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DYNAMICS OF AN SIR EPIDEMIC MODEL WITH HORIZONTAL AND VERTICAL TRANSMISSIONS AND CONSTANT TREATMENT RATES*

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Abstract We investigate the dynamics and bifurcations of SIR epidemic model with horizontal and vertical transmissions and constant treatment rates. It is proved that such SIR epidemic model have up to two positive epidemic equilibria and has no positive disease-free equilibria. We find all the ranges of the parameters involved in the model under which the equilibria of the model are positive. By using the qualitative theory of planar systems and the normal form theory, the phase portraits of each equilibria are obtained. We show that the equilibria of the epidemic system can be saddles, stable nodes, stable or unstable focuses, weak centers or cusps. We prove that the system has the Bogdanov-Takens bifurcations, which exhibit saddle-node bifurcations, Hopf bifurcations and homoclinic bifurcations.

Keywords SIR model, horizontal and vertical transmission, constant treatment rate, Bogdanov-Takens bifurcation.

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1. Introduction

A variety of disease model have been developed by incorporating the control measures or demographic structures into the classic susceptible-infectious-recovered (SIR) model with vital dynamics (birth and death) [7]. In the classic SIR model, the horizontal transmissions: the diseases are transmitted through contact between the infectives and the susceptibles, are considered, but the vertical transmissions: the diseases are transmitted from infective parents to unborn or newly born offsprings are not involved. Treatment including isolation or quarantine is one of control measures for epidemic diseases such as measles, tuberculosis, AIDs, phthisis and flu. There are many infectious diseases such as rubella, herpes simplex, hepatitis

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B, chagas disease, and AIDS can be transmitted by horizontal transmission or by vertical transmission such as eggs of insects or seeds of plants [2,3,12].

In this paper, we incorporate the known constant treatment rate representing the capacity of treatment on infectives into the SIR model with both horizontal and vertical transmissions, and study the dynamics of the model. Under the standard assumptions that the birth rate equals the death rate, all the unborn or newly born offsprings of the infective parents are the susceptible or infective, and the density of the population is 1, the model is governed by the following system

$$\begin{cases} \dot{S} = b - \beta SI - bS - qbI, \\ \dot{I} = \beta SI - bI - rI + qbI - T(I), \\ \dot{R} = rI - bR + T(I), \end{cases} \tag{1.1}$$

where $S \geq 0$, $I \geq 0$ and $R \geq 0$ denote the densities of the populations of the susceptible, infective and removed, respectively; b is the birth rate of susceptible population which equals the death rate, and $r \geq 0$ is the recovery rate of the infective individuals. Note that the disease is considered as a permanent immunity. The parameter β denotes the effective per capita contact rate of infective individuals. The term βSI is called the incidence rate. The parameter $q \in [0,1]$ is the fraction of unborn or newly born offsprings of the infective parents, and $T(I)$ is the constant treatment rate defined by

$$T(I) = \begin{cases} h \text{ for } I > 0, \\ 0 \text{ for } I = 0, \end{cases}$$

and $h > 0$ is a constant representing the capacity of the treatment for a disease in a community. The SIR model (1.1) with $q = h = 0$, $q = 0$ and $h = 0$ were studied, respectively, in [7], [15] and [13].

Under the assumptions that the birth and death rates are equal and the density of the population is 1, that is, $S(t) + I(t) + R(t) = 1$ for $t \geq 0$, from which we can see that the system (1.1) is reduced to the system of the first two equations, namely,

$$\begin{cases} \dot{S} = b - \beta SI - bS - qbI := f(S,I), \\ \dot{I} = \beta SI - bI - rI + qbI - h := g(S,I). \end{cases} \tag{1.2}$$

The system (1.2) with treatments and without vertical transmissions, that is, (1.2) with $h > 0$ and $q = 0$, was studied by Wang and Ruan [15]. They proved that such model undergo saddle-node bifurcation, subcritical Hopf bifurcation, and homoclinic bifurcation, and discussed the existence and nonexistence of limit cycles of the model. We refer to [5,6,8,9,11,16] for other disease model with treatments and without vertical transmissions.

The system (1.2) with vertical transmissions and without treatment, that is, $q \in (0,1)$ and $h = 0$, was studied by Meng and Chen [13]. They showed that when the basic reproductive rate R_0 of the epidemic is greater that 1, (1.2) with $q \in (0,1)$ and $h = 0$ has only one positive infection-free equilibrium, which is unstable, and has only one positive interior (epidemic) equilibrium, which is locally stable. If $R_0 < 1$, then the infection-free equilibrium is locally stable and the interior equilibrium is unstable.

The dynamics of the system (1.2) with $h > 0$ is different from (1.2) with $h = 0$. We shall show that (1.2) with $h > 0$ has no positive infection-free equilibria, has two, one or no positive interior equilibria, for $0 < h < h_1$, $h = h_1$ and $h > h_1$, respectively, where the formula of the critical treatment rate h_1 will be given later. These results are generalizations of the corresponding results obtained in [15] from $q = 0$ to $q \in [0, 1]$. We shall show that the one of positive interior equilibria is a saddle point, which generalizes the corresponding result in [15] from $q = 0$ to $q \in [0, 1]$, and another can be a stable focus or stable or unstable node for small treatment rates, which generalizes the corresponding result in [15], where $q = 0$ and only stability of the equilibrium was obtained. When $h = h_1$, we show that the unique positive interior equilibrium of (1.2) is a saddle-node or a cusp with dimension 2, and (1.2) admits Bogdanov-Takens bifurcations showing that (1.2) can admit saddle-node bifurcations, Hopf bifurcations and homoclinic bifurcations.

2. Positive equilibria

In this section, we find all the equilibria of (1.2) and provide conditions on the parameters involved in (1.2) under which the equilibria are positive. For convenience, we replace S and I in the system (1.2) by x and y, respectively, and obtain the following system

$$\begin{cases} \dot{x} = b - \beta xy - bx - qby := f(x, y), \\ \dot{y} = \beta xy - by - ry + qby - h := g(x, y), \end{cases} \tag{2.1}$$

where $x \geq 0$ and $y \geq 0$ denote the densities of the populations of the susceptible and infective, respectively.

System (2.1) are generalizations of some known model studied in [7,13,15]. (2.1) with $q = h = 0$ is the classic SIR model studied in [7] while (2.1) with $q = 0$ and with $h = 0$ becomes the system (1.3) with $A = b$ studied in [15], where A represents the recruitment rate of the population, and the system (2.3) studied in [13], respectively.

Recall that $(x, y) \in \mathbb{R}^2$ is an equilibrium of (2.1) if it satisfies $f(x, y) = 0$ and $g(x, y) = 0$. An equilibrium point (x, y) is said to be positive if $x, y \geq 0$; to be a positive interior equilibrium if $x, y > 0$. Hence, to find all the equilibria of (2.1), we consider the following system

$$\begin{cases} -\beta xy - bx - qby + b = 0, \\ \beta xy - \eta y - h = 0, \end{cases} \tag{2.2}$$

where $\eta := \eta(r, b, q) = r + (1 - q)b > 0$.

The basic reproductive rate of the epidemic is defined as

$$R_0 = \frac{\beta}{\eta} = \frac{\beta}{r + b - qb}.$$

Theorem 2.1. (1) *Assume that* $b, r, \beta, \eta > 0$ *and* $q, h \geq 0$. *If* $(x, y) \in \mathbb{R}^2$ *is an equilibrium of* (2.1), *then* y *satisfies the following quadrant equation*

$$-\beta(b + r)y^2 + [\beta(b - h) - b\eta]y - bh = 0. \tag{2.3}$$

(2) *Assume that* $b, r > 0$, $q \in [0,1]$ *and* $0 < \beta \le \eta$. *Then equation* (2.1) *with* $h > 0$ *has no positive equilibria.*

Proof. (1) Assume that $(x, y) \in \mathbb{R}^2$ is an equilibrium of (2.1). Then (x, y) is a solution of the system (2.2). Adding the second equation of (2.2) to the first one, we see that the system (2.2) is equivalent to the following system

$$\begin{cases} -bx - (b+r)y + (b-h) = 0, \\ \beta xy - \eta y - h = 0. \end{cases} \tag{2.4}$$

Multiplying the first equation of (2.4) by βy and the second one by b and then adding the two resulting equations implies (2.3).

