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PSEUDO ALMOST AUTOMORPHY OF TWO-TERM FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS*

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Abstract In this paper, by measure theory, we introduce and investigate the concepts of (Stepanov-like) (μ,ν)-pseudo almost automorphic of class r and class infinity, respectively. As applications, we establish some sufficient criteria for the existence, uniqueness of pseudo almost automorphic mild solutions to two-term fractional functional differential equations with finite or infinite delay. The working tools are based on the generalization of semigroup theory, Banach contraction mapping principle and Leray-Schauder alternative theorem. Finally, we explore the same topic for a fractional partial functional differential equation with delay.

Keywords Pseudo almost automorphy, fractional functional differential equations, Leray-Schauder alternative theorem, generalized semigroup theory, sectorial operator.

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1. Introduction

Since Bochner [7] introduced the concept of almost automorphic function, there have been many interesting generalizations of this function in the past few decades. The generalizations include asymptotically almost automorphic function, pseudo almost automorphic function, weighted pseudo almost automorphic function, and these concepts in the Stepanov-like sense, one can see [10,29,34,36] for more details. Recently, a new more general type of almost automorphic function called μ-pseudo almost automorphic function is investigated by Blot et al. [6], which generalize all of above mentioned functions. Subsequently, μ-pseudo almost automorphic function is generalized into (μ,ν)-pseudo almost automorphic function by Diagana et al [20]. The results on the theory or the applications of the (μ,ν)-pseudo almost automorphic function are still few [11,22,33,37] and this topic is to be well explored.

In recent years, fractional differential equations have attracted more and more attentions, used to be described a large number of natural phenomena in various

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fields of the science, such as physics, mechanics, chemistry engineering etc. In particular, the study of abstract semilinear fractional differential equations is one of great interests. Many works have been done to prove existence, uniqueness of the mild solutions with a prescribed qualitative property. Almost automorphy of semilinear fractional differential equations were initiated by Araya and Lizama [1]. In their work, the authors investigated the existence, uniqueness of almost automorphic mild solutions for the following equations

$$D_t^\alpha u(t) - Au(t) = f(t, u(t)), \quad t \in \mathbb{R}, \ \ 1 < \alpha \leq 2, \tag{1.1}$$

when A is a generator of an α-resolvent family. Cuevas and Lizama [13] considered the following semilinear fractional differential equations

$$D_t^\alpha u(t) - Au(t) = D_t^{\alpha-1} f(t, u(t)), \quad t \in \mathbb{R}, \ \ 1 < \alpha \leq 2, \tag{1.2}$$

where A is a linear operator of sectorial negative type on a complex Banach space. In [13], the authors proved the existence and uniqueness of almost automorphic mild solutions to (1.2), then the same topic are explored in [17] but f is almost automorphic function in Stepanov-like sense. Since then, different kinds of almost automorphic mild solutions of (1.2) are investigated by many authors. Existence and uniqueness of asymptotically automorphic automorphic mild solutions of (1.2) are studied in [38]. For pseudo almost automorphy of (1.2), in [14], the authors investigate existence, uniqueness of pseudo almost automorphic mild solutions, and generalized in [28]; weighted pseudo almost automorphic mild solutions are considered in [29], reconsidered in [12] if the nonlinear term is S^p-weighted pseudo almost automorphic perturbation; existence and uniqueness of μ-pseudo almost automorphic mild solutions of (1.2) are studied in [11] by measure theory.

For (1.2) with infinite delay, that is the following fractional functional differential equations

$$D_t^\alpha u(t) - Au(t) = D_t^{\alpha-1} f(t, u_t), \quad t \in \mathbb{R}, \ \ 1 < \alpha \leq 2, \tag{1.3}$$

where the history $x_t : (-\infty, 0] \to X$ defined by $x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathfrak{B}$ (see for instance Hino's et al. [25]) which will be defined later. For almost automorphy of (1.3), the authors investigate the existence, uniqueness of weighted pseudo almost automorphic mild solutions in [2].

Recently, motivated by natural and widespread applicability in several fields of sciences and technology, the following two-term fractional differential equations increasingly begin to receive attention:

$$D_t^{\alpha+1} u(t) + \gamma D_t^\beta u(t) - Au(t) = D_t^\alpha f(t, u(t)), \quad t \in \mathbb{R}, \ \ 0 < \alpha \leq \beta \leq 1, \ \ \gamma \geq 0, \tag{1.4}$$

In [15], by generalization of the semigroup theory, the authors show existence, uniqueness of S-asymptotically ω-periodic mild solution of (1.4). In [3], pseudo asymptotic behavior for mild solutions of (1.4) are studied, the authors analyzed the existence, uniqueness of pseudo asymptotic mild solutions which in particular includes the classes of pseudo periodic, pseudo almost periodic and pseudo (compact) almost automorphic functions. Weighted pseudo almost automorphy of (1.4) are studied in [4] if the forcing term f is S^p-weighted pseudo almost automorphic. However, for (1.4) with delay, that is following two-term fractional functional differential equations (FFDEs)

$$D_t^{\alpha+1}u(t) + \gamma D_t^{\beta}u(t) - Au(t) = D_t^{\alpha} f(t, u_t), \quad t \in \mathbb{R}, \quad 0 < \alpha \leq \beta \leq 1, \quad \gamma \geq 0. \tag{1.5}$$

For (1.5), to the best of our knowledge, there is no work reported in literature, particularly, (μ, ν)-pseudo almost automorphy of (1.5) is quite new and an untreated topic. This is one of the key motivations of this study. In this paper, if the nonlinear term f is (μ, ν)-pseudo almost automorphic perturbation (or in Stepanov-like sense), non-Lipschitz perturbation, (μ, ν)-pseudo almost automorphy of (1.5) with finite delay and infinite delay are discussed, respectively.

The paper is organized as follows. In Section 2, some notations and preliminary results are presented. By measure theory, we introduce the concepts (μ, ν)-pseudo almost automorphic of class r (class infinity), Stepanov-like (μ, ν)-pseudo almost automorphic of class r (class infinity), explore some properties and establish composition theorems, respectively. In Section 3, we investigate the existence, uniqueness of (μ, ν)-pseudo almost automorphic of class r mild solution for (3.1) with finite delay under PAA perturbation, S^pPAA perturbation, and non-Lipschitz perturbation, respectively. In Section 4, we explore the same topic for (3.1) with infinite delay. In Section 5, we present an application to a fractional partial differential equation with delay.

2. Preliminaries and basic results

Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be two complex Banach spaces and $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{C}$ stand for the set of natural numbers, real numbers, nonnegative real numbers, and complex numbers, respectively. $\mathcal{R}(u)$ denotes the range of $u(\cdot)$. For A being a linear operator, $\mathcal{D}(A)$, $\rho(A)$, $R(\lambda, A)$, $\sigma(A)$ stand for the domain, the resolvent set, the resolvent and spectrum of A. In order to facilitate the discussion below, we further introduce the following notations:

- $C(\mathbb{R}, X)$ (resp. $C(\mathbb{R} \times Y, X)$): the set of continuous functions from $\mathbb{R}$ to X (resp. from $\mathbb{R} \times Y$ to X).
- $\mathcal{C} := C([-r, 0], X)$ denotes the space of continuous function from $[-r, 0]$ to X with the supremum norm.
- $BC(\mathbb{R}, X)$ (resp. $BC(\mathbb{R} \times Y, X)$): the Banach space of bounded continuous functions from $\mathbb{R}$ to X (resp. from $\mathbb{R} \times Y$ to X) with the supremum norm.
- $L(X, Y)$: the Banach space of bounded linear operators from X to Y endowed with the operator topology. In particular, we write $L(X)$ when $X = Y$.
- $L^p(\mathbb{R}, X)$: the space of all classes of equivalence (with respect to the equality almost everywhere on $\mathbb{R}$) of measurable functions $f : \mathbb{R} \to X$ such that $\|f\| \in L^p(\mathbb{R}, \mathbb{R})$.
- $L^p_{loc}(\mathbb{R}, X)$: the space of all classes of equivalence of measurable functions $f : \mathbb{R} \to X$ such that the restriction of f to every bounded subinterval of $\mathbb{R}$ is in $L^p(\mathbb{R}, X)$.

2.1. Fraction derivative and sectorial operator

Let $\alpha > 0$ be given, we denote

$$g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \qquad t > 0,$$

where Γ is the Gamma function. Given a vector-valued function $u : \mathbb{R} \to X$, the Weyl fractional integral of order $\alpha > 0$ is defined by

$$D_t^{-\alpha} u(t) := \int_{-\infty}^{t} g_\alpha(t-s)u(s)ds, \qquad t \in \mathbb{R},$$

when the integral is convergent. The Weyl fractional derivative $D_t^\alpha u$ of order $\alpha > 0$ is defined by

$$D_t^\alpha u(t) := \frac{d^n}{dt^n} D_t^{-(n-\alpha)} u(t), \qquad t \in \mathbb{R},$$

where $n = [\alpha] + 1$. It is known that $D_t^\alpha D_t^{-\alpha} u(t) = u(t)$ for any $\alpha > 0$. One can see [31] for more information and further details.

In order to give an operator theoretical approach to fractional functional differential equations, we recall the following definition.

Definition 2.1 ([27]). A closed and densely defined linear operator A is said to ω-sectorial of angle θ if there exist $\theta \in [0, \pi/2)$ and $\omega \in \mathbb{R}$ such that its resolvent exists in the sector

$$\omega + S_\theta := \{\omega + \lambda : \lambda \in \mathbb{C}, |arg(\lambda)| < \pi/2 + \theta\} \setminus \{\omega\}$$
$$\left\|(\lambda I - A)^{-1}\right\| \le \frac{M}{|\lambda - \omega|}, \qquad \lambda \in \omega + S_\theta. \tag{2.1}$$

In the case $\omega = 0$, we merely say that A is sectorial of angle θ.

We should mention that in the general theory of sectorial operator, it is not required that (2.1) holds in a sector of angle $\pi/2$. Our restriction corresponds to the class of operators used in this paper.

Definition 2.2 ([27]). Let $\gamma \ge 0$, $0 \le \alpha, \beta \le 1$ be given. Let A be a closed and linear operator with domain $\mathcal{D}(A)$ defined on a Banach space X. We call A the generator of an $(\alpha, \beta)_\gamma$-regularized family if there exist $\omega \ge 0$ and a strongly continuous function $S_{\alpha,\beta} : \mathbb{R}^+ \to L(X)$ such that $\{\lambda^{\alpha+1} + \gamma\lambda^\beta : Re\lambda > \omega\} \subset \rho(A)$ and

$$\lambda^\alpha(\lambda^{\alpha+1} + \gamma\lambda^\beta - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_{\alpha,\beta}(t)xdt, \qquad Re\lambda > \omega,\ x \in X.$$

Because of the uniqueness theorem for the Laplace transform, if $\gamma = 0, \alpha = 0$, this corresponds to the case of a C_0-semigroup, whereas the case $\gamma = 0, \alpha = 1$ corresponds to the concept of cosine family. For more details on the Laplace transform approach to semigroups and cosine functions, we refer to [5].

Sufficient conditions to existence and the integrability for the generators of an $(\alpha, \beta)_\gamma$-regularized family are given in the following results which corresponds to an extension of Cuesta's theorem [16] in the case $\gamma = 0$.

Theorem 2.1 ([27]). *Let $0 < \alpha \le \beta \le 1$, $\gamma > 0$ and $\omega < 0$. Assume that A is an ω-sectorial of angle $\beta\pi/2$, then A generates an $(\alpha,\beta)_\gamma$-regularized family $S_{\alpha,\beta}(t)$ satisfying*

$$\|S_{\alpha,\beta}(t)\| \le \frac{C}{1+|\omega|(t^{\alpha+1}+\gamma t^{\beta})}, \qquad t \ge 0, \tag{2.2}$$

for constant $C > 0$ depending only on α, β.

Note that

$$\int_0^\infty \frac{1}{1+|\omega|t^{\alpha+1}}dt = \frac{|\omega|^{-1/(\alpha+1)}\pi}{(\alpha+1)\sin(\pi/(\alpha+1))} \tag{2.3}$$

for $0 < \alpha < 1$, therefore $S_{\alpha,\beta}(t)$ is integrable on $(0,\infty)$.

2.2. PAA of class r

Definition 2.3 (Bochner [8]). A function $f \in C(\mathbb{R}, X)$ is said to be almost automorphic in Bochner's sense if for every sequence of real numbers $(s_n^{'})_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that $g(t) := \lim_{n\to\infty} f(t+s_n)$ is well defined for each $t \in \mathbb{R}$, and $\lim_{n\to\infty} g(t-s_n) = f(t)$ for each $t \in \mathbb{R}$.

The almost automorphic functions (denoted by $AA(\mathbb{R}, X)$), which generalize the concept of (Bochner) almost periodic function, constitute a Banach space when endowed with the supremum norm.

Lemma 2.1 ([10]). *If $f, f_1, f_2 \in AA(\mathbb{R}, X)$, then*

- *(i)* $f_1 + f_2 \in AA(\mathbb{R}, X)$;
- *(ii)* $\lambda f \in AA(\mathbb{R}, X)$ *for any scalar* λ;
- *(iii)* $f_\tau \in AA(\mathbb{R}, X)$ *where* $f_\tau : \mathbb{R} \to X$ *is defined by* $f_\tau(\cdot) := f(\cdot + \tau)$;
- *(iv) the range* $\mathcal{R}(f) := \{f(t) : t \in \mathbb{R}\}$ *is relatively compact in* X*, thus* f *is bounded in norm;*
- *(v) if* $f_n \to f$ *uniformly on* $\mathbb{R}$ *where each* $f_n \in AA(\mathbb{R}, X)$*, then* $f \in AA(\mathbb{R}, X)$ *too.*

Next, we introduce the concept of (μ,ν)-pseudo almost automorphic function by the results of measure theory. $\mathcal{B}$ denotes the Lebesgue σ-field of $\mathbb{R}$, $\mathcal{M}$ stands for the set of all positive measure μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = \infty$ and $\mu([a,b]) < \infty$ for all $a, b \in \mathbb{R}$ $(a \le b)$.

Definition 2.4 ([6]). Let $\mu, \nu \in \mathcal{M}$, The measure μ and ν are said to be equivalent (i.e., $\mu \sim \nu$) if there exist constants $c_0, c_1 > 0$ and a bounded interval $I \subset \mathbb{R}$ (eventually $\emptyset$) such that

$$c_0\nu(A) \le \mu(A) \le c_1\nu(A)$$

for all $A \in \mathcal{B}$ satisfying $A \cap I = \emptyset$.

For $\mu \in \mathcal{M}$, $\tau \in \mathbb{R}$, we denote μ_τ the positive measure on $(\mathbb{R}, \mathcal{B})$ defined by

$$\mu_\tau(A) = \mu(\{a+\tau : a \in A\}) \quad \text{ for } A \in \mathcal{B}. \tag{2.4}$$

In this paper, we formulate the following hypotheses:

(M_1) Let $\mu, \nu \in \mathcal{M}$ such that

$$\limsup_{T\to\infty} \frac{\mu([-T,T])}{\nu([-T,T])} < \infty.$$

(M_2) Let $\mu, \nu \in \mathcal{M}$ such that for all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that $\mu_\tau(A) \le \beta\mu(A)$, $\nu_\tau(A) \le \beta\nu(A)$ if $A \in \mathcal{B}$ satisfies $A \cap I = \emptyset$.

Lemma 2.2 ([6]). *Let $\mu, \nu \in \mathcal{M}$, then μ, ν satisfy (M_2) if and only if $\mu \sim \mu_\tau$ and $\nu \sim \nu_\tau$ for all $\tau \in \mathbb{R}$.*

Lemma 2.3 ([6]). *If (M_2) hold, then for all $\sigma > 0$,*

$$\limsup_{T\to\infty} \frac{\nu([-T-\sigma, T+\sigma])}{\nu([-T,T])} < \infty.$$

Let $\mu, \nu \in \mathcal{M}$, define the (μ,ν)-ergodic space

$$PAA_0(\mathbb{R}, X, \mu, \nu) = \left\{ f \in BC(\mathbb{R}, X) : \lim_{T\to\infty} \frac{1}{\nu([-T,T])} \int_{[-T,T]} \|f(t)\| d\mu(t) = 0 \right\}.$$

$$PAA_0(\mathbb{R}\times X, X, \mu, \nu) := \{ f \in BC(\mathbb{R}\times X, X) :$$
$$\left. \lim_{T\to\infty} \frac{1}{\nu([-T,T])} \int_{[-T,T]} \|f(t,u)\| d\mu(t) = 0 \text{ uniformly in } u \in X \right\}.$$

Definition 2.5 ([20]). Let $\mu, \nu \in \mathcal{M}$. A function $f \in C(\mathbb{R}, X)$ (resp. $C(\mathbb{R}\times X, X)$) is called (μ,ν)-pseudo almost automorphic if it can be decomposed as $f = g + h$, where $g \in AA(\mathbb{R}, X)$ (resp. $AA(\mathbb{R}\times X, X)$) and $h \in PAA_0(\mathbb{R}, X, \mu, \nu)$ (resp. $PAA_0(\mathbb{R}\times X, X, \mu, \nu)$). Denote by $PAA(\mathbb{R}, X, \mu, \nu)$ (resp. $PAA(\mathbb{R}\times X, X, \mu, \nu)$) the set of such functions.

