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SYMMETRY ANALYSIS, CONSERVATION LAWS OF A TIME FRACTIONAL FIFTH-ORDER SAWADA-KOTERA EQUATION

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Abstract In this paper, we intend to study the symmetry properties and conservation laws of a time fractional fifth-order Sawada-Kotera (S-K) equation with Riemann-Liouville derivative. Applying the well-known Lie symmetry method, we analysis the symmetry properties of the equation. Based on this, we find that the S-K equation can be reduced to a fractional ordinary differential equation with Erdelyi-Kober derivative by the similarity variable and transformation. Furthermore, we construct some conservation laws for the S-K equation using the idea in the Ibragimov theorem on conservation laws and the fractional generalization of the Noether operators.

Keywords S-K equation, Lie symmetry, conservation laws.

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1. Introduction

In recent years, the study of fractional differential equations has attracted more and more attention in various fields [1, 2, 19]. Not only the theoretical studies of fractional differential equations are interesting, but also their applications are more and more extensively, such as establishing a variety of models for the natural sciences (see [16]). Such equations have recently been proved to be valuable tools in the modelling of many physical phenomena [3, 13].

In this paper, we consider the following time-fractional Sawada-Kotera (S-K) equation

$$D_t^\alpha u + 5u^2 u_x + 5u_x u_{xx} + 5uu_{xxx} + u_{xxxxx} = 0, \tag{1.1}$$

where D_t^α is the Riemann-Liouville fractional derivative of u with respect to t, $\alpha > 0$ is the parameter of the fractional derivative.

The classical Sawada-Kotera equation is an extraordinary unidirectional nonlinear evolution equation, which plays an important role in the field of mathematical models [17], including describing motion of long waves in shallow water under gravity and one-dimensional nonlinear lattice [4]. In recent years, it has been extensively studied [18, 20], such as its Bäcklund transformation, Darboux transformation, bi-Hamiltonian structure, multisoliton solutions and so on. However, to the best of our knowledge, the study of the fractional order S-K equation (1.1) is still on the initial stage, it is meaningful to study the time fractional S-K equation. In this paper, we will investigate the symmetries and conservation laws of (1.1).

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The remainder of this paper is organized as follows. In Section 2, we present some basic definitions and results which will be useful in the following discusses. In Section 3, by means of symmetry analysis, we get the symmetries of S-K equation. As by-product, the similarity transformation and the similarity variable are also obtained by solving a characteristic equation. Based on these, we perform the process of the symmetry reduction. In Section 4, some conservation laws of equation (1.1) are obtained by the idea in the Ibragimov theorem on conservation laws [11] and the fractional generalization of the Noether operators [12].

2. Preliminaries

To begin with, we briefly present some definitions and results which will be used. First, let us recall the definition of the Riemann-Liouville fractional derivative (see [5, 12, 21, 22] and the references therein) as follows:

$$D_t^\alpha u(t,x) = \begin{cases} \dfrac{\partial^n u}{\partial t^n}, & \alpha = n \in \mathbb{N}, \\ \dfrac{1}{\Gamma(n-\alpha)} \dfrac{\partial^n}{\partial t^n} \displaystyle\int_0^t \dfrac{u(s,x)}{(t-s)^{\alpha+1-n}} ds, & n-1 < \alpha < n,\ n \in \mathbb{N}, \end{cases}$$

where $\Gamma(z)$ is the Euler gamma function.

The Riemann-Liouville left-sided time-fractional derivative [10] is given by

$${}_0D_t^\alpha u = D_t^n({}_0I_t^{n-\alpha} u),$$

where D_t is the operator of the derivative with respect to t, $n = [\alpha]+1$, and ${}_0I_t^{n-\alpha}u$ is the left-sided time-fractional integral of order $n-\alpha$, it is defined as

$$({}_0I_t^{n-\alpha}u)(t,x) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u(s,x)}{(t-s)^{\alpha+1-n}} ds.$$

We denote that ${}_tI_T^{n-\alpha}$ is the right-sided operator of $n-\alpha$ order fractional integration [12], the form is given as follows:

$$({}_tI_T^{n-\alpha}f)(t,x) = \frac{1}{\Gamma(n-\alpha)} \int_t^T \frac{f(s,x)}{(s-t)^{\alpha+1-n}} ds.$$

