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ON THE EXACT SOLUTIONS AND CONSERVATION LAWS TO THE BENJAMIN-ONO EQUATION

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Abstract In the present work, we dealt with exact solutions and conservation laws of the Benjamin-Ono equation. We obtained exact solutions of given equation via the $\exp(-\Phi(\xi))$ method. The obtained solutions are included the hyperbolic functions, trigonometric functions and rational functions. By using the multiplier approach, the conservation laws of the mentioned equation was founded.

Keywords Partial differential equations, symbolic computation, exact solutions, conservation laws, multiplier approach.

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1. Introduction

Many significant phenomena and dynamic processes in solid state physics, fluid mechanics, chemical physics and plasma waves, such as water surface gravity waves, acoustic waves in unharmonic crystal, electromagnetic radiation reactions, optical fibers, hydro magnetic waves in cold plasma, the heat flow can be represented by nonlinear partial differential equations (PDEs) [8]. Some of the most attractive features of the physical systems are covert in their nonlinear treatment. These can just be analyzed by using a proper method which is designed to handle nonlinear problems.

The exploration of the exact solutions of nonlinear PDEs plays a vital role in the study of nonlinear physical phenomena and becoming more and more attractive. When we want to have information about the physical mechanism of a phenomena in nature which is described by nonlinear PDEs, we have to explore the exact solutions. Therefore, a large number of powerful methods have been proposed, such as Hirota's direct method [11,29], the homogeneous balance method [9], the inverse scattering method [1], the tanh method [32], the exp-function method [18], the exponential rational function method [4], (G'/G)-expansion method [5,20], the extended (G'/G)-expansion method [33], the $(G'/G, 1/G)$-expansion method [14], the extended trial equation method [10], the first integral method [7,28], the modified simple equation method [15], the sine-cosine method [19,31], the auxiliary equation method [16] and so on [34].

Conservation laws play a crucial role to find solution and reduction of PDEs. They are used for exploring integrability and linearization mappings, for performing

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the existence and uniqueness as well as for analyzing stability and global behaviour of solutions. A number of techniques are developed for constructing conservation laws of differential equations, such as the characteristic method, Noether's method, multiplier approach, symmetry based methods partial Lagrangian and Ibragimov method etc. [12, 13, 21, 22, 26, 30].

The Benjamin-Ono equation is used in the investigation of plenty of other physical applications, for example the percolation of water in the porous subsurface of a horizontal layer of material and also in the analysis of long waves in shallow water. This equation is one of the important nonlinear partial differential equation in physics and written as

$$u_{tt} + \beta(u^2)_{xx} + \gamma u_{xxxx} = 0, \tag{1.1}$$

where the constant coefficient β controlles the nonlinearity and the characteristic speed of the long waves, and the other constant γ is the fluid depth. $u(x,t)$ is the elevation of the free surface of the fluid; the quadratic nonlinearity or the vertical deflection accounts for the curvature of the bending beam [17].

The aim of this paper is to construct exact solutions via the $\exp(-\Phi(\xi))$ method and the conservation laws of the Benjamin-Ono equation via the multiplier approach. The outline of the paper is as follows. In Section 2, firstly we describe the $\exp(-\Phi(\xi))$ method step by step. Then in Section 3, we apply this method to the Benjamin-Ono equation. In Section 4, we construct conservation laws for Eq.(1.1) using the multiplier method. Finally some conclusions are given.

2. The $\exp(-\Phi(\xi))$ method

In the current section, we give an explanation of the $\exp(-\Phi(\xi))$ method to obtain exact solutions of partial differential equations. Consider a general nonlinear PDE, say in two independent variables x and t, in the following form:

$$F(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, ...) = 0. \tag{2.1}$$

We can summarize the basic steps of the $\exp(-\Phi(\xi))$ method as follows [25]:
Step 1: Using the travelling wave transformation

$$\xi = x - vt, \;\; u(x,t) = u(\xi), \tag{2.2}$$

where v is the wave speed, we can rewrite Eq. (2.1) in the following form of nonlinear ordinary differential equation:

$$Q(u, u', u'', ...) = 0. \tag{2.3}$$

Here prime denotes the derivative with respect to ξ. We should integrate Eq.(2.3) term by term as soon as possible.

Step 2: According to the $\exp(-\Phi(\xi))$ method, we suppose that the exact solution of Eq.(2.3) can be expressed in the following form:

$$u(\xi) = \sum_{n=0}^{m} a_n(\exp(-\Phi(\xi)))^n, \tag{2.4}$$

where a_n's $(n = 0, 1, ...m)$ are constants to be determined later, such that $a_m \neq 0$, and $\Phi(\xi)$ satisfies the following auxiliary ordinary differential equation:

$$\Phi'(\xi) = \exp(-\Phi(\xi)) + \mu \exp(\Phi(\xi)) + \lambda \tag{2.5}$$

By the generalized solutions of the auxiliary equation Eq.(2.5), we have the following cases.