Lemma 2.4 ([20]). *Let $\mu, \nu \in \mathcal{M}$, (M_2) hold and $f \in PAA(\mathbb{R}, X, \mu, \nu)$ be such that $f = g + h$, where $g \in AA(\mathbb{R}, X)$, $h \in PAA_0(\mathbb{R}, X, \mu, \nu)$, then $\{g(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}\}}$.*

To deal with the differential equations with finite delay, we introduce the concept of (μ,ν)-pseudo almost automorphic of class r. For each $r > 0$, define (μ,ν)-ergodic space of class r:

$$PAA_0(\mathbb{R}, X, \mu, \nu, r) = \{ f \in BC(\mathbb{R}, X) :$$
$$\left. \lim_{T\to\infty} \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta\in[t-r,t]} \|f(\theta)\| \right) d\mu(t) = 0 \right\}.$$

$$PAA_0(\mathbb{R}\times X, X, \mu, \nu, r) := \{ f \in BC(\mathbb{R}\times X, X) :$$
$$\left. \lim_{T\to\infty} \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta\in[t-r,t]} \|f(\theta,u)\| \right) d\mu(t) = 0 \text{ uniformly in } u \in X \right\}.$$

Definition 2.6. Let $\mu, \nu \in \mathcal{M}$. A function $f \in C(\mathbb{R}, X)$ (resp. $C(\mathbb{R}\times X, X)$) is called (μ,ν)-pseudo almost automorphic of class r if it can be decomposed as $f = g + h$, where $g \in AA(\mathbb{R}, X)$ (resp. $AA(\mathbb{R}\times X, X)$) and $h \in PAA_0(\mathbb{R}, X, \mu, \nu, r)$ (resp. $PAA_0(\mathbb{R}\times X, X, \mu, \nu, r)$). Denote by $PAA(\mathbb{R}, X, \mu, \nu, r)$ (resp. $PAA(\mathbb{R}\times X, X, \mu, \nu, r)$) the set of such functions.

Remark 2.1. (*i*) If $\mu \sim \nu$, (μ,ν)-pseudo almost automorphic function of class r $(PAA(\mathbb{R},X,\mu,\nu,r))$ is μ-pseudo almost automorphic function of class r $(PAA(\mathbb{R},X,\mu,r))$ which defined in [9].

(*ii*) Let $\rho(t)>0$ a.e. on $\mathbb{R}$ for the Lebesgue measure. μ, ν denote the positive measure defined by

$$\mu(A)=\nu(A)=\int_A \rho(t)dt \quad \text{for} \quad A\in\mathcal{B},$$

where dt denotes the Lebesgue measure on $\mathbb{R}$, then (μ,ν)-pseudo almost automorphic function of class r $(PAA(\mathbb{R},X,\mu,\nu,r))$ is weighted pseudo almost automorphic function of class r $(WPAA(\mathbb{R},X,\rho,r))$ which defined in [35].

(*iii*) If $\mu\sim\nu$ and μ,ν are the Lebesgue measures, then (μ,ν)-pseudo almost automorphic function of class r $(PAA(\mathbb{R},X,\mu,\nu,r))$ is pseudo almost automorphic function of class r $(PAA(\mathbb{R},X,r))$ which defined in [23].

Next, we show some properties of the space $PAA(\mathbb{R},X,\mu,\nu,r)$. First, we give the characterization of (μ,ν)-ergodic functions in terms of the measures μ,ν.

Lemma 2.5. *Let I be a bounded interval (eventually $I=\emptyset$). Assume that (M_1) hold and $f\in BC(\mathbb{R},X)$, then the following assertions are equivalent:*

(*i*) $f\in PAA_0(\mathbb{R},X,\mu,\nu,r)$.

(*ii*) $\displaystyle\lim_{T\to\infty}\frac{1}{\nu([-T,T]\backslash I)}\int_{[-T,T]\backslash I}\left(\sup_{\theta\in[t-r,t]}\|f(\theta)\|\right)d\mu(t)=0.$

(*iii*) *For any $\varepsilon>0$,*

$$\lim_{T\to\infty}\frac{\mu\left(\left\{t\in[-T,T]\backslash I:\ \sup\limits_{\theta\in[t-r,t]}\|f(\theta)\|>\varepsilon\right\}\right)}{\nu([-T,T]\backslash I)}=0.$$

Proof. The proof is similar to the one given in [6], but here we deal with the case of finite delay. In fact, we have

Case I: $(i)\Leftrightarrow(ii)$. Denote by

$$\mathbb{A}=\nu(I),\qquad \mathbb{B}=\int_I\left(\sup_{\theta\in[t-r,t]}\|f(\theta)\|\right)d\mu(t),\qquad \mathbb{E}=\mu(I).$$

Since I is bounded and $f\in BC(\mathbb{R},X)$, then $\mathbb{A},\mathbb{B},\mathbb{E}$ are finite. Let $T>0$ be such that $I\subset[-T,T]$ and $\nu([-T,T]\backslash I)>0$, one has

$$\begin{aligned}
&\frac{1}{\nu([-T,T]\backslash I)}\int_{[-T,T]\backslash I}\left(\sup_{\theta\in[t-r,t]}\|f(\theta)\|\right)d\mu(t)\\
&=\frac{1}{\nu([-T,T])-\mathbb{A}}\left(\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\|f(\theta)\|\right)d\mu(t)-\mathbb{B}\right)\\
&=\frac{\nu([-T,T])}{\nu([-T,T])-\mathbb{A}}\left[\frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\|f(\theta)\|\right)d\mu(t)-\frac{\mathbb{B}}{\nu([-T,T])}\right],
\end{aligned}$$

since $\nu(\mathbb{R}) = +\infty$, we deduce that (ii) is equivalent to

$$\lim_{T\to\infty} \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta\in[t-r,t]} \|f(\theta)\| \right) d\mu(t) = 0,$$

that is (i) holds.

Case II: $(ii) \Rightarrow (iii)$. Assume that (ii) holds. Denote by

$$\mathbb{A}_T^{\varepsilon}(f) = \left\{ t \in [-T,T]\backslash I : \sup_{\theta\in[t-r,t]} \|f(\theta)\| > \varepsilon \right\},$$
$$\mathbb{B}_T^{\varepsilon}(f) = \left\{ t \in [-T,T]\backslash I : \sup_{\theta\in[t-r,t]} \|f(\theta)\| \le \varepsilon \right\},$$

then

$$\begin{aligned} \int_{[-T,T]\backslash I} \left(\sup_{\theta\in[t-r,t]} \|f(\theta)\| \right) d\mu(t) &= \int_{\mathbb{A}_T^{\varepsilon}(f)} \left(\sup_{\theta\in[t-r,t]} \|f(\theta)\| \right) d\mu(t) \\ &\quad + \int_{\mathbb{B}_T^{\varepsilon}(f)} \left(\sup_{\theta\in[t-r,t]} \|f(\theta)\| \right) d\mu(t). \end{aligned} \tag{2.5}$$

Suppose the contrary, that there exists $\varepsilon_0 > 0$, such that $\dfrac{\mu(\mathbb{A}_T^{\varepsilon}(f))}{\nu([-T,T]\backslash I)}$ does not converge to 0 as $T \to \infty$, then there exists $\delta > 0$, such that for each n,

$$\frac{\mu(\mathbb{A}_{T_n}^{\varepsilon_0}(f))}{\nu([-T_n,T_n]\backslash I)} \ge \delta \qquad \text{for some} \ \ T_n > n.$$

Hence

$$\begin{aligned} &\frac{1}{\nu([-T_n,T_n]\backslash I)} \int_{[-T_n,T_n]\backslash I} \left(\sup_{\theta\in[t-r,t]} \|f(\theta)\| \right) d\mu(t) \\ &\ge \frac{1}{\nu([-T_n,T_n]\backslash I)} \int_{\mathbb{A}_{T_n}^{\varepsilon_0}(f)} \left(\sup_{\theta\in[t-r,t]} \|f(\theta)\| \right) d\mu(t) \\ &\ge \frac{\mu(\mathbb{A}_{T_n}^{\varepsilon_0}(f))}{\nu([-T_n,T_n]\backslash I)} \varepsilon_0 \\ &\ge \epsilon_0 \delta, \end{aligned}$$

which contradicts the fact that

$$\lim_{T\to\infty} \int_{[-T,T]\backslash I} \left(\sup_{\theta\in[t-r,t]} \|f(\theta)\| \right) d\mu(t) = 0.$$

Thus (iii) holds.

Case III: $(iii) \Rightarrow (ii)$. Assume that (iii) holds, that is

$$\lim_{T\to\infty} \frac{\mu(\mathbb{A}_T^{\varepsilon}(f))}{\nu([-T,T] \setminus I)} = 0. \tag{2.6}$$

By (2.5), for T large enough, one has

$$\begin{aligned}
&\frac{1}{\nu([-T,T]\backslash I)}\int_{[-T,T]\backslash I}\left(\sup_{\theta\in[t-r,t]}\|f(\theta)\|\right)d\mu(t)\\
&\leq \frac{1}{\nu([-T,T]\backslash I)}\int_{\mathbb{A}_T^\varepsilon(f)}\left(\sup_{\theta\in[t-r,t]}\|f(\theta)\|\right)d\mu(t)\\
&\quad+\frac{1}{\nu([-T,T]\backslash I)}\int_{\mathbb{B}_T^\varepsilon(f)}\left(\sup_{\theta\in[t-r,t]}\|f(\theta)\|\right)d\mu(t)\\
&\leq \frac{\|f\|\mu(\mathbb{A}_T^\varepsilon(f))}{\nu([-T,T]\backslash I)}+\frac{\mu(\mathbb{B}_T^\varepsilon(f))\varepsilon}{\nu([-T,T]\backslash I)}\\
&\leq \frac{\|f\|\mu(\mathbb{A}_T^\varepsilon(f))}{\nu([-T,T]\backslash I)}+\frac{\mu([-T,T]\backslash I)\varepsilon}{\nu([-T,T]\backslash I)}\\
&= \frac{\|f\|\mu(\mathbb{A}_T^\varepsilon(f))}{\nu([-T,T]\backslash I)}+\frac{\mu([-T,T])-\mathbb{E}}{\nu([-T,T])-\mathbb{A}}\varepsilon\\
&= \frac{\|f\|\mu(\mathbb{A}_T^\varepsilon(f))}{\nu([-T,T]\backslash I)}+\frac{\mu([-T,T])}{\nu([-T,T])}\times\frac{1-\frac{\mathbb{E}}{\mu([-T,T])}}{1-\frac{\mathbb{A}}{\nu([-T,T])}}\varepsilon.
\end{aligned}$$

Since (2.6), (M_1) hold and $\mu(\mathbb{R})=\nu(\mathbb{R})=\infty$, then for all $\varepsilon>0$, one has

$$\limsup_{T\to\infty}\frac{1}{\nu([-T,T]\backslash I)}\int_{[-T,T]\backslash I}\left(\sup_{\theta\in[t-r,t]}\|f(\theta)\|\right)d\mu(t)\leq\eta\varepsilon,$$

where $\eta:=\limsup_{T\to\infty}\frac{\mu([-T,T])}{\nu([-T,T])}$ is a constant. Hence (ii) holds. □

Proposition 2.1. *Assume that* (M_1) *hold. If* $\mu_i,\nu_i\in\mathcal{M}$, $(i=1,2)$ *and* $\mu_1\sim\mu_2$, $\nu_1\sim\nu_2$, *then* $PAA_0(\mathbb{R},X,\mu_1,\nu_1,r)=PAA_0(\mathbb{R},X,\mu_2,\nu_2,r)$ *and* $PAA(\mathbb{R},X,\mu_1,\nu_1,r)=PAA(\mathbb{R},X,\mu_2,\nu_2,r)$.

Proof. Since $\mu_1\sim\mu_2$, $\nu_1\sim\nu_2$ and $\mathcal{B}$ is the Lebesgue σ-field, by Definition 2.4, for all $A\in\mathcal{B}$ satisfying $A\cap[-T,T]=\emptyset$, there exists $\alpha_i>0$, $\beta_i>0$, $(i=1,2)$ such that

$$\alpha_1\mu_2(A)\leq\mu_1(A)\leq\beta_1\mu_2(A),\qquad \alpha_2\nu_2(A)\leq\nu_1(A)\leq\beta_2\nu_2(A).$$

For T sufficiently large, one has

$$\begin{aligned}
&\frac{\alpha_1}{\beta_2}\times\frac{\mu_2\left(\left\{t\in[-T,T]\backslash I:\sup_{\theta\in[t-r,t]}\|f(\theta)\|>\varepsilon\right\}\right)}{\nu_2([-T,T]\backslash I)}\\
&\leq\frac{\mu_1\left(\left\{t\in[-T,T]\backslash I:\sup_{\theta\in[t-r,t]}\|f(\theta)\|>\varepsilon\right\}\right)}{\nu_1([-T,T]\backslash I)}\\
&\leq\frac{\beta_1}{\alpha_2}\times\frac{\mu_2\left(\left\{t\in[-T,T]\backslash I:\sup_{\theta\in[t-r,t]}\|f(\theta)\|>\varepsilon\right\}\right)}{\nu_2([-T,T]\backslash I)},
\end{aligned}$$

hence $PAA_0(\mathbb{R},X,\mu_1,\nu_1,r)=PAA_0(\mathbb{R},X,\mu_2,\nu_2,r)$ by Lemma 2.5. □

Proposition 2.2. *If* (M_1), (M_2) *hold and* $f \in PAA_0(\mathbb{R}, X, \mu, \nu, r)$, *then* $f(\cdot - \tau) \in PAA_0(\mathbb{R}, X, \mu, \nu, r)$ *for all* $\tau \in \mathbb{R}$.

Proof. For $\nu \in \mathcal{M}$, since $\nu(\mathbb{R}) = +\infty$, there exists $T_0 > 0$ such that $\nu([-T - |\tau|, T + |\tau|]) > 0$ for all $T > T_0$. Let

$$\tau^+ := \max(\tau, 0), \qquad \tau^- := \max(-\tau, 0),$$

then we have $|\tau| + \tau = 2\tau^+$, $|\tau| - \tau = 2\tau^-$, so

$$[-T - |\tau| + \tau, T + |\tau| + \tau] = [-T - 2\tau^-, T + 2\tau^+]. \tag{2.7}$$

For $T > T_0$ and $\tau \in \mathbb{R}$, we have

$$\begin{aligned}
&\frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta \in [t-r,t]} \|f(\theta - \tau)\| \right) d\mu(t) \\
&\leq \frac{1}{\nu([-T,T])} \int_{[-T-2\tau^-, T+2\tau^+]} \left(\sup_{\theta \in [t-r,t]} \|f(\theta - \tau)\| \right) d\mu(t) \\
&= \frac{\nu([-T - 2\tau^-, T + 2\tau^+])}{\nu([-T,T])} \Phi_\tau(T) \\
&\leq \frac{\nu([-T - 2|\tau|, T + 2|\tau|])}{\nu([-T,T])} \Phi_\tau(T),
\end{aligned} \tag{2.8}$$

where

$$\Phi_\tau(T) := \frac{1}{\nu([-T - 2\tau^-, T + 2\tau^+])} \int_{[-T-2\tau^-, T+2\tau^+]} \left(\sup_{\theta \in [t-r,t]} \|f(\theta - \tau)\| \right) d\mu(t).$$

By (2.4) and (2.7), one has

$$\begin{aligned}
&\Phi_\tau(T) \\
&= \frac{1}{\nu([-T - |\tau| + \tau, T + |\tau| + \tau])} \int_{[-T-|\tau|+\tau, T+|\tau|+\tau]} \left(\sup_{\theta \in [t-r,t]} \|f(\theta - \tau)\| \right) d\mu(t) \\
&= \frac{1}{\nu_\tau([-T - |\tau|, T + |\tau|])} \int_{[-T-|\tau|+\tau, T+|\tau|+\tau]} \left(\sup_{\theta \in [t-\tau-r, t-\tau]} \|f(\theta)\| \right) d\mu(t) \\
&= \frac{1}{\nu_\tau([-T - |\tau|, T + |\tau|])} \int_{[-T-|\tau|, T+|\tau|]} \left(\sup_{\theta \in [t-r,t]} \|f(\theta)\| \right) d\mu_\tau(t).
\end{aligned}$$

Note that $\mu \sim \mu_\tau$, $\nu \sim \nu_\tau$ by Lemma 2.2, then $f \in PAA_0(\mathbb{R}, X, \mu_\tau, \nu_\tau, r)$ by Proposition 2.1, so

$$\lim_{T \to \infty} \Phi_\tau(T) = 0.$$

By (2.8) and Lemma 2.3, one has

$$\lim_{T \to \infty} \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta \in [t-r,t]} \|f(\theta - \tau)\| \right) d\mu(t) = 0,$$

that is $f(\cdot - \tau) \in PAA_0(\mathbb{R}, X, \mu, \nu, r)$ for all $\tau \in \mathbb{R}$. $\square$

Remark 2.2. By Proposition 2.2, $PAA_0(\mathbb{R}, X, \mu, \nu, r)$ is translation invariant, thus $PAA(\mathbb{R}, X, \mu, \nu, r)$ is translation invariant.

Proposition 2.3. *If* (M_1), (M_2) *hold and* $r > 0$, *then*

(*i*) $PAA_0(\mathbb{R}, X, \mu, \nu, r) \subset PAA_0(\mathbb{R}, X, \mu, \nu)$, $PAA(\mathbb{R}, X, \mu, \nu, r) \subset PAA(\mathbb{R}, X, \mu, \nu)$.

(*ii*) $PAA_0(\mathbb{R}, X, \mu, \nu, r)$ *is a closed subspace of* $BC(\mathbb{R}, X)$.

(*iii*) $PAA(\mathbb{R}, X, \mu, \nu, r)$ *is a Banach space under the supremum norm.*

Proof. From the estimate

$$\frac{1}{\nu([-T,T])} \int_{[-T,T]} \|f(t)\| d\mu(t) \le \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta \in [t-r,t]} \|f(\theta)\| \right) d\mu(t),$$

hence it is easy to see that (i) holds.

