}[\phi](x,t) + \phi(x,t)(f(\phi) - \delta G * \omega). \tag{2.3}$$

We can show that the upper-lower solutions defined above are ordered.

Theorem 2.1. *Assume that ω and ϕ are a pair of non-negative upper-lower solutions of* (2.1). *Then*

$$\phi(x,t) \leq \omega(x,t) \quad \text{in } \mathbb{R}^N \times [0,T).$$

Proof. Set $\theta(x,t) = \omega(x,t) - \phi(x,t)$, it follows from (2.2)-(2.3) that

$$\begin{aligned}\theta_t(x,t) &\geq \mathscr{D}[\theta](x,t) + \omega(x,t)f(\omega) - \phi(x,t)f(\phi) + \delta\phi(x,t)G*\omega - \delta\omega(x,t)G*\phi \\ &= \mathscr{D}[\theta](x,t) + [f(\omega) + \phi f'(\theta_1) - \delta G*\omega]\theta(x,t) + \delta\omega(x,t)G*\theta,\end{aligned}$$

where θ_1 is between $\phi(x,t)$ and $\omega(x,t)$. Let us denote $a(x,t) = f(\omega) + \phi f'(\theta_1) - \delta G*\omega$, then there holds

$$\begin{cases}\theta_t(x,t) \geq \int_{\mathbb{R}^N}[J(x,y) + \delta\omega G(x-y)]\theta(y,t)dy + (a(x,t)-1)\theta(x,t), \\ \theta(x,0) \geq 0.\end{cases}$$

Note that G is non-negative and $\int_{\mathbb{R}^N} G(x)dx = 1$, by the comparison principle (see [24]), we get $\theta \geq 0$. □

Theorem 2.2. *Suppose that ω and ϕ are a pair of non-negative upper-lower solutions of* (2.1) *in $\mathbb{R}^N\times[0,T)$. Then* (2.1) *admits a unique solution $u\in C^1((0,T);L^\infty(\mathbb{R}^N))$ which satisfies the relation*

$$\phi(x,t) \leq u(x,t) \leq \omega(x,t) \quad \text{in } \mathbb{R}^N \times [0,T).$$

Proof. Note that ω and ϕ are bounded, we can choose a constant $L > 0$ such that $f(s) + sf'(s) + L > 0$ for $s \in [0, \|\omega\|_\infty + \|\phi\|_\infty]$. We give the main proof in the following steps.

Step 1. Consider the linear nonlocal dispersal equations

$$\begin{cases}\phi_t^k(x,t) = \mathscr{D}[\phi^k](x,t) + \phi^{k-1}(x,t)f(\phi^{k-1}) - \delta\phi^k G*\omega^{k-1} - L(\phi^k - \phi^{k-1}), \\ \phi^k(x,0) = v_0(x)\end{cases} \tag{2.4}$$

and

$$\begin{cases}\omega_t^k(x,t) = \mathscr{D}[\omega^k](x,t) + \omega^{k-1}(x,t)f(\omega^{k-1}) - \delta\omega^k G*\phi^{k-1} - L(\omega^k - \omega^{k-1}), \\ \omega^k(x,0) = v_0(x),\end{cases} \tag{2.5}$$

where $k = 1, 2, \ldots$, $\phi^0(x,t) = \phi(x,t)$ and $\omega^0(x,t) = \omega(x,t)$. For each $k \geq 1$, we know that the sequences $\{\phi^k\}$ and $\{\omega^k\}$ are well defined in $\mathbb{R}^N \times (0,T)$ (see [1]).

Step 2. We show that the sequences defined above satisfy

$$\phi(x,t) \leq \phi^k(x,t) \leq \phi^{k+1}(x,t) \leq \omega^{k+1}(x,t) \leq \omega^k(x,t) \leq \omega(x,t) \tag{2.6}$$

for $k = 1, 2, \ldots$ and $(x,t) \in \mathbb{R}^N \times [0,T)$.