Now we give a brief review of the Lie symmetry analysis concerning the fractional differential operator which is a nonlocal operator. For a time fractional differential equations with two independent variables

$$F(t, x, u, u_x, u_{xx}, \cdots, D_t^\alpha u(t,x)) = 0, \quad \alpha > 0, \tag{2.1}$$

we consider a one parameter Lie group of point transformations

$$\begin{aligned} t^* &= t + \varepsilon\tau(x,t,u) + O(\varepsilon^2), \\ x^* &= x + \varepsilon\xi(x,t,u) + O(\varepsilon^2), \\ u^* &= u + \varepsilon\eta(x,t,u) + O(\varepsilon^2), \\ \frac{\partial^\alpha u^*}{\partial t^{*\alpha}} &= \frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon\eta^{\alpha t}(x,t,u) + O(\varepsilon^2), \\ \frac{\partial^k u^*}{\partial x^{*k}} &= \frac{\partial^k u}{\partial x^k} + \varepsilon\eta^{kx}(x,t,u) + O(\varepsilon^2), \end{aligned} \tag{2.2}$$

where ε is a group parameter, ξ, τ, η are infinitesimals, η^{kx}, $\eta^{\alpha t}$ are extended infinitesimals and the associated generator has the form:

$$V = \tau(x,t,u)\partial_t + \xi(x,t,u)\partial_x + \eta(x,t,u)\partial_u. \tag{2.3}$$

Let us illuminate the extended infinitesimals in what follows.

From the classical Lie group analysis [15], we know that η^{kx} $(k = 1, 2, \cdots)$ are given by

$$\begin{aligned} \eta^{x} &= D_x\eta - u_x D_x(\xi) - u_t D_t(\tau), \\ \eta^{xx} &= D_x\eta^{x} - u_{xx} D_x(\xi) - u_{xt} D_t(\tau), \\ \eta^{kx} &= D_x\eta^{(k-1)x} - u_{kx} D_x(\xi) - u_{(k-1)x,t} D_t(\tau), \end{aligned} \tag{2.4}$$

where D_x is the total derivative operator with respect to x.

For $\eta^{\alpha t}$, according to [5], the α-th order extended infinitesimal has the form:

$$\begin{aligned} \eta^{\alpha t} =& D_t^{\alpha}(\eta) + \xi D_t^{\alpha}(u_x) - D_t^{\alpha}(\xi u_x) + D_t^{\alpha}(D_t(\tau)u) \\ & - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u), \end{aligned} \tag{2.5}$$

where D_t^{α} denotes the total fractional derivative operator. Recalling the generalized Leibnitz rule [14]

$$D_t^{\alpha}[u(t)v(t)] = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n} u(t) D_t^{n} v(t), \qquad \alpha > 0,$$

where

$$\binom{\alpha}{n} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)},$$

we find that (2.5) becomes

$$\begin{aligned} \eta^{\alpha t} = & D_t^{\alpha}(\eta) - \alpha D_t(\tau)\frac{\partial^{\alpha} u}{\partial t^{\alpha}} - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^{n}(\xi) D_t^{\alpha-n}(u_x) \\ & - \sum_{n=1}^{\infty} \binom{\alpha}{n+1} D_t^{n+1}(\tau) D_t^{\alpha-n}(u). \end{aligned} \tag{2.6}$$

According to the chain rule for composite functions

$$\frac{d^m f(g(t))}{dt^m} = \sum_{k=0}^{m}\sum_{r=0}^{k} \binom{k}{r} \frac{1}{k!}[-g(t)]^r \frac{d^m}{dt^m}[g(t)^{k-r}] \frac{d^k f(g)}{dg^k},$$

and generalized Leibnitz rule, we can write

$$D_t^{\alpha}(\eta) = \frac{\partial^{\alpha}\eta}{\partial t^{\alpha}} + \eta_u \frac{\partial^{\alpha} u}{\partial t^{\alpha}} - u\frac{\partial^{\alpha}\eta_u}{\partial t^{\alpha}} + \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{\partial^{n}\eta_u}{\partial t^{n}} D_t^{\alpha-n}(u) + \mu, \tag{2.7}$$

where

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1} \binom{\alpha}{n}\binom{n}{m}\binom{k}{r} \frac{1}{k!} \frac{t^{n-\alpha}(-u)^r}{\Gamma(n+1-\alpha)} \frac{\partial^m}{\partial t^m}(u^{k-r}) \frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}.$$