Let $f_n \in PAA_0(\mathbb{R}, X, \mu, \nu, r)$ and $f_n \to f$ in $BC(\mathbb{R}, X)$, then

$$\begin{aligned}
&\frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta \in [t-r,t]} \|f(\theta)\| \right) d\mu(t) \\
&= \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta \in [t-r,t]} \|f_n(\theta)\| \right) d\mu(t) \\
&\quad + \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta \in [t-r,t]} \|f(\theta)\| - \sup_{\theta \in [t-r,t]} \|f_n(\theta)\| \right) d\mu(t) \\
&\le \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta \in [t-r,t]} \|f_n(\theta)\| \right) d\mu(t) \\
&\quad + \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta \in [t-r,t]} \|f(\theta) - f_n(\theta)\| \right) d\mu(t),
\end{aligned}$$

which yields that $f \in PAA_0(\mathbb{R}, X, \mu, \nu, r)$ since (M_1) holds, then (ii) holds.

Let $f_n = g_n + h_n$ be a Cauchy sequence in $PAA(\mathbb{R}, X, \mu, \nu, r)$, where $g_n \in AA(\mathbb{R}, X)$, $h_n \in PAA_0(\mathbb{R}, X, \mu, \nu, r)$. By (i) and Lemma 2.4, one has g_n, h_n are Cauchy sequences. Noting that $AA(\mathbb{R}, X)$ and $PAA_0(\mathbb{R}, X, \mu, \nu, r)$ are closed subspaces of $BC(\mathbb{R}, X)$, there exist $g \in AA(\mathbb{R}, X)$, $h \in PAA_0(\mathbb{R}, X, \mu, \nu, r)$ such that

$$g_n \to g, \quad h_n \to h, \quad n \to \infty.$$

Let $f = g + h$, then $f_n \to f$ and $f \in PAA(\mathbb{R}, X, \mu, \nu, r)$. Therefore (iii) holds. □

Proposition 2.4. *If* (M_1), (M_2) *hold and* $r_1 > 0$, $r_2 > 0$, *then*

$$PAA_0(\mathbb{R}, X, \mu, \nu, r_1) = PAA_0(\mathbb{R}, X, \mu, \nu, r_2), \ (\mathbb{R}, X, \mu, \nu, r_1) = PAA(\mathbb{R}, X, \mu, \nu, r_2).$$

Proof. Let $r > 0$, first we show that

$$PAA_0(\mathbb{R}, X, \mu, \nu, r) \subset PAA_0(\mathbb{R}, X, \mu, \nu, 2r). \tag{2.9}$$

For $f \in PAA_0(\mathbb{R}, X, \mu, \nu, r)$, one has

$$\frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta \in [t-2r,t]} \|f(\theta)\| \right) d\mu(t)$$

$$\begin{aligned}
&\leq \frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-2r,t-r]}\|f(\theta)\|\right)d\mu(t)\\
&\quad+\frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\|f(\theta)\|\right)d\mu(t)\\
&\leq \frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\|f(\theta-r)\|\right)d\mu(t)\\
&\quad+\frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\|f(\theta)\|\right)d\mu(t),
\end{aligned}$$

by Proposition 2.2, one has

$$\lim_{T\to\infty}\frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-2r,t]}\|f(\theta)\|\right)d\mu(t)=0,$$

thus $f\in PAA_0(\mathbb{R},X,\mu,\nu,2r)$. Hence (2.9) holds.

Now, let $r_1>r_2>0$, if $f\in PAA_0(\mathbb{R},X,\mu,\nu,r_1)$, then

$$\lim_{T\to\infty}\frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r_1,t]}\|f(\theta)\|\right)d\mu(t)=0.$$

From

$$\begin{aligned}
&\frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r_2,t]}\|f(\theta)\|\right)d\mu(t)\\
&\qquad\leq \frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r_1,t]}\|f(\theta)\|\right)d\mu(t),
\end{aligned}$$

then

$$\lim_{T\to\infty}\frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r_2,t]}\|f(\theta)\|\right)d\mu(t)=0,$$

so $f\in PAA_0(\mathbb{R},X,\mu,\nu,r_2)$, i.e., one has

$$PAA_0(\mathbb{R},X,\mu,\nu,r_1)\subset PAA_0(\mathbb{R},X,\mu,\nu,r_2). \tag{2.10}$$

On the other hand, since $r_1>r_2$, there exists $k\in\mathbb{N}$ such that $2^k r_2>r_1$. By (2.9), (2.10), one has

$$PAA_0(\mathbb{R},X,\mu,\nu,r_2)\subset PAA_0(\mathbb{R},X,\mu,\nu,2^k r_2)\subset PAA_0(\mathbb{R},X,\mu,\nu,r_1).$$

Thus,

$$PAA_0(\mathbb{R},X,\mu,\nu,r_1)=PAA_0(\mathbb{R},X,\mu,\nu,r_2)$$

and

$$PAA(\mathbb{R},X,\mu,\nu,r_1)=PAA(\mathbb{R},X,\mu,\nu,r_2).$$

The proof is complete. $\square$

Remark 2.3. It is interesting that $PAA_0(\mathbb{R}, X, \mu, \nu, r) = PAA_0(\mathbb{R}, X, \mu, \nu, 1)$ for all $r > 0$ by Proposition 2.4, but for $r = 0$, it is not necessarily holds, i.e., $PAA_0(\mathbb{R}, X, \mu, \nu, 0) = PAA_0(\mathbb{R}, X, \mu, \nu, 1)$ is not true. The similarly results hold for $PAA(\mathbb{R}, X, \mu, \nu, r)$.

Now, we establish the composition theorem for $PAA(\mathbb{R}, X, \mu, \nu, r)$.

Theorem 2.2. *Assume that* (M_1) *holds,* $f = g+h \in PAA(\mathbb{R}\times Y, X, \mu, \nu, r)$*, where* $g \in AA(\mathbb{R} \times Y, X)$*,* $h \in PAA_0(\mathbb{R} \times Y, X, \mu, \nu, r)$ *and*

($\mathcal{I}_1$) *There exists a continuous function* $L_f : \mathbb{R} \to \mathbb{R}^+$ *such that*

$$\limsup_{T\to\infty} \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta\in[t-r,t]} L_f(\theta) \right) d\mu(t) < +\infty.$$

($\mathcal{I}_2$) $\forall\varepsilon > 0$*, there exists* $\delta > 0$ *such that for* $u, v \in Y$ *with* $\|u - v\| < \delta$*, one has*

$$\|f(t, u) - f(t, v)\| \le L_f(t)\varepsilon, \qquad t \in \mathbb{R},$$

($\mathcal{I}_3$) *For all bounded subset* B *of* Y*,* f *is bounded on* $\mathbb{R} \times B$*.*

($\mathcal{I}_4$) $g(t, x)$ *is uniformly continuous in any compact subset* $K \subset Y$ *uniformly for* $t \in \mathbb{R}$*.*

Then $f(\cdot, x(\cdot)) \in PAA(\mathbb{R}, X, \mu, \nu, r)$ *if* $x \in PAA(\mathbb{R}, Y, \mu, \nu, r)$*.*

Proof. Let $f = g + h$, $x = \alpha + \beta$, where $\alpha \in AA(\mathbb{R}, Y)$, $\beta \in PAA_0(\mathbb{R}, Y, \mu, \nu, r)$, $g \in AA(\mathbb{R} \times Y, X)$, and $h \in PAA_0(\mathbb{R} \times Y, X, \mu, \nu, r)$. Now the function f can be decomposed as

$$\begin{aligned} f(t, x(t)) &= g(t, \alpha(t)) + f(t, x(t)) - g(t, \alpha(t)) \\ &= g(t, \alpha(t)) + f(t, x(t)) - f(t, \alpha(t)) + h(t, \alpha(t)). \end{aligned}$$

Set

$$F(t) = g(t, \alpha(t)), \quad G(t) = f(t, x(t)) - f(t, \alpha(t)), \quad H(t) = h(t, \alpha(t)).$$

Let

$$\mathfrak{F} = \sup_{t\in\mathbb{R}, u\in B} \|f(t, u)\|, \qquad \mathfrak{L} = \limsup_{T\to\infty} \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta\in[t-r,t]} L_f(\theta) \right) d\mu(t),$$

by the assumptions, $\mathfrak{F}, \mathfrak{L} < +\infty$. It is not difficult to see that the function $t \to f(t, x(t))$ is continuous and bounded, and $F(\cdot) \in AA(\mathbb{R}, X)$.

For any $\varepsilon > 0$, let δ be as in the assumptions, and

$$\mathbb{M}_T^\delta(\beta) = \left\{ t \in [-T, T] : \sup_{\theta\in[t-r,t]} \|\beta(\theta)\| > \delta \right\}.$$

Note that $\beta \in PAA_0(\mathbb{R}, Y, \mu, \nu, r)$, for above $\delta > 0$, by Lemma 2.5, one has

$$\lim_{T\to\infty} \frac{\mu(\mathbb{M}_T^\delta(\beta))}{\nu([-T,T])} = 0. \tag{2.11}$$

In addition, for all $t \in [-T,T]\backslash \mathbb{M}_T^{\delta}(\beta)$, one has

$$\sup_{\theta\in[t-r,t]} \|\beta(\theta)\| < \delta,$$

which means that

$$\sup_{\theta\in[t-r,t]} \|G(\theta)\| = \sup_{\theta\in[t-r,t]} \|f(\theta,x(\theta)) - f(\theta,\alpha(\theta))\| \le \left(\sup_{\theta\in[t-r,t]} L_f(\theta)\right)\varepsilon,$$

then

$$\begin{aligned}
&\frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\|G(\theta)\|\right)d\mu(t)\\
&\le \frac{1}{\nu([-T,T])}\int_{\mathbb{M}_T^{\delta}(\beta)}\left(\sup_{\theta\in[t-r,t]}\|G(\theta)\|\right)d\mu(t)\\
&\quad + \frac{1}{\nu([-T,T])}\int_{[-T,T]\backslash\mathbb{M}_T^{\delta}(\beta)}\left(\sup_{\theta\in[t-r,t]}\|G(\theta)\|\right)d\mu(t)\\
&\le \frac{2\mathfrak{F}\mu(\mathbb{M}_T^{\delta}(\beta))}{\nu([-T,T])} + \frac{\varepsilon}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]} L_f(\theta)\right)d\mu(t),
\end{aligned}$$

by (2.11), one has

$$\limsup_{T\to\infty}\frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\|G(\theta)\|\right)d\mu(t) \le \mathfrak{L}\varepsilon,$$

by the arbitrariness of ε, one has

$$\lim_{T\to\infty}\frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\|G(\theta)\|\right)d\mu(t) = 0,$$

that is $G(\cdot) \in PAA_0(\mathbb{R}, X, \mu, \nu, r)$.

It remains to show that $H(\cdot) \in PAA_0(\mathbb{R}, X, \mu, \nu, r)$. Let $K = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$, then K is compact and g is uniformly continuous on $\mathbb{R} \times K$. Thus, for any $\varepsilon > 0$, there exists $\delta' > 0$ such that

$$\|h(t,u) - h(t,v)\| \le \|f(t,u) - f(t,v)\| + \|g(t,u) - g(t,v)\| \le (L_f(t)+1)\varepsilon$$

for all $t \in \mathbb{R}$, $u, v \in K$ with $\|u - v\| \le \delta'$. Let $x_1, x_2, \ldots, x_m \in K$ be such that

$$K \subset \bigcup_{i=1}^{m} B(x_i, \delta').$$

Then, for all $t \in \mathbb{R}$, there exists x_i such that $\|\alpha(t) - x_i\| < \delta'$, which gives that

$$\|H(t)\| = \|h(t,\alpha(t))\| \le \|h(t,\alpha(t)) - h(t,x_i)\| + \|h(t,x_i)\| \le (L_f(t)+1)\varepsilon + \sum_{i=1}^{m}\|h(t,x_i)\|.$$

Since

$$\begin{aligned}
&\frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\|H(\theta)\|\right)d\mu(t)\\
&\leq \frac{\varepsilon}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}L_f(\theta)+1\right)d\mu(t)\\
&\quad+\sum_{i=1}^{m}\frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\|h(\theta,x_i)\|\right)d\mu(t),
\end{aligned}$$

and $h(\cdot,x_i)\in PAA_0(\mathbb{R},X,\mu,\nu,r)$ for each $i=1,2\ldots,m$, we have

$$\limsup_{T\to\infty}\frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\|H(\theta)\|\right)d\mu(t)\leq(\mathfrak{L}+\eta)\varepsilon,$$

which yields that

$$\lim_{T\to\infty}\frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\|H(\theta)\|\right)d\mu(t)=0,$$

that is $H(\cdot)\in PAA_0(\mathbb{R},X,\mu,\nu,r)$. Therefore, $f(\cdot,x(\cdot))\in PAA(\mathbb{R},X,\mu,\nu,r)$. □

2.3. S^pPAA of class r

First, we introduce the space of Stepanov bounded functions [32]. Let $p\in[1,\infty)$, the space $BS^p(\mathbb{R},X)$ of all Stepanov bounded functions with the exponent p, consists of all measurable functions $f:\mathbb{R}\to X$ such that $f^b\in L^\infty(\mathbb{R},L^p([0,1];X))$, where f^b is the Bochner transform of f defined by $f^b(t,s):=f(t+s),t\in\mathbb{R},s\in[0,1]$. $BS^p(\mathbb{R},X)$ is a Banach space with the norm

$$\|f\|_{S^p}=\|f^b\|_{L^\infty(\mathbb{R},L^p)}=\sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\|f(\tau)\|^p\mathrm{d}\tau\right)^{1/p}.$$

It is obvious that $L^p(\mathbb{R},X)\subset BS^p(\mathbb{R},X)\subset L^p_{loc}(\mathbb{R},X)$ and $BS^p(\mathbb{R},X)\subset BS^q(\mathbb{R},X)$ for $p\geq q\geq 1$.

Definition 2.7. The space $S^pAA(\mathbb{R},X)$ of Stepanov-like almost automorphic functions (or S^p-almost automorphic functions) consists of all $f\in BS^p(\mathbb{R},X)$ such that $f^b\in AA(\mathbb{R},L^p([0,1],X))$.

In other words, a function $f\in L^p_{loc}(\mathbb{R},X)$ is said to be Stepanov-like almost automorphic if its Bochner transform $f^b:\mathbb{R}\to L^p([0,1],X)$ is almost automorphic in the sense that for every sequence of real numbers $(s_n^{'})_{n\in\mathbb{N}}$, there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $g\in L^p_{loc}(\mathbb{R},X)$ such that

$$\lim_{n\to\infty}\left(\int_0^1\|f(t+s+s_n)-g(t+s)\|^pds\right)^{1/p}=0,$$
$$\lim_{n\to\infty}\left(\int_0^1\|g(t+s-s_n)-f(t+s)\|^pds\right)^{1/p}=0$$

pointwisely on $\mathbb{R}$. The collection of all such functions will be denoted by $S^pAA(\mathbb{R}, X)$.

It is clear that, for $1 \le p < q < \infty$, if $f \in L^q_{loc}(\mathbb{R}, X)$ is S^q-almost automorphic, then f is S^p-almost automorphic. In addition, if $f \in AA(\mathbb{R}, X)$, then f is S^p-almost automorphic for any $1 \le p < \infty$.

Definition 2.8. A function $f : \mathbb{R} \times X \to X, (t, u) \to f(t, u)$ with $f(\cdot, u) \in L^p_{loc}(\mathbb{R}, X)$ for each $u \in X$ is said to be S^p-almost automorphic in $t \in \mathbb{R}$ uniformly for $u \in X$ if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $g : \mathbb{R} \times X \to X$ with $g(\cdot, u) \in L^p_{loc}(\mathbb{R}, X)$ such that

$$\lim_{n\to\infty} \left(\int_0^1 \|f(t + s + s_n, u) - g(t + s, u)\|^p ds \right)^{1/p} = 0,$$

and

$$\lim_{n\to\infty} \left(\int_0^1 \|g(t + s - s_n, u) - f(t + s, u)\|^p ds \right)^{1/p} = 0,$$

for each $t \in \mathbb{R}$ and for each $u \in X$. We denote by $S^pAA(\mathbb{R} \times X, X)$ the set of all such functions.

Next, we introduce the concept of Stepanov-like (μ, ν)-pseudo almost automorphic function by the results of measure theory.

For $r > 0$, define

$$\begin{aligned} S^pPAA_0(\mathbb{R}, X, \mu, \nu) = \Big\{ & f \in BS^p(\mathbb{R}, X) : \\ & \lim_{T\to\infty} \frac{1}{\nu([-T, T])} \int_{[-T,T]} \left(\int_t^{t+1} \|f(\sigma)\|^p d\sigma \right)^{1/p} d\mu(t) = 0 \Big\}, \\ S^pPAA_0(\mathbb{R}, X, \mu, \nu, r) = \Big\{ & f \in BS^p(\mathbb{R}, X) : \\ & \lim_{T\to\infty} \frac{1}{\nu([-T, T])} \int_{[-T,T]} \left(\sup_{\theta\in[t-r,t]} \left(\int_\theta^{\theta+1} \|f(\sigma)\|^p d\sigma \right)^{1/p} \right) d\mu(t) = 0 \Big\}. \end{aligned}$$

Definition 2.9. Let $\mu, \nu \in \mathcal{M}$. A function $f \in BS^p(\mathbb{R}, X)$ is said to be Stepanov-like (μ, ν)-pseudo almost automorphic (or S^p-(μ, ν)-pseudo almost automorphic) if it can be decomposed as $f = g+h$, where $g \in S^pAA(\mathbb{R}, X)$ and $h \in S^pPAA_0(\mathbb{R}, X, \mu, \nu)$. Denote by $S^pPAA(\mathbb{R}, X, \mu, \nu)$ the collection of such functions.