Case 1 (Hyperbolic function solutions): When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$,

$$\Phi_1(\xi) = \ln\left(\frac{-\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C)\right) - \lambda}{2\mu}\right). \tag{2.6}$$

Case 2 (Trigonometric function solutions): When $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$,

$$\Phi_2(\xi) = \ln\left(\frac{\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi + C)\right) - \lambda}{2\mu}\right). \tag{2.7}$$

Case 3 (Hyperbolic function solutions): When $\lambda^2 - 4\mu > 0$, $\mu = 0$ and $\lambda \neq 0$,

$$\Phi_3(\xi) = -\ln\left(\frac{\lambda}{\cosh(\lambda(\xi + C)) + \sinh(\lambda(\xi + C)) - 1}\right). \tag{2.8}$$

Case 4 (Rational function solutions): When $\lambda^2 - 4\mu = 0, \mu \neq 0$ and $\lambda \neq 0$,

$$\Phi_4(\xi) = \ln\left(-\frac{2(\lambda(\xi + C) + 2)}{\lambda^2(\xi + C)}\right). \tag{2.9}$$

Case 5: When $\lambda^2 - 4\mu = 0, \mu = 0$ and $\lambda = 0$,

$$\Phi_5(\xi) = \ln(\xi + C). \tag{2.10}$$

Here C is the integration constant and m is a positive integer which is determined by considering the homogeneous balance principle. Namely, we balance the highest order derivative term and nonlinear term in Eq. (2.3).

Step 3: By substituting Eq. (2.4) into Eq. (2.3) along with Eq.(2.5) and then collecting all the coefficients of $\exp(-\Phi(\xi))$ together, Eq. (2.3) is converted into another polynomial in $\exp(-\Phi(\xi))$. Afterwards we equate each coefficient of this polynomial to zero and we find a set of algebraic equations for $a_n(n = 0, 1, 2, ...), v$, λ, μ.

Step 4: Solving the system of algebraic equations obtained in Step 3 and subsequently substituting the constants $a_n(n = 0, 1, 2, ..., m), v, \lambda$ and μ, and also solutions of Eq. (2.5) into Eq. (2.4), we can get the exact solutions of Eq. (2.1) in terms of trigonometric, hyperbolic and rational functions.

3. Exact Solutions

By considering the traveling wave transformation Eq.(2.2), the Benjamin-Ono equation (1.1) can be reduced to following ODE:

$$v^2u'' + 2\beta((u')^2 + uu'') + \gamma u'''' = 0, \tag{3.1}$$

where β and γ are constants and prime denotes the derivation with respect to ξ. Considering the homogeneous balance principle, we get the balancing number as $m = 2$. By the way, we can seek the exact solutions as:

$$u(\xi) = a_0 + a_1\exp(-\Phi(\xi)) + a_2(\exp(-\Phi(\xi)))^2. \tag{3.2}$$

We substitute Eq.(3.2) into Eq.(3.1), then we collect all terms with the same order of $\exp(-\Phi(\xi))^n$ together, Eq.(3.1) converts into another polynomial in $\exp(-\Phi(\xi))^n (n = 0, 1, ..., 6)$. Then we equate each coefficient of this polynomial to zero and we get a set of algebraic equations of $v, \beta, \gamma, a_0, a_1, a_2, \lambda$ and μ as follows:

$$\begin{aligned}
e^{6\xi} &: 16\gamma a_2\mu^3 + v^2 a_1\mu\lambda + 2\beta a_1^2\mu^2 + 14\gamma a_2\mu^2\lambda^2 + 2v^2 a_2\mu^2 \\
&+ 8\gamma a_1\mu^2\lambda + 2\beta a_0 a_1\mu\lambda + \gamma a_1\mu\lambda^3 + 4\beta a_0 a_2\mu^2 = 0, \\
e^{5\xi} &: v^2 a_1\lambda^2 + 2v^2 a_1\mu + 2\beta a_0 a_1\lambda^2 + 6\beta a_1^2\mu\lambda + 6v^2 a_2\mu\lambda \\
&+ 22\gamma a_1\mu\lambda^2 + 4\beta a_0 a_1\mu + 12\beta a_0 a_2\mu\lambda + 12\beta a_1\mu^2 a_2 \\
&+ 120\gamma a_2\mu^2\lambda + 16\gamma a_1\mu^2 + 30\gamma a_2\mu\lambda^3 + \gamma a_1\lambda^4 = 0, \\
e^{4\xi} &: 15\gamma a_1\lambda^3 + 8\beta a_1^2\mu + 4\beta a_1^2\lambda^2 + 4v^2 a_2\lambda^2 + 16\gamma a_2\lambda^4 + 8v^2 a_2\mu \\
&+ 232\gamma a_2\mu\lambda^2 + 3v^2 a_1\lambda + 30\beta a_1\mu a_2\lambda + 12\beta a_2^2\mu^2 + 16\beta a_0 a_2\mu \\
&+ 136\gamma a_2\mu^2 + 60\gamma a_1\mu\lambda + 6\beta a_0 a_1\lambda^4 + 8\beta a_0 a_2\lambda^2 = 0, \\
e^{3\xi} &: 50\gamma a_1\lambda^2 + 10v^2 a_2\lambda + 40\gamma a_1\mu + 28\beta a_2^2\mu\lambda + 10\beta a_1^2\lambda \\
&+ 20\beta a_0 a_2\lambda + 4\beta a_0 a_1 + 130\gamma a_2\lambda^3 \\
&+ 18\beta a_1\lambda^2 a_2 + 2v^2 a_1 + 440\gamma a_2\mu\lambda + 36\beta a_1 a_2\mu = 0, \\
e^{2\xi} &: 32\beta a_2^2\mu + 42\beta a_1 a_2\lambda + 60\gamma a_1\lambda + 12\beta a_0 a_2 + 240\gamma a_2\mu \\
&+ 16\beta a_2^2\lambda^2 + 6v^2 a_2 + 330\gamma a_2\lambda^2 + 6\beta a_1^2 = 0, \\
e^{\xi} &: 36\beta a_2^2\lambda + 24\beta a_1 a_2 + 24\gamma a_1 + 336\gamma a_2\lambda = 0, \\
e^{0\xi} &: 120\gamma a_2 + 20\beta a_2^2 = 0.
\end{aligned}$$