Consequently, we obtain that

$$\eta^{\alpha t} = \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x) + \sum_{n=1}^{\infty}\Big[\binom{\alpha}{n}\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \binom{\alpha}{n+1} D_t^{n+1}(\tau)\Big] D_t^{\alpha-n}(u). \tag{2.8}$$

By the infinitesimal invariance criterion [5], we can see that

$$pr^{(\alpha,t)}V(F)|_{F=0} = 0, \tag{2.9}$$

where $pr^{(\alpha,t)}V$ is the prolongation of the generator V given by

$$pr^{(\alpha,t)}V = V + \partial_{D_t^\alpha u} + \eta^x \partial_{u_x} + \eta^{xx}\partial_{u_{xx}} + \cdots + \eta^{\alpha t}.$$

We will analysis the symmetry properties of the time fractional S-K equation using the above method in the next section.

3. Lie symmetries of S-K equation (1.1)

We assume that the time fractional S-K equation (1.1) is invariant under the one parameter Lie group of point transformations (2.2), and the associated generator has the form (2.3). The fifth order prolongation of the generator V is

$$pr^{(\alpha,t)}V = V + \eta^{\alpha t}\partial_{D_t^\alpha u} + \eta^x \partial_{u_x} + \eta^{xx}\partial_{u_{xx}} + \eta^{3x}\partial_{u_{3x}} + \eta^{4x}\partial_{u_{4x}} + \eta^{5x}\partial_{u_{5x}}. \tag{3.1}$$

For the S-K equation, one inserts (3.1) into (2.9) and gets

$$\eta^{\alpha t} + 10\eta u u_x + 5u^2\eta^x + 5\eta^x u_{xx} + 5\eta^{xx}u_x + 5\eta u_{xxx} + 5u\eta^{xxx} + \eta^{xxxxx} = 0. \tag{3.2}$$

Substituting (2.4), (2.8) into (3.2), separating with respect to derivatives of u and equating the coefficients of various powers of partial derivatives of u to zero we obtain the following linear over-determined system of differential equations

$$\begin{aligned}
&\xi_u = \tau_u = \xi_t = \tau_x = \eta_{uu} = 0,\\
&D_t^\alpha \eta - u D_t^\alpha \eta_u + 5u^2\eta_x + 5u\eta_{xxx} + \eta_{xxxxx} = 0,\\
&10\eta u + 5u^2(\alpha\tau_t - \xi_x) + 5\eta_{xx} + 5u(3\eta_{xxu} - \xi_{xxx}) + 5\eta_{xxxxu} - \xi_{xxxxx} = 0,\\
&5\eta + 5u(\alpha\tau_t - 3\xi_x) - 3\tau_{xxu} + 10\eta_{xxu} - 10\xi_{xxx} = 0,\\
&\alpha\tau_t - 5\xi_x = 0, \eta_{xu} - 2\xi_{xx} = 0,\\
&\binom{\alpha}{n} D_t^\alpha \eta_u - \binom{\alpha}{n+1} D_t^{n+1}\tau = 0, n = 1, 2, \ldots .
\end{aligned} \tag{3.3}$$

With the help of Maple, after some tedious and lengthy calculations, we can find some solutions of the above system as follows:

$$\xi = c_1 x + c_2, \quad \tau = \frac{5c_1 t}{\alpha}, \quad \eta = -2c_1 u,$$

where c_1, c_2 are arbitrary constants. So we obtain the symmetry generator admitted by (1.1)

$$V = (c_1x + c_2)\frac{\partial}{\partial x} + \frac{5c_1t}{\alpha}\frac{\partial}{\partial t} - 2c_1u\frac{\partial}{\partial u}.$$

This implies that (1.1) has two Lie point symmetry operators

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = x\frac{\partial}{\partial x} + \frac{5t}{\alpha}\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}. \tag{3.4}$$

In particular, for the symmetry operator V_2, the corresponding characteristic equation is

$$\frac{dx}{x} = \frac{\alpha dt}{5t} = \frac{du}{-2u}.$$

Solving it gives the similarity variable and transformation

$$\rho = xt^{-\frac{\alpha}{5}}, \quad u = t^{-\frac{2\alpha}{5}}g(\rho), \tag{3.5}$$

where $g(\rho)$ is an arbitrary differential function of ρ. Therefore, we have the following result.