Definition 2.10. Let $\mu, \nu \in \mathcal{M}$. A function $f : \mathbb{R} \times X \to X$, $(t, u) \to f(t, u)$ with $f(\cdot, u) \in BS^p(\mathbb{R}, X)$ for each $u \in X$ is said to be S^p-(μ, ν)-pseudo almost automorphic if it can be decomposed as $f = g + h$, where $g^b \in AA(\mathbb{R} \times X, L^p([0, 1], X))$ and $h^b \in PAA_0(\mathbb{R} \times X, L^p([0, 1], X), \mu, \nu)$. The collection of such functions will be denoted by $S^pPAA(\mathbb{R} \times X, X, \mu, \nu)$.

Definition 2.11. Let $\mu, \nu \in \mathcal{M}$. A function $f \in BS^p(\mathbb{R}, X)$ is said to be Stepanov-like (μ, ν)-pseudo almost automorphic of class r (or S^p-(μ, ν)-pseudo almost automorphic of class r) if it can be decomposed as $f = g + h$, where $g \in S^pAA(\mathbb{R}, X)$ and $h \in S^pPAA_0(\mathbb{R}, X, \mu, \nu, r)$. Denote by $S^pPAA(\mathbb{R}, X, \mu, \nu, r)$ the collection of such functions.

In other words, a function $f \in L^p_{loc}(\mathbb{R}, X)$ is said to be S^p-(μ, ν)-pseudo almost automorphic of class r if its Bochner transform $f^b : \mathbb{R} \to L^p([0, 1], X)$ is (ν, ν)-pseudo almost automorphic of class r in the sense that there exist two functions

$g, h : \mathbb{R} \to X$ such that $f = g + h$, where $g^b \in AA(\mathbb{R}, L^p([0,1], X))$ and $h^b \in PAA_0(\mathbb{R}, L^p([0,1], X), \mu, \nu, r)$, i.e.,

$$\lim_{T\to\infty} \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta\in[t-r,t]} \left(\int_\theta^{\theta+1} \|h(\sigma)\|^p d\sigma \right)^{1/p} \right) d\mu(t) = 0.$$

Remark 2.4. (*i*) If $\mu \sim \nu$, then S^p-(μ,ν)-pseudo almost automorphic function of class r ($S^pPAA(\mathbb{R}, X, \mu, \nu, r)$) is S^p-μ-pseudo almost automorphic of class r ($S^pPAA(\mathbb{R}, X, \mu, r)$) which defined in [9].

(*ii*) Let $\rho(t) > 0$ a.e. on $\mathbb{R}$ for the Lebesgue measure. μ, ν denote the positive measure defined by

$$\mu(A) = \nu(A) = \int_A \rho(t)dt \quad \text{for} \quad A \in \mathcal{B},$$

where dt denotes the Lebesgue measure on $\mathbb{R}$, then S^p-(μ,ν)-pseudo almost automorphic function of class r ($S^pPAA(\mathbb{R}, X, \mu, \nu, r)$) is S^p-weighted pseudo almost automorphic function of class r ($S^pWPAA(\mathbb{R}, X, \rho, r)$) defined in [39].

(*iii*) If $\mu \sim \nu$ and μ, ν are the Lebesgue measures, then S^p-(μ,ν)-pseudo almost automorphic of class r ($S^pPAA(\mathbb{R}, X, \mu, \nu, r)$) is Stepanov-like pseudo almost automorphic of class r ($S^pPAA(\mathbb{R}, X, r)$).

Similarly as the proof of [18, 19], the following result holds.

Lemma 2.6. *If* (M_1), (M_2) *hold, then* $PAA_0(\mathbb{R}, X, \mu, \nu) \subset S^pPAA_0(\mathbb{R}, X, \mu, \nu)$ *for each* $1 \le p < \infty$.

For S^p-(μ,ν)-pseudo almost automorphic of class r, one has

Proposition 2.5. *If* (M_1), (M_2) *hold, then* $PAA(\mathbb{R}, X, \mu, \nu, r) \subset S^pPAA(\mathbb{R}, X, \mu, \nu, r)$ *for each* $1 \le p < \infty$.

Proof. It suffices to show that $PAA_0(\mathbb{R}, X, \mu, \nu, r) \subset S^pPAA_0(\mathbb{R}, X, \mu, \nu, r)$. Let $f \in PAA_0(\mathbb{R}, X, \mu, \nu, r)$, then $g \in PAA_0(\mathbb{R}, \mathbb{R}, \mu, \nu)$, where

$$g(t) = \sup_{\theta\in[t-r,t]} \|f(\theta)\| = \sup_{\theta\in[-r,0]} \|f(t+\theta)\|, \quad t \in \mathbb{R}.$$

It follows from Lemma 2.6 that $g \in S^pPAA_0(\mathbb{R}, \mathbb{R}, \mu, \nu)$. Thus, one has

$$\begin{aligned}
&\frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta\in[t-r,t]} \left(\int_\theta^{\theta+1} \|f(\sigma)\|^p d\sigma \right)^{1/p} \right) d\mu(t) \\
&= \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta\in[-r,0]} \left(\int_0^1 \|f(t+\theta+\sigma)\|^p d\sigma \right)^{1/p} \right) d\mu(t) \\
&\le \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left[\int_0^1 \left(\sup_{\theta\in[-r,0]} \|f(t+\theta+\sigma)\| \right)^p d\sigma \right]^{1/p} d\mu(t) \\
&\le \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left[\int_0^1 |g(t+\sigma)|^p \, d\sigma \right]^{1/p} d\mu(t)
\end{aligned}$$

$$= \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left[\int_t^{t+1} |g(\sigma)|^p \, d\sigma \right]^{1/p} d\mu(t) \to 0, \qquad T \to \infty,$$

which means that $f \in S^p PAA(\mathbb{R}, X, \mu, \nu, r)$. $\square$

Remark 2.5. By Proposition 2.5, it is not difficult to see that

$$\begin{array}{ccccccc} PAA(\mathbb{R},X,r) & \Rightarrow & WPAA(\mathbb{R},X,\rho,r) & \Rightarrow & PAA(\mathbb{R},X,\mu,r) & \Rightarrow & PAA(\mathbb{R},X,\mu,\nu,r) \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ S^pPAA(\mathbb{R},X,r) & \Rightarrow & S^pWPAA(\mathbb{R},X,\rho,r) & \Rightarrow & S^pPAA(\mathbb{R},X,\mu,r) & \Rightarrow & S^pPAA(\mathbb{R},X,\mu,\nu,r) \end{array}$$

where “ $\Rightarrow$ ” denotes subset relation “ $\subset$ ”.

Similarly as the proof of [9], the following composition theorems hold for $S^pPAA(\mathbb{R}, X, \mu, \nu, r)$ under Lipschitz condition and non-Lipschitz condition, respectively.

Theorem 2.3. *Assume that* (M_1) *holds,* $f = g+h \in S^pPAA(\mathbb{R}\times Y, X, \mu, \nu, r)$ *with* $g^b \in AA(\mathbb{R} \times Y, L^p([0,1], X))$, $h^b \in PAA_0(\mathbb{R} \times Y, L^p([0,1], X), \mu, \nu, r)$ *and*

$(\mathcal{J}_1)$ *There exists a nonnegative function* $L_f \in L^p(\mathbb{R}, \mathbb{R}^+)$, $p > 1$ *such that*

$$\limsup_{T\to\infty} \frac{1}{(\nu([-T,T]))^{1/p}} \left[\int_{[-T,T]} \left(\sup_{\theta\in[t-r,t]} L_f(\theta) \right)^p d\mu(t) \right]^{1/p} < +\infty.$$

$(\mathcal{J}_2)$ *For all* $u, v \in Y$ *and* $t \in \mathbb{R}$, *one has*

$$\left(\int_t^{t+1} \|f(s,u) - f(s,v)\|^p ds \right)^{1/p} \leq L_f(t)\|u - v\|.$$

$(\mathcal{J}_3)$ $g(t,x)$ *is uniformly continuous in any bounded subset* $B \subset Y$ *uniformly for* $t \in \mathbb{R}$.

If $x = \alpha + \beta \in S^pPAA(\mathbb{R}, Y, \mu, \nu, r)$ *with* $\alpha^b \in AA(\mathbb{R}, L^p([0,1], Y))$ *and* $\beta^b \in PAA_0(\mathbb{R}, L^p([0,1], Y), \mu, \nu, r)$, $K = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ *is compact in* Y. *Then* $f(\cdot, x(\cdot)) \in S^pPAA(\mathbb{R}, X, \mu, \nu, r)$.

Theorem 2.4. *Assume that* (M_1) *holds,* $f = g + h \in S^pPAA(\mathbb{R} \times Y, X, \mu, \nu, r)$ *with* $g^b \in AA(\mathbb{R} \times Y, L^p([0,1], X))$, $h^b \in PAA_0(\mathbb{R} \times Y, L^p([0,1], X), \mu, \nu, r)$ *and the following conditions satisfied:*

$(\mathcal{N}_1)$ $f(t,x)$ *is uniformly continuous in any bounded subset* $B \subset Y$ *uniformly for* $t \in \mathbb{R}$.

$(\mathcal{N}_2)$ $g(t,x)$ *is uniformly continuous in any bounded subset* $B \subset Y$ *uniformly for* $t \in \mathbb{R}$.

$(\mathcal{N}_3)$ *For every bounded subset* $B \subset Y$, $\{f(\cdot, x) : x \in B\}$ *is bounded in* $S^pPAA(\mathbb{R} \times Y, X, \mu, \nu, r)$.

If $x = \alpha + \beta \in S^pPAA(\mathbb{R}, Y, \mu, \nu, r)$ *with* $\alpha^b \in AA(\mathbb{R}, L^p([0,1], Y))$ *and* $\beta^b \in PAA_0(\mathbb{R}, L^p([0,1], Y), \mu, \nu, r)$, $K = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ *is compact in* Y. *Then* $f(\cdot, x(\cdot)) \in S^pPAA(\mathbb{R}, X, \mu, \nu, r)$.

2.4. $PAA(S^pPAA)$ of class infinity

To deal with infinite delays, we introduce the following new spaces of functions:

$$
\begin{aligned}
&PAA_0(\mathbb{R}, X, \mu, \nu, \infty) := \bigcap_{r>0} PAA_0(\mathbb{R}, X, \mu, \nu, r),\\
&PAA_0(\mathbb{R}\times X, X, \mu, \nu, \infty) := \bigcap_{r>0} PAA_0(\mathbb{R}\times X, X, \mu, \nu, r),\\
&PAA_0(\mathbb{R}, L^p([0,1], X), \mu, \nu, \infty) := \bigcap_{r>0} PAA_0(\mathbb{R}, L^p([0,1], X), \mu, \nu, r),\\
&PAA_0(\mathbb{R}\times X, L^p([0,1], X), \mu, \nu, \infty) := \bigcap_{r>0} PAA_0(\mathbb{R}\times X, L^p([0,1], X), \mu, \nu, r),
\end{aligned}
$$

It is not difficult to see that $PAA_0(\mathbb{R}, X, \mu, \nu, \infty)$ and $PAA_0(\mathbb{R}, L^p([0,1], X), \mu, \nu, \infty)$ are closed subspaces of $PAA_0(\mathbb{R}, X, \mu, \nu, r)$, $PAA_0(\mathbb{R}, L^p([0,1], X), \mu, \nu, r)$, respectively.

Definition 2.12. Let $\mu, \nu \in \mathcal{M}$. A function $f \in C(\mathbb{R}, X)$ (resp. $C(\mathbb{R}\times X, X)$) is called (μ, ν)-pseudo almost automorphic of class infinity if it can be decomposed as $f = g+h$, where $g \in AA(\mathbb{R}, X)$ (resp. $AA(\mathbb{R}\times X, X)$) and $h \in PAA_0(\mathbb{R}, X, \mu, \nu, \infty)$ (resp. $PAA_0(\mathbb{R}\times X, X, \mu, \nu, \infty)$). Denote by $PAA(\mathbb{R}, X, \mu, \nu, \infty)$ (resp. $PAA(\mathbb{R}\times X, X, \mu, \nu, \infty)$) the set of such functions. It is not difficult to see that $PAA(\mathbb{R}, X, \mu, \nu, \infty)$ is a Banach space under the supremum norm.

Definition 2.13. Let $\mu, \nu \in \mathcal{M}$. A function $f \in BS^p(\mathbb{R}, X)$ is said to be Stepanov-like (μ, ν)-pseudo almost automorphic of class infinity (or S^p-(μ, ν)-pseudo almost automorphic of class infinity) if it can be decomposed as $f = g + h$, where $g^b \in AA(\mathbb{R}, L^p([0,1], X))$ and $h^b \in PAA_0(\mathbb{R}, L^p([0,1], X), \mu, \nu, \infty)$. $S^pPAA(\mathbb{R}, X, \mu, \nu, \infty)$ stands for the collection of such functions. It is easy to see that $PAA(\mathbb{R}, X, \mu, \nu, \infty) \subset S^pPAA(\mathbb{R}, X, \mu, \nu, \infty)$.

Similar as the proof of Theorem 2.2, Theorem 2.3, the following composition theorems are hold for $PAA(\mathbb{R}\times Y, X, \mu, \nu, \infty)$, $S^pPAA(\mathbb{R}\times Y, X, \mu, \nu, \infty)$, respectively, that is

Theorem 2.5. *Assume that (M_1) holds, $f = g + h \in PAA(\mathbb{R}\times Y, X, \mu, \nu, \infty)$, where $g \in AA(\mathbb{R}\times Y, X)$, $h \in PAA_0(\mathbb{R}\times Y, X, \mu, \nu, \infty)$ and $(\mathcal{I}_1)$-$(\mathcal{I}_4)$ hold, then $f(\cdot, x(\cdot)) \in PAA(\mathbb{R}, X, \mu, \nu, \infty)$ if $x \in PAA(\mathbb{R}, Y, \mu, \nu, \infty)$.*

Theorem 2.6. *Assume that (M_1) holds, $f = g+h \in S^pPAA(\mathbb{R}\times Y, X, \mu, \nu, \infty)$ with $g^b \in AA(\mathbb{R}\times Y, L^p([0,1], X))$, $h^b \in PAA_0(\mathbb{R}\times Y, L^p([0,1], X), \mu, \nu, \infty)$ and $(\mathcal{J}_1)$-$(\mathcal{J}_3)$ hold. If $x = \alpha + \beta \in S^pPAA(\mathbb{R}, Y, \mu, \nu, \infty)$ with $\alpha^b \in AA(\mathbb{R}, L^p([0,1], Y))$, $\beta^b \in PAA_0(\mathbb{R}, L^p([0,1], Y), \mu, \nu, \infty)$ and $K = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact in Y. Then $f(\cdot, x(\cdot)) \in S^pPAA(\mathbb{R}, X, \mu, \nu, \infty)$.*

3. FFDEs with finite delay

In this section, we establish some sufficient criteria for the existence and uniqueness of $PAA(\mathbb{R}, X, \mu, \nu, r)$ mild solutions for the following fraction differential equations with finite delay:

$$D_t^{\alpha+1}u(t)+\gamma D_t^{\beta}u(t)-Au(t)=D_t^{\alpha}f(t,u_t),\qquad t\in\mathbb{R},\quad 0<\alpha\le\beta\le 1,\quad \gamma\ge 0,\tag{3.1}$$

where $u_t(\theta):=u(t+\theta)$ for $\theta\in[-r,0]$, $r\ge 0$ is a fixed constant. The fractional derivative is understood in the Weyl sense.

To establish our results, we introduce the following conditions:

(H_1) A is an ω-sectorial operator of angle $\beta\pi/2$ with $\omega<0$.

(H_{21}) $f=g+h\in PAA(\mathbb{R}\times\mathcal{C},X,\mu,\nu,r)$, where $g\in AA(\mathbb{R}\times\mathcal{C},X)$, $h\in PAA_0(\mathbb{R}\times\mathcal{C},X,\mu,\nu,r)$.

(H_{22}) $f=g+h\in S^pPAA(\mathbb{R}\times\mathcal{C},X,\mu,\nu,r)$, where $g^b\in AA(\mathbb{R}\times\mathcal{C},L^p([0,1],X))$, $h^b\in PAA_0(\mathbb{R}\times\mathcal{C},L^p([0,1],X),\mu,\nu,r)$.

(H_{23}) $f=g+h\in PAA(\mathbb{R}\times\mathcal{C},X,\mu,\nu,\infty)$, where $g\in AA(\mathbb{R}\times\mathcal{C},X)$, $h\in PAA_0(\mathbb{R}\times\mathcal{C},X,\mu,\nu,\infty)$.

(H_{24}) $f=g+h\in S^pPAA(\mathbb{R}\times\mathcal{C},X,\mu,\nu,\infty)$, where $g^b\in AA(\mathbb{R}\times\mathcal{C},L^p([0,1],X))$, $h^b\in PAA_0(\mathbb{R}\times\mathcal{C},L^p([0,1],X),\mu,\nu,\infty)$.

(H_{31}) f satisfies the Lipschitz condition

$$\|f(t,\phi)-f(t,\psi)\|\le L_f\|\phi-\psi\|_{\mathcal{C}},\qquad \phi,\psi\in\mathcal{C},\ t\in\mathbb{R},$$

where $L_f>0$ is a constant.

