Journal of Applied Analysis and Computation
Volume 7, Number 2, May 2017, 632–643
Website:http://jaac-online.com/
DOI:10.11948/2017039

EXISTENCE OF KINK AND UNBOUNDED TRAVELING WAVE SOLUTIONS OF THE CASIMIR EQUATION FOR THE ITO SYSTEM*

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Abstract This paper study the traveling wave solutions of the Casimir equation for the Ito system. Since the derivative function of the wave function is a solution of a planar dynamical system, from which the exact parametric representations of solutions and bifurcations of phase portraits can be obtained. Thus, we show that corresponding to the compacton solutions of the derivative function system, there exist uncountably infinite kink wave solutions of the wave equation. Corresponding to the positive or negative periodic solutions and homoclinic solutions of the derivative function system, there exist unbounded wave solutions of the wave function equation.

Keywords Kink wave solution, unbounded wave solution, bifurcation, exact solution, Casimir equation for the Ito system.

MSC(2010) 34A05, 34C25-28, 34M55, 35Q51, 35Q53, 58F05, 58F14, 58F30, 35C05, 35C07, 34C60.

1. Introduction

Abbasbandy etc [1] consider the traveling wave solutions for the following partial differential equation:

$$\frac{\partial^2 \mathrm{w}}{\partial t^2} = \left(\frac{\partial \mathrm{w}}{\partial t}\right)^2 \frac{\partial}{\partial x}\left[\left(\frac{\partial \mathrm{w}}{\partial t}\right)^2\left(\frac{\partial \mathrm{w}}{\partial x} \pm \frac{\partial^3 \mathrm{w}}{\partial x^3}\right)\right], \tag{1.1}$$

which was called the Casimir equation for the Ito system. The equation was originally derived by Olver [11] to study the Ito system [8] which is given as:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial^3 U}{\partial x^3} + 3U\frac{\partial U}{\partial x} + V\frac{\partial V}{\partial x}, \\ \frac{\partial V}{\partial t} &= \frac{\partial}{\partial x}(UV). \end{aligned} \tag{1.2}$$

Ito showed that (1.2) is highly symmetric and possesses infinitely many conservation laws. It is an extension of the KdV equation, [5, 12], with an additional field variable V. [2, 6, 7] introduced a dual bi-Hamiltonian system for system (1.2), which admits a Casimir functional and associated Casimir equations. Introducing a stream function for the Casimir equations, they then obtained the single partial differential equation (1.1).

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*The authors were supported by National Natural Science Foundation of China (11471289, 11571318).

To investigate the traveling wave solutions of (1.1), let $\mathrm{w}(x,t) = \phi(\xi)$, where $\xi = x - \beta^{-\frac{1}{2}}t$ is the wave variable and $\beta > 0$. (1.1) reduces to the following ordinary differential equation:

$$\phi'' = \frac{1}{\beta}\phi'^2\left[\phi'^2(\phi' + \epsilon\phi''')\right]', \tag{1.3}$$

where prime denotes differentiation with respect to ξ and either $\epsilon = -1$ or $\epsilon = 1$ depending on the sign $(+)$ or $(-)$ in (1.1).

For the wave function $\phi(\xi)$, introducing the derivative function $\psi(\xi) = \phi'(\xi)$, we have

$$\frac{\psi'}{\psi^2} = \frac{1}{\beta}[\psi^2(\psi + \epsilon\psi'')]'. \tag{1.4}$$

Integrating equation (1.4) once, we obtain

$$\alpha - \frac{\beta}{\psi} = (\psi + \epsilon\psi'')\psi^2, \tag{1.5}$$

where α is a constant of integration. Equation (1.5) is equivalent to the planar dynamical system [4] as follows:

$$\frac{d\psi}{d\xi} = y, \qquad \frac{dy}{d\xi} = \pm\frac{\beta - \alpha\psi + \psi^4}{\psi^3}. \tag*{(1.6)$^{\pm}$}$$

with the first integral

$$H_{\pm}(\psi, y) = \frac{1}{2}y^2 \mp \left(\frac{1}{2}\psi^2 + \frac{\alpha}{\psi} - \frac{\beta}{2\psi^2}\right) = h. \tag*{(1.7)$^{\pm}$}$$

It is easy to see that systems of $(1.6)^{\pm}$ are singular traveling wave systems of the first class defined in [9, 10] with one singular straight line $\phi = 0$.

If we have an exact solution $\psi(\xi)$ of the derivative function systems $(1.6)^{\pm}$, then we obtain wave function

$$\phi(\xi) = \int_{\xi_0}^{\xi} \psi(\xi)d\xi. \tag{1.8}$$

We notice that the study results in Abbasbandy etc [1] are not complete. Depending on the change of parameter group (α, β), all possible dynamical behavior of traveling wave solutions of system $(1.6)^{\pm}$ have not be discussed. For the derivative function $\psi(\xi)$, any exact explicit solution of systems $(1.6)^{\pm}$ has not be given. In addition, Abbasbandy etc [1] did not say any words about the wave function $\phi(\xi)$.

