Journal of Applied Analysis and Computation
Volume 7, Number 3, August 2017, 977–991
Website:http://jaac-online.com/
DOI:10.11948/2017061

A NEW HALF-DISCRETE HILBERT-TYPE INEQUALITY IN THE WHOLE PLANE*

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Abstract By the use of Hermite-Hadamard's inequality and weight functions, a new half-discrete Hilbert-type inequality in the whole plane with multi-parameters is given. The constant factor related to the gamma function is proved to be the best possible. The equivalent forms, two kinds of particular inequalities, and the operator expressions are considered.

Keywords Half-discrete Hilbert-type inequality, Hermite-Hadamard's inequality, weight function, equivalent form, gamma function.

MSC(2010) 26D15, 47A05.

1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n > 0, 0 < \sum_{m=1}^{\infty} a_m^p < \infty, 0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then we have the following discrete Hardy-Hilbert's inequality (cf. [3]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}, \tag{1.1}$$

where, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

Assuming that $f(x), g(y) \geq 0$, satisfying $0 < \int_0^{\infty} f^p(x)dx < \infty$ and $0 < \int_0^{\infty} g^q(y)dy < \infty$, we have the following Hardy-Hilbert's integral inequality with the same best possible constant factor (cf. [4]):

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^{\infty}g^q(y)dy\right)^{\frac{1}{q}}. \tag{1.2}$$

Recently, a half-discrete Hardy-Hilbert's inequality with the same best possible constant factor was given as follows (cf. [20]):

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{b_n f(x)}{x+n}dx < \frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty}f^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}. \tag{1.3}$$

Inequalities (1.1), (1.2) and (1.3) are important in analysis and its applications (cf. [4, 9, 21, 22]).

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*The authors were supported by National Natural Science Foundation (No. 61370186), and Appropriative Researching Fund for Professors and Doctors, Guangdong University of Education (No. 2015ARF25).

Noticing that inequalities (1.1)-(1.3) are with the homogenous kernel of degree -1, in 2009, a survey of the study of Hilbert-type inequalities with the homogeneous kernels of degree negative numbers is given by [23]. A few inequalities with the homogenous kernels of degree 0 and non-homogenous kernels have been studied by [1, 16, 17, 24, 25]. Some other kinds of Hilbert-type inequalities are provided by [10–13]. All of the above inequalities are built in the quarter plane of the first quadrant.

In 2007, a Hilbert-type integral inequality in the whole plane was given by Yang [26]. Another Hilbert-type integral inequality in the whole plane was proved by [27] as follows:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{|1+xy|^{\lambda}}f(x)g(y)dxdy$$
$$< k_{\lambda}\left[\int_{-\infty}^{\infty}|x|^{p(1-\frac{\lambda}{2})-1}f^{p}(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1-\frac{\lambda}{2})-1}g^{q}(y)dy\right]^{\frac{1}{q}}, \tag{1.4}$$

where, the constant factor $k_{\lambda}=B(\frac{\lambda}{2},\frac{\lambda}{2})+2B(1-\lambda,\frac{\lambda}{2})(0<\lambda<1)$ is the best possible. And He *et al.* [2,5,6,14,15,18,19,29,30] also published some integral and discrete Hilbert-type inequalities in the whole plane.

In this paper, by the use of Hermite-Hadamard's inequality and the way of weight functions, a new half-discrete Hilbert-type inequality in the whole plane with a best possible constant factor related to the gamma function is built as follows:

$$\sum_{|n|=1}^{\infty}\int_{-\infty}^{\infty}e^{-\rho\left(\frac{|n|}{|x|}\right)^{\gamma}}f(x)b_{n}dx$$
$$< \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma\rho^{\sigma/\gamma}}\left[\int_{-\infty}^{\infty}|x|^{p(1+\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty}|n|^{q(1-\sigma)-1}b_{n}^{q}\right]^{\frac{1}{q}}, \tag{1.5}$$

where, $\rho>0,\gamma,\sigma\in(0,1]$. Moreover, an extension of (1.5) with multi-parameters is given. The equivalent forms, two kinds of particular inequalities and the operator expressions are also considered.

2. Some lemmas

In the following, we make appointment that $\delta\in\{-1,1\},\alpha,\beta\in(0,\pi),\rho>0,\gamma\in(0,1]$, $\xi\in(-\infty,\infty),\eta\in[0,\frac{1}{2}]$,

$$h(x,y):=e^{-\rho\left\{\frac{|y-\eta|+(y-\eta)\cos\beta}{[|x-\xi|+(x-\xi)\cos\alpha]^{\delta}}\right\}^{\gamma}}(x\neq\xi,y\neq\eta), \tag{2.1}$$

wherefrom,

$$h(x,y)=e^{-\rho\left\{\frac{(y-\eta)(1+\cos\beta)}{[|x-\xi|+(x-\xi)\cos\alpha]^{\delta}}\right\}^{\gamma}}(y>\eta),$$
$$h(x,y)=e^{-\rho\left\{\frac{|y-\eta|+(y-\eta)\cos\beta}{[(x-\xi)(1+\cos\alpha)]^{\delta}}\right\}^{\gamma}}(x>\xi),$$
$$h(-x,y)=e^{-\rho\left\{\frac{|y-\eta|+(y-\eta)\cos\beta}{[|(x+\xi)(1-\cos\alpha)]^{\delta}}\right\}^{\gamma}}(x>-\xi),$$
$$h(x,-y)=e^{-\rho\left\{\frac{(y+\eta)(1-\cos\beta)}{[|x-\xi|+(x-\xi)\cos\alpha]^{\delta}}\right\}^{\gamma}}(y>-\eta).$$