Theorem 3.1. *Upon the similarity transformation $u = t^{-\frac{2\alpha}{5}}g(\rho)$ with $\rho = xt^{-\frac{\alpha}{5}}$, S-K equation (1.1) can be reduced to a fractional ordinary differential equation as follows*

$$\left(P^{1-\frac{7\alpha}{5},\alpha}_{\frac{5}{\alpha}}g\right)(\rho) + 5g^2g_\rho + 5g_\rho g_{\rho\rho} + 5gg_{\rho\rho\rho} + g_{\rho\rho\rho\rho\rho} = 0, \tag{3.6}$$

with the Erdelyi-Kober fraction differential operator

$$(P^{\tau,\alpha}_\beta g)(\rho) := \prod_{j=0}^{n-1}(\tau + j - \frac{1}{\beta}\rho\frac{d}{d\rho})(K^{\tau+\alpha,n-\alpha}_\beta g)(\rho), \quad n = \begin{cases} [\alpha]+1, & \alpha \notin \mathbb{N}, \\ \alpha, & \alpha \in \mathbb{N}, \end{cases}$$

where

$$(K^{\tau,\alpha}_\beta g)(\rho) := \begin{cases} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^\infty (v-1)^{\alpha-1}v^{-(\tau+\alpha)}g(\rho v^{\frac{1}{\beta}})dv, & \alpha > 0, \\ g(\rho), & \alpha = 0. \end{cases}$$

is the Erdelyi-Kober fractional integral operator.

Proof. According to the Riemann-Liouville fractional derivative and the similarity transformation, we get

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}s^{-\frac{2\alpha}{5}}g(xs^{-\frac{\alpha}{5}})ds\right]. \tag{3.7}$$

Let $v = \frac{t}{s}$, then we have $ds = -\frac{t}{v^2}dv$. Then, (3.7) can be transformed to

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\partial^n}{\partial t^n}\left[t^{n-\frac{7\alpha}{5}}\frac{1}{\Gamma(n-\alpha)}\int_1^\infty (v-1)^{n-\alpha-1}v^{-\left(n-\frac{7\alpha}{5}+1\right)}g\left(\rho v^{\frac{\alpha}{5}}\right)dv\right] \\ &= \frac{\partial^n}{\partial t^n}\left[t^{n-\frac{7\alpha}{5}}(K^{1-\frac{2\alpha}{5},n-\alpha}_{\frac{5}{\alpha}}g)(\rho)\right].\end{aligned} \tag{3.8}$$

Taking advantage of the relation $\rho = xt^{-\frac{\alpha}{5}}$ and supposing that ϕ is an arbitrary differential function of ρ, we obtain that

$$t\frac{\partial}{\partial t}\phi(\rho) = tx\Big(-\frac{\alpha}{5}\Big)t^{-\frac{\alpha}{5}-1}\phi^{'}(\rho) = -\frac{\alpha}{5}\rho\frac{d}{d\rho}\phi(\rho).$$

Note that

$$\begin{aligned}
&\frac{\partial^n}{\partial t^n}\Big[t^{n-\frac{7\alpha}{5}}\Big(K^{1-\frac{2\alpha}{5},n-\alpha}_{\frac{5}{\alpha}}g\Big)(\rho)\Big]\\
=&\frac{\partial^{n-1}}{\partial t^{n-1}}\Big[\frac{\partial}{\partial t}\Big(t^{n-\frac{7\alpha}{5}}\Big(K^{1-\frac{2\alpha}{5},n-\alpha}_{\frac{5}{\alpha}}g\Big)(\rho)\Big)\Big]\\
=&\frac{\partial^{n-1}}{\partial t^{n-1}}\Big[t^{n-\frac{7\alpha}{5}-1}\Big(n-\frac{7\alpha}{5}-\frac{\alpha}{5}\rho\frac{d}{d\rho}\Big)\Big(K^{1-\frac{2\alpha}{5},n-\alpha}_{\frac{5}{\alpha}}g\Big)(\rho)\Big],
\end{aligned}$$

then repeating above ways $n-1$ times yields that

$$\begin{aligned}
&\frac{\partial^n}{\partial t^n}\Big[t^{n-\frac{7\alpha}{5}}\Big(K^{1-\frac{2\alpha}{5},n-\alpha}_{\frac{5}{\alpha}}g\Big)(\rho)\Big]\\
=&\,t^{n-\frac{7\alpha}{5}}\prod_{j=0}^{n-1}\Big(1-\frac{7\alpha}{5}+j-\frac{\alpha}{5}\rho\frac{d}{d\rho}\Big)\Big(K^{1-\frac{2\alpha}{5},n-\alpha}_{\frac{5}{\alpha}}g\Big)(\rho).
\end{aligned} \tag{3.9}$$