The aim of this paper is to give complete and new study results for system $(1.6)^{\pm}$ and for the traveling wave solutions of equation (1.1).

This paper is organized as follows. In section 2, we discuss the bifurcations of phase portraits of systems $(1.6)^{\pm}$ under different parametric conditions. In section 3, we give exact parametric representations for all bounded solutions of system $(1.6)^{\pm}$. In section 4, we study the existence of kink wave solutions and unbounded traveling wave solutions of (1.1) by giving the main results of system $(1.6)^{\pm}$.

2. The bifurcations of phase portraits of systems $(1.6)^{\pm}$

We know from section 1 that the parameter $\beta > 0$. For mathematical completion, in this section, we assume that $\beta \in (-\infty, \infty)$.

Write $F(\psi) = \psi^4 - \alpha\psi + \beta$. Because every equilibrium point $(\psi_j, 0)$ of systems $(1.6)^{\pm}$ satisfies $F(\psi_j) = 0$. To investigate the equilibrium points of system $(1.6)^{\pm}$, we must discuss the number of real zeros of function $F(\psi)$.

Clearly, we have $F'(\psi) = 4\psi^3 - \alpha$, for which it has only one real zero $\psi = \left(\frac{\alpha}{4}\right)^{\frac{1}{3}} \equiv \psi_0$. Notice that $F(\psi_0) = \beta - 3\left(\frac{\alpha}{4}\right)^{\frac{4}{3}}, F(0) = \beta$. Therefore, when $F(\psi_0) < 0$ and $\beta > 0$, there exist two positive real zeros $\psi_j, j = 1, 2$ of $F(\psi)$ with $0 < \psi_1 < \psi_0 < \psi_2$. When $F(\psi_0) = 0, \beta > 0$ there exists a double zeros of $F(\psi)$ at $\psi = \psi_0$. When $F(\psi_0) < 0$ and $\beta = 0$, there exist two real zeros 0 and ψ_1 of $F(\psi)$ with $0 < \psi_0 < \psi_1$. When $F(\psi_0) < 0$ and $\beta < 0$, there exist two positive real zeros $\psi_j, j = 1, 2$ of $F(\psi)$ satisfying $\psi_1 < 0 < \psi_2$. If $F(\psi_0) > 0$, then systems $(1.6)^{\pm}$ have no equilibrium point.

Following [9], we consider the associated regular systems of systems $(1.6)^{\pm}$ as follows:

$$\frac{d\psi}{d\zeta} = y\psi^3, \quad \frac{dy}{d\zeta} = \pm F(\psi), \tag{2.1}$$

where $d\xi = \psi^3 d\zeta$, for $\psi \neq 0$.

Let $M(\psi_j, y)$ be the coefficient matrix of the linearized system for equation $(1.6)^{\pm}$ at an equilibrium point. we have $J(\psi_j, 0) = detM(\psi_j, 0) = -\psi^3 F'(\psi_j)$, for $j = 1, 2$. And $J(0, 0) = detM(0, 0) = 0$. In the $(\alpha, \beta)-$parameter plane, there are two parameter curves $(L_1) : \beta = 3\left(\frac{\alpha}{4}\right)^{\frac{4}{3}}$ and $(L_2) : \beta = 0$, which partition this parameter plane into four regions: $(II), (IV), (VI)$, and $(VIII)$ shown in Fig.1.

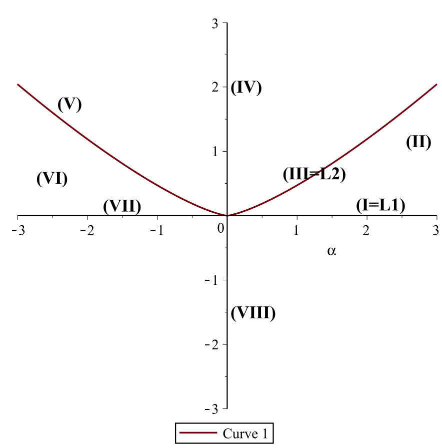
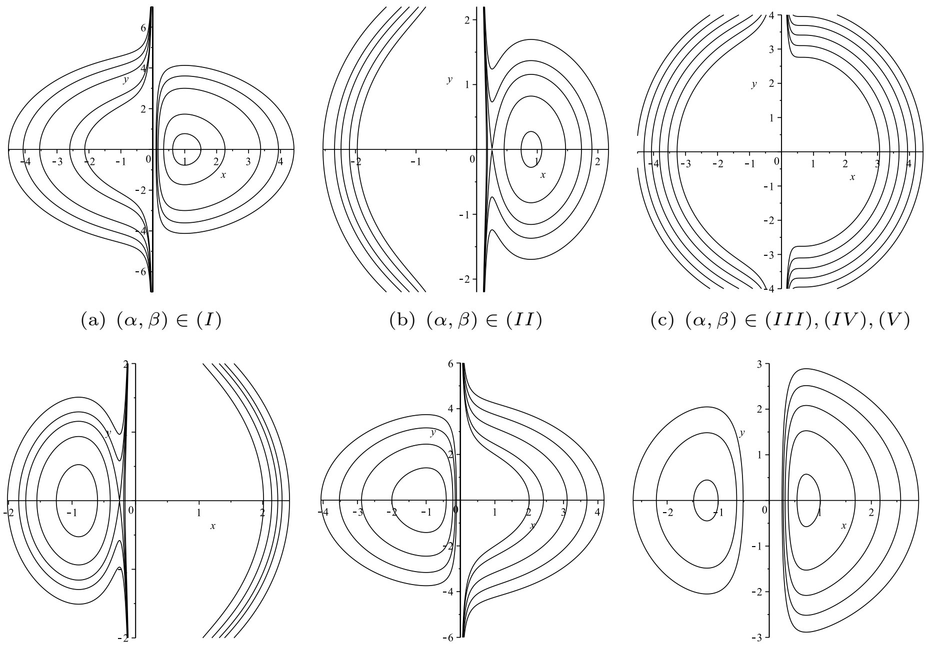