Lemma 2.1. *We define two weight functions* $\omega(\sigma,n)$ *and* $\varpi(\sigma,x)$ *as follow:*

$$\omega(\sigma,n) := \int_{-\infty}^{\infty} h(x,n)\frac{[|n-\eta|+(n-\eta)\cos\beta]^{\sigma}dx}{[|x-\xi|+(x-\xi)\cos\alpha]^{1+\delta\sigma}}\ (|n|\in\mathbf{N}), \tag{2.2}$$

$$\varpi(\sigma,x) := \sum_{|n|=1}^{\infty} h(x,n)\frac{[|x-\xi|+(x-\xi)\cos\alpha]^{-\delta\sigma}}{[|n-\eta|+(n-\eta)\cos\beta]^{1-\sigma}}\ (x\in\mathbf{R}\backslash\{\xi\}). \tag{2.3}$$

Then for $\sigma>0$, *we have*

$$\omega(\sigma,n) = k_{\alpha}(\sigma) := \frac{2\csc^2\alpha}{\gamma\rho^{\sigma/\gamma}}\Gamma(\frac{\sigma}{\gamma}) \in \mathbf{R}_{+}\ (|n|\in\mathbf{N}); \tag{2.4}$$

for $\sigma\in(0,1]$, *we have*

$$k_{\beta}(\sigma)(1-\theta(\sigma,x)) < \varpi(\sigma,x) < k_{\beta}(\sigma)\ (x\in\mathbf{R}\backslash\{\xi\}), \tag{2.5}$$

where,

$$\begin{aligned}\theta(\sigma,x) &:= \frac{1}{\Gamma(\frac{\sigma}{\gamma})}\int_0^{\rho\left\{\frac{(1+\eta)(1-\cos\beta)}{[|x-\xi|+(x-\xi)\cos\alpha]^{\delta}}\right\}^{\gamma}} e^{-u}u^{\frac{\sigma}{\gamma}-1}du\\ &= O\left(\frac{1}{[|x-\xi|+(x-\xi)\cos\alpha]^{\delta\sigma}}\right)\in(0,1).\end{aligned} \tag{2.6}$$

Proof. We find

$$\begin{aligned}\omega(\sigma,n) &= \int_{-\infty}^{\xi} h(x,n)\frac{[|n-\eta|+(n-\eta)\cos\beta]^{\sigma}}{[(x-\xi)(\cos\alpha-1)]^{1+\delta\sigma}}dx\\ &\quad+\int_{\xi}^{\infty} h(x,n)\frac{[|n-\eta|+(n-\eta)\cos\beta]^{\sigma}}{[(x-\xi)(\cos\alpha+1)]^{1+\delta\sigma}}dx\\ &= \int_{-\xi}^{\infty} h(-x,n)\frac{[|n-\eta|+(n-\eta)\cos\beta]^{\sigma}}{[(x+\xi)(1-\cos\alpha)]^{1+\delta\sigma}}dx\\ &\quad+\int_{\xi}^{\infty} h(x,n)\frac{[|n-\eta|+(n-\eta)\cos\beta]^{\sigma}}{[(x-\xi)(1+\cos\alpha)]^{1+\delta\sigma}}dx.\end{aligned}$$

Setting $u=\rho\left\{\frac{[|n-\eta|+(n-\eta)\cos\beta]}{[(x+\xi)(1-\cos\alpha)]^{\delta}}\right\}^{\gamma}$ ($u=\rho\left\{\frac{[|n-\eta|+(n-\eta)\cos\beta]}{[(x-\xi)(1+\cos\alpha)]^{\delta}}\right\}^{\gamma}$) in the above first (second) integral, by simplifications, we have

$$\begin{aligned}\omega(\sigma,n) &= \frac{1}{(1-\cos\alpha)\gamma\rho^{\sigma/\gamma}}\int_0^{\infty} e^{-u}u^{\frac{\sigma}{\gamma}-1}du\\ &\quad+\frac{1}{(1+\cos\alpha)\gamma\rho^{\sigma/\gamma}}\int_0^{\infty} e^{-u}u^{\frac{\sigma}{\gamma}-1}du = \frac{2\csc^2\alpha}{\gamma\rho^{\sigma/\gamma}}\Gamma(\frac{\sigma}{\gamma}).\end{aligned}$$

Hence, (2.4) follows.

We obtain

$$\begin{aligned}
\varpi(\sigma,x) &= \sum_{n=-1}^{-\infty} h(x,n)\frac{[|x-\xi|+(x-\xi)\cos\alpha]^{-\delta\sigma}}{[|n-\eta|+(n-\eta)\cos\beta]^{1-\sigma}} \\
&\quad+\sum_{n=1}^{\infty} h(x,n)\frac{[|x-\xi|+(x-\xi)\cos\alpha]^{-\delta\sigma}}{[|n-\eta|+(n-\eta)\cos\beta]^{1-\sigma}} \\
&=\frac{[|x-\xi|+(x-\xi)\cos\alpha]^{-\delta\sigma}}{(1-\cos\beta)^{1-\sigma}}\sum_{n=1}^{\infty}\frac{h(x,-n)}{(n+\eta)^{1-\sigma}} \\
&\quad+\frac{[|x-\xi|+(x-\xi)\cos\alpha]^{-\delta\sigma}}{(1+\cos\beta)^{1-\sigma}}\sum_{n=1}^{\infty}\frac{h(x,n)}{(n-\eta)^{1-\sigma}}. \qquad (2.7)
\end{aligned}$$

For $\gamma,\sigma\in(0,1]$, we find both $\frac{h(x,-y)}{(y+\eta)^{1-\sigma}}$ and $\frac{h(x,y)}{(y-\eta)^{1-\sigma}}$ are strictly decreasing and strictly convex in $y\in(\frac{1}{2},\infty)$, satisfying

$$(-1)^i\frac{d^{(i)}}{dy^{(i)}}\frac{h(x,-y)}{(y+\eta)^{1-\sigma}}>0,(-1)^i\frac{d^{(i)}}{dy^{(i)}}\frac{h(x,y)}{(y-\eta)^{1-\sigma}}>0\ (i=1,2).$$