(2) The proof is by contradiction. Assume that $(x, y) \in \mathbb{R}^2$ is a positive equilibrium of (2.1). Then $y \ge 0$ and y satisfies (2.3). Since $b, r > 0$, $q \in [0,1]$ and $0 < \beta \le \eta$, we have $b(\eta - \beta) + \beta h \ge 0$. By (2.3), we have

$$\begin{aligned} 0 &= -\beta(b+r)y^2 + [\beta(b-h) - b\eta]y - bh \\ &= -\beta(b+r)y^2 - [b(\eta - \beta) + \beta h]y - bh \le -bh < 0, \end{aligned}$$

a contradiction. □

Theorem 2.1 (2) is a generalization of a result given in [15, page 779], where $q = 0$. Theorem 2.1 (2) shows that if the basic reproductive rate R_0 is less than or equal to 1 and the treatment is introduced, then equation (2.1) with $h > 0$ has no positive equilibria. Hence, the epidemic cannot maintain itself. This result is different from that of (2.1) with $h = 0$. It is known that equation (2.1) with $h = 0$ has a disease-free equilibrium $(1, 0)$ for $b, r, \beta > 0$ and $q \in [0, 1]$, and a positive interior epidemic equilibrium $\left(\frac{\eta}{\beta}, \frac{b}{b+r}(1 - \frac{\eta}{\beta})\right)$ for $b, r > 0$, $q \in [0, 1]$ and $\eta < \beta$, see [7, section 6], where (2.1) with $h = q = 0$ is considered and [13, page 584], where (2.1) with $h = 0$ and $q \in (0, 1)$ is considered. Hence, when $R_0 \le 1$, (2.1) with $h = 0$ always has a disease-free equilibrium $(1, 0)$ and when $R_0 > 1$, (2.1) with $h = 0$ always has the disease-free equilibrium $(1, 0)$ and a positive interior epidemic equilibrium $\left(\frac{\eta}{\beta}, \frac{b}{b+r}(1 - \frac{\eta}{\beta})\right)$.

However, when the treatment is introduced, (2.1) with $h > 0$ has no disease-free equilibria and we shall show in Theorem 2.2 below that if the basic reproductive rate R_0 is greater than 1, that is, $\beta > \eta$, then (2.1) with $h > 0$ has up to two positive interior equilibria depending on the value h, and when the treatment rate h is greater than the value h_1 to be given below, (2.1) has no positive equilibria and the epidemic is eradicated.

Notation 1. Let

$$\begin{aligned} & h_0 = \frac{b(\beta - \eta)}{\beta}, \qquad h^* = h_0 + \frac{2b(b+r)}{\beta}, \\ & h_1 := h_1(q) = h^* - \frac{2b}{\beta}\sqrt{(b+r)(\beta + qb)}, \quad h_2 = h^* + \frac{2b}{\beta}\sqrt{(b+r)(\beta + qb)}, \\ & \Delta := \Delta(h) = [\beta(b-h) - b\eta]^2 - 4bh\beta(b+r) = \beta^2(h_0 - h)^2 - 4bh\beta(b+r). \end{aligned}$$

Lemma 2.1. *Assume that* $b, r > 0$, $q \in [0,1]$ *and* $\beta > \eta$. *Then* $0 < h_1 < h_0$.

Proof. By the definition of h_1, we have

$$\begin{aligned} h_1 &= h^* - \frac{2b}{\beta}\sqrt{(b+r)(\beta+qb)} \\ &= \frac{b}{\beta}\Big[(\beta+b+r+qb) - 2\sqrt{(b+r)(\beta+qb)}\Big] \\ &= \frac{b}{\beta}\big[\sqrt{\beta+qb} - \sqrt{b+r}\big]^2 \\ &= \frac{b(\beta-\eta)^2}{\beta\big[\sqrt{\beta+qb}+\sqrt{b+r}\big]^2} > 0. \end{aligned}$$

By the definition of h_0 and h_1, we have

$$\begin{aligned} h_0 - h_1 &= \frac{2b}{\beta}\Big[\sqrt{(b+r)(\beta+qb)} - (b+r)\Big] \\ &= \frac{2b[(b+r)(\beta+qb) - (b+r)^2]}{\beta\Big[b+r+\sqrt{(b+r)(\beta+qb)}\Big]} \\ &= \frac{2b(b+r)(\beta-\eta)}{\beta\Big[b+r+\sqrt{(b+r)(\beta+qb)}\Big]} > 0 \end{aligned} \tag{2.5}$$

and $h_1 < h_0$. □

Theorem 2.2. *Assume that* $b, r > 0$, $q \in [0,1]$ *and* $\beta > \eta$. *Then the following assertions hold.*

(1) *If* $0 < h < h_1$, *then system* (2.1) *has two positive interior equilibria* (x_1, y_1) *and* (x_2, y_2), *where*

$$y_1 = \frac{[\beta(b-h) - b\eta] - \sqrt{\Delta}}{2\beta(b+r)}, \quad y_2 = \frac{[\beta(b-h) - b\eta] + \sqrt{\Delta}}{2\beta(b+r)} \tag{2.6}$$

and

$$x_1 = \frac{\eta y_1 + h}{\beta y_1}, \quad x_2 = \frac{\eta y_2 + h}{\beta y_2}. \tag{2.7}$$

(2) *If* $h = h_1$, *then system* (2.1) *has one positive interior equilibrium* (x^*, y^*), *where*

$$x^* = \frac{\eta y^* + h_1}{\beta y^*} = \frac{\eta}{\beta} + \frac{(b+r)(\beta-\eta)}{\beta\big[b+r+\sqrt{(b+r)(\beta+qb)}\big]}$$

and

$$y^* = \frac{b(\beta-\eta)}{\beta\big[b+r+\sqrt{(b+r)(\beta+qb)}\big]}.$$

(3) *If* $h > h_1$, *then system* (2.1) *has no positive equilibria.*

Proof. By (2.2) and (2.3), we see that if (x, y) is an equilibrium of (2.1), then y satisfies (2.3) and if $y > 0$, then

$$x = \frac{\eta y + h}{\beta y}. \tag{2.8}$$

We prove the following

$$(y - y^*(h))^2 = \frac{\Delta}{4\beta^2(b+r)^2} = \frac{(h-h_1)(h-h_2)}{4(b+r)^2}, \tag{2.9}$$

where

$$y^*(h) = \frac{b(\beta-\eta)-\beta h}{2\beta(b+r)} = \frac{h_0 - h}{2(b+r)}. \tag{2.10}$$

Indeed, by (2.3) and the definition of Δ, we have

$$\begin{aligned}
(y - y^*(h))^2 &= \frac{\Delta}{4\beta^2(b+r)^2} \\
&= \frac{1}{4\beta^2(b+r)^2}\Big[\beta^2(b-h)^2 - 2b\beta(b-h)\eta + b^2\eta^2 - 4bh\beta(b+r)\Big] \\
&= \frac{1}{4\beta^2(b+r)^2}\Big[\beta^2h^2 - 2b\beta(\beta-\eta+2r+2b)h + b^2(\beta-\eta)^2\Big] \\
&= \frac{1}{4(b+r)^2}\Big\{(h-h^*)^2 - \frac{b^2[(\beta-\eta+2r+2b)^2-(\beta-\eta)^2]}{\beta^2}\Big\} \\
&= \frac{1}{4(b+r)^2}\Big[(h-h^*)^2 - \frac{4b^2(b+r)(\beta+qb)}{\beta^2}\Big] \\
&= \frac{(h-h_1)(h-h_2)}{4(b+r)^2}.
\end{aligned}$$

(1) If $0 < h < h_1$, then by Lemma 2.1, $h < h_0 < h_2$ and $\Delta(h) > 0$. Solving (2.9) gets the two solutions y_1 and y_2 given in (2.6). Substituting y_1 and y_2 in (2.6) into (2.8) obtains x_1 and x_2 given in (2.7). By (2.9) and the definition of Δ, we have

$$\begin{aligned}
y_1 = y^*(h) - \frac{\sqrt{\Delta}}{2\beta(b+r)} &= \frac{\beta(h_0-h) - \sqrt{\beta^2(h-h_0)^2 - 4bh\beta(b+r)}}{2\beta(b+r)} \\
&= \frac{4bh\beta(b+r)}{2\beta(b+r)[\beta(h_0-h) + \sqrt{\beta^2(h_0-h)^2 - 4bh\beta(b+r)}]} > 0.
\end{aligned}$$

Since $y_1 < y_2$, we have $y_2 > 0$. By (2.7), we see that $x_1, x_2 > 0$. Hence, (x_1, y_1) and (x_2, y_2) are two positive interior equilibria of equation (2.1).