Figure 1. The parameter regions partitioned by bifurcation curves in the (α, β)-plane.

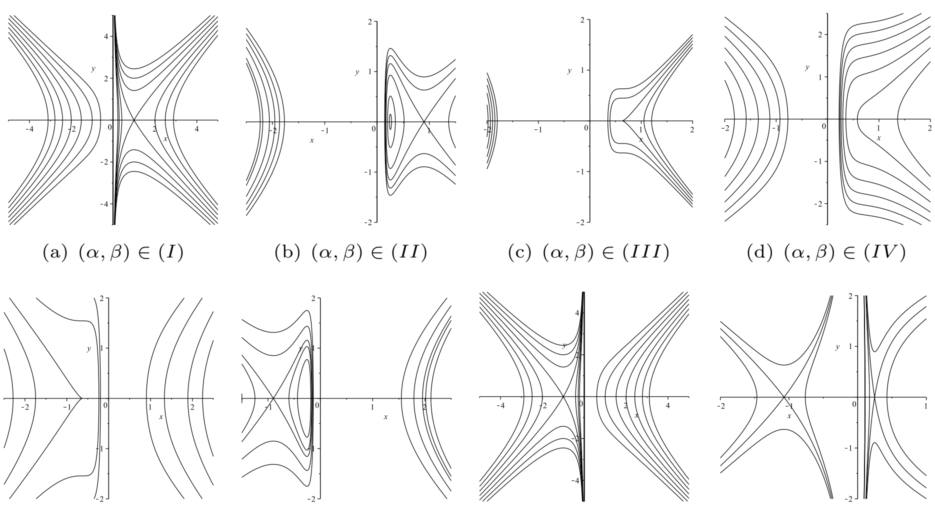
By using the above information to do qualitative analysis, we have the following bifurcations of phase portraits of system $(1.6)^{\pm}$ shown in Fig. 2 and Fig. 3, respectively.



(a) $(\alpha, \beta) \in (I)$ (b) $(\alpha, \beta) \in (II)$ (c) $(\alpha, \beta) \in (III), (IV), (V)$

(d) $(\alpha, \beta) \in (VI)$ (e) $(\alpha, \beta) \in (VII)$ (f) $(\alpha, \beta) \in (VIII)$

Figure 2. Bifurcations of phase portraits of system $(1.6)^{-}$.



(a) $(\alpha, \beta) \in (I)$ (b) $(\alpha, \beta) \in (II)$ (c) $(\alpha, \beta) \in (III)$ (d) $(\alpha, \beta) \in (IV)$

(e) $(\alpha, \beta) \in (V)$ (f) $(\alpha, \beta) \in (VI)$ (g) $(\alpha, \beta) \in (VII)$ (h) $(\alpha, \beta) \in (VIII)$

Figure 3. Bifurcations of phase portraits of system $(1.6)^{+}$.

3. The exact explicit bounded solutions of systems $(1.6)^{\mp}$

In this section, we are interested in determining the bounded solutions of systems $(1.6)^{\pm}$ and considering the exact parametric representations of the solutions of system $(1.6)^{\pm}$.

First, we consider system $(1.6)^{-}$. We see from $(1.7)^{-}$ that $y^2 = \frac{\beta-2\alpha\psi+2h\psi^2-\psi^4}{\psi^2}$. By using the first equation of $(1.6)^{\pm}$, we have

$$\xi = \int_{\psi_o}^{\psi} \sqrt{\frac{\psi^2}{\beta - 2\alpha\psi + 2h\psi^2 - \psi^4}} d\psi. \tag{3.1}$$

Then from (3.1), we may obtain the parametric representations of solutions of system $(1.6)^{-}$.

3.1. $(\alpha, \beta) \in (I)$ i.e., $\alpha > 0, \beta = 0$ (see Fig. 2(a)).