(H_{32}) f satisfies the Lipschitz condition

$$\|f(t,\phi)-f(t,\psi)\|\le L_f(t)\|\phi-\psi\|_{\mathcal{C}},\qquad \phi,\psi\in\mathcal{C},\ t\in\mathbb{R},$$

where $L_f\in C(\mathbb{R},\mathbb{R}^+)$.

(H_{33}) f satisfies the Lipschitz condition

$$\|f(t,\phi)-f(t,\psi)\|\le L_f(t)\|\phi-\psi\|_{\mathcal{C}},\qquad \phi,\psi\in\mathcal{C},\ t\in\mathbb{R},$$

where $L_f\in BS^p(\mathbb{R},\mathbb{R}^+)$.

(H_{34}) f satisfies the Lipschitz condition

$$\|f(t,\phi)-f(t,\psi)\|\le L_f(t)\|\phi-\psi\|_{\mathcal{C}},\qquad \phi,\psi\in\mathcal{C},\ t\in\mathbb{R},$$

where $L_f\in BS^p(\mathbb{R},\mathbb{R}^+)\cap L^1(\mathbb{R},\mathbb{R}^+)$.

(H_4) g satisfies the Lipschitz condition

$$\|g(t,\phi)-g(t,\psi)\|\le L_g\|\phi-\psi\|_{\mathcal{C}},\qquad \phi,\psi\in\mathcal{C},\ t\in\mathbb{R}.$$

where $L_g>0$ is a constant.

For (3.1), we adopt the following concept of mild solution.

Definition 3.1. Assume that the operator A generates an integrable $(\alpha,\beta)_\gamma$-regularized family $\{S_{\alpha,\beta}(t)\}_{t\ge 0}$. A function $u\in C([-r,\infty),X)$ is said to be a mild solution of (3.1) if satisfying the following integral equation

$$u(t)=\int_{-\infty}^{t}S_{\alpha,\beta}(t-s)f(s,u_s)ds,\qquad t\in\mathbb{R}.\tag{3.2}$$

3.1. PAA perturbation

In this subsection, if f satisfies the Lipschitz condition, we investigate existence, uniqueness of $PAA(\mathbb{R}, X, \mu, \nu, r)$ mild solutions for (3.1) under PAA perturbation, i.e., (H_{21}) holds.

Lemma 3.1. *If* (M_1), (M_2) *hold,* $u \in PAA(\mathbb{R}, X, \mu, \nu, r)$, *then* $u_t \in PAA(\mathbb{R}, \mathcal{C}, \mu, \nu, r)$.

Proof. Suppose that $u = \alpha + \beta$, where $\alpha \in AA(\mathbb{R}, X)$ and $\beta \in PAA_0(\mathbb{R}, X, \mu, \nu, r)$, then $u_t = \alpha_t + \beta_t$ and $\alpha_t \in AA(\mathbb{R}, \mathcal{C})$ by Lemma 2.1. Next, we will show that $\beta_t \in PAA_0(\mathbb{R}, \mathcal{C}, \mu, \nu, r)$. In fact, for $\nu \in \mathcal{M}$, $r \geq 0$, since $\nu(\mathbb{R}) = +\infty$, there exists $T_0 > 0$ such that $\nu([-T - r, T + r]) > 0$ for all $T > T_0$. Hence, for $T > T_0$, one has

$$\begin{aligned}
&\frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta \in [t-2r, t-r]} \|\beta(\theta)\| \right) d\mu(t) \\
&\leq \frac{1}{\nu([-T,T])} \int_{[-T-r,T-r]} \left(\sup_{\theta \in [t-r, t]} \|\beta(\theta)\| \right) d\mu_r(t) \\
&\leq \frac{1}{\nu([-T,T])} \int_{[-T-r,T+r]} \left(\sup_{\theta \in [t-r, t]} \|\beta(\theta)\| \right) d\mu_r(t) \\
&\leq \frac{\nu_r([-T-r,T+r])}{\nu([-T,T])} \times \frac{1}{\nu_r([-T-r,T+r])} \int_{[-T-r,T+r]} \left(\sup_{\theta \in [t-r, t]} \|\beta(\theta)\| \right) d\mu_r(t).
\end{aligned}$$

Since $\mu \sim \mu_r$, $\nu \sim \nu_r$ by Lemma 2.2, then $\beta \in PAA_0(\mathbb{R}, X, \mu_r, \nu_r, r)$ by Proposition 2.1, so

$$\lim_{T \to \infty} \frac{1}{\nu_r([-T-r,T+r])} \int_{[-T-r,T+r]} \left(\sup_{\theta \in [t-r, t]} \|\beta(\theta)\| \right) d\mu_r(t) = 0,$$

then

$$\lim_{T \to \infty} \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta \in [t-2r, t-r]} \|\beta(\theta)\| \right) d\mu(t) = 0 \tag{3.3}$$

by Lemma 2.3. For $T > T_0$, we see that

$$\begin{aligned}
&\frac{1}{\nu([-T,T])} \int_{[-T,T]} \left[\sup_{\theta \in [t-r, t]} \left(\sup_{\tau \in [-r, 0]} \|\beta(\theta + \tau)\| \right) \right] d\mu(t) \\
&\leq \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta \in [t-2r, t]} \|\beta(\theta)\| \right) d\mu(t) \\
&\leq \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta \in [t-2r, t-r]} \|\beta(\theta)\| \right) d\mu(t) \\
&\quad + \frac{1}{\nu([-T,T])} \int_{[-T,T]} \left(\sup_{\theta \in [t-r, t]} \|\beta(\theta)\| \right) d\mu(t),
\end{aligned}$$

since $\beta \in PAA_0(\mathbb{R}, X, \mu, \nu, r)$ and (3.3) holds, we have $\beta_t \in PAA_0(\mathbb{R}, \mathcal{C}, \mu, \nu, r)$. Therefore, $u_t \in PAA(\mathbb{R}, \mathcal{C}, \mu, \nu, r)$. $\square$

Lemma 3.2. *If $(H_1), (M_1), (M_2)$ hold and $f \in PAA(\mathbb{R}, X, \mu, \nu, r)$, then*

$$(\Lambda f)(t) := \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)f(s)ds \in PAA(\mathbb{R}, X, \mu, \nu, r), \qquad t \in \mathbb{R}.$$

Proof. It is clear that $\Lambda f \in BC(\mathbb{R}, X)$ since

$$\|\Lambda f\| \leq \frac{C|\omega|^{-1/(\alpha+1)}\pi\|f\|}{(\alpha+1)\sin(\pi/(\alpha+1))}.$$

Let $f(t) = f_1(t) + f_2(t)$, where $f_1 \in AA(\mathbb{R}, X)$ and $f_2 \in PAA_0(\mathbb{R}, X, \mu, \nu, r)$, then

$$\begin{aligned}
(\Lambda f)(t) &= \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)f(s)ds \\
&= \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)f_1(s)ds + \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)f_2(s)ds \\
&:= \Lambda_1(t) + \Lambda_2(t), \quad t \in \mathbb{R}.
\end{aligned}$$

Let $(s'_n)_{n\in\mathbb{N}}$ be any sequence of real numbers, then $f_1 \in AA(\mathbb{R}, X)$ implies that there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ of $(s'_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} f_1(t+s_n) = g_1(t), \quad \lim_{n\to\infty} g_1(t-s_n) = f_1(t), \quad t \in \mathbb{R}.$$

Define

$$G(t) = \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)g_1(s)ds,$$

and consider

$$\Lambda_1(t+s_n) = \int_{-\infty}^{t+s_n} S_{\alpha,\beta}(t+s_n-s)f_1(s)ds = \int_0^{\infty} S_{\alpha,\beta}(\sigma)f_1(t+s_n-\sigma)d\sigma.$$

Note that

$$\|\Lambda_1(t+s_n)\| \leq \frac{C|\omega|^{-1/(\alpha+1)}\pi\|f_1\|}{(\alpha+1)\sin(\pi/(\alpha+1))}, \quad \|G(t)\| \leq \frac{C|\omega|^{-1/(\alpha+1)}\pi\|g_1\|}{(\alpha+1)\sin(\pi/(\alpha+1))},$$

and by the strong continuity of $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$, for any fixed $\sigma \in \mathbb{R}$ and any $t \geq \sigma$, one has $S_{\alpha,\beta}(t-\sigma)f_1(\sigma+s_n) \to S_{\alpha,\beta}(t-\sigma)g_1(\sigma)$ as $n \to \infty$. Then by the Lebesgue dominated convergence theorem, for any $t \in \mathbb{R}$, $\Lambda_1(t+s_n) \to G(t)$ as $n \to \infty$, and similarly, $G(t-s_n) \to \Lambda_1(t)$ as $n \to \infty$. Therefore, $\Lambda_1 \in AA(\mathbb{R}, X)$.

To complete the proof, we show that $\Lambda_2 \in PAA_0(\mathbb{R}, X, \mu, \nu, r)$. In fact, for $T > 0$, one has

$$\begin{aligned}
&\frac{1}{\nu([-T,T])}\int_{[-T,T]} \left(\sup_{\theta\in[t-r,t]} \|\Lambda_2(\theta)\|\right) d\mu(t) \\
&= \frac{1}{\nu([-T,T])}\int_{[-T,T]} \left(\sup_{\theta\in[t-r,t]} \left\|\int_{-\infty}^{\theta} S_{\alpha,\beta}(\theta-s)f_2(s)ds\right\|\right) d\mu(t) \\
&= \frac{1}{\nu([-T,T])}\int_{[-T,T]} \left(\sup_{\theta\in[t-r,t]} \left\|\int_0^{\infty} S_{\alpha,\beta}(s)f_2(\theta-s)ds\right\|\right) d\mu(t)
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\int_0^\infty \|S_{\alpha,\beta}(s)\|\,\|f_2(\theta-s)\|\,ds\right)d\mu(t)\\
&\le \frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\int_0^\infty \frac{C}{1+|\omega|s^{\alpha+1}}\sup_{\theta\in[t-r,t]}\|f_2(\theta-s)\|\,ds\right)d\mu(t)\\
&\le C\int_0^\infty \frac{1}{1+|\omega|s^{\alpha+1}}\left(\frac{1}{\nu([-T,T])}\int_{[-T,T]}\sup_{\theta\in[t-r,t]}\|f_2(\theta-s)\|d\mu(t)\right)ds\\
&= C\int_0^\infty \frac{\Phi_T(s)}{1+|\omega|s^{\alpha+1}}ds,
\end{aligned}$$

where

$$\Phi_T(s)=\frac{1}{\nu([-T,T])}\int_{[-T,T]}\sup_{\theta\in[t-r,t]}\|f_2(\theta-s)\|d\mu(t).$$

Since (M_2) holds, from Proposition 2.2, it follows that $f_2(\cdot-s)\in PAA_0(\mathbb{R},X,\mu,\nu,r)$ for $s\in\mathbb{R}$. Hence $\Phi_T(s)\to 0$ as $T\to\infty$. Note that Φ_T is bounded by (M_1) and $1/(1+|\omega|s^{\alpha+1})$ is integrable on $[0,\infty)$ by (2.3), from Lebesgue dominated convergence theorem, it follows that

$$\lim_{T\to\infty}\int_0^\infty \frac{\Phi_T(s)}{1+|\omega|s^{\alpha+1}}ds=0,$$

then

$$\lim_{T\to\infty}\frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\|\Lambda_2(\theta)\|\right)d\mu(t)=0.$$

The proof is complete. □

Theorem 3.1. *Assume that (M_1), (M_2), (H_1), (H_{21}), (H_{31}) hold, then* (3.1) *has a unique mild solution $u\in PAA(\mathbb{R},X,\mu,\nu,r)$ if $CL_f|\omega|^{-1/(\alpha+1)}\pi<(\alpha+1)\sin(\pi/(\alpha+1))$.*

Proof. Define the operator $\mathcal{F}:PAA(\mathbb{R},X,\mu,\nu,r)\to PAA(\mathbb{R},X,\mu,\nu,r)$ by

$$(\mathcal{F}u)(t)=\int_{-\infty}^t S_{\alpha,\beta}(t-s)f(s,u_s)ds,\qquad t\in\mathbb{R}. \tag{3.4}$$

For $u\in PAA(\mathbb{R},X,\mu,\nu,r)$, $u_s\in PAA(\mathbb{R},\mathcal{C},\mu,\nu,r)$ by Lemma 3.1. Since $f(t,0)\in PAA(\mathbb{R},X,\mu,\nu,r)$ and $PAA(\mathbb{R},X,\mu,\nu,r)\subset BC(\mathbb{R},X)$, then one has

$$\|f(t,u_t)\|\le L_f\|u\|+\|f(t,0)\|\le L_f\|u\|+\sup_{t\in\mathbb{R}}\|f(t,0)\|$$

with $\sup_{t\in\mathbb{R}}\|f(t,0)\|<\infty$. Therefore for all bounded subset B of $\mathcal{C}$, f is bounded on $R\times\mathcal{C}$. In view of Theorem 2.2, $f(\cdot,u_\cdot)\in PAA(\mathbb{R},X,\mu,\nu,r)$. Hence $\mathcal{F}$ is well defined by Lemma 3.2.

For any $u,v\in PAA(\mathbb{R},X,\mu,\nu,r)$,

$$\begin{aligned}
\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\| &\le \int_{-\infty}^t \|S_{\alpha,\beta}(t-s)\|\|f(s,u_s)-f(s,v_s)\|ds\\
&\le L_f\|u-v\|\int_{-\infty}^t \|S_{\alpha,\beta}(t-s)\|ds
\end{aligned}$$

$$\le L_f\|u-v\|\int_0^\infty \|S_{\alpha,\beta}(s)\|ds$$
$$\le L_f\|u-v\|\left(\int_0^\infty \frac{C}{1+|\omega|(s^{\alpha+1}+\gamma s^\beta)}ds\right)$$
$$\le \frac{CL_f|\omega|^{-1/(\alpha+1)}\pi}{(\alpha+1)\sin(\pi/(\alpha+1))}\|u-v\|,$$

hence by the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $PAA(\mathbb{R}, X, \mu, \nu, r)$, which is the unique (μ,ν)-pseudo almost automorphic of class r mild solution to (3.1). □

A different Lipschitz condition is considered in the following result.

Theorem 3.2. *Assume that* (M_1), (M_2), (H_1), (H_{21}), (H_{32}), $(\mathcal{I}_1)$, $(\mathcal{I}_3)$, $(\mathcal{I}_4)$ *hold, then* (3.1) *has a unique mild solution* $u\in PAA(\mathbb{R}, X, \mu, \nu, r)$ *provided that*

$$\sup_{t\in\mathbb{R}}\int_{-\infty}^t \frac{L_f(s)}{1+|\omega|(t-s)^{\alpha+1}}ds<\frac{1}{C}.$$

Proof. Define the operator $\mathcal{F}: PAA(\mathbb{R}, X, \mu, \nu, r)\to PAA(\mathbb{R}, X, \mu, \nu, r)$ as (3.4). It is not difficult to see that $\mathcal{F}$ is well defined by Lemma 3.1, Lemma 3.2 and Theorem 2.2. For any $u, v\in PAA(\mathbb{R}, X, \mu, \nu, r)$,

$$\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\|\le\int_{-\infty}^t \|S_{\alpha,\beta}(t-s)\|\|f(s,u_s)-f(s,v_s)\|ds$$
$$\le\|u-v\|\int_{-\infty}^t L_f(s)\|S_{\alpha,\beta}(t-s)\|ds$$
$$\le\|u-v\|\int_{-\infty}^t \frac{CL_f(s)}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^\beta)}ds$$
$$\le C\sup_{t\in\mathbb{R}}\int_{-\infty}^t \frac{L_f(s)}{1+|\omega|(t-s)^{\alpha+1}}ds\cdot\|u-v\|,$$

hence by the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $PAA(\mathbb{R}, X, \mu, \nu, r)$, which is the unique (μ,ν)-pseudo almost automorphic of class r mild solution to (3.1). □

3.2. S^pPAA perturbation

In this subsection, if f satisfies the Lipschitz condition, we investigate existence, uniqueness of $PAA(\mathbb{R}, X, \mu, \nu, r)$ mild solutions for (3.1) under S^pPAA perturbation, i.e., (H_{22}) holds.

Lemma 3.3. *Let* $\{S(t)\}_{t\ge 0}\subset L(X)$ *be a strongly continuous family of bounded and linear operators such that* $\|S(t)\|\le\phi(t), t\in\mathbb{R}^+$, *where* $\phi\in L^1(\mathbb{R}^+)$ *is nonincreasing. If* $f\in S^pPAA(\mathbb{R}, X, \mu, \nu, r)$, *then*

$$(\Pi f)(t):=\int_{-\infty}^t S(t-s)f(s)ds\in PAA(\mathbb{R}, X, \mu, \nu, r),\qquad t\in\mathbb{R}.$$

Proof. Since $f\in S^pPAA(\mathbb{R}, X, \mu, \nu, r)$, let $f(s)=f_1(s)+f_2(s)$, where $f_1^b\in AA(\mathbb{R}, L^p([0,1], X)$ and $f_2^b\in PAA_0(\mathbb{R}, L^p([0,1], X), \mu, \nu, r)$. Consider the integrals

$$v_n(t)=\int_{t-n}^{t-n+1} S(t-s)f(s)ds$$

$$= \int_{t-n}^{t-n+1} S(t-s)f_1(s)ds + \int_{t-n}^{t-n+1} S(t-s)f_2(s)ds, \quad n = 1, 2, \ldots,$$

and set

$$X_n(t) = \int_{t-n}^{t-n+1} S(t-s)f_1(s)ds, \quad Y_n(t) = \int_{t-n}^{t-n+1} S(t-s)f_2(s)ds.$$

First, we show that $X_n \in AA(\mathbb{R}, X)$. For each $n \in \mathbb{N}$, by the the uniform boundedness principle or Banach-Steinhaus theorem, $L_n := \sup\limits_{n-1 \le t \le n} \|S(t)\| < +\infty$. Fix $n \in \mathbb{N}$ and $t \in \mathbb{R}$, one has

$$\begin{aligned} \|X_n(t+h) - X_n(t)\| &\le \int_{n-1}^{n} \|S(s)\| \|f_1(t+h-s) - f_1(t-s)\| ds \\ &\le L_n \int_{t-n}^{t-n+1} \|f_1(s+h) - f_1(s)\| ds \\ &\le L_n \left(\int_{t-n}^{t-n+1} \|f_1(s+h) - f_1(s)\|^p ds \right)^{1/p}. \end{aligned}$$

Since $f_1 \in L^p_{loc}(\mathbb{R}, X)$, we have

$$\lim_{h \to 0} \int_{t-n}^{t-n+1} \|f_1(s+h) - f_1(s)\|^p ds = 0,$$

then

$$\lim_{h \to 0} \|X_n(t+h) - X_n(t)\| = 0,$$

so $X_n(t)$ is continuous.