(2) If $h = h_1$, then by (2.5), (2.9) and (2.10), we have $\Delta(h_1) = 0$ and

$$\begin{aligned}
y^* = y^*(h_1) &= \frac{b(\beta-\eta)-\beta h_1}{2\beta(b+r)} = \frac{1}{2(b+r)}(h_0 - h_1) \\
&= \frac{1}{2(b+r)} \frac{2b(b+r)(\beta-\eta)}{\beta\Big[b+r+\sqrt{(b+r)(\beta+qb)}\Big]} \\
&= \frac{b(\beta-\eta)}{\beta\Big[b+r+\sqrt{(b+r)(\beta+qb)}\Big]} > 0.
\end{aligned} \tag{2.11}$$

Using the above formulas for h_1 and y^*, we have

$$\begin{aligned}
x^* &= \frac{\eta y^* + h_1}{\beta y^*} = \frac{\eta}{\beta} + \frac{h_1}{\beta y^*} = \frac{\eta}{\beta} + \frac{b(\beta-\eta) + 2b[b+r-\sqrt{(b+r)(\beta+qb)}]}{\beta^2 y^*} \\
&= \frac{\eta}{\beta} + \frac{(\beta-\eta)[b+r+\sqrt{(b+r)(\beta+qb)}] + 2[(b+r)^2 - (b+r)(\beta+qb)]}{\beta(\beta-\eta)} \\
&= \frac{\eta}{\beta} + \frac{(\beta-\eta)[b+r+\sqrt{(b+r)(\beta+qb)}] - 2(b+r)(\beta-\eta)}{\beta(\beta-\eta)} \\
&= \frac{\eta}{\beta} + \frac{-(b+r) + \sqrt{(b+r)(\beta+qb)}}{\beta} \\
&= \frac{\eta}{\beta} + \frac{(b+r)(\beta-\eta)}{\beta[b+r+\sqrt{(b+r)(\beta+qb)}]}.
\end{aligned}$$

(3) We consider the following three cases.

(*i*) If $h_1 < h < h_2$, then $(h-h_1)(h-h_2) < 0$ and by (2.9), (2.3) has no solutions. Hence, (2.1) has no positive equilibria.

(*ii*) If $h = h_2$, then by (2.9) and y^*, we have

$$\begin{aligned}
y = y^*(h_2) &= \frac{1}{2(b+r)}(h_0 - h_2) \\
&= \frac{b}{2\beta(b+r)}\left[-2(b+r) - 2\sqrt{(b+r)(\beta+qb)}\right] < 0
\end{aligned}$$

and (2.1) has no positive equilibria.

(*iii*) If $h > h_2$, then by (2.9) and y^*, we have

$$\begin{aligned}
y_2 &= y^*(h) + \frac{\sqrt{\Delta}}{2(b+r)} \\
&= \frac{\beta(h_0 - h) + \sqrt{\beta^2(h-h_0)^2 - 4bh\beta(b+r)}}{2\beta(b+r)} \\
&= \frac{-4bh\beta(b+r)}{2\beta(b+r)[\beta(h-h_0) + \sqrt{\beta^2(h-h_0)^2 - 4bh\beta(b+r)}]} < 0.
\end{aligned}$$

Hence, $y_1 \le y_2 < 0$ and (2.1) has no positive equilibria. □

Theorem 2.2 generalizes the corresponding results given in [15, page 778] from $q = 0$ to $q \in [0,1]$. Note that the conditions $0 < h < h_1(0)$, $h = h_1(0)$ and $h > h_1(0)$) in Theorem 2.2 with $q = 0$ are equivalent to the conditions $0 < H < (\sqrt{R_0} - 1)^2$, $H = (\sqrt{R_0} - 1)^2$ and $H > (\sqrt{R_0} - 1)^2$ in [15, page 778], where we let the recruitment rate A of the population used in [15] be the birth rate b. As mentioned above, if $b, r > 0$, $q \in [0,1]$ and $\beta > \eta$, then system (2.1) with $h = 0$ has only one interior equilibrium, see [13, page 584] while Theorem 2.2 shows that system (2.1) with $h > 0$ has up to two interior equilibria.

Corollary 2.1. *Assume that* $b, r > 0$, $q \in [0,1]$ *and* $\beta > \eta$. *Then equation* (2.1) *has no positive equilibria for* $h > b$.

Proof. By Lemma 2.1, $0 < h_1 < h_0 < b$ and by Theorem 2.2 (3), we see that equation (2.1) has no positive equilibria for $h > b$. □

Corollary 2.1 shows that if the treatment rate is greater that the death rate, then the epidemic can be eradicated.

3. The phase portraits of the equilibria

In this section, we study the phase portraits of each positive equilibrium of (2.1). We recall some results on phase portraits of planar systems near equilibria in the qualitative theory [1, 14]. We consider the following planar system

$$\begin{cases} \dot{x} = f(x,y), \\ \dot{y} = g(x,y), \end{cases} \tag{3.1}$$

where $f, g : X \to \mathbb{R}$ are functions having continuous first partial derivatives and X is an open subset in $\mathbb{R}^2$. We denote by $A(x,y)$ the Jacobian matrix of f and g at (x,y), that is,

$$A(x,y) = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix} \tag{3.2}$$

and by $|A(x,y)|$ and $Tr(A(x,y))$ the determinant and the trace of $A(x,y)$, respectively. Recall that a map $T : X \to Y$ defined by $T(x,y) = (f(x,y), g(x,y))$ is said to be regular if T is one to one and onto, T and T^{-1} are continuous and $|A(x,y)| \neq 0$ on X (see [1]). If T is regular, then the following transformation

$$\begin{cases} u = f(x,y), \\ v = g(x,y) \end{cases} \tag{3.3}$$

is said to be a regular transformation. If (3.1) is changed into another system under suitable regular transformations, then the two systems are said to be equivalent. It is known that the topological structures of solutions of a planar system near equilibria including a variety of dynamics like saddles, topological saddles, nodes, saddle-nodes, foci, centers, or cusps remain unchanged under regular transformations.

It is well known that the topological structures of solutions of a planar system near its equilibria (x_0, y_0) can be studied by the eigenvalues of $A(x_0, y_0)$, which are essentially determined by $|A(x_0, y_0)|$ and $Tr(A(x_0, y_0))$.

The following results can be found in [14] and have been used, for example in [10, 19].

Lemma 3.1. *If* (x_0, y_0) *is an equilibrium of* (3.1)*, then the following assertions hold.*
(i) If $|A(x_0, y_0)| < 0$*, then* (x_0, y_0) *is a saddle of* (3.1).
(ii) If $|A(x_0, y_0)| > 0$, $(Tr(A(x_0, y_0)))^2 - 4|A(x_0, y_0)| \geq 0$, $Tr(A(x_0, y_0)) \neq 0$, *then* (x_0, y_0) *is a node of* (3.1)*; it is stable if* $Tr(A(x_0, y_0)) < 0$ *and unstable if* $Tr(A(x_0, y_0)) > 0$.
(iii) If $|A(x_0, y_0)| > 0$, $(Tr(A(x_0, y_0)))^2 - 4|A(x_0, y_0)| < 0$, $Tr(A(x_0, y_0)) \neq 0$, *then* (x_0, y_0) *is a focus of* (3.1)*; it is stable if* $Tr(A(x_0, y_0)) < 0$ *and unstable if* $Tr(A(x_0, y_0)) > 0$.