Let us begin to show that (2.6) holds for $k = 1$. Take $z(x,t) = \phi^1(x,t) - \phi(x,t)$, it follows from (2.3) and (2.4) that

$$\begin{cases} z_t(x,t) \geq \mathscr{D}[z](x,t) - [\delta G * \omega(x,t) + L]z(x,t) & \text{in } \mathbb{R}^N, \\ z(x,0) \geq 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{2.7}$$

Thus we know that $z(x,t) \geq 0$ in $\mathbb{R}^N \times [0,T)$, since 0 is a lower-solution to (2.7). A similar discussion gives that $\omega^1(x,t) \leq \omega(x,t)$ in $\mathbb{R}^N \times [0,T)$.

Now denote $z^1(x,t) = \omega^1(x,t) - \phi^1(x,t)$. We know from Theorem 2.1 that $\phi(x,t) \leq \omega(x,t)$. Then by (2.4)-(2.5), we have

$$\begin{aligned} z^1_t =& \mathscr{D}[z^1](x,t) - Lz^1(x,t) - \delta G * \omega(x,t) z^1(x,t) \\ &+ [\theta(x,t) f'(\theta) + f(\theta) + L](\omega(x,t) - \phi(x,t)) \\ \geq& \mathscr{D}[z^1](x,t) - Lz^1(x,t) - \delta G * \omega(x,t) z^1(x,t), \end{aligned}$$

and $z^1(x,0) \geq 0$, here θ is between $\phi(x,t)$ and $\omega(x,t)$. Thus we get $v^1(x,t) \leq \omega^1(x,t)$ in $\mathbb{R}^N \times [0,T)$.

We then show that $\phi^1(x,t)$ and $\omega^1(x,t)$ are a pair of lower-upper solutions to (2.1). Since $\phi(x,t) \leq \phi^1(x,t)$ and $\omega^1(x,t) \leq \omega(x,t)$, we have

$$\begin{aligned} & \phi^1_t(x,t) - \mathscr{D}[\phi^1](x,t) - \phi^1(x,t) f(\phi^1) + \delta \phi^1(x,t) G * \omega^1 \\ =& \phi(x,t) f(\phi) - \phi^1(x,t) f(\phi^1) + \delta \phi^1(x,t) G * \omega^1 - \delta \phi^1(x,t) G * \omega - L(\phi^1 - \phi) \\ =& \delta \phi^1(x,t) G * (\omega^1 - \omega) - [f(\theta^1) + \theta^1(x,t) f'(\theta^1) + L](\phi^1 - \phi) \\ \leq& 0, \end{aligned}$$

here θ^1 is between $\phi(x,t)$ and $\omega(x,t)$. Similarly, we get

$$\begin{aligned} & \omega^1_t(x,t) - \mathscr{D}[\omega^1](x,t) - \omega^1(x,t) f(\omega^1) + \delta \omega^1(x,t) G * \phi^1 \\ =& \omega(x,t) f(\omega) - \omega^1(x,t) f(\omega^1) + \delta \omega^1(x,t) G * \phi^1 - \delta \omega^1(x,t) G * \phi - L(\omega^1 - \omega) \\ \geq& 0. \end{aligned}$$

By choosing ω^1 and ϕ^1 as upper-lower solutions, after the similar above argument, we have

$$\phi^1(x,t) \leq \phi^2(x,t) \leq \omega^2(x,t) \leq \omega^1(x,t) \text{ in } \mathbb{R}^N \times [0,T).$$

Also $\phi^2(x,t)$ and $\omega^2(x,t)$ are lower-upper solutions of (2.1). The conclusion of (2.6) is followed by induction argument.