Let $(s_m)_{m \in \mathbb{N}}$ be a sequence of real numbers. From $f_1^b \in AA(\mathbb{R}, L^p([0,1], X))$, it follows that there exist a subsequence $(s_{m_k})_{k \in \mathbb{N}}$ and a function $v \in L^p_{loc}(\mathbb{R}, X)$ such that for any $t \in \mathbb{R}$

$$\left(\int_t^{t+1} \|f_1(s+s_{m_k}) - v(s)\|^p \, ds \right)^{1/p} \to 0, \quad k \to \infty.$$

Note that

$$X_n(t) = \int_{t-n}^{t-n+1} S(t-s)f_1(s)ds = \int_{n-1}^{n} S(\xi)f_1(t-\xi)d\xi,$$

and define $w_n(t) = \int_{n-1}^{n} S(\xi)v(t-\xi)d\xi$, then by the Hölder inequality, we have

$$\begin{aligned} \|X_n(t+s_{m_k}) - w_n(t)\| &= \left\| \int_{n-1}^{n} S(\xi) \left[f_1(t+s_{m_k}-\xi) - v(t-\xi) \right] d\xi \right\| \\ &\le L_n \int_{n-1}^{n} \|f_1(t+s_{m_k}-\xi) - v(t-\xi)\| \, d\xi \\ &\le L_n \left(\int_{t-n}^{t-n+1} \|f_1(s+s_{m_k}) - v(s)\|^p \, ds \right)^{1/p} \to 0, \ k \to \infty. \end{aligned}$$

Similarly, $\|w_n(t-s_{m_k})-X_n(t)\| \to 0, k \to \infty$. Therefore, $X_n \in AA(\mathbb{R}, X)$ for $n \in \mathbb{N}$.

By the Hölder inequality, one has

$$\begin{aligned}\|X_n(t)\| &\le \int_{n-1}^{n} \phi(s)\|f_1(t-s)\|ds \\ &\le \phi(n-1)\int_{n-1}^{n} \|f_1(t-s)\|ds \\ &= \phi(n-1)\int_{t-n}^{t-n+1} \|f_1(s)\|ds \\ &\le \phi(n-1)\left(\int_{t-n}^{t-n+1} \|f_1(s)\|^p ds\right)^{1/p} \le \phi(n-1)\|f_1\|_{S^p}\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} \phi(n-1)\|f_1\|_{S^p} &\le \left(\phi(0)+\sum_{n=2}^{\infty}\int_{n-2}^{n-1}\phi(t)dt\right)\|f_1\|_{S^p} \\ &\le \left(\phi(0)+\|\phi\|_{L^1}\right)\|f_1\|_{S^p} < +\infty,\end{aligned}$$

then $\sum\limits_{n=1}^{\infty} X_n(t)$ is uniformly convergent on $\mathbb{R}$.

Let $X(t) = \sum\limits_{n=1}^{\infty} X_n(t), t \in \mathbb{R}$, then

$$X(t) = \int_{-\infty}^{t} S(t-s)f_1(s)ds \quad t \in \mathbb{R},$$

by Lemma 2.1, we have $X(t) = \sum\limits_{n=1}^{\infty} X_n(t) \in AA(\mathbb{R}, X)$.

To complete the proof, we only need to prove that $Y_n \in PAA_0(\mathbb{R}, X, \mu, \nu, r)$. By carrying out similar arguments as above, we know that $\sum\limits_{n=1}^{\infty} Y_n(t)$ is uniformly convergent on $\mathbb{R}$. Let $Y(t) = \sum\limits_{n=1}^{\infty} Y_n(t)$, then

$$Y(t) = \int_{-\infty}^{t} S(t-s)f_2(s)ds, t \in \mathbb{R}.$$

It is obvious that $Y \in BC(\mathbb{R}, X)$. So, we only need to show that

$$\lim_{T\to\infty} \frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\|Y(\theta)\|\right)d\mu(t) = 0.$$

In fact, by the Hölder inequality, one has

$$\begin{aligned}\sup_{\theta\in[t-r,t]}\|Y_n(\theta)\| &\le \sup_{\theta\in[t-r,t]}\int_{n-1}^{n}\|S(\xi)\|\|f_2(\theta-\xi)\|d\xi \\ &\le \sup_{\theta\in[t-r,t]}\int_{n-1}^{n}\|\phi(\xi)\|\|f_2(\theta-\xi)\|d\xi\end{aligned}$$

$$\le \phi(n-1)\cdot \sup_{\theta\in[t-r,t]}\int_{t-n}^{t-n+1}\|f_2(s)\|\,ds$$
$$\le \phi(0)\cdot \sup_{\theta\in[t-r,t]}\left(\int_{t-n}^{t-n+1}\|f_2(s)\|^p\,ds\right)^{1/p},$$

then

$$\frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\|Y_n(\theta)\|\right)d\mu(t)$$
$$\le \frac{\phi(0)}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\left(\int_{t-n}^{t-n+1}\|f_2(s)\|^p ds\right)^{1/p}\right)d\mu(t),$$

hence $Y_n\in PAA_0(\mathbb{R},X,\mu,\nu,r)$ since $f_2^b\in PAA_0(\mathbb{R},L^p([0,1],X),\mu,\nu,r)$. From $Y_n\in PAA_0(\mathbb{R},X,\mu,\nu,r)$ and

$$\frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\|Y(\theta)\|\right)d\mu(t)$$
$$\le \frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\left\|Y(\theta)-\sum_{n=1}^{N}Y_n(\theta)\right\|\right)d\mu(t)$$
$$+\sum_{n=1}^{N}\frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\|Y_n(\theta)\|\right)d\mu(t),$$

it follows that $Y\in PAA_0(\mathbb{R},X,\mu,\nu,r)$, whence $\Pi f\in PAA(\mathbb{R},X,\mu,\nu,r)$. □

Theorem 3.3. *Assume that* (M_1), (M_2), (H_1), (H_{22}), (H_{31}), (H_4) *hold, then* (3.1) *has a unique mild solution* $u\in PAA(\mathbb{R},X,\mu,\nu,r)$ *if* $CL_f|\omega|^{-1/(\alpha+1)}\pi<(\alpha+1)\sin(\pi/(\alpha+1))$.

Proof. Let $u\in PAA(\mathbb{R},X,\mu,\nu,r)$, define the operator $\mathcal{F}:PAA(\mathbb{R},X,\mu,\nu,r)\to PAA(\mathbb{R},X,\mu,\nu,r)$ as (3.4). By Lemma 3.1, we have $u_s\in PAA(\mathbb{R},\mathcal{C},\mu,\nu,r)\subset S^pPAA(\mathbb{R},\mathcal{C},\mu,\nu,r)$. Let $u_s=u_1+u_2$, where $u_1\in AA(\mathbb{R},\mathcal{C})$, $u_2\in PAA_0(\mathbb{R},\mathcal{C},\mu,\nu,r)$, then $K=\overline{\{u_1:t\in\mathbb{R}\}}$ is compact in $\mathcal{C}$ by Lemma 2.1. In view of Theorem 2.3, $f(\cdot,u_\cdot)\in S^pPAA(\mathbb{R},X,\mu,\nu,r)$, so $\mathcal{F}$ is well defined by Lemma 3.3.

For any $u,v\in PAA(\mathbb{R},X,\mu,\nu,r)$,

$$\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\|\le\int_{-\infty}^{t}\|S_{\alpha,\beta}(t-s)\|\|f(s,u_s)-f(s,v_s)\|ds$$
$$\le L_f\|u-v\|\int_0^\infty\|S_{\alpha,\beta}(s)\|ds$$
$$\le\frac{CL_f|\omega|^{-1/(\alpha+1)}\pi}{(\alpha+1)\sin(\pi/(\alpha+1))}\|u-v\|,$$

hence by the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $PAA(\mathbb{R},X,\mu,\nu,r)$, which is the unique (μ,ν)-pseudo almost automorphic of class r mild solution to (3.1). □

Theorem 3.4. *Assume that* (M_1), (M_2), (H_1), (H_{22}), (H_{33}), $(\mathcal{J}_1)$, $(\mathcal{J}_3)$ *hold, if*

$$C\left(1+\frac{|\omega|^{-1/(\alpha+1)}\pi}{(\alpha+1)\sin(\pi/(\alpha+1))}\right)\|L_f\|_{S^p}<1,$$

then (3.1) *has a unique mild solution* $u \in PAA(\mathbb{R}, X, \mu, \nu, r)$.

Proof. Define the operator $\mathcal{F} : PAA(\mathbb{R}, X, \mu, \nu, r) \to PAA(\mathbb{R}, X, \mu, \nu, r)$ as (3.4). It is easy to see that $\mathcal{F}$ is well defined similar as the proof of Theorem 3.3.

For $u, v \in PAA(\mathbb{R}, X, \mu, \nu, r)$, one has

$$\begin{aligned}
\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\| &\le \int_{-\infty}^{t}\|S_{\alpha,\beta}(t-s)\|\|f(s,u_s)-f(s,v_s)\|ds\\
&\le \|u-v\|\cdot\int_{-\infty}^{t}\frac{CL_f(s)}{1+|\omega|[(t-s)^{\alpha+1}+\gamma(t-s)^{\beta}]}ds\\
&\le \|u-v\|\cdot\int_{-\infty}^{t}\frac{CL_f(s)}{1+|\omega|(t-s)^{\alpha+1}}ds\\
&\le C\|u-v\|\cdot\int_{0}^{+\infty}\frac{L_f(t-s)}{1+|\omega|s^{\alpha+1}}ds\\
&\le C\|u-v\|\cdot\sum_{k=0}^{\infty}\int_{k}^{k+1}\frac{L_f(t-s)}{1+|\omega|s^{\alpha+1}}ds\\
&\le C\|u-v\|\cdot\sum_{k=0}^{\infty}\frac{1}{1+|\omega|k^{\alpha+1}}\int_{k}^{k+1}L_f(t-s)ds\\
&\le C\|u-v\|\cdot\sum_{k=0}^{\infty}\frac{1}{1+|\omega|k^{\alpha+1}}\left(\int_{t-k-1}^{t-k}\|L_f(s)\|^p ds\right)^{1/p}\\
&\le C\|L_f\|_{S^p}\|u-v\|\cdot\sum_{k=0}^{\infty}\frac{1}{1+|\omega|k^{\alpha+1}}\\
&\le C\|L_f\|_{S^p}\|u-v\|\cdot\left(1+\int_{0}^{\infty}\frac{1}{1+|\omega|t^{\alpha+1}}dt\right)\\
&\le C\|L_f\|_{S^p}\left(1+\frac{|\omega|^{-1/(\alpha+1)}\pi}{(\alpha+1)\sin(\pi/(\alpha+1))}\right)\cdot\|u-v\|,
\end{aligned}$$

by the contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $PAA(\mathbb{R}, X, \mu, \nu, r)$, which is the unique (μ, ν)-pseudo almost automorphic of class r mild solution to (3.1). □

Theorem 3.5. *Assume that* (M_1), (M_2), (H_1), (H_{22}), (H_{34}), $(\mathcal{J}_1)$, $(\mathcal{J}_3)$ *hold, then* (3.1) *has a unique mild solution* $u \in PAA(\mathbb{R}, X, \mu, \nu, r)$.

Proof. Define the operator $\mathcal{F}$ as in (3.4). Let $u, v \in PAA(\mathbb{R}, X, \mu, \nu, r)$, one has

$$\begin{aligned}
\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\| &\le \int_{-\infty}^{t}\|S_{\alpha,\beta}(t-s)\|\|f(s,u_s)-f(s,v_s)\|ds\\
&\le \|u-v\|\cdot\int_{-\infty}^{t}\frac{CL_f(s)}{1+|\omega|[(t-s)^{\alpha+1}+\gamma(t-s)^{\beta}]}ds
\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_{-\infty}^{t} L_f(s)ds\cdot\|u-v\|\\
&\leq C\|L_f\|_{L^1}\cdot\|u-v\|.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\|(\mathcal{F}^2u)(t)-(\mathcal{F}^2v)(t)\| &\leq \int_{-\infty}^{t}\frac{CL_f(s)}{1+|\omega|[(t-s)^{\alpha+1}+\mu(t-s)^{\beta}]}\|(\mathcal{F}u)(s)-(\mathcal{F}v)(s)\|ds\\
&\leq C\int_{-\infty}^{t} L_f(s)\|(\mathcal{F}u)(s)-(\mathcal{F}v)(s)\|ds\\
&\leq C^2\|u-v\|\int_{-\infty}^{t} L_f(s)\left(\int_{-\infty}^{s} L_f(\tau)d\tau\right)ds\\
&= C^2\|u-v\|\int_{-\infty}^{t}\left(\int_{-\infty}^{s} L_f(\tau)d\tau\right)d\left(\int_{-\infty}^{s} L_f(\tau)d\tau\right)\\
&\leq \frac{C^2}{2!}\|u-v\|\left(\int_{-\infty}^{t} L_f(\tau)d\tau\right)^2\\
&\leq \frac{(C\|L_f\|_{L^1})^2}{2!}\|u-v\|.
\end{aligned}
$$

By the method of mathematical induction, we have

$$
\|(\mathcal{F}^nu)(t)-(\mathcal{F}^nv)(t)\| \leq \frac{C^n}{n!}\|u-v\|\left(\int_{-\infty}^{t} L_f(\tau)d\tau\right)^n.
$$

Moreover, since $L_f\in L^1(\mathbb{R},\mathbb{R}^+)$,

$$
\|(\mathcal{F}^nu)(t)-(\mathcal{F}^nv)(t)\| \leq \frac{(C\|L_f\|_{L^1})^n}{n!}\|u-v\|,
$$

which implies that

$$
\|\mathcal{F}^nu-\mathcal{F}^nv\| \leq \frac{(C\|L_f\|_{L^1})^n}{n!}\|u-v\|.
$$

For sufficiently large n, we have $(C\|L_f\|_{L^1})^n/n! < 1$, by the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $PAA(\mathbb{R},X,\mu,\nu,r)$, which is the unique $PAA(\mathbb{R},X,\mu,\nu,r)$ mild solution of (3.1). □

3.3. Non-Lipschitz case

In this subsection, we study the existence of $PAA(\mathbb{R},X,\mu,\nu,r)$ mild solution of (3.1) when f is not satisfies Lipschitz condition. First, we recall a useful compactness criterion and nonlinear Leray-Schauder alternative theorem.

Let $h^*:\mathbb{R}\to\mathbb{R}$ be a continuous nondecreasing function such that $h^*(t)\geq 1$ for all $t\in\mathbb{R}$, and $h^*(t)\to\infty$ as $|t|\to\infty$. Define

$$
C_{h^*}(\mathbb{R},X) := \{u\in C(\mathbb{R},X) : \lim_{|t|\to\infty}\frac{u(t)}{h^*(t)} = 0\}
$$

endowed with the norm $\|u\|_{h^*} = \sup_{t\in\mathbb{R}}(\|u(t)\|/h^*(t))$.

Lemma 3.4 ([26]). *A set $K \subseteq C_{h^*}(\mathbb{R}, X)$ is relatively compact in $C_{h^*}(\mathbb{R}, X)$ if it verifies the following conditions:*

(c_1) *The set $K(t) := \{u(t) : u \in K\}$ is relatively compact in X for each $t \in \mathbb{R}$.*

(c_2) *The set K is equicontinuous.*

(c_3) *For each $\varepsilon > 0$, there exists $\vartheta > 0$ such that $\|u(t)\| \leq \varepsilon h^*(t)$ for all $u \in K$ and all $|t| > \vartheta$.*

Theorem 3.6 ([24] Leray-Schauder Alternative Theorem). *Let Ω be a closed convex subset of a Banach space X such that $0 \in \Omega$. Let $\mathcal{F} : \Omega \to \Omega$ be a completely continuous map. Then the set $\{x \in \Omega : \ x = \lambda \mathcal{F}(x), \ 0 < \lambda < 1\}$ is unbounded or the map $\mathcal{F}$ has a fixed point in Ω.*

Now, we are in a position to establish the following result of the existence of $PAA(\mathbb{R}, X, \mu, \nu, r)$ mild solutions. The result is based upon nonlinear Leray-Schauder alternative theorem and the proof is similar as [4], one can see [4] for more details.