Lemma 3.2 ([10]). *Assume (x_0, y_0) is an equilibrium of (3.1), $|A(x_0, y_0)| = 0$, $Tr(A(x_0, y_0)) \neq 0$ and (3.1) is equivalent to the following system*

$$\begin{cases} \dot{u} = p(u, v), \\ \dot{v} = \varrho v + q(u, v) \end{cases} \tag{3.4}$$

with an equilibrium $(0, 0)$, where $\varrho \neq 0$, $p(u, v) = \sum_{i+j=2,i,j\geq 0}^{\infty} a_{ij} u^i v^j$ and $q(u, v) = \sum_{i+j=2,i,j\geq 0}^{\infty} b_{ij} u^i v^j$ are convergent power series. If $a_{20} \neq 0$, then (x_0, y_0) is a saddle-node of (3.1).

Recall that $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is said to be *analytic* in an open set $\Omega \subset \mathbb{R}^2$ if f has a convergent Taylor series in some neighborhood of every point in Ω (see [14, page 69]).

When $|A(x_0, y_0)| = 0$, $Tr(A(x_0, y_0)) = 0$ and $A(x_0, y_0) \neq 0$, under suitable regular transformations, (3.1) is equivalent to the following form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = a_k x^k[1 + h(x)] + b_n x^n y[1 + g(x)] + y^2 R(x, y) \end{cases} \tag{3.5}$$

with an equilibrium point $(0, 0)$, where h, g and R are analytic in a neighborhood of $(0, 0)$, $h(0) = g(0) = 0$, $k \geq 2$, $a_k \neq 0$ and $n \geq 1$.

The following well-known result from [1, 14] has been used in [10, 19].

Lemma 3.3. *Assume that (x_0, y_0) is an equilibrium of (3.1),*

$$|A(x_0, y_0)| = Tr(A(x_0, y_0)) = 0, \quad A(x_0, y_0) \neq 0$$

and (3.1) is equivalent to (3.5). If $k = 2m \in \mathbb{N}$ and $n \geq m$, then (x_0, y_0) is a cusp of (3.1).

It is known that if the equilibrium (x_0, y_0) of (3.1) satisfies all the conditions of Lemma 3.3, then $A(x_0, y_0)$ has two zero eigenvalues. We refer to [14, page 456, Section 4.12] or [4, page 229, Section 4.1] for the study of bifurcations of systems having double-zero eigenvalues.

Now, we use the theory mentioned above to study the phase portraits of (2.1) near its positive equilibria. Assume that (x, y) is an equilibrium of (2.1). Then by (3.2) and (2.1), we have

$$A(x, y) = \begin{pmatrix} -b - \beta y & -\beta x - qb \\ \beta y & \beta x - \eta \end{pmatrix}.$$

It follows that

$$|A(x, y)| = \beta(b + r)y - b\beta x + b\eta \tag{3.6}$$

and

$$Tr(A(x, y)) = \beta(x - y) - (b + \eta). \tag{3.7}$$

Lemma 3.4. *Assume that $b, r > 0$, $q \in [0,1]$ and $\eta < \beta$. Then the following assertions hold.*

(1) *If $0 < h < h_1$, then $|A(x_1, y_1)| = -\sqrt{\Delta}$ and $|A(x_2, y_2)| = \sqrt{\Delta}$.*

(2) *If $h = h_1$, then $|A(x^*, y^*)| = 0$.*

(3) *If $0 < h < h_1$, then*

$$Tr(A(x_2, y_2)) = \frac{-(2b+r)\sqrt{\Delta} + r[\beta(b-h) - b\eta] - 2b^2(b+r)}{2b(b+r)}.$$

(4) *If $h = h_1$, then*

$$Tr(A(x_2, y_2)) = Tr(A(x^*, y^*)) = \frac{r^2(\beta - \beta_0)}{\left[r\sqrt{(b+r)(\beta+qb)} + (b+r)^2\right]},$$

where

$$\beta_0 := \beta_0(b, r, q) = \eta + \frac{b(b+r)(b+2r)}{r^2}. \tag{3.8}$$

Proof. (1) By the formulas of y_1 and Δ, we have

$$\beta(b+r)y_1 = \frac{\beta(b-h) - b\eta - \sqrt{\Delta}}{2}$$

and

$$\begin{aligned}
\frac{1}{y_1} &= \frac{2\beta(b+r)}{[\beta(b-h) - b\eta] - \sqrt{\Delta}} \\
&= \frac{2\beta(b+r)[\beta(b-h) - b\eta + \sqrt{\Delta}]}{[\beta(b-h) - b\eta]^2 - \Delta} \\
&= \frac{2\beta(b+r)[\beta(b-h) - b\eta + \sqrt{\Delta}]}{4bh\beta(b+r)} \\
&= \frac{\beta(b-h) - b\eta + \sqrt{\Delta}}{2bh}.
\end{aligned}$$

By (2.6), (2.7) and (3.6), we have

$$\begin{aligned}
|A(x_1, y_1)| =& \beta(b+r)y_1 - b\beta x_1 + b\eta \\
=& \beta(b+r)y_1 - b\beta\Big[\frac{\eta y_1 + h}{\beta y_1}\Big] + b\eta \\
=& \beta(b+r)y_1 - \frac{bh}{y_1} \\
=& \frac{\beta(b-h) - b\eta - \sqrt{\Delta}}{2} - \frac{\beta(b-h) - b\eta + \sqrt{\Delta}}{2} \\
=& -\sqrt{\Delta} < 0.
\end{aligned}$$

Similarly, we can show $|A(x_2, y_2)| = \sqrt{\Delta}$.

(2) Since $h = h_1$, by (2.9), $\Delta(h_1) = 0$. A computation similar to $|A(x_1, y_1)|$ shows $|A(x^*, y^*)| = \sqrt{\Delta(h_1)} = 0$.

(3) By (2.6), (2.7) and (3.7), we have

$$\begin{aligned} Tr(A(x_2,y_2)) =& \frac{\beta(b-h)-b\eta-\sqrt{\Delta}}{2b} - \frac{\beta(b-h)-b\eta+\sqrt{\Delta}}{2(b+r)} - b \\ =& \frac{(b+r)[\beta(b-h)-b\eta-\sqrt{\Delta}] - b[\beta(b-h)-b\eta+\sqrt{\Delta}] - 2b^2(b+r)}{2b(b+r)} \\ =& \frac{-(2b+r)\sqrt{\Delta} + r[\beta(b-h)-b\eta] - 2b^2(b+r)}{2b(b+r)}. \end{aligned}$$

(4) Let $\nu = r[\beta(b-h_1)-b\eta] - 2b^2(b+r)$. Then by the definition of h_1, we have

$$\begin{aligned} \nu =& r\Big\{\beta\big[b - \frac{b(\beta-\eta+2b+2r)}{\beta} + \frac{2b\sqrt{(b+r)(\beta+qb)}}{\beta}\big] - b\eta\Big\} \\ &- 2b^2(b+r) = 2br\big[\sqrt{(b+r)(\beta+qb)} - (b+r)\big] - 2b^2(b+r) \\ =& 2b\Big[r\sqrt{(b+r)(\beta+qb)} - (b+r)^2\Big] \\ =& \frac{2b(b+r)\Big[r^2(\beta+qb) - (b+r)^3\Big]}{\Big[r\sqrt{(b+r)(\beta+qb)} + (b+r)^2\Big]} \\ =& \frac{2b(b+r)r^2\Big[(\beta-\eta+b+r) - \frac{(b+r)^3}{r^2}\Big]}{\Big[r\sqrt{(b+r)(\beta+qb)} + (b+r)^2\Big]} \\ =& \frac{2b(b+r)r^2\Big[\beta-\eta - \frac{b(b+r)(b+2r)}{r^2}\Big]}{\Big[r\xi + (b+r)^2\Big]} \\ =& \frac{2b(b+r)r^2(\beta-\beta_0)}{\Big[r\xi + (b+r)^2\Big]}, \end{aligned}$$

where $\xi = \sqrt{(b+r)(\beta+qb)}$. This, together with the above formula for $Tr(A(x_2,y_2))$ with $\Delta = 0$, implies

$$Tr(A(x_2,y_2))\big|_{h=h_1} = \frac{\nu}{2b(b+r)} = \frac{r^2(\beta-\beta_0)}{\Big[r\sqrt{(b+r)(\beta+qb)} + (b+r)^2\Big]}.$$

It is clear that $Tr(A(x_2,y_2))\big|_{h=h_1} = Tr(A(x^*,y^*))$ since when $h = h_1$, $\Delta(h_1) = 0$ and $x_2 = x^*$ and $y_2 = y^*$. □

By Lemma 3.1 (1) and Lemma 3.4 (1), we obtain the following result.