Step 3. Since the sequences $\{\phi^k\}_{k=1}^\infty$ and $\{\omega^k\}_{k=1}^\infty$ are monotone and bounded, there exist two function $\bar{\phi}$ and $\bar{\omega}$ such that

$$\lim_{k\to\infty} \phi^k(x,t) = \bar{\phi}(x,t) \text{ and } \lim_{k\to\infty} \omega^k(x,t) = \bar{\omega}(x,t)$$

pointwise in $\mathbb{R}^N \times [0,T)$. It is trivial to see that $\bar{\phi} \le \bar{\omega}$,

$$\begin{cases} \bar{\omega}_t(x,t) = \mathscr{D}[\bar{\omega}](x,t) + \bar{\omega}(x,t)f(\bar{\omega}) - \delta\bar{\omega}G * \bar{\phi}, \\ \bar{\omega}(x,0) = v_0(x) \end{cases}$$

and

$$\begin{cases} \bar{\phi}_t(x,t) = \mathscr{D}[\bar{\phi}](x,t) + \bar{\phi}(x,t)f(\bar{\phi}) - \delta\bar{\phi}G * \bar{\omega}, \\ \bar{\phi}(x,0) = v_0(x). \end{cases}$$

Meanwhile, we can treat $\bar{\omega}$ and $\bar{\phi}$ as a pair of upper-lower solutions to (2.1). Thus we have $\bar{\phi}(x,t) \ge \bar{\omega}(x,t)$ and then $\bar{\phi}(x,t) = \bar{\omega}(x,t)$ in $\mathbb{R}^N \times [0,T)$. This gives $\bar{\phi}$ is a solution to (2.1).

Step 4. We show that $\bar{\phi}(x,\cdot) \in C^1((0,T)) \bigcap C([0,T))$. Note that $\bar{\phi}$ is a solution of (2.1), we get

$$\bar{\phi}(x,t) = v_0(x) + \int_0^t [\mathscr{D}[\bar{\phi}](x,s) + \bar{\phi}(x,s)(f(x,s,\bar{\phi}) - \delta G * \bar{\phi}(x,s))]ds,$$

and so

$$\bar{\phi}(x,t+\varepsilon) - \bar{\phi}(x,t) = \int_t^{t+\varepsilon} [\mathscr{D}[\bar{\phi}](x,s) + \bar{\phi}(x,s)(f(x,s,\bar{\phi}) - \delta G * \bar{\phi}(x,s))]ds,$$

here ε is a parameter. Then we know that

$$\begin{aligned} |\bar{\phi}(x,t+\varepsilon) - \bar{\phi}(x,t)| &\le \left| \int_t^{t+\varepsilon} [\mathscr{D}[\bar{\phi}](x,s) + \bar{\phi}(x,s)(f(x,s,\bar{\phi}) - \delta G * \bar{\phi}(x,s))]ds \right| \\ &\le |\varepsilon| C, \end{aligned}$$

where $C > 0$ is constant which is independent of ε. This gives that $\bar{\phi}(x,\cdot) \in C([0,T))$. Furthermore, we have

$$\begin{aligned} &\lim_{\varepsilon\to 0} \frac{\bar{\phi}(x,t+\varepsilon) - \bar{\phi}(x,t)}{\varepsilon} \\ =&\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} [\mathscr{D}[\bar{\phi}](x,s) + \bar{\phi}(x,s)(f(x,s,\bar{\phi}) - \delta G * \bar{\phi}(x,s))]ds \\ =&\lim_{\varepsilon\to 0} [\mathscr{D}[\bar{\phi}](x,t+\varepsilon) + \bar{\phi}(x,t+\varepsilon)(f(x,t+\varepsilon,\bar{\phi}) - \delta G * \bar{\phi}(x,t+\varepsilon))] \\ =&\mathscr{D}[\bar{\phi}](x,t) + \bar{\phi}(x,t)(f(x,t,\bar{\phi}) - \delta G * \bar{\phi}(x,t)). \end{aligned}$$

Thus $\bar{\phi}(x,\cdot) \in C^1((0,T))$.