By (2.7) and Hermite-Hadamard's inequality (cf. [7]), in view of $\eta\in[0,\frac{1}{2}]$, we have

$$\begin{aligned}
\varpi(\sigma,x) &< \frac{[|x-\xi|+(x-\xi)\cos\alpha]^{-\delta\sigma}}{(1-\cos\beta)^{1-\sigma}}\int_{\frac{1}{2}}^{\infty}\frac{h(x,-y)dy}{(y+\eta)^{1-\sigma}} \\
&\quad+\frac{[|x-\xi|+(x-\xi)\cos\alpha]^{-\delta\sigma}}{(1+\cos\beta)^{1-\sigma}}\int_{\frac{1}{2}}^{\infty}\frac{h(x,y)dy}{(y-\eta)^{1-\sigma}} \\
&\le \frac{[|x-\xi|+(x-\xi)\cos\alpha]^{-\delta\sigma}}{(1-\cos\beta)^{1-\sigma}}\int_{-\eta}^{\infty}\frac{h(x,-y)dy}{(y+\eta)^{1-\sigma}} \\
&\quad+\frac{[|x-\xi|+(x-\xi)\cos\alpha]^{-\delta\sigma}}{(1+\cos\beta)^{1-\sigma}}\int_{\eta}^{\infty}\frac{h(x,y)dy}{(y-\eta)^{1-\sigma}}.
\end{aligned}$$

Setting $u=\rho\left\{\frac{(y+\eta)(1-\cos\beta)}{[|x-\xi|+(x-\xi)\cos\alpha]^{\delta}}\right\}^{\gamma}$ $(u=\rho\left\{\frac{(y-\eta)(1+\cos\beta)}{[|x-\xi|+(x-\xi)\cos\alpha]^{\delta}}\right\}^{\gamma})$ in the above first (second) integral, by simplifications, we have

$$\varpi(\sigma,x)<\frac{2\csc^2\beta}{\gamma\rho^{\sigma/\gamma}}\Gamma(\frac{\sigma}{\gamma})=k_\beta(\sigma).$$

By (2.7) and the decreasing property, we still have

$$\begin{aligned}
\varpi(\sigma,x) &> \frac{[|x-\xi|+(x-\xi)\cos\alpha]^{-\delta\sigma}}{(1-\cos\beta)^{1-\sigma}}\int_{1}^{\infty}\frac{h(x,-y)dy}{(y+\eta)^{1-\sigma}} \\
&\quad+\frac{[|x-\xi|+(x-\xi)\cos\alpha]^{-\delta\sigma}}{(1+\cos\beta)^{1-\sigma}}\int_{1}^{\infty}\frac{h(x,y)dy}{(y-\eta)^{1-\sigma}}.
\end{aligned}$$

Setting $u=\rho\left\{\frac{(y+\eta)(1-\cos\beta)}{[|x-\xi|+(x-\xi)\cos\alpha]^{\delta}}\right\}^{\gamma}$ $(u=\rho\left\{\frac{(y-\eta)(1+\cos\beta)}{[|x-\xi|+(x-\xi)\cos\alpha]^{\delta}}\right\}^{\gamma})$ in the above first (second) integral, by simplifications, we have

$$\begin{aligned}\varpi(\sigma,x) &> \frac{1}{\gamma\rho^{\sigma/\gamma}(1-\cos\beta)}\int_{\rho\left\{\frac{(1+\eta)(1-\cos\beta)}{[|x-\xi|+(x-\xi)\cos\alpha]^{\delta}}\right\}^{\gamma}}^{\infty} e^{-u}u^{\frac{\sigma}{\gamma}-1}du \\ &\quad+\frac{1}{\gamma\rho^{\sigma/\gamma}(1+\cos\beta)}\int_{\rho\left\{\frac{(1-\eta)(1+\cos\beta)}{[|x-\xi|+(x-\xi)\cos\alpha]^{\delta}}\right\}^{\gamma}}^{\infty} e^{-u}u^{\frac{\sigma}{\gamma}-1}du \\ &\geq \frac{2\csc^2\beta}{\gamma\rho^{\sigma/\gamma}}\int_{\rho\left\{\frac{(1+\eta)(1-\cos\beta)}{[|x-\xi|+(x-\xi)\cos\alpha]^{\delta}}\right\}^{\gamma}}^{\infty} e^{-u}u^{\frac{\sigma}{\gamma}-1}du \\ &= k_\beta(\sigma)(1-\theta(\sigma,x))>0.\end{aligned}$$

We find

$$\begin{aligned}0<\theta(\sigma,x) &= \frac{1}{\Gamma(\frac{\sigma}{\gamma})}\int_0^{\rho\left\{\frac{(1+\eta)(1-\cos\beta)}{[|x-\xi|+(x-\xi)\cos\alpha]^{\delta}}\right\}^{\gamma}} e^{-u}u^{\frac{\sigma}{\gamma}-1}du \\ &\leq \frac{1}{\Gamma(\frac{\sigma}{\gamma})}\int_0^{\rho\left\{\frac{(1+\eta)(1-\cos\beta)}{[|x-\xi|+(x-\xi)\cos\alpha]^{\delta}}\right\}^{\gamma}} u^{\frac{\sigma}{\gamma}-1}du \\ &= \frac{\gamma\rho^{\sigma/\gamma}}{\sigma\Gamma(\frac{\sigma}{\gamma})}\left\{\frac{(1+\eta)(1+\cos\beta)}{[|x-\xi|+(x-\xi)\cos\alpha]^{\delta}}\right\}^{\sigma},\end{aligned}$$

and then (2.5) and (2.6) follow. □

Lemma 2.2. *For $\varepsilon>0$, $H_\varepsilon(\beta):=\sum_{|n|=1}^{\infty}\frac{1}{[|n-\eta|+(n-\eta)\cos\beta]^{1+\varepsilon}}$, we have*

$$H_\varepsilon(\beta)\leq\frac{1}{\varepsilon}(2\csc^2\beta+o_1(1))(1+o_2(1))\ (\varepsilon\to0^+). \tag{2.8}$$