Consequently, we arrive at

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-\frac{7\alpha}{5}}\left(P^{1-\frac{7\alpha}{5},\alpha}_{\frac{5}{\alpha}}g\right)(\rho).$$

Thus, we get the following fractional order ordinary differential S-K equation

$$\left(P^{1-\frac{7\alpha}{5},\alpha}_{\frac{5}{\alpha}}g\right)(\rho) + 5g^2g_\rho + 5g_\rho g_{\rho\rho} + 5gg_{\rho\rho\rho} + g_{\rho\rho\rho\rho\rho} = 0.$$

This completes the proof. □

4. Conservation laws

In this section, we construct conservation laws of (1.1) by Ibragimov theorem [11]. This theorem is usually applied to integer order differential equations, see [7,9] and the references therein. Moreover, it is also applicable for some fractional differential equations [12, 21].

The formal Lagrangian function for (1.1) is given by

$$L = v(x,t)(D^\alpha_t u + 5u^2u_x + 5u_xu_{xx} + 5uu_{xxx} + u_{xxxxx}), \tag{4.1}$$

where $v(x,t)$ is a new dependent variable. Now, let us recall the form of the Euler-Lagrange operator

$$\frac{\delta}{\delta u} = \partial_u + (D^\alpha_t)^*\partial_{D^\alpha_t} - D_x\partial_{u_x} + D^2_x\partial_{u_{xx}} - D^3_x\partial_{u_{xxx}} + D^4_x\partial u_{xxxx} - D^5_x\partial_{u_{xxxxx}}, \tag{4.2}$$

where $(D^\alpha_t)^*$ is the adjoint operator of D^α_t and it has the form:

$$(D^\alpha_t)^* = (-1)^n{}_tI^{n-\alpha}_T(D^n_t).$$

So, we get the adjoint equation to (1.1) as the Euler-Lagrange equation

$$\frac{\delta L}{\delta u} = \left(D_t^{\alpha}\right)^{*} v - 5u^2 v_x - 10u_{xx}v_x - 10u_x v_{xx} - 5uv_{xxx} - v_{xxxxx} = 0. \tag{4.3}$$

Nonlinear self-adjointness of differential equations was presented in [8,11] and the references therein for constructing conservation laws. This concept can be extended to fractional differential equations. We say that S-K equation (1.1) is nonlinearly self-adjoint if (4.3) is satisfied for all solutions u of S-K equation (1.1) upon a substitution $v = \varphi(t,x,u)$, and $\varphi(t,x,u) \neq 0$ [6].

Let us substitute the function $v = \varphi(t,x,u)$ and its derivatives into (4.3) and take into account the self-adjoint condition

$$\frac{\delta L}{\delta u}\Big|_{v=\varphi(x,t,u)} = \lambda E,$$

for a certain function λ, where $E = D_t^{\alpha}u + 5u^2u_x + 5u_xu_{xx} + 5uu_{xxx} + u_{xxxxx}$. Then it is not difficult to find that

$$\varphi(x,t,u) = c\psi(x) \tag{4.4}$$

is a substitution function of the adjointed system (1.1) and (4.3), where c is arbitrary constant except zero and $\psi(x)$ satisfies

$$\psi^{(5)} + 5u\psi^{(3)} + 10u_x\psi^{''} + 10u_{xx}\psi^{'} + 5u^2\psi^{'} = 0.$$

This tells us S-K equation (1.1) is nonlinear self-adjoint with the substitution (4.4).

Note that the S-K equation does not involve fractional derivatives with respect to x, so the x-component of conserved vector can be given by the formula for differential equations of integer orders [11], that is

$$\begin{aligned} C_i^x = \xi L &+ W_i\left(\frac{\partial L}{\partial u_x} - D_x\frac{\partial L}{\partial u_{xx}} + D_x^2\frac{\partial L}{\partial u_{xxx}} - D_x^3\frac{\partial L}{\partial u_{xxxx}} + D_x^4\frac{\partial L}{\partial u_{xxxxx}}\right) \\ &+ D_x\left(W_i\right)\left(\frac{\partial L}{\partial u_{xx}} - D_x\frac{\partial L}{\partial u_{xxx}} + D_x^2\frac{\partial L}{\partial u_{xxxx}} - D_x^3\frac{\partial L}{\partial u_{xxxxx}}\right) \\ &+ D_x^2\left(W_i\right)\left(\frac{\partial L}{\partial u_{xxx}} - D_x\frac{\partial L}{\partial u_{xxxx}} + D_x^2\frac{\partial L}{\partial u_{xxxxx}}\right) \\ &+ D_x^3\left(W_i\right)\left(\frac{\partial L}{\partial u_{xxxx}} - D_x\frac{\partial L}{\partial u_{xxxxx}}\right) + D_x^4\left(W_i\right)\frac{\partial L}{\partial u_{xxxxx}}, \end{aligned}$$