Step 5. We give the uniqueness by comparison argument. Assume that $v_1(x,t)$ and $v_2(x,t)$ are two bounded solutions to (2.1). Let $\psi(x,t) = v_1(x,t) - v_2(x,t)$, then we get

$$\begin{aligned} \psi_t = &\int_{\mathbb{R}^N} [J(x,y) + \delta v_2 G(x-y)]\psi(y,t)dy - \delta\psi(x,t)G * v_1(x,s) \\ &+ (f(\hat{v}) + \hat{v}f'(\hat{v}))\psi(x,t), \end{aligned}$$

where $\hat{v}$ is between $v_1(x,t)$ and $v_2(x,t)$. But $\psi(x,0) = 0$, thus we have $\psi \equiv 0$ and this gives the uniqueness of the solution. □

Remark 2.1. In Definition 2.1, the upper-lower solutions satisfy the inequalities (2.2) and (2.3), respectively. We shall point out that (2.2)-(2.3) can be replaced by

$$\omega_t(x,t) \geq \mathscr{D}[\omega](x,t) + \omega(x,t)f(\omega) - \delta\phi(x,t)G * \phi,$$

and

$$\phi_t(x,t) \leq \mathscr{D}[\phi](x,t) + \phi(x,t)f(\phi) - \delta\omega(x,t)G * \omega.$$

In this case, the conclusion of Theorem 2.2 still holds.

Now we give the main result of this section.

Theorem 2.3. *Assume that there exists $P > 0$ such that $f(s) \leq 0$ for $s \geq P$. Then* (2.1) *admits a unique solution $v(x,t)$ in $\mathbb{R}^N \times [0,\infty)$.*

Proof. We can see that $\omega = P + \|v_0\|_\infty$ and $\phi = 0$ are a pair of upper-lower solutions of (2.1). It follows from Theorem 2.2 that (2.1) admits a unique solution. □

3. Stability of stationary solution

In this section, we establish the long time dynamic behavior of the nonlocal dispersal equation (1.1). In view of Theorem 2.3, we have the following result.

Theorem 3.1. *Assume that $(A1)-(A3)$ hold. Then* (1.1) *admits a unique solution $u \in C^1((0,\infty); L^\infty(\mathbb{R}^N)) \cap C([0,\infty); L^\infty(\mathbb{R}^N))$ in $\mathbb{R}^N \times [0,\infty)$.*

We can see that $u^* = 1$ is a constant stationary solution of (1.1). Set $d = \frac{\alpha+\sqrt{\alpha^2+4\beta}}{2\beta}$, we know that $d > 0$ and $1 + \alpha d - \beta d^2 = 0$. Let $F(s) = 1 + \alpha s - \beta s^2 - \delta s$ for $s \in [0,\infty)$, then we have $F(0) > 0$ and $F(d) < 0$. Thus we get $u^* < d$. Assume that $\delta d < 1$. We can choose $d^* > d$ such that $\delta d^* < 1$. Then we have the following technical lemma, see [8].

Lemma 3.1. *Assume that $\delta d < 1$. For any fixed $l \in [0, d^*]$, there exists $\omega^* = \omega^*(l) \in (\alpha/2\beta, d]$ such that $1 + \alpha\omega^* - \beta[\omega^*]^2 - \delta l = 0$. Furthermore, $\omega^*(l)$ is continuous with respect to l and strictly decreasing on $[0, d^*]$. Let ω be the unique solution of*

$$\begin{cases} \omega_t(t) = \omega(t)(1 + \alpha\omega(t) - \beta[\omega(t)]^2 - \delta l), \\ \omega(0) = 1/\epsilon + \alpha/2\beta. \end{cases} \tag{3.1}$$

Then

$$\lim_{t\to\infty} \omega(t) = \omega^*(l).$$

At the end of this section, we prove the global stability of the constant stationary solution u^* of (1.1).