Theorem 3.1. *Assume that $b, r > 0$, $q \in [0,1]$, $\eta < \beta$ and $0 < h < h_1$. Then (x_1, y_1) is a saddle of* (2.1).

Remark 3.1. Theorem 3.1 generalizes the result given in [15, page 779], where $q = 0$.

Let

$$\beta_1 = \eta + \frac{(b+r)\big[(b+2r) - 2\sqrt{(2b+r)(b+r)}\big]}{b}$$
$$\text{and} \quad \beta_2 = \eta + \frac{(b+r)\big[(b+2r) + 2\sqrt{(2b+r)(b+r)}\big]}{b}.$$

It is easy to show $\beta_1 < \eta$. Indeed, $\beta_1 < \eta$ if and only if $(b+2r) < 2\sqrt{(2b+r)(b+r)}$ if and only if $b^2 + 4br + 4r^2 < 4(2b^2 + 3br + r^2)$ if and only if $7b^2 + 8br > 0$.

Theorem 3.2. *Let $b, r > 0$ and $q \in [0,1]$. Then the following assertions hold.*

(i) If $\eta < \beta < \beta_2$, then there exists $\overline{h}_1 \in (0, h_1)$ such that (x_2, y_2) is a stable focus for $0 \le h < \overline{h}_1$.

(ii) If $\beta > \beta_2$, then there exists $\overline{h}_2 \in (0, h_1)$ such that (x_2, y_2) is a stable node for $0 \le h < \overline{h}_2$.

(iii) If $\eta < \beta < \beta_0$, then there exists $\bar{h}_3 \in (0, h_1)$ such that (x_2, y_2) is a stable node for $h \in (\bar{h}_3, h_1)$.

(iv) If $\beta > \beta_0$, then there exists $\bar{h}_4 \in (0, h_1)$ such that (x_2, y_2) is an unstable node for $h \in (\bar{h}_4, h_1)$.

Proof. By Lemma 3.4, we have

$$|A(x_2, y_2)| = \sqrt{\Delta(h)} > 0 \quad \text{for } 0 < h < h_1. \tag{3.9}$$

If $h = 0$, then $|A(x_2, y_2)| = \sqrt{\Delta(0)} = b(\beta - \eta)$ and by Lemma 3.4 (3), we have

$$\begin{aligned} Tr(A(x_2, y_2))\big|_{h=0} &= \frac{-b(2b+r)(\beta-\eta) + rb(\beta-\eta) - 2b^2(b+r)}{2b(b+r)} \\ &= \frac{-2b(\beta-\eta) - 2b(b+r)}{2(b+r)} \\ &= -\frac{b(\beta - \eta + b + r)}{b+r} < 0. \end{aligned}$$

Hence, there exists $h^* \in (0, h_1)$ such that

$$Tr(A(x_2, y_2)) < 0 \quad \text{for } h \in (0, h^*). \tag{3.10}$$

Let

$$\zeta(h) = [Tr(A(x_2, y_2))]^2 - 4|A(x_2, y_2)|. \tag{3.11}$$

We prove the following assertion.

(H) $\zeta(0) < 0$ if and only if $\beta_1 < \beta < \beta_2$.

Indeed, $\zeta(0) = \dfrac{b^2(\beta - \eta - b - r)^2}{(b+r)^2} - 4b(\beta - \eta) < 0$ if and only if

$$b(\beta - \eta - b - r)^2 < 4(\beta - \eta)(b+r)^2$$

if and only if

$$(\beta - \eta)^2 - 2\frac{(b+r)(3b+2r)}{b}(\beta - \eta) < -(b+r)^2$$

if and only if

$$\Big[\beta - \eta - \frac{(b+r)(3b+2r)}{b}\Big]^2 < \frac{4(b+r)^3(2b+r)}{b^2}$$

if and only if

$$\Big|\beta - \eta - \frac{(b+r)(b+2r)}{b}\Big| < \frac{2(b+r)\sqrt{(b+r)(2b+r)}}{b}$$

if and only if $\beta_1 < \beta < \beta_2$.

(i) Note that $\beta_1 < \eta$. Hence, $\eta < \beta < \beta_2$. By (H), $\zeta(0) < 0$. It follows from the continuity of ζ that there exists $\overline{h}_1 \in (0, h^*)$ such that $\zeta(h) < 0$ for $0 < h < \overline{h}_1$. The result follows from Lemma 3.1 (iii).

(ii) When $\beta > \beta_2$, by (H), we have $\zeta(0) > 0$. It follows from the continuity of ζ that there exists $\overline{h}_2 \in (0, h^*)$ such that $\zeta(h) > 0$ for $0 < h < \overline{h}_2$. The result follows from Lemma 3.1 (ii).

(iii) By Lemma 3.4 (4), we see that if $\eta < \beta < \beta_0$, then $Tr(A(x_2, y_2))\big|_{h=h_1} < 0$. By Lemma 3.4, $|A(x_2, y_2)|)|_{h=h_1} = 0$. It follows that

$$\zeta(h_1) = ([Tr(A(x_2, y_2)]^2 - 4|A(x_2, y_2)|)|_{h=h_1} = (Tr(A(x_2, y_2))|_{h=h_1})^2 > 0.$$

Hence, there exists $\bar{h}_3 \in (0, h_1)$ such that for $h \in (\bar{h}_3, h_1)$, $Tr(A(x_2, y_2)) < 0$ and $\zeta(h) > 0$. It follows from Lemma 3.1 (ii) that (x_2, y_2) is a stable node for $h \in (\bar{h}_3, h_1)$.

(iv) By Lemma 3.4 (4), we see that if $\beta > \beta_0$, then $Tr(A(x_2, y_2))\big|_{h=h_1} > 0$ and $\zeta(h_1) > 0$. Hence, there exists $\bar{h}_4 \in (0, h_1)$ such that for $h \in (\bar{h}_4, h_1)$, $Tr(A(x_2, y_2)) > 0$ and $\zeta(h) > 0$. It follows from Lemma 3.1 (ii) that (x_2, y_2) is an unstable node for $h \in (\bar{h}_4, h_1)$. □

Remark 3.2. Theorem 3.2 not only obtains the stability of the interior equilibrium (x_2, y_2), but also determines that (x_2, y_2) is a stable focus or stable or unstable node. Hence, Theorem 3.2 improves Theorem 2.1 in [15], where $q = 0$ and only stability of (x_2, y_2) is determined.

Now, we consider bifurcations of the equilibrium (x^*, y^*).

Theorem 3.3. *Let $b, r > 0$ and $q \in [0, 1]$. Then the following assertions hold.*

(1) *If $\beta > \eta$, $\beta \neq \beta_0$ and $h = h_1$, then (x^*, y^*) is a saddle-node of* (2.1).

(2) *If $\beta = \beta_0$ and $h = h_1$, then (x^*, y^*) is a cusp of* (2.1) *with codimension 2.*

Proof. Since $\beta > \eta$, by Lemma 3.4 (2), we have $|A(x^*, y^*)| = 0$. By Lemma 3.4 (4), if $\eta < \beta$ and $\beta \neq \beta_0$, then $Tr(A(x^*, y^*)) \neq 0$ and if $\beta = \beta_0$, then $Tr(A(x^*, y^*)) = 0$.