(i) Corresponding to the close branch of the level curves defined by $H_{-}(\psi, y) = h, h > h_1$, there exists a family of periodic orbits enclosing the center $E_1(\psi_1, 0)$ in the right half-phase plane, which gives rise to a family of periodic solutions of system $(1.6)^{-}$. Now, $(1.7)^{-}$ can be written as $y^2 = \frac{\beta-2\alpha\psi+2h_2\psi^2-\psi^4}{\psi^2} = \frac{(r_1-\psi)(\psi-r_2)\psi(\psi-r_3)}{\psi^2}$, where $r_3 < 0 < r_2 < \psi_0 < r_1$. and (3.1) becomes

$$\xi = \int_{r_2}^{\psi} \frac{\psi d\psi}{\sqrt{(r_1-\psi)(\psi-r_2)\psi(\psi-r_3)}}.$$

Thus, we have the following parametric representation of the periodic orbits:

$$\begin{aligned} \psi(\chi) &= \frac{r_2}{1-\alpha_1^2 \mathrm{sn}^2\chi}, \\ \xi(\chi) &= \frac{2r_2}{\sqrt{r_1(r_2-r_3)}} \Pi(\arcsin(\mathrm{sn}(\chi,k)), \alpha_1^2, k), \end{aligned} \tag{3.2}$$

where $\alpha_1^2 = \frac{r_1-r_2}{r_1}$, $k = \sqrt{\frac{-r_3(r_1-r_2)}{r_1(r_2-r_3)}}$ and $\mathrm{sn}(\chi, k)$ is a Jacobian elliptic function [3].

(ii) Corresponding to the open branch of the level curves defined by $H_{-}(\psi, y) = h, h > h_1$, there is an open curve family passing through the point $(r_3, 0)$ and tending to the singular straight line $\psi = 0$ when $|y| \to \infty$ in the left half-phase plane, which gives rise to a family of compacton solutions of system $(1.6)^{-}$. In this case, (3.1) becomes

$$\xi = \int_{r_3}^{\psi} \frac{\psi d\psi}{\sqrt{(r_1-\psi)(r_2-\psi)\psi(\psi-r_3)}}.$$

Hence, we have the following parametric representation of the open orbits:

$$\begin{aligned} \psi(\chi) &= r_1 - \frac{r_1-r_3}{1-\alpha_2^2\mathrm{sn}^2\chi}, \quad \chi \in (0, 2K(k)), \\ \xi(\chi) &= \frac{2}{\sqrt{r_1(r_2-r_3)}} \left[r_1\chi - (r_1-r_3)\Pi(\arcsin(\mathrm{sn}(\chi,k)), \alpha_2^2, k) \right], \end{aligned} \tag{3.3}$$

where $\alpha_2^2 = \frac{r_3}{r_1}$, $\quad k = \sqrt{\frac{-r_3(r_1-r_2)}{r_1(r_2-r_3)}}$ and $K(k)$ is the elliptic integral of the first kind.

3.2. $(\alpha, \beta) \in (II)$, i.e., $\alpha > 0, \beta < 3\left(\frac{\alpha}{4}\right)^{\frac{4}{3}}$ (see Fig. 2(b)).

(i) Corresponding to the level curves defined by $H_-(\psi, y) = h, h \in (h_2, h_1)$, there exist a family of close orbits and two open orbit families, which tend to the singular straight line $\psi = 0$ as $y \to \pm\infty$. In this case, $H_-(\psi, y) = h$ can be written as $y^2 = \frac{(r_1-\psi)(\psi-r_2)(\psi-r_3)(\psi-r_4)}{\psi^2}$, where $r_4 < 0 < r_3 < \psi_1 < r_2 < \psi_2 < r_1$. Now, for the periodic orbits, (3.1) becomes

$$\xi = \int_{r_2}^{\psi} \frac{\psi d\psi}{\sqrt{(r_1-\psi)(\psi-r_2)(\psi-r_3)(\psi-r_4)}}.$$

Hence, we have the following parametric representation of the periodic orbits:

$$\begin{aligned} \psi(\chi) &= r_3 + \frac{r_2 - r_3}{1 - \alpha_3^2 \mathrm{sn}^2\chi}, \\ \xi(\chi) &= \frac{2}{\sqrt{(r_1-r_3)(r_2-r_4)}}[r_3\chi + (r_2-r_3)\Pi(\arcsin(\mathrm{sn}(\chi,k)), \alpha_3^2, k)], \end{aligned} \tag{3.4}$$

where $\alpha_3^2 = \frac{r_1-r_2}{r_1-r_3},\ k = \sqrt{\frac{(r_3-r_4)(r_1-r_2)}{(r_1-r_3)(r_2-r_4)}}$.