Theorem 3.2. *Assume that $\delta d < 1$. Let $u(x,t)$ be the unique global solution of* (1.1)*. Then*

$$\lim_{t\to\infty} u(x,t) = u^* \text{ uniformly in } \mathbb{R}^N.$$

Proof. Denote

$$h(t) = \inf_{x\in\mathbb{R}^N} u(x,t) \text{ and } H(t) = \sup_{x\in\mathbb{R}^N} u(x,t),$$

and $I = [\liminf_{t\to\infty} h(t), \limsup_{t\to\infty} H(t)]$. Note $\phi^0 = 0$ and $\omega^0 = d + 1/\epsilon$ are a pair of upper-lower solutions of (1.1), by Theorem 2.3, we have $0 \le u(x,t) \le d+1/\epsilon$ and so I is a subset of $[0, d+1/\epsilon]$. Let $\psi(x,t)$ be the unique solution of

$$\begin{cases} \psi_t(x,t) = \mathscr{D}[\psi](x,t) + \phi^0(1-\beta[\phi^0]^2) - \delta\psi G * \omega^0 - L(\psi - \phi^0), \\ \psi(x,0) = u_0(x), \end{cases}$$

here L given in Thoerem 2.1. By the iteration equation (2.4), we have

$$u(x,t) \ge \psi(x,t) \ge 0$$

for $(x,t) \in \mathbb{R}^N \times [0,\infty)$. But

$$\psi_t(x,t) \ge -(L + \delta\omega^0 + 1)\psi(x,t) \text{ and } u_0(x) \ge \epsilon,$$

we get

$$h(t) = \inf_{x\in\mathbb{R}^N} u(x,t) \ge \inf_{x\in\mathbb{R}^N} \psi(x,t) \ge \epsilon e^{-(L+\delta\omega^0+1)t} \tag{3.2}$$

for $t \ge 0$.

Let $\phi^1 = 0$ and ω^1 be the unique solution of

$$\begin{cases} \omega^1_t(t) = \omega^1(t)(1 + \alpha\omega^1(t) - \beta[\omega^1(t)]^2), \\ \omega^1(0) = 1/\epsilon + \alpha/2\beta, \end{cases}$$

a simple computation gives that ϕ^1 and ω^1 are a pair of lower-upper solutions of (1.1). From Theorem 2.1 we know that

$$0 \le u(x,t) \le \omega^1(t).$$

Then by Lemma 3.1 we have

$$\lim_{t\to\infty} \omega^1(t) = \omega^*(0) = d$$

and I is a subset of $[0,d]$.

Take $\varepsilon_0 > 0$ such that $d^* > d + \varepsilon_0$. Since $\limsup_{t\to\infty} H(t) \le d$, we can find $t_2 > 0$ such that

$$u(x,t) \le H(t) \le d + \varepsilon_0$$

for $(x,t) \in \mathbb{R}^N \times [t_2,\infty)$. By (3.2), we have $h(t_2) > 0$. Let ϕ^2 be the unique solution of

$$\begin{cases} \phi^2_t(t) = \phi^2(t)(1 + \alpha\phi^2(t) - \beta[\phi^2(t)]^2 - \delta(d+\varepsilon)), \\ \phi^2(t_2) = h(t_2). \end{cases}$$

We can see that $\phi^2(t) > 0$ for $t \ge t_2$. Take $\omega^2(x,t) = d + \varepsilon_0$, we can check that ω^2 and ϕ^2 are a pair of upper-lower solutions of (1.1) and so

$$\phi^2(x,t) \le u(x,t) \le \omega^2(x,t)$$

for $(x,t) \in \mathbb{R}^N \times [t_2,\infty)$. Again by Lemma 3.1, we have

$$\lim_{t\to\infty} \phi^2(t) = \omega^*(d + \varepsilon_0)$$

and I is a subset of $[\omega^*(d+\varepsilon_0), d]$. Letting $\varepsilon_0 \to 0$, we get $I \subset [\omega^*(d), d]$.