Proof. We have

$$\begin{aligned}H_\varepsilon(\beta) &= \sum_{n=-1}^{-\infty}\frac{1}{[(n-\eta)(\cos\beta-1)]^{1+\varepsilon}}+\sum_{n=1}^{\infty}\frac{1}{[(n-\eta)(\cos\beta+1)]^{1+\varepsilon}} \\ &= \frac{1}{(1-\cos\beta)^{1+\varepsilon}}\sum_{n=1}^{\infty}\frac{1}{(n+\eta)^{1+\varepsilon}}+\frac{1}{(1+\cos\beta)^{1+\varepsilon}}\sum_{n=1}^{\infty}\frac{1}{(n-\eta)^{1+\varepsilon}}.\end{aligned} \tag{2.9}$$

By (2.9) and the decreasing property, we find

$$\begin{aligned}H_\varepsilon(\beta) &\leq \left[\frac{1}{(1+\cos\beta)^{1+\varepsilon}}+\frac{1}{(1-\cos\beta)^{1+\varepsilon}}\right]\sum_{n=1}^{\infty}\frac{1}{(n-\eta)^{1+\varepsilon}} \\ &= \left[\frac{1}{(1+\cos\beta)^{1+\varepsilon}}+\frac{1}{(1-\cos\beta)^{1+\varepsilon}}\right]\times\left[\frac{1}{(1-\eta)^{1+\varepsilon}}+\sum_{n=2}^{\infty}\frac{1}{(n-\eta)^{1+\varepsilon}}\right] \\ &< \left[\frac{1}{(1+\cos\beta)^{1+\varepsilon}}+\frac{1}{(1-\cos\beta)^{1+\varepsilon}}\right]\times\left[\frac{1}{(1-\eta)^{1+\varepsilon}}+\int_1^{\infty}\frac{dy}{(y-\eta)^{1+\varepsilon}}\right] \\ &= \frac{1}{\varepsilon}\left[\frac{1}{(1+\cos\beta)^{1+\varepsilon}}+\frac{1}{(1-\cos\beta)^{1+\varepsilon}}\right] \\ &\quad\times\left\{1+\left[\frac{\varepsilon}{(1-\eta)^{1+\varepsilon}}+\frac{1}{(1-\eta)^{\varepsilon}}-1\right]\right\}.\end{aligned}$$

Hence we have (2.8). □

Lemma 2.3. *For $\varepsilon > 0$, setting $E_\delta := \{x \in \mathbf{R}\backslash\{\xi\}; \frac{1}{[|x-\xi|+(x-\xi)\cos\alpha]^\delta} \geq 1\}$, we have*

$$H_\delta := \int_{E_\delta} \frac{1}{[|x-\xi| + (x-\xi)\cos\alpha]^{1+\delta\varepsilon}} dx = \frac{2}{\varepsilon}\csc^2\alpha. \tag{2.10}$$

Proof. Setting

$$E_\delta^+ := \{x > \xi; \frac{1}{[(x-\xi)(1+\cos\alpha)]^\delta} \geq 1\},$$
$$E_\delta^- := \{x < \xi; \frac{1}{[(\xi-x)(1-\cos\alpha)]^\delta} \geq 1\},$$

it follows that $E_\delta = E_\delta^+ \cup E_\delta^-$. We find

$$H_\delta = \frac{1}{(1+\cos\alpha)^{1+\delta\varepsilon}} \int_{E_\delta^+} \frac{1}{(x-\xi)^{1+\delta\varepsilon}} dx + \frac{1}{(1-\cos\alpha)^{1+\delta\varepsilon}} \int_{E_\delta^-} \frac{1}{(\xi-x)^{1+\delta\varepsilon}} dx.$$

Setting $u = [(x-\xi)(1+\cos\alpha)]^\delta$ ($u = [(\xi-x)(1-\cos\alpha)]^\delta$) in the above first (second) integral, we obtain

$$H_\delta = \left(\frac{1}{1+\cos\alpha} + \frac{1}{1-\cos\alpha}\right) \int_1^\infty \frac{du}{u^{1+\varepsilon}} = \frac{2}{\varepsilon}\csc^2\alpha.$$

Hence we have (2.3). $\square$

3. Main results

Theorem 3.1. *Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1,\ 0 < \sigma \leq 1$,*

$$K_{\alpha,\beta}(\sigma) := k_\alpha^{\frac{1}{q}}(\sigma) k_\beta^{\frac{1}{p}}(\sigma) = \frac{2}{\gamma\rho^{\sigma/\gamma}} \Gamma(\frac{\sigma}{\gamma}) \csc^{\frac{2}{q}}\alpha \csc^{\frac{2}{p}}\beta. \tag{3.1}$$

If $f(x), b_n \geq 0$, satisfying

$$0 < \int_{-\infty}^\infty [|x-\xi| + (x-\xi)\cos\alpha]^{p(1+\delta\sigma)-1} f^p(x) dx < \infty,$$
$$0 < \sum_{|n|=1}^\infty [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\sigma)-1} b_n^q < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} I :&= \sum_{|n|=1}^\infty \int_{-\infty}^\infty h(x,n) f(x) b_n dx \\ &< K_{\alpha,\beta}(\sigma) \left\{\int_{-\infty}^\infty [|x-\xi| + (x-\xi)\cos\alpha]^{p(1+\delta\sigma)-1} f^p(x) dx\right\}^{\frac{1}{p}} \\ &\quad \times \left\{\sum_{|n|=1}^\infty [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\sigma)-1} b_n^q\right\}^{\frac{1}{q}}, \end{aligned} \tag{3.2}$$

$$J_1 := \left\{ \sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{p\sigma-1} \left[\int_{-\infty}^{\infty} h(x,n) f(x) dx \right]^p \right\}^{\frac{1}{p}}$$
$$< K_{\alpha,\beta}(\sigma) \left\{ \int_{-\infty}^{\infty} [|x-\xi| + (x-\xi)\cos\alpha]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}}, \tag{3.3}$$

$$J_2 := \left\{ \int_{-\infty}^{\infty} [|x-\xi| + (x-\xi)\cos\alpha]^{-q\delta\sigma-1} \left[\sum_{|n|=1}^{\infty} h(x,n) b_n \right]^q dx \right\}^{\frac{1}{q}}$$
$$< K_{\alpha,\beta}(\sigma) \left\{ \sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\sigma)-1} b_n^q \right\}^{\frac{1}{q}}. \tag{3.4}$$