where $W_i = \eta_i - \tau_i u_t - \xi_i u_x$ $(i = 1,2)$ are the Lie characteristic functions corresponding to the Lie symmetries V_1 and V_2. In view of (4.1) and (4.4), we can reduce the component C_i^x to

$$\begin{aligned} C_i^x = {} & cW_i\left(5\psi u^2 + 5\psi^{''}u + 5\psi^{'}u_x + 5\psi u_{xx} + \psi^{(4)}\right) - cD_x\left(W_i\right)\left(5\psi^{'}u + \psi^{(3)}\right) \\ & + cD_x^2\left(W_i\right)\times\left(5\psi u + \psi^{''}\right) - cD_x^3\left(W_i\right)\psi^{'} + cD_x^4\left(W_i\right)\psi. \end{aligned} \tag{4.5}$$

By the fractional generalizations of the Noether operators (see [12] for the details), the t-component of conserved vector can be given by

$$C_i^t = \sum_{k=0}^{n-1}(-1)^k\left({}_0D_t^{\alpha-1-k}\right)\left(W_i\right)D_t^k\frac{\partial L}{\partial({}_0D_t^{\alpha}u)} - (-1)^n J\left(W_i, D_t^n\frac{\partial L}{\partial({}_0D_t^{\alpha}u)}\right),$$

where $n=[\alpha]+1$, $i=1,2$ and J is the integral

$$J(f,g)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T\frac{f(s,x)g(\mu,x)}{(\mu-s)^{\alpha+1-n}}d\mu ds.$$

For S-K equation (1.1), using (4.1) and (4.4), the component C_i^t can be reduced to

$$\begin{aligned}C_i^t&=\tau L+(-1)^0\ {}_0D_t^{\alpha-1}(W_i)\,D_t^0\frac{\partial L}{\partial_0 D_t^\alpha u}-(-1)^1 J\left(W_i,D_t^1\frac{\partial L}{\partial_0 D_t^\alpha u}\right)\\&=c({}_0D_t^{\alpha-1})\,(W_i)\,\psi.\end{aligned}\tag{4.6}$$

Recalling Lie point symmetries (3.4), W_i can be expressed as

$$W_1=-u_x,\quad W_2=-2u-\frac{5t}{\alpha}u_t-xu_x.$$

Therefore, for Lie point symmetry $V_1=\frac{\partial}{\partial_x}$, substituting W_1 into (4.5) and (4.6), one can easily get the components of conservation law for (1.1) as follows:

$$\begin{aligned}C_1^x&=-u_x(5\psi u^2+5\psi^{''}u+5\psi^{'}u_x+5\psi u_{xx}+\psi^{(4)})+u_{xx}(5\psi^{'}u+\psi^{(3)})\\&\quad-u_{xxx}\times(5\psi u+\psi^{''})+u_{xxxx}\psi^{'}+u_{xxxxx}\psi,\\C_1^t&=-\psi\ {}_0D_t^{\alpha-1}u_x.\end{aligned}$$

Similarly, for the symmetry operator V_2, the components C_2^x, C_2^t of conservation law for (1.1) can be obtained by the same way, we omit the details.

5. Conclusions

In this paper, we extend the classical Lie symmetry analysis method to fractional differential equations and investigate the symmetry properties, conservation laws for the time-fractional fifth-order Sawada-Kotera with Riemann-Liouville derivative. By the symmetries of (1.1), we reduce this fractional partial differential equation to a fractional ordinary differential equation, and construct some conservation laws using the idea from the Ibragimov theorem on conservation laws and the fractional generalization of the Noether operators. The obtained results may be useful to our further study on fractional differential equations. Although there are more and more works on the study of fractional differential equations, we still have very little knowledge about these equations. Basically, we don't know whether the methods adopted in study of partial differential equations can be extended to fractional ones, so there are several issues which need to be pursued in the future.

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