(1) We change the equilibrium (x^*, y^*) to the origin $(0, 0)$ by the change of

variables: $u = x - x^*, v = y - y^*$. Then the system (2.1) becomes

$$\begin{aligned}
\dot{u} &= \dot{x} = -\beta(u + x^*)(v + y^*) - b(u + x^*) - qd(v + y^*) + b \\
&= -(\beta y^* + b)u - (\beta x^* + qb)v - \beta uv \\
\dot{v} &= \dot{y} = \beta(u + x^*)(v + y^*) - \eta(v + y^*) - h_1 \\
&= \beta(uv + x^* v + v^* u) - \eta v \\
&= \beta y^* u + (\beta x^* - \eta)v + \beta uv.
\end{aligned}$$

Let $u_1 = u,\ v_1 = (\beta x^* - \eta)u + (\beta x^* + qb)v$. Then $u = u_1$ and

$$v = \frac{v_1 - (\beta x^* - \eta)u_1}{\beta x^* + qb} = \frac{1}{(b + r)y^* + h_1}(y^* v_1 - h_1 u_1),$$

where we have used $x^* = (\eta y^* + h_1)/(\beta y^*)$. Hence

$$\begin{aligned}
\dot{u}_1 =& \dot{u} = -(\beta y^* + b)u - (\beta x^* + qb)v - \beta uv \\
=& -(\beta y^* + b)u_1 - v_1 + (\beta x^* - \eta)u_1 - \frac{\beta}{(b + r)y^* + h_1}(y^* v_1 u_1 - h_1 u_1^2) \\
=& -Tr(A(x^*, y^*))u_1 - v_1 + \frac{\beta h_1}{(b + r)y^* + h_1}u_1^2 - \frac{\beta y^*}{(b + r)y^* + h_1}v_1 u_1, \\
\dot{v}_1 =& (\beta x^* - \eta)\dot{u} + (\beta x^* + qb)\dot{v} \\
=& (\beta x^* - \eta)[-(\beta y^* + b)u - (\beta x^* + qb)v - \beta uv] \\
&+ (\beta x^* + qb)[\beta y^* u + (\beta x^* - \eta)v + \beta uv] \\
=& \beta(\eta + qb)uv \\
=& \beta(b + r)uv \\
=& -\frac{\beta(b + r)h_1}{(b + r)y^* + h_1}u_1^2 + \frac{\beta(b + r)y^*}{(b + r)y^* + h_1}u_1 v_1.
\end{aligned}$$

If $\beta \neq \beta_0$, we let $u_2 = -Tr(A(x^*, y^*))u_1 - v_1,\ v_2 = v_1$.Then $u_1 = -(u_2 + v_2)/Tr(A(x^*, y^*)),\ v_1 = v_2$. Hence we have

$$\begin{aligned}
\dot{u}_2 =& -Tr(A(x^*, y^*))\dot{u}_1 - \dot{v}_1 \\
=& -Tr(A(x^*, y^*))\big[-Tr(A(x^*, y^*))u_1 - v_1 + \frac{\beta h_1}{(b + r)y^* + h_1}u_1^2 \\
&- \frac{\beta y^*}{(b + r)y^* + h_1}v_1 u_1\big] - \frac{\beta(b + r)h_1}{(b + r)y^* + h_1}u_1^2 + \frac{\beta(b + r)y^*}{(b + r)y^* + h_1}u_1 v_1 \\
=& -Tr(A(x^*, y^*))u_2 - \frac{\beta h_1\big(b + r + Tr(A(x^*, y^*))\big)}{(b + r)y^* + h_1}\frac{u_2^2 + 2u_2 v_2 + v_2^2}{Tr(A(x^*, y^*))^2} \\
&+ \frac{\beta y^*\big(b + r + Tr(A(x^*, y^*))\big)}{(b + r)y^* + h_1}\frac{u_2 v_2 + v_2^2}{Tr(A(x^*, y^*))} \\
=& -Tr(A(x^*, y^*))u_2 + a_1 u_2^2 + a_2 u_2 v_2 + a_3 v_2^2, \\
\dot{v}_2 =& \dot{v}_1 = -\frac{\beta h_1}{y^*}u_1^2 + \beta u_1 v_1 \\
=& -\frac{\beta h_1}{y^*}\frac{y^{*2}}{(h_1 - by^*)^2}(u_2 + v_2)^2 + \beta\frac{y^*(u_2 + v_2)v_2}{(h_1 - by^*)}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\beta h_1 y^*}{(h_1-by^*)^2}(u_2^2+2u_2v_2+v_2^2)+\frac{\beta y^*(u_2v_2+v_2^2)}{(h_1-by^*)}\\
&= b_1u_2^2+b_2u_2v_2+b_3v_2^2,
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{a}_1 &= \frac{\big(b+r+Tr(A(x^*,y^*))\big)}{\big((b+r)y^*+h_1\big)},\\
a_1 &= -\frac{\beta h_1\tilde{a}_1}{Tr(A(x^*,y^*))^2},\\
a_2 &= -\frac{2\beta h_1\tilde{a}_1}{Tr(A(x^*,y^*))^2}+\frac{\beta y^*\tilde{a}_1}{Tr(A(x^*,y^*))},\\
a_3 &= -\frac{\beta h_1\tilde{a}_1}{Tr(A(x^*,y^*))^2}+\frac{\beta y^*\tilde{a}_1}{Tr(A(x^*,y^*))};\\
b_1 &= -\frac{\beta h_1 y^*}{(h_1-by^*)^2},\\
b_2 &= -\frac{\beta y^*(h_1+by^*)}{(h_1-by^*)^2},\\
b_3 &= -\frac{\beta by^*}{(h_1-by^*)^2}.
\end{aligned}
$$

It follows that

$$
\begin{cases}
\dot{u}_2 = -Tr(A(x^*,y^*))u_2+a_1u_2^2+a_2u_2v_2+a_3v_2^2,\\
\dot{v}_2 = b_1u_2^2+b_2u_2v_2+b_3v_2^2.
\end{cases}
$$

By Lemma 3.2 with $\varrho = -Tr(A(x^*,y^*))$ and $a_{20} = b_3 \neq 0$, (x^*,y^*) is a saddle-node of (2.1).

(2) If $\beta = \beta_0$, then $Tr(A(x^*,y^*)) = 0$. From the proof of (1), we have

$$
\begin{cases}
\dot{u}_1 = -v_1+\dfrac{\beta_0 h_1}{(b+r)y^*+h_1}u_1^2-\dfrac{\beta_0 y^*}{(b+r)y^*+h_1}v_1u_1 := -v_1+f_1(u_1,v_1),\\
\dot{v}_1 = -\dfrac{\beta_0(b+r)h_1}{(b+r)y^*+h_1}u_1^2+\dfrac{\beta_0(b+r)y^*}{(b+r)y^*+h_1}u_1v_1 := f_2(u_1,v_1),
\end{cases}
\tag{3.12}
$$

where f_1, f_2 are polynomials of degree 2. Let $u_2 = u_1, v_2 = -v_1 + f_1(u_1, v_1)$. Then $\dot{u}_2 = \dot{u}_1 = v_2$ and

$$
\begin{aligned}
\dot{v}_2 =& -\dot{v}_1+\frac{2\beta_0 h_1}{(b+r)y^*+h_1}u_1\dot{u}_1-\frac{\beta_0 y^*}{(b+r)y^*+h_1}u_1\dot{v}_1-\frac{\beta_0 y^*}{(b+r)y^*+h_1}v_1\dot{u}_1\\
=& -f_2(u_1,v_1)+\frac{2\beta_0 h_1}{(b+r)y^*+h_1}u_1[-v_1+f_1(u_1,v_1)]\\
&-\frac{\beta_0 y^*}{(b+r)y^*+h_1}u_1f_2(u_1,v_1)-\frac{\beta_0 y^*}{(b+r)y^*+h_1}v_1[-v_1+f_1(u_1,v_1)]\\
=& -f_2(u_1,v_1)-\frac{2\beta_0 h_1}{(b+r)y^*+h_1}u_1v_1+\frac{\beta_0 y^*}{(b+r)y^*+h_1}v_1^2+f_3(u_1,v_1)\\
=& \frac{\beta_0(b+r)h_1}{(b+r)y^*+h_1}u_2^2+\frac{\beta_0 y^*}{(b+r)y^*+h_1}v_1^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\beta_0\big((b+r)y^*+2h_1\big)}{(b+r)y^*+h_1}u_2\big(-v_2-f_1(u_1,v_1)\big)+f_3(u_1,v_1)\\
=&b_1u_2^2+b_2u_2v_2+b_3v_2^2+f_4(u_2,v_2),
\end{aligned}
$$

where $b_0 = (b+r)y^* + h_1$, $f_3(u_1,v_1) = \frac{2\beta_0 h_1}{b_0}u_1 f_1(u_1,v_1) - \frac{\beta_0 y^*}{b_0}u_1 f_2(u_1,v_1) - \frac{\beta_0 y^*}{b_0}v_1 f_1(u_1,v_1)$ is a polynomial of degree 3 in u_1 and v_1, $b_1 = \frac{\beta_0(b+r)h_1}{b_0}$, $b_2 = \frac{\beta_0((b+r)y^*+2h_1)}{b_0}$, $b_3 = \frac{\beta_0 y^*}{(b+r)y^*+h_1}$, and $f_4(u_2,v_2) = b_2u_2f_1(u_1,v_1)) + f_3(u_1,v_1)$. By Lemma 3.3 with $m=1$ and $n=1$, (x^*,y^*) is a cusp of (2.1). □

Remark 3.3. Theorem 3.3 (1) is new, which provides the conditions under which (2.1) has a saddle-node. By (3.8), we see that if $q=0$, then $\beta_0 = (b+r)^3r^{-2}$. Hence, it is easy to verify that when $q=0$, the condition $\beta=\beta_0$ in Theorem 3.3 (2) is equivalent to the condition $(H3)$ with $A=b$ in [15, page 783]. This shows that Theorem 3.3 (2) generalizes Theorem 2.3 in [15].