In particular, for $\alpha = \beta = \frac{\pi}{2}$, we have the following equivalent inequalities:

$$\sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} e^{-\rho\left(\frac{|n-\eta|}{|x-\xi|^{\delta}}\right)^{\gamma}} f(x) b_n dx$$
$$< \frac{2}{\gamma\rho^{\sigma/\gamma}} \Gamma\left(\frac{\sigma}{\gamma}\right) \left\{ \int_{-\infty}^{\infty} |x-\xi|^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}}$$
$$\times \left\{ \sum_{|n|=1}^{\infty} |n-\eta|^{q(1-\sigma)-1} b_n^q \right\}^{\frac{1}{q}}, \tag{3.5}$$

$$\left\{ \sum_{|n|=1}^{\infty} |n-\eta|^{p\sigma-1} \left[\int_{-\infty}^{\infty} e^{-\rho\left(\frac{|n-\eta|}{|x-\xi|^{\delta}}\right)^{\gamma}} f(x) dx \right]^p \right\}^{\frac{1}{p}}$$
$$< \frac{2}{\gamma\rho^{\sigma/\gamma}} \Gamma\left(\frac{\sigma}{\gamma}\right) \left\{ \int_{-\infty}^{\infty} |x-\xi|^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}}, \tag{3.6}$$

$$\left\{ \int_{-\infty}^{\infty} |x-\xi|^{-q\delta\sigma-1} \left[\sum_{|n|=1}^{\infty} e^{-\rho\left(\frac{|n-\eta|}{|x-\xi|^{\delta}}\right)^{\gamma}} b_n \right]^q dx \right\}^{\frac{1}{q}}$$
$$< \frac{2}{\gamma\rho^{\sigma/\gamma}} \Gamma\left(\frac{\sigma}{\gamma}\right) \left\{ \sum_{|n|=1}^{\infty} |n-\eta|^{q(1-\sigma)-1} b_n^q \right\}^{\frac{1}{q}}. \tag{3.7}$$

Proof. By Hölder's inequality (cf. [7]) and (2.2), we find

$$\left[\int_{-\infty}^{\infty} h(x,n) f(x) dx \right]^p$$
$$= \left\{ \int_{-\infty}^{\infty} h(x,n) \frac{[|x-\xi| + (x-\xi)\cos\alpha]^{(1+\delta\sigma)/q}}{[|n-\eta| + (n-\eta)\cos\beta]^{(1-\sigma)/p}} f(x) \right.$$
$$\left. \times \frac{[|n-\eta| + (n-\eta)\cos\beta]^{(1-\sigma)/p}}{[|x-\xi| + (x-\xi)\cos\alpha]^{(1+\delta\sigma)/q}} dx \right\}^p$$

$$\leq \int_{-\infty}^{\infty} h(x,n) \frac{[|x-\xi| + (x-\xi)\cos\alpha]^{(1+\delta\sigma)(p-1)}}{[|n-\eta| + (n-\eta)\cos\beta]^{1-\sigma}} f^p(x)dx$$
$$\times \left\{ \int_{-\infty}^{\infty} h(x,n) \frac{[|n-\eta| + (n-\eta)\cos\beta]^{(1-\sigma)(q-1)}}{[|x-\xi| + (x-\xi)\cos\alpha]^{1+\delta\sigma}} dx \right\}^{p-1}$$
$$= \frac{\omega^{p-1}(\sigma,n)}{[|n-\eta| + (n-\eta)\cos\beta]^{p\sigma-1}}$$
$$\times \int_{-\infty}^{\infty} h(x,n) \frac{[|x-\xi| + (x-\xi)\cos\alpha]^{(1+\delta\sigma)(p-1)}}{[|n-\eta| + (n-\eta)\cos\beta]^{1-\sigma}} f^p(x)dx.$$

Then by (2.4) and Lebesgue term by term integration theorem (cf. [8]), in view of (2.3), we find

$$\begin{aligned} J_1 \leq & k_\alpha^{\frac{1}{q}}(\sigma) \left\{ \sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} h(x,n) \frac{[|x-\xi| + (x-\xi)\cos\alpha]^{(1+\delta\sigma)(p-1)}}{[|n-\eta| + (n-\eta)\cos\beta]^{1-\sigma}} f^p(x)dx \right\}^{\frac{1}{p}} \\ = & k_\alpha^{\frac{1}{q}}(\sigma) \left\{ \int_{-\infty}^{\infty} \sum_{|n|=1}^{\infty} h(x,n) \frac{[|x-\xi| + (x-\xi)\cos\alpha]^{(1+\delta\sigma)(p-1)}}{[|n-\eta| + (n-\eta)\cos\beta]^{1-\sigma}} f^p(x)dx \right\}^{\frac{1}{p}} \\ = & k_\alpha^{\frac{1}{q}}(\sigma) \left\{ \int_{-\infty}^{\infty} \varpi(\sigma,x)[|x-\xi| + (x-\xi)\cos\alpha]^{p(1+\delta\sigma)-1} f^p(x)dx \right\}^{\frac{1}{p}}. \end{aligned} \tag{3.8}$$

Hence, by (2.5), we have (3.3).

By Hölder's inequality (cf. [7]), we have

$$\begin{aligned} I = & \sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{\frac{-1}{p}+\sigma} \int_{-\infty}^{\infty} h(x,n)f(x)dx \\ & \times [|n-\eta| + (n-\eta)\cos\beta]^{\frac{1}{p}-\sigma} b_n \\ \leq & J_1 \left[\sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{q}}. \end{aligned} \tag{3.9}$$

Then by (3.3), we have (3.2). On the other hand, assuming that (3.2) is valid, we set

$$b_n := [|n-\eta| + (n-\eta)\cos\beta]^{p\sigma-1} \left[\int_{-\infty}^{\infty} h(x,n)f(x)dx \right]^{p-1} \ (|n| \in \mathbf{N}).$$

Then we find

$$J_1 = \left[\sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{p}}.$$

In view of (3.8), it follows that $J_1 < \infty$. If $J_1 = 0$, then (3.3) is trivially valid; if

$J_1 > 0$, then by (3.2), we have

$$\begin{aligned}
&\sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\sigma)-1} b_n^q \\
&= J_1^p = I < K_{\alpha,\beta}(\sigma) \left\{ \int_{-\infty}^{\infty} [|x-\xi| + (x-\xi)\cos\alpha]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\sigma)-1} b_n^q \right\}^{\frac{1}{q}}, \\
&\left\{ \sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\sigma)-1} b_n^q \right\}^{\frac{1}{p}} \\
&= J_1 < K_{\alpha,\beta}(\sigma) \left\{ \int_{-\infty}^{\infty} [|x-\xi| + (x-\xi)\cos\alpha]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}},
\end{aligned}$$

namely, (3.3) holds, which is equivalent to (3.2).