Theorem 3.7. *Assume (M_1), (M_2), (H_1), (H_{22}), $(\mathcal{N}_1)$-$(\mathcal{N}_3)$ hold and satisfies the following conditions:*

($\mathcal{K}_1$) *There exists a continuous nondecreasing function $W : [0, +\infty) \to [0, +\infty)$ such that $\|f(t, u_t)\| \leq W(\|u\|)$ for all $t \in \mathbb{R}, u \in X$.*

($\mathcal{K}_2$) *For each $\varpi \geq 0$,* $\displaystyle\lim_{|t|\to\infty} \frac{1}{h^*(t)} \int_{-\infty}^{t} \frac{W(\varpi h^*(s))}{1 + |\omega|((t-s)^{\alpha+1} + \gamma(t-s)^{\beta})} ds = 0.$

($\mathcal{K}_3$) *For each $\varepsilon > 0$, there exists $\delta > 0$ such that for $u, v \in C_{h^*}(\mathbb{R}, X)$, $\|u - v\|_{h^*} \leq \delta$ implies that*

$$\int_{-\infty}^{t} \frac{\|f(s, u_s) - \|f(s, v_s)\|}{1 + |\omega|((t-s)^{\alpha+1} + \gamma(t-s)^{\beta})} ds \leq \varepsilon, \qquad \textit{for all} \ \ t \in \mathbb{R}.$$

($\mathcal{K}_4$) *For all $a, b \in \mathbb{R}, a \leq b$ and $\lambda \geq 0$, the set $\{f(s, u_s) : a \leq s \leq b, u \in C_{h^*}(\mathbb{R}, X), \|u\|_{h^*} \leq \lambda\}$ is relatively compact in X.*

($\mathcal{K}_5$) $\displaystyle\liminf_{\xi\to\infty} \frac{\xi}{\Phi(\xi)} > 1$, *where*

$$\Phi(z) := C \left\| \int_{-\infty}^{t} \frac{W(zh^*(s))}{1 + |\omega|((t-s)^{\alpha+1} + \gamma(t-s)^{\beta})} ds \right\|_{h^*}, \qquad \textit{for} \ \ z \geq 0,$$

C is a constant given in (2.2).

Then (3.1) has a mild solution $u \in PAA(\mathbb{R}, X, \mu, \nu, r)$.

Proof. Define $\Gamma : C_{h^*}(\mathbb{R}, X) \to C(\mathbb{R}, X)$ by

$$(\Gamma u)(t) = \int_{-\infty}^{t} S_{\alpha,\beta}(t-s) f(s, u_s) ds, \qquad t \in \mathbb{R}.$$

Next, we prove that Γ has a fixed point in $PAA(\mathbb{R}, X, \mu, \nu, r)$ and divide the proof in several steps.

(i) For $u \in C_{h^*}(\mathbb{R}, X)$, by $(\mathcal{K}_1)$, one has

$$\begin{aligned}\frac{\|\Gamma u(t)\|}{h^*(t)} &\le \int_{-\infty}^{t} \|S_{\alpha,\beta}(t-s)\|\|f(s,u_s)\|ds \\ &\le \int_{-\infty}^{t} \frac{C\|f(s,u_s)\|}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^{\beta})}ds \\ &\le \int_{-\infty}^{t} \frac{CW(\|u(s)\|)}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^{\beta})}ds \\ &\le \int_{-\infty}^{t} \frac{CW(\|u\|_{h^*}h^*(s))}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^{\beta})}ds.\end{aligned}$$

It follows from $(\mathcal{K}_2)$ that $\Gamma : C_{h^*}(\mathbb{R}, X) \to C_{h^*}(\mathbb{R}, X)$.

(ii) Γ is continuous. In fact, for each $\varepsilon > 0$, by $(\mathcal{K}_3)$, there exits $\delta > 0$, for $u, v \in C_{h^*}(\mathbb{R}, X)$ and $\|u - v\|_{h^*} \le \delta$, one has

$$\begin{aligned}\|\Gamma u - \Gamma v\| &\le \int_{-\infty}^{t} \|S_{\alpha}(t-s)\|\|f(s,u_s) - f(s,v_s)\|ds \\ &\le \int_{-\infty}^{t} \frac{C\|f(s,u_s) - \|f(s,v_s)\|}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^{\beta})}ds \\ &\le C\varepsilon, \quad \text{for all} \ \ t \in \mathbb{R},\end{aligned}$$

take into account that $h^*(t) \ge 1$,

$$\frac{\|\Gamma u - \Gamma v\|}{h^*(t)} \le C\varepsilon,$$

which implies that $\|\Gamma u - \Gamma v\|_{h^*} \le C\varepsilon$, so Γ is continuous.

(iii) Γ is completely continuous. Set $B_r(Z)$ for the closed ball with center at 0 and radius $r > 0$ in the space Z. Let $V = \Gamma(B_{\varpi}(C_{h^*}(\mathbb{R}, X)))$ and $v = \Gamma(u)$ for $u \in B_{\varpi}(C_{h^*}(\mathbb{R}, X))$.

Initially, we prove that V is a relatively compact subset of X for each $t \in \mathbb{R}$. Let $\varepsilon > 0$, since $h^*(t) \to \infty$ as $|t| \to \infty$, it follow $(\mathcal{K}_2)$ that there exists $a \ge 0$ such that

$$C\int_{a}^{\infty} \frac{W(\varpi h^*(t-s))}{1+|\omega|(s^{\alpha+1}+\gamma s^{\beta})}ds \le \varepsilon.$$

Since

$$\begin{aligned}v(t) &= \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)f(s,u_s)ds \\ &= \int_{0}^{\infty} S_{\alpha,\beta}(s)f(t-s,u_{t-s})ds \\ &= \int_{0}^{a} S_{\alpha,\beta}(s)f(t-s,u_{t-s})ds + \int_{a}^{\infty} S_{\alpha,\beta}(s)f(t-s,u_{t-s})ds,\end{aligned}$$

and

$$\left\|\int_{a}^{\infty} S_{\alpha,\beta}(s)f(t-s,u_{t-s})ds\right\| \le C\int_{a}^{\infty} \frac{W(\varpi h^*(t-s))}{1+|\omega|(s^{\alpha+1}+\gamma s^{\beta})}ds \le \varepsilon,$$

hence $v(t) \in a\overline{c_0(N)} + B_\varepsilon(X)$, where $c_0(N)$ denotes the convex hull of N and $N = \{S_{\alpha,\beta}(s)f(\xi, u_\xi) : 0 \leq s \leq a, t-a \leq \xi \leq t, \|u\|_{h^*} \leq \lambda\}$. Using the fact that $S_{\alpha,\beta}(\cdot)$ is strong continuous and $(\mathcal{K}_4)$, we infer that N is a relatively compact set, and $V(t) \subset a\overline{c_0(N)} + B_\varepsilon(X)$ is also a relatively compact set.

Next, we show that V is equicontinuous. In fact, for each $\varepsilon > 0$, we can choose $a > 0$, $\delta_1 > 0$ such that

$$\begin{aligned}
&\left\| \int_0^\tau S_{\alpha,\beta}(\sigma) f(t+\tau-\sigma, u_{t+\tau-\sigma}) d\sigma + \int_a^\infty [S_{\alpha,\beta}(\sigma+\tau) - S_{\alpha,\beta}(\sigma)] f(t-\sigma, u_{t-\sigma}) d\sigma \right\| \\
&\leq C \left(\int_0^\tau \frac{W(\varpi h^*(t+\tau-\sigma))}{1+|\omega|(\sigma^{\alpha+1}+\gamma\sigma^\beta)} d\sigma + \int_a^\infty \frac{W(\varpi h^*(t-\sigma))}{1+|\omega|((\sigma+\tau)^{\alpha+1}+\gamma(\sigma+\tau)^\beta)} d\sigma \right. \\
&\quad \left. + \int_a^\infty \frac{W(\varpi h^*(t-\sigma))}{1+|\omega|(\sigma^{\alpha+1}+\gamma\sigma^\beta)} d\sigma \right) \\
&\leq \frac{\varepsilon}{2}, \qquad \text{for } \tau \leq \delta_1.
\end{aligned}$$

Moreover, since $\{f(t-\sigma, u_{t-\sigma}) : 0 < \sigma < a, u \in B_\varpi(C_{h^*}(\mathbb{R}, X))\}$ is a relatively compact set and $S_{\alpha,\beta}$ is strong continuous, we can choose $\delta_2 > 0$ such that

$$\|[S_{\alpha,\beta}(\sigma+\tau) - S_{\alpha,\beta}(\sigma)] f(t-\sigma, u_{t-\sigma})\| \leq \frac{\varepsilon}{2a}, \qquad \text{for } \tau \leq \delta_2.$$

Note that

$$\begin{aligned}
v(t+\tau) - v(t) &= \int_{-\infty}^{t+\tau} S_{\alpha,\beta}(t+\tau-s) f(s, u_s) ds - \int_{-\infty}^t S_{\alpha,\beta}(t-s) f(s, u_s) ds \\
&= \int_{-\infty}^t S_{\alpha,\beta}(t+\tau-s) f(s, u_s) ds + \int_t^{t+\tau} S_{\alpha,\beta}(t+\tau-s) f(s, u_s) ds \\
&\quad - \int_{-\infty}^t S_{\alpha,\beta}(t-s) f(s, u_s) ds \\
&= \int_{-\infty}^t [S_{\alpha,\beta}(t+\tau-s) - S_{\alpha,\beta}(t-s)] f(s, u_s) ds \\
&\quad + \int_t^{t+\tau} S_{\alpha,\beta}(t+\tau-s) f(s, u_s) ds \\
&= \int_0^\infty [S_{\alpha,\beta}(\sigma+\tau) - S_{\alpha,\beta}(\sigma)] f(t-\sigma, u_{t-\sigma}) d\sigma \\
&\quad + \int_0^\tau S_{\alpha,\beta}(\sigma) f(t+\tau-\sigma, u_{t+\tau-\sigma}) d\sigma \\
&= \int_0^a [S_{\alpha,\beta}(\sigma+\tau) - S_{\alpha,\beta}(\sigma)] f(t-\sigma, u_{t-\sigma}) d\sigma \\
&\quad + \int_a^\infty [S_{\alpha,\beta}(\sigma+\tau) - S_{\alpha,\beta}(\sigma)] f(t-\sigma, u_{t-\sigma}) d\sigma \\
&\quad + \int_0^\tau S_{\alpha,\beta}(\sigma) f(t+\tau-\sigma, u_{t+\tau-\sigma}) d\sigma,
\end{aligned}$$

then we have $\|v(t+\tau) - v(t)\| \leq \varepsilon$ for τ small enough and independent of $u \in B_\varpi(C_{h^*}(\mathbb{R}, X))$.

Finally, by $(\mathcal{K}_2)$, one has

$$\frac{\|v(t)\|}{h^*(t)} \leq \int_{-\infty}^{t} \frac{CW(\|u\|_{h^*}h^*(s))}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^{\beta})}ds \to 0 \quad \text{for} \ \ |t| \to \infty,$$

and this convergence is independent of $u \in B_{\varpi}(C_{h^*}(\mathbb{R},X))$. Hence, V is relatively compact set in $C_{h^*}(\mathbb{R},X)$ by Lemma 3.4.

(iv) If u^{λ} is a solution of the equation $u^{\lambda} = \lambda\Gamma(u^{\lambda})$ for some $0 < \lambda < 1$, then

$$\begin{aligned}\|u^{\lambda}\| &= \lambda \left\| \int_{-\infty}^{t} S_{\alpha,\beta}(t-s) f(s, u_s^{\lambda}) ds \right\| \\ &\leq \int_{-\infty}^{t} \frac{CW(\|u^{\lambda}\|_{h^*}h^*(s))}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^{\beta})}ds \\ &\leq \Phi(\|u^{\lambda}\|_{h^*})h^*(t).\end{aligned}$$

Hence, one has

$$\frac{\|u^{\lambda}\|_{h^*}}{\Phi(\|u^{\lambda}\|_{h^*})} \leq 1$$

and by $(\mathcal{K}_5)$, we conclude that the set $\{u^{\lambda} : u^{\lambda} = \lambda\Gamma(u^{\lambda}), \lambda \in (0,1)\}$ is bounded.

(v) It is easy to see that there exists $r_0 > 0$ such that $\Gamma(B_{r_0}(C_{h^*}(\mathbb{R},X))) \subset B_{r_0}(C_{h^*}(\mathbb{R},X))$. It follows from Lemma 3.1, Lemma 3.3 and Theorem 2.4 that

$$\Gamma(PAA(\mathbb{R},X,\mu,\nu,r)) \subseteq PAA(\mathbb{R},X,\mu,\nu,r),$$

consequently, we consider

$$\begin{aligned}\Gamma : &\overline{B_{r_0}(C_{h^*}(\mathbb{R},X)) \cap PAA(\mathbb{R},X,\mu,\nu,r)}^{C_{h^*}(\mathbb{R},X)} \\ &\to \overline{B_{r_0}(C_{h^*}(\mathbb{R},X)) \cap PAA(\mathbb{R},X,\mu,\nu,r)}^{C_{h^*}(\mathbb{R},X)}\end{aligned}$$

where $\overline{B}^{C_{h^*}(\mathbb{R},X)}$ denotes the closure of a set B in the space $C_{h^*}(\mathbb{R},X)$. Using (i)-(iii), we have that the map is completely continuous. By (iv) and Theorem 3.6, we deduce that Γ has a fixed point $u \in \overline{B_{r_0}(C_{h^*}(\mathbb{R},X)) \cap PAA(\mathbb{R},X,\mu,\nu,r)}^{C_{h^*}(\mathbb{R},X)}$.

Let u^n be a sequence in $B_{r_0}(C_{h^*}(\mathbb{R},X)) \cap PAA(\mathbb{R},X,\mu,\nu,r)$ such that it converges to u in the norm $C_{h^*}(\mathbb{R},X)$. For $\varepsilon > 0$, let $\delta > 0$ be the constant in $(\mathcal{K}_3)$, there exists $n_0 \in \mathbb{N}$ such that $\|u^n - u\|_{h^*} \leq \delta$ for all $n \geq n_0$. For $n \geq n_0$,

$$\begin{aligned}\|\Gamma u^n - \Gamma u\| &\leq \int_{-\infty}^{t} \|S_{\alpha,\beta}(t-s)\| \|f(s,u_s^n) - f(s,u_s)\| ds \\ &\leq C \int_{-\infty}^{t} \frac{\|f(s,u_s^n) - f(s,u_s)\|}{1+|\omega|((t-s)^{\alpha+1}+\gamma(t-s)^{\beta})}ds \leq C\varepsilon.\end{aligned}$$

Hence, Γu^n converges to $\Gamma u = u$ uniformly in $\mathbb{R}$. That is $u \in PAA(\mathbb{R},X,\mu,\nu,r)$ and completes the proof. □

Corollary 3.1. *Assume* (M_1), (M_2), (H_1), (H_{22}), $(\mathcal{N}_1)$-$(\mathcal{N}_3)$ *hold and satisfies the following conditions:*

(a_1) $f(t,0) = q(t)$.

(a_2) *f satisfies the Hölder type condition:*

$$\|f(t,\phi)-f(t,\psi)\| \le C_1\|\phi-\psi\|_{\mathcal{C}}^{\theta}, \qquad \phi,\psi\in\mathcal{C}, \quad t\in\mathbb{R},$$

where $0<\theta<1$, $C_1>0$ is a constant.

(a_3) *For all $a,b\in\mathbb{R}, a\le b$ and $\lambda\ge 0$, the set $\{f(s,u_s): a\le s\le b, u\in C_{h^*}(\mathbb{R},X), \|u\|_{h^*}\le\lambda\}$ is relatively compact in X.*

Then (3.1) *has a mild solution $u\in PAA(\mathbb{R},X,\mu,\nu,r)$.*

Proof. By (a_2), it is easy to see that $(\mathcal{N}_1)$ hold. Let $C_0=\|q\|$ and $W(\xi)=C_0+C_1\xi^\theta$, then $(\mathcal{K}_1)$ is satisfied. Take a function h^* such that $\sup\limits_{t\in\mathbb{R}}\int_{-\infty}^{t}\frac{h^*(s)^\theta}{1+|\omega|(t-s)^{\alpha+1}}ds := C_2<\infty$, it is not difficult to see that $(\mathcal{K}_2)$ is satisfied. To verify $(\mathcal{K}_3)$, note that for each $\varepsilon>0$, there exists $0<\delta<\left(\frac{\varepsilon}{C_1C_2}\right)^{1/\theta}$, such that for every $u,v\in C_{h^*}(\mathbb{R},X)$, $\|u-v\|_{h^*}\le\delta$ implies that

$$\begin{aligned}\int_{-\infty}^{t}\frac{\|f(s,u_s)-f(s,v_s)\|}{1+|\omega|((t-s)^{\alpha+1}+(t-s)^{\beta})}ds &\le \int_{-\infty}^{t}\frac{C_1h^*(s)^\theta\|u-v\|_{h^*}^\theta}{1+|\omega|((t-s)^{\alpha+1}+(t-s)^{\beta})}ds\\ &\le \int_{-\infty}^{t}\frac{C_1h^*(s)^\theta\|u-v\|_{h^*}^\theta}{1+|\omega|(t-s)^{\alpha+1}}ds\\ &\le C_1C_2\delta^\theta\le\varepsilon, \quad \text{for all } \ t\in\mathbb{R}.\end{aligned}$$

On the other hand, $(\mathcal{K}_5)$ can be easily verified using the definition of W. By Theorem 3.7, (3.1) has a mild solution $u\in PAA(\mathbb{R},X,\mu,\nu,r)$. □

4. FFDEs with infinite delay

In this section, we establish some sufficient criteria for the existence, uniqueness of $PAA(\mathbb{R},X,\mu,\nu,\infty)$ solutions for (3.1) with infinite delay.