4. Bogdanov-Takens bifurcations

In this section, we use Lemma 4.1 to study existence of Bogdanov-Takens bifurcations of (2.1) at the cusp (x^*,y^*) given in Theorem 3.3 (2). We denote by $B_r(x,y)$ the ball in $\mathbb{R}^2$ centered at (x,y) with radius r. We outline part of the theory of Bogdanov-Takens bifurcations which we need below. To study Bogdanov-Takens bifurcations, one needs to consider suitable small perturbations of systems involving suitable parameters. Here, we consider the system (3.1) with five parameters b,r,β,η,h used in section 2. Therefore, we can rewrite (3.1) as follows:

$$
\begin{cases}\dot{x} = f(x,y) := f^*(x,y,b,r,\beta),\\ \dot{y} = g(x,y) := g^*(x,y,b,r,\beta,\eta,h).\end{cases} \tag{4.1}
$$

For each fixed set of parameters (b,r,β,η,h), if (x_0,y_0) is an equilibrium of (4.1) and satisfies all the conditions of Lemma 3.3, then we consider the following perturbations of (4.1):

$$
\begin{cases}\dot{x} = f^*(x,y,b,r,\beta),\\ \dot{y} = g^*(x,y,b,r,\beta,\eta+\rho_1,h+\rho_2),\end{cases} \tag{4.2}
$$

where only the two parameters β and η in the second equation of (4.1) are replaced by $\beta+\rho_1$ and $\eta+\rho_2$, respectively and $(\rho_1,\rho_2)\in B_r(0)$, the open ball in $\mathbb{R}^2$ centered at origin with radius r. From the analysis given in [4, pages 229–252, Section 4.1], we see that under suitable regular transformations, (4.2) can be transformed into the following equivalent system

$$
\begin{cases}\dot{u} = v,\\ \dot{v} = \lambda_1(\rho_1,\rho_2)+\lambda_2(\rho_1,\rho_2)v+u^2+uv\phi(u,\rho_1,\rho_2)+v^2\psi(u,v,\rho_1,\rho_2),\end{cases} \tag{4.3}
$$

where $\lambda_1,\lambda_2,\phi,\psi$ are C^∞ functions satisfying the following conditions

$$
\lambda_1(0,0)=\lambda_2(0,0)=0 \quad \text{and} \quad \phi(0,0,0)=1.
$$

The further analysis given in [4, pages 252–259, Section 4.1] shows that the following result holds.

Lemma 4.1. *Let* (x_0, y_0) *is an equilibrium of* (3.1)*. Assume that* (x_0, y_0) *satisfies all the conditions of Lemma 3.3. If* $\left|\frac{\partial(\lambda_1,\ \lambda_2)}{\partial(\rho_1,\ \rho_2)}\right|_{(0,0)} \neq 0$*, then* (4.2) *undergos Bogdanov-Takens bifurcations.*

By [14, Section 4.12], we see that if the system (4.2) exhibits Bogdanov-Takens bifurcations, then it has saddle-node bifurcations, Hopf bifurcations and homoclinic-loop bifurcations depending on the choices of the parameters (ρ_1, ρ_2) in different regions near the origin $(0, 0)$.

The definition of the value h_1 is given in section. Here, we write $h_1 = h_1(b, r, \beta, q)$ as a function of b, r, β, q.

Theorem 4.1. *If* $b^*, r^* > 0$ $q^* \geq 0$, $\beta_0^* = \beta_0(b^*, r^*, q^*)$, $h_1^* = h_1(b^*, r^*, \beta^*, q^*)$ *and* $\eta^* = r^* + b^* - q^* d^*$*, then there exists* $\rho_0 > 0$ *such that* (2.1) *with* $(\eta, h) \in B_{\rho_0}(\eta^*, h_1^*)$ *and* $(b, r, q, \beta) = (b^*, r^*, q^*, \beta_0^*)$ *admits a Bogdanov-Takens bifurcation.*

Proof. We consider the following perturbed system:

$$\begin{cases} \dot{x} = -\beta_0^* yx - b^* x - q^* d^* y + b^*, \\ \dot{y} = \beta_0^* xy - (\eta^* + \rho_1) y - (h_1^* + \rho_2), \end{cases} \tag{4.4}$$

where $\rho_1, \rho_2 \in \mathbb{R}$. Now, we make regular transformations to change (4.4) into the form of (4.3). Note that (x^*, y^*) is an equilibrium of (4.4) with $(\rho_1, \rho_2) = (0, 0)$. Let $u = x - x^*$ and $v = y - y^*$. Then (4.4) becomes

$$\begin{cases} \dot{u} = -\beta_0^*(u + x^*)(v + y^*) - b^*(u + x^*) - q^* d^*(v + y^*) + b^*, \\ \dot{v} = \beta_0^*(u + x^*)(v + y^*) - (\eta^* + \rho_1)(v + y^*) - (h_1^* + \rho_2). \end{cases} \tag{4.5}$$

Hence,

$$\begin{aligned} \dot{u} =& (-\beta_0^* x^* y^* - b^* x^* - q^* d^* y^* + b^*) - b^* u - q^* d^* v - \beta_0^* y^* u - \beta_0^* x^* v - \beta_0^* uv \\ =& -(b^* + \beta_0^* y^*) u - (\beta_0^* x^* + q^* d^*) v - \beta_0^* uv\dot{v} \\ =& \beta_0^*(uv + vx^* + uy^* + x^* y^*) - (\eta^* + \rho_1)(v + y^*) - (h_1^* + \rho_2) \\ =& \big((\beta_0^* x^* y^* - \eta^* y^* - h_1^*) - (\rho_1 y^* + \rho_2)\big) + \beta_0^* x^* v + \beta_0^* y^* u - (\eta^* + \rho_1) v + \beta_0^* uv \\ =& -(\rho_1 y^* + \rho_2) + (\beta_0^* x^* - \eta^* - \rho_1) v + \beta_0^* y^* u + \beta_0^* uv \\ =& -(\rho_1 y^* + \rho_2) + \beta_0^* y^* u + (\beta_0^* x^* - \eta^* - \rho_1) v + \beta_0^* uv. \end{aligned}$$