For the open orbits passing through the points $(r_3, 0)$, (3.1) becomes

$$\xi = \int_{\psi}^{r_3} \frac{\psi d\psi}{\sqrt{(r_1-\psi)(r_2-\psi)(r_3-\psi)(\psi-r_4)}}.$$

Thus, we have the following parametric representation of the compacton solutions:

$$\begin{aligned} \psi(\chi) &= r_2 - \frac{r_2 - r_3}{1 - \alpha_4^2 \mathrm{sn}^2\chi}, \quad \chi \in \left(-\mathrm{sn}^{-1}\sqrt{\frac{r_3(r_2-r_4)}{r_2(r_3-r_4)}}, 0\right), \\ \xi(\chi) &= \frac{2}{\sqrt{(r_1-r_3)(r_2-r_4)}}[r_2\chi - (r_2-r_3)\Pi(\arcsin(\mathrm{sn}(\chi,k)), \alpha_4^2, k)], \end{aligned} \tag{3.5}$$

where $\alpha_4^2 = \frac{r_3-r_4}{r_2-r_4},\ k = \sqrt{\frac{(r_3-r_4)(r_1-r_2)}{(r_1-r_3)(r_2-r_4)}}$.

For the open orbits passing through the points $(r_4, 0)$, (3.1) becomes

$$\xi = \int_{r_4}^{\psi} \frac{\psi d\psi}{\sqrt{(r_1-\psi)(r_2-\psi)(r_3-\psi)(\psi-r_4)}}.$$

Therefore, we have the following parametric representation of the compacton solutions:

$$\begin{aligned} \psi(\chi) &= r_1 - \frac{r_1 - r_4}{1 - \alpha_5^2 \mathrm{sn}^2\chi}, \quad \chi \in \left(0, \mathrm{sn}^{-1}\sqrt{\frac{-r_4(r_1-r_3)}{r_2(r_3-r_4)}}\right), \\ \xi(\chi) &= \frac{2r_1}{\sqrt{(r_1-r_3)(r_2-r_4)}}[r_1\chi - (r_1-r_4)\Pi(\arcsin(\mathrm{sn}(\chi,k)), \alpha_5^2, k)], \end{aligned} \tag{3.6}$$

where $\alpha_5^2 = \frac{r_4-r_3}{r_1-r_3},\ k = \sqrt{\frac{(r_3-r_4)(r_1-r_2)}{(r_1-r_3)(r_2-r_4)}}$.

(ii) Corresponding to the level curves defined by $H_-(\psi, y) = h_1$, there exist a homoclinic orbit to the equilibrium point $E_1(\psi_1, 0)$ enclosing the center $E_2(\psi_2, 0)$

and a open curve, which tends to the singular straight line $\psi = 0$ as $y \to \pm\infty$. The homoclinic orbit gives rise to a bright solitary wave solution of system $(1.6)^-$ (see Fig. 4(a)).

For the homoclinic orbit, we see from $(1.7)^-$ that $y^2 = \frac{(r_M-\psi)(\psi-\psi_1)^2(\psi-r_m)}{\psi^2}$, where $r_m < 0 < \psi_1 < \psi_2 < r_M$. Thus by using the first equation of $(1.6)^-$, we have

$$\xi = \int_{\psi}^{r_M} \frac{d\psi}{\sqrt{(r_M-\psi)(\psi-r_m)}} + \psi_1 \int_{\psi}^{r_M} \frac{d\psi}{(\psi-\psi_1)\sqrt{(r_M-\psi)(\psi-r_m)}}.$$

It follows the parametric representation of the bright solitary wave solution of system $(1.6)^-$:

$$\begin{aligned} \psi(\chi) &= \psi_1 + \frac{2(r_M-\psi_1)(\psi_1-r_m)}{(r_M-r_m)\cosh(\sqrt{(r_M-\psi_1)(\psi_1-r_m)}\chi) - (r_M+r_m-2\psi_1)}, \\ \xi(\chi) &= \psi_1\chi + \arcsin\left(\frac{r_M+r_m-2\psi(\chi)}{r_M-r_m}\right) + \frac{1}{2}\pi. \end{aligned} \tag{3.7}$$

For the open orbit passing through the point $(r_m, 0)$, we have that

$$\xi = \int_{r_m}^{\psi} \frac{d\psi}{\sqrt{(r_M-\psi)(\psi-r_m)}} + \psi_1 \int_{r_m}^{\psi} \frac{d\psi}{(\psi-\psi_1)\sqrt{(r_M-\psi)(\psi-r_m)}}.$$

Thus, we have the following parametric representation of the compacton solution (see Fig. 4(b)):

$$\begin{aligned} \psi(\chi) &= \psi_1 - \frac{2(r_M-\psi_1)(\psi_1-r_m)}{(r_M-r_m)\cosh\left(\sqrt{(r_M-\psi_1)(\psi_1-r_m)}\chi\right) + (r_M+r_m-2\psi_1)}, \\ \xi(\chi) &= \psi_1\chi - \arcsin\left(\frac{r_M+r_m-2\psi(\chi)}{r_M-r_m}\right) - \frac{1}{2}\pi. \end{aligned} \tag{3.8}$$

(iii) Corresponding to the level curves defined by $H_-(\psi, y) = h, h > h_1$, there exist two open orbit families laying in left and right half-phase planes, respectively, which tend to the singular straight line $\psi = 0$ as $y \to \pm\infty$. We consider the left open orbits. Now (3.1) becomes:

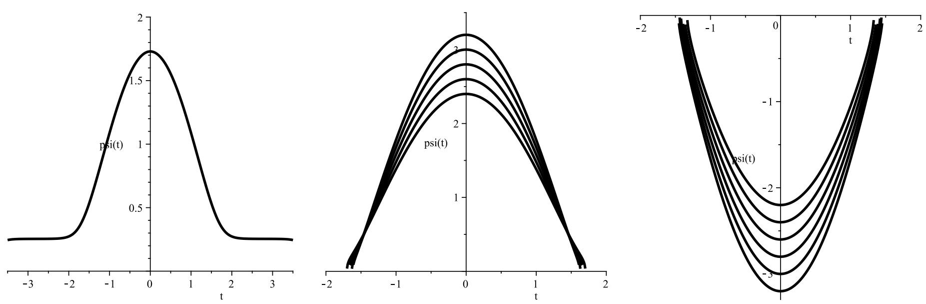
$$\begin{aligned} \xi &= \int_{r_2}^{\psi} \sqrt{\frac{\psi^2}{(r_1-\psi)(\psi-r_2)(\psi-\psi_c)(\psi-\bar{\psi}_c)}} d\psi \\ &= \int_{r_2}^{\psi} \frac{\psi d\psi}{\sqrt{(r_1-\psi)(\psi-r_2)[(\psi-b_1)^2+a_1^2]}}. \end{aligned}$$

Thus, we obtain the following explicit exact family of compacton solutions (see Fig. 4(c)):

$$\begin{aligned} \psi(\chi) &= \frac{(r_1B_1+r_2A_1) - (r_1B_1-r_2A_1)\mathrm{cn}(\chi,k)}{(A_1+B_1)+(A_1-B_1)\mathrm{cn}(\chi,k)}, \\ \xi(\chi) &= \frac{r_1B_1-r_2A_1}{(A_1-B_1)\sqrt{A_1B_1}} \left[\hat{\alpha}_2\chi + \frac{\alpha-\hat{\alpha}_2}{1-\alpha^2}\Pi\left(\arccos(\mathrm{cn}(\chi,k)), \frac{\alpha^2}{\alpha^2-1}, k\right) \right. \\ &\quad \left. -\frac{\alpha(\alpha-\hat{\alpha}_2)}{1-\alpha^2}F_1(\chi)\right], \end{aligned} \tag{3.9}$$

where $A_1^2=(r_1-b_1)^2+a_1^2,\ B_1^2=(r_2-b_1)^2+a_1^2,\ \alpha=\frac{A_1-B_1}{A_1+B_1},\ \hat{\alpha}_2=\frac{r_2A_1-r_1B_1}{r_2A_1+r_1B_1},\ k^2=\frac{(r_1-r_2)^2-(A_1-B_1)^2}{4A_1B_1}, k'=\sqrt{1-k^2}$ and $\mathrm{sn}(\cdot,k),\mathrm{cn}(\cdot,k),\mathrm{dn}(\cdot,k)$ are Jacobin elliptic functions [3]., $\Pi(\cdot,\cdot,k), E(\cdot,k)$ are the elliptic integrals of the third kind and second kind, respectively.

$$F_1(\chi)=\begin{cases}\sqrt{\dfrac{1-\alpha^2}{k^2+(k')^2\alpha^2}}\tan^{-1}\left(\sqrt{\dfrac{k^2+(k')^2\alpha^2}{1-\alpha^2}}\mathrm{sd}(\chi,k)\right), & \text{if } \dfrac{\alpha^2}{\alpha^2-1}<k^2,\\ \mathrm{sd}(\chi,k), \quad \text{if } \dfrac{\alpha^2}{\alpha^2-1}=k^2, \\ \dfrac{1}{2}\sqrt{\dfrac{1-\alpha^2}{k^2+(k')^2\alpha^2}}\ln\left(\dfrac{\sqrt{k^2+(k')^2\alpha^2}\mathrm{dn}(\chi,k)+\sqrt{\alpha^2-1}\mathrm{sn}(\chi,k)}{\sqrt{k^2+(k')^2\alpha^2}\mathrm{dn}(\chi,k)-\sqrt{\alpha^2-1}\mathrm{sn}(\chi,k)}\right), \\ \qquad \text{if } \dfrac{\alpha^2}{\alpha^2-1}>k^2.\end{cases} \tag{3.10}$$



(a) Solitary wave (b) Upper compactons (c) Lower compactons

Figure 4. The profiles of the solitary wave and compatons of system $(1.6)^-$.

Similarly, we can calculate the parametric representation for the right family of open orbits.

3.3. $(\alpha,\beta)\in(III),(IV),(V)$ (see Fig. 2(c)).

Corresponding to the level curves defined by $H_-(\psi,y)=h, h\in(-\infty,\infty)$, there exist two open orbit families laying in left and right half-phase planes, respectively, which tend to the singular straight line $\psi=0$ as $y\to\pm\infty$.