Since $\omega^*(d) \in (\alpha/2\beta, d]$, we can find $\theta \in (0,1)$ such that $\lambda_3 = \theta\omega^*(d) \geq \alpha/2\beta$. Set $\mu_3 = d$, we define

$$\lambda_k = \omega^*(\mu_{k-1}) \text{ and } \mu_k = \omega^*(\lambda_{k-1}), \tag{3.3}$$

where $k > 3$. We know that

$$0 < \lambda_3 = \theta\omega^*(d) < \lambda_4 = \omega^*(d) < \omega^*(0) = d = \mu_3.$$

Since $\omega^*(u^*) = u^* < d$ and ω^* is decreasing, then

$$\lambda_4 = \omega^*(d) < \omega^*(u^*) = u^*.$$

Similarly, we have

$$\lambda_3 = \theta\omega^*(d) < \omega^*(u^*) = u^*$$

and so

$$u^* = \omega^*(u^*) < \mu_4 = \omega^*(\lambda_3) < \omega^*(0) = d = \mu_3.$$

Thus we obtain that

$$0 < \lambda_3 < \lambda_4 < u^* < \mu_4 < \mu_3 = d.$$

Inductively, we know that

$$0 < \lambda_k < \lambda_{k+1} < u^* < \mu_{k+1} < \mu_k \tag{3.4}$$

for $k > 3$.

Note that $I \subset [\omega^*(d), d]$, we have $I \subset [\lambda_3, \mu_3]$. Assume that $I \subset [\lambda_{k-1}, \mu_{k-1}]$ for some $k > 4$. We show that $I \subset [\lambda_k, \mu_k]$. By (3.3) and (3.4), we can choose $\varepsilon > 0$ small such that

$$\lambda_{k-1} \leq \omega^*(\mu_{k-1}+\varepsilon) \leq \omega^*(\lambda_{k-1}-\varepsilon) \leq \mu_{k-1}. \tag{3.5}$$

Since $I \subset [\lambda_{k-1}, \mu_{k-1}]$, we can find $t_k > 0$ such that

$$\lambda_{k-1} - \varepsilon \leq h(t) \leq H(t) \leq \mu_{k-1} + \varepsilon$$

for $t \geq t_k$. Let ϕ^k and ω^k satisfy

$$\begin{cases} \phi^k_t(t) = \phi^k(t)(1 - \beta[\phi^k(t)]^2 - \delta(\mu_{k-1}+\varepsilon)), \\ \phi^k(t_k) = \lambda_{k-1} - \varepsilon \end{cases}$$

and

$$\begin{cases} \omega^k_t(t) = \omega^k(t)(1 - \beta[\omega^k(t)]^2 - \delta(\lambda_{k-1}-\varepsilon)), \\ \omega^k(t_2) = \mu_{k-1} + \varepsilon, \end{cases}$$

respectively. Since

$$\lim_{t\to\infty} \phi^k(t) = \omega^*(\mu_{k-1}+\varepsilon) \text{ and } \lim_{t\to\infty} \omega^k(t) = \omega^*(\lambda_{k-1}-\varepsilon), \tag{3.6}$$

it follows from (3.5) that

$$\phi^k(t) \geq \lambda_{k-1} - \varepsilon \text{ and } \omega^k(t) \leq \mu_{k-1} + \varepsilon, \tag{3.7}$$

provided t is sufficiently large, say $t \geq t_k^*$. By (3.7), we know that ϕ^k and ω^k are a pair of lower-upper solutions to (1.1) and so

$$\phi^k(t) \leq u(x,t) \leq \omega^k(t)$$

for $(x,t) \in \mathbb{R}^N \times [t_k^*, \infty)$. Letting $\varepsilon \to 0$, by (3.6) we get

$$I \subset [\omega^*(\mu_{k-1}), \omega^*(\mu_{k-1})] = [\lambda_k, \mu_k].$$

On the other hand, it follows from (3.4) that there exist λ and μ such that $\lambda_k \to \lambda$ and $\mu_k \to \mu$ as $k \to \infty$. Since ω^* is continuous, we get

$$\omega^*(\lambda) = \mu \text{ and } \omega^*(\mu) = \lambda.$$

and so

$$1 + \alpha\lambda - \beta\lambda^2 - \delta\mu = 1 + \alpha\mu - \beta\mu^2 - \delta\lambda = 0.$$

But $\delta d < 1$, we know that $\lambda = \mu = u^*$, (see [8]). Thus $I = u^*$.

□

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