In this work, we will employ an axiomatic definition of the phase space $\mathfrak{B}$ which is similar to the one introduced in [25]. More precisely, $\mathfrak{B}$ is a vector space of functions mapping $(-\infty,0]$ into X endowed with seminorm $\|\cdot\|_{\mathfrak{B}}$ such that the next axioms hold:

(A) If $x:(-\infty,\sigma+a)\to X$, $a>0$, $\sigma\in\mathbb{R}$ is continuous on $[\sigma,\sigma+a)$ and $x_\sigma\in\mathfrak{B}$, then for every $t\in[\sigma,\sigma+a)$, the following hold:

- (i) $x_t\in\mathfrak{B}$;
- (ii) $\|x(t)\|\le H\|x_t\|_{\mathfrak{B}}$;
- (iii) $\|x_t\|_{\mathfrak{B}}\le K(t-\sigma)\sup\{\|x(s)\|:\sigma\le s\le t\}+M(t-\sigma)\|x_\sigma\|_{\mathfrak{B}}$, where $H>0$ is a constant; $K,M:[0,\infty)\to[1,\infty)$, K is continuous, M is locally bounded and H,K,M are independent of $x(\cdot)$.

(A_1) For the function $x(\cdot)$ in (A), the function $t\to x_t$ is continuous from $[\sigma,\sigma+a)$ into $\mathfrak{B}$.

(B) The space $\mathfrak{B}$ is complete.

(C) If $(\varphi^n)_{n\in\mathbb{N}}$ is a bounded sequence in $C((-\infty,0],X)$ given by functions with compact support and $\varphi^n\to\varphi$ in the compact-open topology, then $\varphi\in\mathfrak{B}$ and $\|\varphi^n-\varphi\|_{\mathfrak{B}}\to 0$ as $n\to\infty$.

Definition 4.1 ([21]). Let $\mathfrak{B}_0 = \{\varphi \in \mathfrak{B} : \varphi(0) = 0\}$ and $S(t) : \mathfrak{B} \to \mathfrak{B}$ be the C_0-semigroup defined by $S(t)\varphi(\theta) = \varphi(0)$ on $[-t, 0]$ and $S(t)\varphi(\theta) = \varphi(t+\theta)$ on $(-\infty, -t]$. The phase space $\mathfrak{B}$ is called a fading memory space if $\|S(t)\varphi\|_{\mathfrak{B}} \to 0$ as $t \to \infty$ for every $\varphi \in \mathfrak{B}_0$. We said that $\mathfrak{B}$ is a uniform fading memory space if $\|S(t)\|_{L(\mathfrak{B}_0)} \to 0$ as $t \to \infty$.

Remark 4.1 ([21]). Assume that $\varsigma > 0$ such that $\|\varphi\|_{\mathfrak{B}} \leq \varsigma \sup_{\theta\leq 0} \|\varphi(\theta)\|$ for each $\varphi \in \mathfrak{B} \cap BC((-\infty, 0], X)$, see [25] for more details. Moreover, if $\mathfrak{B}$ is a fading memory, we assume that $\max\{K(t), M(t)\} \leq \mathcal{R}$ for all $t \geq 0$, see [25].

Lemma 4.1 ([25]). *The space $\mathfrak{B}$ is a uniform fading memory space if and only if axiom (C) holds, the function K is bounded and $\lim\limits_{t\to\infty} M(t) = 0$.*

Lemma 4.2. *If (M_1) holds, $u \in PAA(\mathbb{R}, X, \mu, \nu, \infty)$ and $\mathfrak{B}$ is a uniform fading memory space, then $u_t \in PAA(\mathbb{R}, \mathfrak{B}, \mu, \nu, \infty)$.*

Proof. Let $u = g + h$, where $h \in AA(\mathbb{R}, X)$, $h \in PAA_0(\mathbb{R}, X, \mu, \nu, \infty)$, then $u_t = g_t + h_t$ and clearly $g_t \in AA(\mathbb{R}, \mathfrak{B})$. Next, we show that $h_t \in PAA_0(\mathbb{R}, \mathfrak{B}, \mu, \nu, \infty)$. Let $r > 0, \varepsilon > 0$, since $\mathfrak{B}$ is a uniform fading memory space, by Lemma 4.1, there exists $\tau_\varepsilon > r$ such that $M(\tau) < \varepsilon$ for every $\tau > \tau_\varepsilon$. Hence, for $r > 0$ and $\tau > \tau_\varepsilon$, one has

$$
\begin{aligned}
&\frac{1}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{\theta\in[t-r,t]}\|h_\theta\|_{\mathfrak{B}}\right)d\mu(t)\\
&\leq \frac{1}{\nu([-T,T])}\int_{[-T,T]}\left[\sup_{\theta\in[t-r,t]} M(\theta-\sigma)\|h_\sigma\|_{\mathfrak{B}} + \sup_{\theta\in[t-r,t]} K(\theta-\sigma)\sup_{s\in[\sigma,\theta]}\|h(s)\|\right]d\mu(t)\\
&\leq \frac{1}{\nu([-T,T])}\int_{[-T,T]}\left[\sup_{\theta\in[t-r,t]} M(\tau)\|h_{\theta-\tau}\|_{\mathfrak{B}} + \sup_{\theta\in[t-r,t]} K(\tau)\sup_{s\in[\theta-\tau,\theta]}\|h(s)\|\right]d\mu(t)\\
&\leq \frac{\mu([-T,T])}{\nu([-T,T])}\varsigma\|h\|\varepsilon + \frac{\mathcal{R}}{\nu([-T,T])}\int_{[-T,T]}\left(\sup_{s\in[t-\tau-r,t]}\|h(s)\|\right)d\mu(t),
\end{aligned}
$$

which complete the proof since ε is arbitrary and $h \in PAA_0(\mathbb{R}, X, \mu, \nu, \infty) \subset PAA_0(\mathbb{R}, X, \mu, \nu, r+\tau)$. $\square$

Similar as the proof of Lemma 3.2 and Lemma 3.3, one has

Lemma 4.3. *If $(M_1), (M_2)$ hold and $f \in PAA(\mathbb{R}, X, \mu, \nu, \infty)$, then*

$$(\Lambda f)(t) = \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)f(s)ds \in PAA(\mathbb{R}, X, \mu, \nu, \infty), \qquad t \in \mathbb{R}.$$

Lemma 4.4. *Let $\{S(t)\}_{t\geq 0} \subset L(X)$ be a strongly continuous family of bounded and linear operators such that $\|S(t)\| \leq \phi(t), t \in \mathbb{R}^+$, where $\phi \in L^1(\mathbb{R}^+)$ is nonincreasing. If $f \in S^pPAA(\mathbb{R}, X, \mu, \nu, \infty)$, then*

$$(\Pi f)(t) = \int_{-\infty}^{t} S(t-s)f(s)ds \in PAA(\mathbb{R}, X, \mu, \nu, \infty), \qquad t \in \mathbb{R}.$$

Similar as the proof of Theorems 3.1-3.7 in Section 3, by Lemma 4.2, Lemma 4.3 and Lemma 4.4, one has the following results.

- PAA perturbation

Theorem 4.1. *Assume that* (M_1), (M_2), (H_1), (H_{23}), (H_{31}) *hold, where* $\mathfrak{B}$ *instead of* $\mathcal{C}$ *in* (H_{31}), *then* (3.1) *has a unique mild solution* $u \in PAA(\mathbb{R}, X, \mu, \nu, \infty)$, *if*

$$\varsigma C L_f |\omega|^{-1/(\alpha+1)} \pi < (\alpha+1)\sin(\pi/(\alpha+1)),$$

where ς *is defined as in Remark 4.1.*

Proof. Define the operator $\mathcal{F} : PAA(\mathbb{R}, X, \mu, \nu, \infty) \to PAA(\mathbb{R}, X, \mu, \nu, \infty)$ as in (3.4). By Lemma 4.2 and Theorem 2.5, one has $f(t, u_t) \in PAA(\mathbb{R}, X, \mu, \nu, \infty)$, hence $\mathcal{F}$ is well defined by Lemma 4.3. For any $u, v \in PAA(\mathbb{R}, X, \mu, \nu, \infty)$, one has

$$\begin{aligned}
\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\| &\le \int_{-\infty}^{t} \|S_{\alpha,\beta}(t-s)\| \|f(s, u_s) - f(s, v_s)\| ds \\
&\le L_f \int_{-\infty}^{t} \|S_{\alpha,\beta}(t-s)\| \|u_s - v_s\|_{\mathfrak{B}} ds \\
&\le \varsigma L_f \|u - v\| \left(\int_0^{\infty} \frac{C}{1 + |\omega|(s^{\alpha+1} + \gamma s^{\beta})} ds \right) \\
&\le \frac{\varsigma C L_f |\omega|^{-1/(\alpha+1)} \pi}{(\alpha+1)\sin(\pi/(\alpha+1))} \|u - v\|,
\end{aligned}$$

hence by the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $PAA(\mathbb{R}, X, \mu, \nu, \infty)$. □

Theorem 4.2. *Assume that* (M_1), (M_2), (H_1), (H_{23}), (H_{32}), $(\mathcal{I}_1)$, $(\mathcal{I}_3)$, $(\mathcal{I}_4)$ *hold, where* $\mathfrak{B}$ *instead of* $\mathcal{C}$ *in* (H_{32}), *then* (3.1) *has a unique mild solution* $u \in PAA(\mathbb{R}, X, \mu, \nu, \infty)$ *provided that*

$$\sup_{t \in \mathbb{R}} \int_{-\infty}^{t} \frac{L_f(s)}{1 + |\omega|(t-s)^{\alpha+1}} ds < \frac{1}{\varsigma C}.$$

- $S^p PAA$ perturbation

Theorem 4.3. *Assume that* (M_1), (M_2), (H_1), (H_{24}), (H_{31}), (H_4) *hold, where* $\mathfrak{B}$ *instead of* $\mathcal{C}$ *in* (H_{31}), *if* $\varsigma C L_f |\omega|^{-1/(\alpha+1)} \pi < (\alpha+1)\sin(\pi/(\alpha+1))$, *then* (3.1) *has a unique mild solution* $u \in PAA(\mathbb{R}, X, \mu, \nu, \infty)$.

Theorem 4.4. *Assume that* (M_1), (M_2), (H_1), (H_{24}), (H_{33}), $(\mathcal{J}_1)$, $(\mathcal{J}_3)$ *hold, where* $\mathfrak{B}$ *instead of* $\mathcal{C}$ *in* (H_{33}), *if*

$$\varsigma C \left(1 + \frac{|\omega|^{-1/(\alpha+1)} \pi}{(\alpha+1)\sin(\pi/(\alpha+1))} \right) \|L_f\|_{S^p} < 1,$$

then (3.1) *has a unique mild solution* $u \in PAA(\mathbb{R}, X, \mu, \nu, \infty)$.

Theorem 4.5. *Assume that the conditions* (M_1), (M_2), (H_1), (H_{24}), (H_{34}), $(\mathcal{J}_1)$, $(\mathcal{J}_3)$ *hold, where* $\mathfrak{B}$ *instead of* $\mathcal{C}$ *in* (H_{34}), *then* (3.1) *has a unique mild solution* $u \in PAA(\mathbb{R}, X, \mu, \nu, \infty)$.

- Non-Lipschitz case

Theorem 4.6. *Assume* (M_1), (M_2), (H_1), (H_{24}), $(\mathcal{N}_1)$-$(\mathcal{N}_3)$, $(\mathcal{K}_1)$-$(\mathcal{K}_5)$ *hold, then* (3.1) *has a mild solution* $u \in PAA(\mathbb{R}, X, \mu, \nu, \infty)$.

5. Example

Consider the following fractional partial differential equation with delay

$$D_t^{\alpha+1}u(t,x)+\gamma D_t^{\beta}u(t,x)=\frac{\partial^2}{\partial x^2}u(t,x)-\delta u(t,x)+D_t^{\alpha}\left[a(t)\int_{-1}^{0}b(s)\sin[u(t+s,x)]ds\right], \tag{5.1}$$

with initial and zero boundary conditions: $u(0,t)=u(1,t)=0$, where $t\in\mathbb{R}, x\in[0,1]$, $0<\alpha\leq\beta\leq 1$, $\gamma>0$, $\delta>0$.

Let $X=(L^2([0,1],\mathbb{R}),\|\cdot\|_{L^2})$ and define the operator A on X by

$$Au=\frac{\partial^2}{\partial x^2}u-\delta u$$

with

$$\mathcal{D}(A)=\{u\in L^2([0,1],\mathbb{R}): u''\in L^2[0,1], u(0)=u(1)=0\},$$

and

$$f(t,\phi)(x)=a(t)\int_{-1}^{0}b(s)\sin[\phi(s)(x)]ds,\ \ t\in\mathbb{R},\ \ \phi\in\mathcal{C}_1:=C([-1,0],X),\ \ x\in[0,1].$$

It is well know that A is a ω-sectorial operator with $\omega=-\delta<0$ and angle $\pi/2$ (and hence of angle $\beta\pi/2$ with $\beta\leq 1$) [27]. Let $u(t)=u(t,\cdot)$, then (5.1) can be rewritten as an abstract system of the form (3.1).

(i) Let

$$f(t,\phi)(x)=a(t)\int_{-1}^{0}b(s)\sin[\phi(s)(x)]ds,\ \ t\in\mathbb{R},\ \ \phi\in\mathcal{C}_1,\ \ x\in[0,1],$$

where $a(t)\in PAA(\mathbb{R},\mathbb{R},\mu,\nu,1)$, $\mu=\nu$ and suppose that its Radon-Nikodym derivative is given

$$\rho(t)=\begin{cases}e^t, & \text{if}\ \ t\leq 0,\\ 1, & \text{if}\ \ t>0,\end{cases}$$

then $\mu,\nu\in\mathcal{M}$ and satisfy (M_1), (M_2) [6]. In addition, since

$$\|f(t,\phi)-f(t,\psi)\|\leq|a|\left(\int_{-1}^{0}|b(s)|^2ds\right)^{1/2}\|\phi-\psi\|_{\mathcal{C}_1},\ \ \text{for all}\ \ \phi,\psi\in\mathcal{C}_1,$$

so (H_{31}) holds with $L_f\equiv|a|\left(\int_{-1}^{0}|b(s)|^2ds\right)^{1/2}$. By Theorem 3.1, we conclude that (5.1) has a unique solution $u\in PAA(\mathbb{R},\mathbb{R},\mu,\nu,1)$ if $CL_f\delta^{-1/(\alpha+1)}\pi<(\alpha+1)\sin(\pi/(\alpha+1))$.

(ii) Let

$$f(t,\phi)(s)=m(t)\sin(\phi(s))+m(t)e^{-t}\cos(\phi(s)),\qquad \phi\in\mathcal{C}_1,$$

where

$$m(t)=\begin{cases}\sin\left(\dfrac{1}{\cos n+\cos\pi n+2}\right), & t\in(n-\varepsilon,n+\varepsilon),\ n\in\mathbb{Z},\\ 0, & \text{otherwise},\end{cases}$$

for some $\varepsilon \in (0, a)$ and

$$a = \min\{1/2, (\alpha+1)\sin(\pi/(\alpha+1))/[4C((\alpha+1)\sin(\pi/(\alpha+1)) + |\delta|^{-1/\alpha}\pi)]\}.$$

By [30], $m(t) \in S^2AA(\mathbb{R}, \mathbb{R})$, then $m(t)\sin\phi \in S^2AA(\mathbb{R} \times \mathcal{C}_1, \mathbb{R})$, whence $f \in S^2PAA(\mathbb{R} \times \mathcal{C}_1, \mathbb{R}, \mu, \nu, 1)$, where $\mu = \nu$ and its Radon-Nikodym derivative is given by $\rho(t) = e^t$. In addition, for each $t \in \mathbb{R}$ and $\phi, \psi \in \mathcal{C}_1$, one has

$$\begin{aligned}\|m(t)\sin\phi - m(t)\sin\psi\|_{L^2} &= \left(\int_0^1 |m(t)\sin(\phi(s)) - m(t)\sin(\psi(s))|^2 ds\right)^{1/2} \\ &\leq |m(t)| \|\phi - \psi\|_{L^2},\end{aligned}$$

and

$$\begin{aligned}\|f(t, \phi) - f(t, \psi)\|_{L^2} &\leq \left(\int_0^1 |m(t)\sin(\phi(s)) - m(t)\sin(\psi(s))|^2 ds\right)^{1/2} \\ &+ \left(\int_0^1 |m(t)e^{-t}\cos(\phi(s)) - m(t)e^{-t}\cos(\psi(s))|^2 ds\right)^{1/2} \leq 2|m(t)| \|\phi - \psi\|_{L^2}.\end{aligned}$$

Since

$$\||m(\cdot)|\|_{S^1} = \sup_{t \in \mathbb{R}} \int_t^{t+1} |m(s)| ds \leq 2\varepsilon,$$

then

$$\begin{aligned}&C\left(1 + \frac{\delta^{-1/(\alpha+1)}\pi}{(\alpha+1)\sin(\pi/(\alpha+1))}\right)\|L_f\|_{S^1} \\ &= C\left(1 + \frac{\delta^{-1/(\alpha+1)}\pi}{(\alpha+1)\sin(\pi/(\alpha+1))}\right) \cdot 2\|m(\cdot)\|_{S^1} \\ &\leq 4C\varepsilon\left(1 + \frac{\delta^{-1/(\alpha+1)}\pi}{(\alpha+1)\sin(\pi/(\alpha+1))}\right) < 1.\end{aligned}$$

By Theorem 3.4, there exists a unique $PAA(\mathbb{R}, X, \mu, \nu, 1)$ mild solution to (5.1).

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