The left family of the open orbits has the same parametric representation as (3.10).

Because the orbits in Fig. 2(d) and Fig. 2(e) are symmetric with respect to $y-$axis, so that we do not need to consider the exact parametric representations for these orbits.

3.4. $(\alpha,\beta)\in(VIII)$ (see Fig. 2(f)).

In this case, system $(1.6)^-$ has two centers $E_1(\psi_1,0)$ and $E_2(\psi_2,0)$. When $\alpha<0$, we have $h_2<h_1$. Corresponding to the level curves defined by $H_-(\psi,y)=h, h>$

h_1, there exist two close orbit families laying in left and right half-phase planes, respectively. For the right orbit family, we have

$$\xi = \int_{r_2}^{\psi} \frac{\psi d\psi}{\sqrt{(r_1-\psi)(\psi-r_2)(\psi-r_3)(\psi-r_4)}}.$$

Thus, we have the parametric representation of the periodic solutions as (3.5).

For the left orbit family, we have

$$\xi = \int_{r_4}^{\psi} \frac{\psi d\psi}{\sqrt{(r_1-\psi)(r_2-\psi)(r_3-\psi)(\psi-r_4)}}.$$

Hence, we have the following parametric representation of the periodic solutions:

$$\begin{aligned} \psi(\chi) &= r_1 - \frac{r_1-r_4}{1-\alpha_5^2 \mathrm{sn}^2\chi}, \\ \xi(\chi) &= \frac{2}{\sqrt{(r_1-r_3)(r_2-r_4)}}[r_1\chi - (r_1-r_4)\Pi(\arcsin(\mathrm{sn}(\chi,k)),\alpha_6^2,k)], \end{aligned} \tag{3.11}$$

where $\alpha_6^2 = \frac{r_4-r_3}{r_1-r_3}$, $k = \sqrt{\frac{(r_3-r_4)(r_1-r_2)}{(r_1-r_3)(r_2-r_4)}}$.

We next discuss the bounded orbits of system $(1.6)^+$.

3.5. $(\alpha,\beta) \in (II)$ (see Fig. 3(b)).

(i) Corresponding to the level curves defined by $H_+(\psi,y) = h, h \in (h_1,h_2)$, there exists a family of close orbits of system $(1.6)^+$. In this case, we see from $(1.7)^+$ and the first equation of $(1.6)^+$ that

$$\xi = \int_{r_3}^{\psi} \frac{\psi d\psi}{\sqrt{(r_1-\psi)(r_2-\psi)(\psi-r_3)(\psi-r_4)}}.$$

Thus, we have parametric representations for the family of periodic orbits as follows (see Fig. 5(a)):

$$\begin{aligned} \psi(\chi) &= r_4 + \frac{r_3-r_4}{1-\alpha_6^2 \mathrm{sn}^2\chi}, \\ \xi(\chi) &= \frac{2}{\sqrt{(r_1-r_3)(r_2-r_4)}}[r_4\chi + (r_3-r_4)\Pi(\arcsin(\mathrm{sn}(\chi,k)),\alpha_7^2,k)], \end{aligned} \tag{3.12}$$

where $\alpha_7^2 = \frac{r_2-r_3}{r_2-r_4}$, $k = \sqrt{\frac{(r_2-r_3)(r_1-r_4)}{(r_1-r_3)(r_2-r_4)}}$.

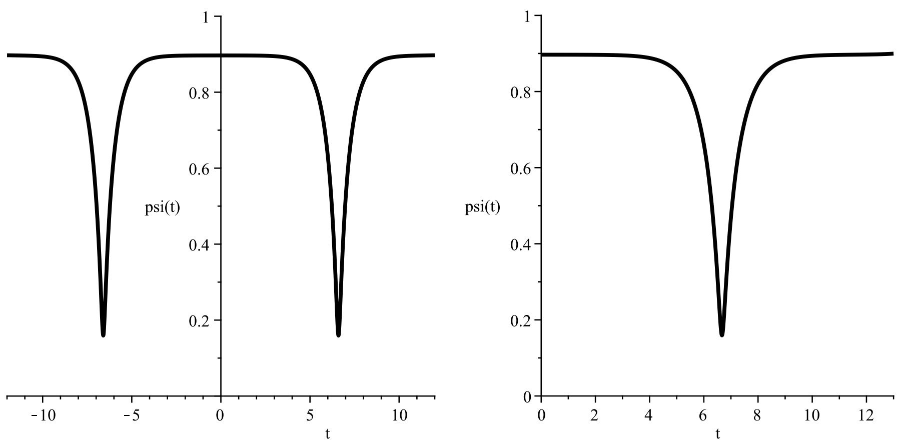
(ii) Corresponding to the level curves defined by $H_+(\psi,y) = h_2$, there exists a homoclinic orbit to the equilibrium point $E_2(\psi_2,0)$ enclosing the center, $E_1(\psi_1,0)$, which gives rise to a dark solitary wave solution of system $(1.6)^+$ (see Fig. 5(b)).