In the same way of obtaining (3.8), we have

$$J_2 \le k_\beta^{\frac{1}{p}}(\sigma) \left\{ \sum_{|n|=1}^{\infty} \omega(\sigma, n) [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\sigma)-1} b_n^q \right\}^{\frac{1}{q}}. \tag{3.10}$$

We have proved that (3.2) is valid. Setting

$$f(x) := \int_{-\infty}^{\infty} [|x-\xi| + (x-\xi)\cos\alpha]^{-q\delta\sigma-1} \left[\sum_{|n|=1}^{\infty} h(x,n) b_n \right]^{q-1} \quad (x \in \mathbf{R} \backslash \{\xi\}),$$

then it follows that

$$J_2 = \left\{ \int_{-\infty}^{\infty} [|x-\xi| + (x-\xi)\cos\alpha]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{q}},$$

and in view of (3.10), $J_2 < \infty$. If $J_2 = 0$, then (3.4) is trivially valid; if $J_2 > 0$, then by (3.2), we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} [|x-\xi| + (x-\xi)\cos\alpha]^{p(1+\delta\sigma)-1} f^p(x) dx \\
&= J_2^q = I < K_{\alpha,\beta}(\sigma) \left\{ \int_{-\infty}^{\infty} [|x-\xi| + (x-\xi)\cos\alpha]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\sigma)-1} b_n^q \right\}^{\frac{1}{q}}, \\
&\left\{ \int_{-\infty}^{\infty} [|x-\xi| + (x-\xi)\cos\alpha]^{p(1+\delta\sigma)-1} f^p(x) dx \right\}^{\frac{1}{q}} \\
&= J_2 < K_{\alpha,\beta}(\sigma) \left\{ \sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\sigma)-1} b_n^q \right\}^{\frac{1}{q}},
\end{aligned}$$

namely, (3.4) follows. On the other hand, assuming that (3.4) is valid, by Hölder's inequality (cf. [7]) and in the same way of obtaining (3.9), we have

$$I \leq \left\{\int_{-\infty}^{\infty} [|x-\xi| + (x-\xi)\cos\alpha]^{p(1+\delta\sigma)-1} f^p(x)dx\right\}^{\frac{1}{p}} J_2. \tag{3.11}$$

Then by (3.4), we have (3.2), which is equivalent to (3.4).

Therefore, inequalities (3.2), (3.3) and (3.4) are equivalent. □

Theorem 3.2. *As regards to the assumptions of Theorem 3.1, the constant factor $K_{\alpha,\beta}(\sigma)$ in (3.2), (3.3) and (3.4) is the best possible.*

Proof. For $0 < \varepsilon < q\sigma$, we set $\widetilde{\sigma} = \sigma - \frac{\varepsilon}{q}$ $(\in (0,1))$,

$$\widetilde{f}(x) := \begin{cases} \frac{1}{[|x-\xi|+(x-\xi)\cos\alpha]^{\delta(\sigma+\frac{\varepsilon}{p})+1}}, & x \in E_\delta, \\ 0, \quad x \in \mathbf{R}\backslash E_\delta, \end{cases}$$

and $\widetilde{b}_n := [|n-\eta| + (n-\eta)\cos\beta]^{(\sigma-\frac{\varepsilon}{q})-1}$, $|n| \in \mathbf{N}$. Then by (2.8) and (2.3), we find

$$\begin{aligned} \widetilde{I}_1 :&= \left\{\int_{-\infty}^{\infty} [|x-\xi| + (x-\xi)\cos\alpha]^{p(1+\delta\sigma)-1} \widetilde{f}^p(x)dx\right\}^{\frac{1}{p}} \\ &\quad \times \left\{\sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\sigma)-1}\widetilde{b}_n^q\right\}^{\frac{1}{q}} \\ &= \left\{\int_{-\infty}^{\infty} \frac{dx}{[|x-\xi| + (x-\xi)\cos\alpha]^{\delta\varepsilon+1}}\right\}^{\frac{1}{p}} \\ &\quad \times \left\{\sum_{|n|=1}^{\infty} \frac{1}{[|n-\eta| + (n-\eta)\cos\beta]^{\varepsilon+1}}\right\}^{\frac{1}{q}} \\ &\leq \frac{1}{\varepsilon}\left(2\csc^2\alpha\right)^{\frac{1}{p}} \left[(2\csc^2\beta + o_1(1))(1+o_2(1))\right]^{\frac{1}{q}}. \end{aligned}$$

By (2.5), we still have

$$\begin{aligned} \widetilde{I} :&= \sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} h(x,n)\widetilde{f}(x)\widetilde{b}_n dx \\ &= \int_{E_\delta} \sum_{|n|=1}^{\infty} h(x,n) \frac{[|x-\xi| + (x-\xi)\cos\alpha]^{-\delta(\widetilde{\sigma}+\varepsilon)-1}}{[|n-\eta| + (n-\eta)\cos\beta]^{1-\widetilde{\sigma}}} dx \\ &= \int_{E_\delta} \frac{\varpi(\widetilde{\sigma},x)}{[|x-\xi| + (x-\xi)\cos\alpha]^{\delta\varepsilon+1}} dx \\ &\geq k_\beta(\widetilde{\sigma}) \int_{E_\delta} \frac{1-\theta(\widetilde{\sigma},x)}{[|x-\xi| + (x-\xi)\cos\alpha]^{\delta\varepsilon+1}} dx \\ &= k_\beta(\widetilde{\sigma}) \left\{\int_{E_\delta} \frac{1}{[|x-\xi| + (x-\xi)\cos\alpha]^{\delta\varepsilon+1}} dx\right. \end{aligned}$$

$$
\left.- \int_{E_\delta} \frac{1}{O([|x-\xi| + (x-\xi)\cos\alpha]^{\delta(\sigma+\frac{\varepsilon}{p})+1})} dx \right\}
$$
$$
= \frac{1}{\varepsilon} k_\beta(\sigma - \frac{\varepsilon}{q})(2\csc^2\alpha - \varepsilon O(1)).
$$