Let $u_1 = u, v_1 = -(b^* + \beta_0^* y^*) u - (\beta_0^* x^* + q^* d^*) v$. Since

$$Tr(A(x^*, y^*)) = -(b^* + \beta_0^* y^*) + (\beta_0^* x^* - \eta^*) = 0$$

and $\eta^* = r^* + b^* - q^* d^*, -(\beta_0^* x^* + q^* d^*) < 0$ and

$$u = u_1, v = ((b^* + \beta_0^* y^*) u_1 + v_1)/(\beta_0^* x^* + q^* d^*).$$

Hence,

$$\begin{aligned}
\dot{u}_1 =& \dot{u} = v_1 - \beta_0^* u_1 \frac{1}{\beta_0^* x^* + q^* d^*}\big((b^* + \beta_0^* y^*)u_1 + v_1\big) \\
=& v_1 + a_1 u_1^2 + a_2 u_1 v_1, \\
\dot{v}_1 =& -(b^* + \beta_0^* y^*)\dot{u} - (\beta_0^* x^* + q^* d^*)\dot{v} \\
=& -(b^* + \beta_0^* y^*)[-(b^* + \beta_0^* y^*)u - (\beta_0^* x^* + q^* d^*)v - \beta_0^* uv] \\
& - (\beta_0^* x^* + q^* d^*)[-(\rho_1 y^* + \rho_2) + \beta_0^* y^* u + (\beta_0^* x^* - \eta^* - \rho_1)v + \beta_0^* uv] \\
=& (\beta_0^* x^* + q^* d^*)(\rho_1 y^* + \rho_2) + (\beta_0^* x^* + q^* d^*)\rho_1 v - (\beta_0^* x^* - \beta_0^* y^* - (1 - q^*)b^*)\beta_0^* uv \\
=& (\beta_0^* x^* + q^* d^*)(\rho_1 y^* + \rho_2) + \rho_1((b^* + \beta_0^* y^*)u_1 + v_1) \\
& - (\beta_0^* x^* - \beta_0^* y^* - (1 - q^*)b^*)\beta_0^* u_1 \frac{\big((b^* + \beta_0^* y^*)u_1 + v_1\big)}{(\beta_0^* x^* + q^* d^*)} \\
=& (\beta_0^* x^* + q^* d^*)(\rho_1 y^* + \rho_2) + \rho_1(b^* + \beta_0^* y^*)u_1 + \rho_1 v_1 + b_1 u_1^2 + b_2 u_1 v_1,
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= -\frac{\beta_0^*(b^* + \beta_0^* y^*)}{\beta_0^* x^* + q^* d^*}, \quad a_2 = -\frac{\beta_0^*}{\beta_0^* x^* + q^* d^*}, \\
b_1 &= -\beta_0^*(b^* + \beta_0^* y^*)(\beta_0^* x^* - \beta_0^* y^* - (1 - q^*)b^*)/(\beta_0^* x^* + q^* d^*), \\
b_2 &= -\beta_0^*(\beta_0^* x^* - \beta_0^* y^* - (1 - q^*)b^*)/(\beta_0^* x^* + q^* d^*).
\end{aligned}$$

Let $u_2 = u_1, v_2 = v_1 + a_1 u_1^2 + a_2 u_1 v_1$. Then the last system becomes

$$\begin{aligned}
\dot{u}_2 =& \dot{u}_1 = v_2, \\
\dot{v}_2 =& \dot{v}_1 + 2a_1 u_1 \dot{u}_1 + a_2 \dot{u}_1 v_1 + a_2 u_1 \dot{v}_1 \\
=& (1 + a_2 u_1)\dot{v}_1 + (2a_1 u_1 + a_2 v_1)\dot{u}_1 \\
=& (1 + a_2 u_1)[(\beta_0^* x^* + q^* d^*)(\rho_1 y^* + \rho_2) + \rho_1(b^* + \beta_0^* y^*)u_1 + \rho_1 v_1 \\
& + b_1 u_1^2 + b_2 u_1 v_1] + (2a_1 u_1 + a_2 v_1)[v_1 + a_1 u_1^2 + a_2 u_1 v_1] \\
=& (\beta_0^* x^* + q^* d^*)(\rho_1 y^* + \rho_2) + \rho_1(b^* + \beta_0^* y^*)u_1 + \rho_1 v_1 + b_1 u_1^2 \\
& + b_2 u_1 v_1 + a_2 u_1[(\beta_0^* x^* + q^* d^*)(\rho_1 y^* + \rho_2) + u_1 + \rho_1 v_1] + f_1(u_1, v_1) \\
=& \tilde{a}_0 + a_3 u_1 + \rho_1 v_1 + b_3 u_1^2 + (b_2 + a_2 \rho_1)u_1 v_1 + f_1(u_1, v_1) \\
=& \tilde{a}_0 + a_3 u_2 + \rho_1 v_2 + b_3 u_2^2 + (b_2 + a_2 \rho_1)u_2 v_2 + f_2(u_2, v_2),
\end{aligned}$$

where $f_1(u_1, v_1) = a_2 u_1[b_1 u_1^2 + b_2 u_1 v_1] + (2a_1 u_1 + a_2 v_1)[a_1 u_1^2 + a_2 u_1 v_1]$,

$$\begin{aligned}
f_2(u_2, v_2) &= -(b_2 + a_2 \rho_1)u_2(a_1 u_1^2 + a_2 u_1 v_1) + f_1(u_1, v_1), \\
\tilde{a}_0 &= (\beta_0^* x^* + q^* d^*)(\rho_1 y^* + \rho_2), \\
a_3 &= \rho_1(b^* + \beta_0^* y^*) + a_2 \tilde{a}_0, \\
b_3 &= b_1 + a_2 \rho_1(b^* + \beta_0^* y^*).
\end{aligned}$$

Let $u_3 = u_2 + \frac{a_3}{2b_3}, v_3 = v_2$. Then $\dot{u}_3 = \dot{u}_2 = v_3$ and

$$\begin{aligned}
\dot{v}_3 = \dot{v}_2 &= \tilde{a}_0 + a_3 u_2 + \rho_1 v_2 + b_3 u_2^2 + (b_2 + a_2 \rho_1)u_2 v_2 + f_2(u_2, v_2) \\
&= \mu_1(\rho_1, \rho_2) + \mu_1(\rho_1, \rho_2)\rho_1 v_3 + b_3 u_2^2 + (b_2 + a_2 \rho_1)u_3 v_3 + f_2(u_3 - \frac{a_3}{2b_3}, v_3),
\end{aligned}$$

where $\mu_1(\rho_1,\rho_2) = \tilde{a}_0 + \frac{a_3^2}{4b_3}$ and $\mu_2(\rho_1,\rho_2) = \rho_1 - \frac{a_3}{2b_3}(b_2 + a_2\rho_1)$. Hence,

$$\begin{cases} \dot{u}_3 = v_3, \\ \dot{v}_3 = \mu_1(\rho_1,\rho_2) + \mu_1(\rho_1,\rho_2)\rho_1 v_3 + b_3 u_2^2 \\ \qquad + (b_2 + a_2\rho_1)u_3 v_3 + f_2(u_3 - \frac{a_3}{2b_3}, v_3). \end{cases}$$

By the formulas of μ_1, μ_2, we have

$$\Big|\frac{\partial(\mu_1,\mu_2)}{\partial(\rho_1,\rho_2)}\Big| = \begin{vmatrix} (\beta_0^* x^* + q^* d^*)y^* & \beta_0^* x^* + q^* d^* \\ 1 + \frac{b^*}{b^* + \beta_0^* y^*} & -\frac{\beta_0^*}{b^* + \beta_0^* y^*} \end{vmatrix} = -2(\beta_0^* x^* + q^* d^*) \neq 0.$$

By Lemma 4.1, (2.1) undergos Bogdanov-Takens bifurcations. □

5. Discussion

We investigate the dynamics and bifurcations of SIR epidemic model with transmissions and treatments. The transmissions contains the horizontal transmissions: the diseases are transmitted through contact between the infectives and the susceptibles and the vertical transmissions: the diseases are transmitted from infective parents to unborn or newly born offsprings. The treatment includes isolation or quarantine. In this paper we consider the constant treatment rate and the horizontal transmissions and the vertical transmissions. It is proved that such SIR epidemic model have up to two positive epidemic equilibria and has no positive disease-free equilibria. And the equilibria of the epidemic system can be saddles, stable nodes, stable or unstable focuses, weak centers or cusps. We prove that the system has the Bogdanov-Takens bifurcations, which exhibit saddle-node bifurcations, Hopf bifurcations and homoclinic bifurcations. Without the treatment, Meng and Chen [13] considered the dynamics of an epidemic model with vertical transmissions. They showed that when the basic reproductive rate R_0 of the epidemic is greater that 1, (1.2) with $q \in (0,1)$ and $h = 0$ has only one positive infection-free equilibrium, which is unstable, and has only one positive interior (epidemic) equilibrium, which is locally stable. If $R_0 < 1$, then the infection-free equilibrium is locally stable and the interior equilibrium is unstable. This shows that the system has richer dynamics when the constant treatment rate is introduced.

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