Now $(1.7)^+$ implies that $y^2 = \frac{(\psi_2-\psi)^2(\psi-\psi_M)(\psi-\psi_l)}{\psi^2}$, where $\psi_l < 0 < \psi_M < \psi_1 < \psi_2$. Then, from the first equation of $(1.6)^+$, we have

$$\xi = -\int_{\psi_M}^{\psi} \frac{d\psi}{\sqrt{(\psi-\psi_M)(\psi-\psi_l)}} + \psi_2 \int_{\psi_M}^{\psi} \frac{d\psi}{(\psi_2-\psi)\sqrt{(\psi-\psi_M)(\psi-\psi_l)}}.$$

Hence, we obtain the following parametric representation:

$$\begin{aligned}\psi(\chi) =&\psi_2 - \frac{2(\psi_2-\psi_M)(\psi_2-\psi_l)}{(\psi_M-\psi_l)\cosh\left(\sqrt{(\psi_2-\psi_M)(\psi_2-\psi_l)}\chi\right)+(2\psi_2-\psi_M-\psi_l)},\\ \xi(\chi) =&-\psi_2\chi+\ln\left(\sqrt{(\psi(\chi)-\psi_M)(\psi(\chi)-\psi_l)}+\psi(\chi)-\frac{1}{2}(\psi_M+\psi_l)\right)\\ &-\ln\left(\frac{1}{2}(\psi_M-\psi_l)\right).\end{aligned} \tag{3.13}$$



(a) Periodic wave (b) Solitary wave

Figure 5. The profiles of the solitary wave and periodic wave of system $(1.6)^+$.

4. The existence of kink wave solutions and unbounded traveling wave solutions of equation $(1.6)^\pm$

In section 3, we have obtained the parametric representations of the derivative function $\psi(\xi)$ given by $\psi=\psi(\chi), \xi=\xi(\chi)$. Because the hight order nonlinearity of system $(1.6)^\pm$, we can not use (1.8) to calculate explicitly $\phi(\xi)$. By using the results in section 3, we can obtain the qualitative results for the traveling wave solution $\phi(\xi)$ of equation (1.1).

We notice that the indefinite integral of a periodic function may be not a periodic function. We next need to use the following result [10].

Proposition 4.1. *If the function $f(x)$ is a periodic function with the period $2T$, i.e., $f(x+2T)=f(x)$ then*

$$\int_0^x f(s)ds = mx + p(x),$$

where $p(x+2T)=p(x)$, $\quad m=\frac{1}{2T}\int_0^{2T} f(s)ds.$

This proposition tell us that if the mean value m of a periodic function is not zero, then its indefinite integral is a sum of a linear function and a periodic function, namely, it is an unbounded function.

We see in section 3 that systems $(1.6)^{\pm}$ only have positive or negative periodic solutions. Therefore, the mean values of these periodic solutions are not zero. In addition, for the homoclinic orbits of system $(1.6)^{\pm}$, because $\phi(\xi)$ converges to $\phi_1 \neq 0$ or $\phi_2 \neq 0$, so that, the integration $\int_0^{\infty} \psi(\xi)d\xi$ is not convergent.

Thus, by proposition 4.1, we have the following conclusion.

Theorem 4.1. *Suppose that* $0 < \beta < 3\left(\frac{\alpha}{4}\right)^{\frac{4}{3}}$ *and* $\alpha > 0$ *(or* $\alpha < 0$*).*

(i) Corresponding to the periodic solution family of systems $(1.6)^{\pm}$ *defined by* $H_{\pm}(\psi, y) = h,\ h \in (h_1, h_2)$*, equation* (1.1) *has uncountably infinite many unbounded solutions.*

(ii) Corresponding to the homoclinic orbits of systems $(1.6)^{\pm}$ *defined by* $H_{\pm}(\psi, y) = h_1$ *or* h_2*, equation* (1.1) *has unbounded solutions.*

Suppose that $\mathrm{w}(x,t) = \phi(\xi)$ is a continuous solution of equation (1.1) (i.e., a traveling wave solution) for $\xi \in (-\infty, \infty)$ and $\lim_{\xi\to\infty} \phi(\xi) = a$, $\lim_{\xi\to-\infty} \phi(\xi) = b$. Recall that $\phi(\xi)$ is called a kink or anti-kink solution if $a \neq b$.

We see from section 2 that for every orbit of a compacton family of systems $(1.6)^{\pm}$, the value of function $\psi(\xi)$ converges to zero. Therefore, there exist finite values a such that $\lim_{\xi\to\infty} \phi(\xi) = a$ and $\lim_{\xi\to-\infty} \phi(\xi) = -a$ where $\phi(\xi)$ is defined by (1.8).

Hence, the following conclusion holds.

Theorem 4.2. *For* $\beta > 0, \alpha \neq 0$*, corresponding to the compacton solution families of systems* $(1.6)^{-}$*, equation* (1.1) *has uncountably infinite many kink wave solutions or anti-kink wave solutions.*

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