If the constant factor $K_{\alpha,\beta}(\sigma)$ in (3.2) is not the best possible, then, there exists a positive number k, with $K_{\alpha,\beta}(\sigma) > k$, such that (3.2) is valid when replacing $K_{\alpha,\beta}(\sigma)$ by k. Then in particular, we have $\varepsilon\tilde{I} < \varepsilon k \widetilde{I}_1$, namely,

$$
\begin{aligned}
& k_\beta(\sigma - \frac{\varepsilon}{q})(2\csc^2\alpha - \varepsilon O(1)) \\
< {} & k \left(2\csc^2\alpha\right)^{\frac{1}{p}} \left[(2\csc^2\beta + o_1(1))(1 + o_2(1))\right]^{\frac{1}{q}}.
\end{aligned}
$$

It follows that $2k_\beta(\sigma)\csc^2\alpha \le 2k\csc^{\frac{2}{p}}\alpha\csc^{\frac{2}{q}}\beta$ $(\varepsilon \to 0^+)$, namely,

$$
K_{\alpha,\beta}(\sigma) = \frac{2}{\gamma\rho^{\sigma/\gamma}} \Gamma(\frac{\sigma}{\gamma}) \csc^{\frac{2}{q}}\alpha \csc^{\frac{2}{p}}\beta \le k.
$$

This is a contradiction. Hence, the constant factor $K_{\alpha,\beta}(\sigma)$ in (3.2) is the best possible.

The constant factor $K_{\alpha,\beta}(\sigma)$ in (3.3) ((3.4)) is still the best possible. Otherwise, we would reach a contradiction by (3.9) ((3.11)) that the constant factor $K_{\alpha,\beta}(\sigma)$ in (3.2) is not the best possible. $\square$

4. Operator expressions

Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1$. We set the following functions:

$$
\begin{aligned}
\Phi(x) &:= [|x-\xi| + (x-\xi)\cos\alpha]^{p(1+\delta\sigma)-1}, \\
\Psi(n) &:= [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\sigma)-1},
\end{aligned}
$$

wherefrom, $\Phi^{1-q}(x) = [|x-\xi| + (x-\xi)\cos\alpha]^{-q\delta\sigma-1}$, $\Psi^{1-p}(n) = [|n-\eta| + (n-\eta)\cos\beta]^{p\sigma-1}$ $(x \in \mathbf{R}\backslash\{\xi\}, |n| \in \mathbf{N})$. Define the following real weight normed linear spaces:

$$
\begin{aligned}
L_{p,\Phi}(\mathbf{R}) &:= \left\{ f; ||f||_{p,\Phi} := \left(\int_{-\infty}^{\infty} \Phi(x)|f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\}, \\
L_{q,\Phi^{1-q}}(\mathbf{R}) &:= \left\{ h; ||h||_{q,\Phi^{1-q}} := \left(\int_{-\infty}^{\infty} \Phi^{1-q}(x)|h(x)|^q dx \right)^{\frac{1}{q}} < \infty \right\}, \\
l_{q,\Psi} &:= \left\{ b = \{b_n\}_{|n|=1}^{\infty}; ||b||_{q,\Psi} := \left(\sum_{|n|=1}^{\infty} \Psi(n)|b_n|^q \right)^{\frac{1}{q}} < \infty \right\}, \\
l_{p,\Psi^{1-p}} &:= \left\{ c = \{c_n\}_{|n|=1}^{\infty}; ||c||_{p,\Psi^{1-p}} := \left(\sum_{|n|=1}^{\infty} \Psi^{1-p}(n)|c_n|^p \right)^{\frac{1}{p}} < \infty \right\}.
\end{aligned}
$$

(a) In view of Theorem 3.1, for $f \in L_{p,\Phi}(\mathbf{R})$, setting

$$H^{(1)}(n) := \int_{-\infty}^{\infty} h(x,n)|f(x)|dx \ (|n| \in \mathbf{N}),$$

by (3.3), we have

$$||H^{(1)}||_{p,\Psi^{1-p}} = \left[\sum_{|n|=1}^{\infty} \Psi^{1-p}(n)(H^{(1)}(n))^p\right]^{\frac{1}{p}} < K_{\alpha,\beta}(\sigma)||f||_{p,\Phi} < \infty, \tag{4.1}$$

namely, $H^{(1)} \in l_{p,\Psi^{1-p}}$.

Definition 4.1. Define a Hilbert-type operator in the whole plane $T^{(1)} : L_{p,\Phi}(\mathbf{R}) \to l_{p,\Psi^{1-p}}$ as follows: For any $f \in L_{p,\Phi}(\mathbf{R})$, there exists a unique representation $T^{(1)}f = H^{(1)} \in l_{p,\Psi^{1-p}}$, satisfying for any $|n| \in \mathbf{N}, (T^{(1)}f)(n) = H^{(1)}(n)$.

In view of (4.1), it follows that $||T^{(1)}f||_{p,\Psi^{1-p}} = ||H^{(1)}||_{p,\Psi^{1-p}} \leq K_{\alpha,\beta}||f||_{p,\Phi}$, and then the operator $T^{(1)}$ is bounded satisfying

$$||T^{(1)}|| = \sup_{f(\neq\theta)\in L_{p,\Phi}(\mathbf{R})} \frac{||T^{(1)}f||_{p,\Psi^{1-p}}}{||f||_{p,\Phi}} \leq K_{\alpha,\beta}(\sigma).$$

Since the constant factor $K_{\alpha,\beta}(\sigma)$ in (4.1) is the best possible, we have

$$||T^{(1)}|| = K_{\alpha,\beta}(\sigma) = \frac{2}{\gamma\rho^{\sigma/\gamma}}\Gamma(\frac{\sigma}{\gamma}) \csc^{\frac{2}{q}} \alpha \csc^{\frac{2}{p}} \beta. \tag{4.2}$$

If we define the formal inner product of $T^{(1)}f$ and b $(\in l_{q,\Psi})$ as follows:

$$(T^{(1)}f, b) := \sum_{|n|=1}^{\infty} (\int_{-\infty}^{\infty} h(x,n)f(x)dx)b_n$$

then we can rewrite (3.2) and (3.3) as follows:

$$(T^{(1)}f, b) < ||T^{(1)}|| \cdot ||f||_{p,\Psi}||b||_{q,\Phi}, ||T^{(1)}f||_{p,\Psi^{1-p}} < ||T^{(1)}|| \cdot ||f||_{p,\Phi}. \tag{4.3}$$

(b) In view of Theorem 3.1, for $b \in l_{q,\Psi}$, setting

$$H^{(2)}(x) := \sum_{|n|=1}^{\infty} h(x,n)b_n \ (x \in \mathbf{R}),$$

then by (3.4), we have

$$||H^{(2)}||_{q,\Phi^{1-q}} = \left[\int_{-\infty}^{\infty} \Phi^{1-q}(x)(H^{(2)}(x))^q dx\right]^{\frac{1}{q}} < K_{\alpha,\beta}(\sigma)||b||_{q,\Psi} < \infty, \tag{4.4}$$

namely $H^{(2)} \in L_{q,\Psi^{1-q}}(\mathbf{R})$.

Definition 4.2. Define a Hilbert-type operator in the whole plane $T^{(2)} : l_{q,\Psi} \to L_{q,\Psi^{1-q}}(\mathbf{R})$ as follows: For any $b \in l_{q,\Psi}$, there exists a unique representation $T^{(2)}b = H^{(2)} \in L_{q,\Psi^{1-q}}(\mathbf{R})$, satisfying for any $x \in \mathbf{R}, (T^{(2)}b)(x) = H^{(2)}(x)$.

In view of (4.4), we have $||T^{(2)}b||_{q,\Phi^{1-q}} = ||H^{(2)}||_{q,\Phi^{1-q}} \le K_{\alpha,\beta}(\sigma)||b||_{q,\Psi}$, and then the operator $T^{(2)}$ is bounded satisfying

$$||T^{(2)}|| = \sup_{b(\neq\theta)\in l_{q,\Psi}} \frac{||T^{(2)}b||_{q,\Phi^{1-q}}}{||b||_{q,\Psi}} \le K_{\alpha,\beta}(\sigma).$$

Since the constant factor $K_{\alpha,\beta}(\sigma)$ in (4.4) is the best possible, we have

$$||T^{(2)}|| = K_{\alpha,\beta}(\sigma) = ||T^{(1)}||. \tag{4.5}$$

If we define the formal inner product of $T^{(2)}b$ and f $(\in L_{p,\Phi}(\mathbf{R}))$ as follows:

$$(T^{(2)}b, f) := \int_{-\infty}^{\infty} \sum_{|n|=1}^{\infty} h(x,n) b_n f(x) dx,$$

then we can rewrite (3.2) and (3.4) as follows:

$$(T^{(2)}b, f) < ||T^{(2)}|| \cdot ||f||_{p,\Psi} ||b||_{q,\Phi}, ||T^{(2)}b||_{q,\Phi^{1-q}} < ||T^{(2)}|| \cdot ||b||_{q,\Psi}. \tag{4.6}$$

Remark 4.1. (i) For $\xi = \eta = 0$, $\delta = 1$, (3.5) reduces to (1.5). If $f(-x) = f(x)$ $(x > 0), b_{-n} = b_n$ $(n \in \mathbf{N})$, then (1.5) reduces to the following half-discrete Hilbert-type inequality (cf. [22]):

$$\begin{aligned} &\sum_{n=1}^{\infty} \int_0^{\infty} e^{-\rho(\frac{n}{x})^{\gamma}} f(x) b_n dx \\ &< \frac{\Gamma(\frac{\sigma}{\gamma})}{\gamma \rho^{\sigma/\gamma}} \left[\int_0^{\infty} x^{p(1+\sigma)-1} f^p(x) dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{q(1-\sigma)-1} b_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{4.7}$$

(ii) For $\delta = 1$, (3.2) reduces to the following particular inequality with the homogeneous kernel of degree-0:

$$\begin{aligned} &\sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} e^{-\rho\{\frac{|n-\eta|+(n-\eta)\cos\beta}{|x-\xi|+(x-\xi)\cos\alpha}\}^{\gamma}} f(x) b_n dx \\ &< K_{\alpha,\beta}(\sigma) \left\{\int_{-\infty}^{\infty} [|x-\xi| + (x-\xi)\cos\alpha]^{p(1+\sigma)-1} f^p(x) dx\right\}^{\frac{1}{p}} \\ &\times \left\{\sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\sigma)-1} b_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{4.8}$$

(iii) For $\delta = -1$, (3.2) reduces to the following particular inequality with the non-homogeneous kernel:

$$\begin{aligned} &\sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} e^{-\rho\{[|x-\xi|+(x-\xi)\cos\alpha][|n-\eta|+(n-\eta)\cos\beta]\}^{\gamma}} f(x) b_n dx \\ &< K_{\alpha,\beta}(\sigma) \left\{\int_{-\infty}^{\infty} [|x-\xi| + (x-\xi)\cos\alpha]^{p(1-\sigma)-1} f^p(x) dx\right\}^{\frac{1}{p}} \\ &\times \left\{\sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\sigma)-1} b_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{4.9}$$

The constant factors in the above inequalities are the best possible.

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