Solving the above system with the help of Maple, we obtain:

$$a_0 = -\frac{v^2 + \gamma\lambda^2 + 8\gamma\mu}{2\beta}, a_1 = -\frac{6\gamma\lambda}{\beta}, a_2 = -\frac{6\gamma}{\beta}.$$

Consequently, we have the following different cases for the exact solutions of Benjamin-Ono equation:

Case 1 (Hyperbolic function solutions): When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$,

$$\begin{aligned}
u_1(\xi) = -\frac{v^2 + \gamma\lambda^2 + 8\gamma\mu}{2\beta} &+ \frac{12\mu\gamma\lambda}{\beta\left(\sqrt{\lambda^2 - 4\mu}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C)) + \lambda\right)} \\
&- \frac{24\mu^2\gamma}{\beta\left(\sqrt{\lambda^2 - 4\mu}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C)) + \lambda\right)^2}.
\end{aligned}$$

Case 2 (Trigonometric function solutions): When $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$,

$$\begin{aligned}
u_2(\xi) = -\frac{v^2 + \gamma\lambda^2 + 8\gamma\mu}{2\beta} &+ \frac{12\mu\gamma\lambda}{\beta\left(-\sqrt{4\mu - \lambda^2}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi + C)) + \lambda\right)} \\
&- \frac{24\mu^2\gamma}{\beta\left(\sqrt{4\mu - \lambda^2}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi + C)) - \lambda\right)^2}.
\end{aligned}$$

Case 3 (Hyperbolic function solutions): When $\lambda^2 - 4\mu > 0$, $\mu = 0$ and $\lambda \neq 0$,

$$u_3(\xi) = -\frac{v^2 + \gamma\lambda^2 + 8\gamma\mu}{2\beta} - \left(\frac{6\gamma\lambda^2}{\beta(\cosh(\lambda(\xi + C)) + \sinh(\lambda(\xi + C)) - 1)}\right) - \frac{6\gamma\lambda^2}{\beta(\cosh(\lambda(\xi + C)) + \sinh(\lambda(\xi + C)) - 1)^2}.$$

Case 4 (Rational function solutions): When $\lambda^2 - 4\mu = 0, \mu \neq 0$ and $\lambda \neq 0$,

$$u_4(\xi) = -\frac{v^2 + \gamma\lambda^2 + 8\gamma\mu}{2\beta} + \frac{3\gamma\lambda^3(\xi + C)}{\beta(\lambda(\xi + C) + 2)} - \frac{6\gamma}{\beta}\left(\frac{\lambda^2(\xi + C)}{2(\lambda(\xi + C) + 2)}\right)^2.$$

Case 5: When $\lambda^2 - 4\mu = 0, \mu = 0$ and $\lambda = 0$,

$$u_5(\xi) = -\frac{v^2 + \gamma\lambda^2 + 8\gamma\mu}{2\beta} - \frac{6\gamma\lambda(\xi + C + \gamma)}{\beta(\xi + C)^2},$$

where $\xi = x - vt$ and C is the integration constant.

Note that, we have compared our solutions with the other solutions in literature. We can state that; whereas our solutions are different from the given ones in [35], they are similar to founded in [27].

4. Conservation Laws

Conservation laws play a significant roles to understanding physical properties and interpretations about the assorted systems. The presence of a number of conservation laws of a system of partial differential equations (PDEs) is a strong evidence of its integrability [2]. They have been utilized for the development of convenient numerical methods and construction of exact solutions of partial differential equations.

In this section adopting the multiplier approach (also known characteristic method), we construct local conservation laws for Eq(1.1). Consider the k^{th} order system of PDEs of n independent variables $x = (x^1, x^2, ..., x^n)$ and m dependent variables $u = (u^1, u^2, ..., u^m)$

$$E^\alpha\left(x, u, u_{(1)}, ..., u_{(k)}\right) = 0, \quad \alpha = 1, ..., m, \tag{4.1}$$

where $u_{(i)}$ is the collection of i th-order partial derivatives, $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$, respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + ..., \quad i = 1, ..., n, \tag{4.2}$$

in which the summation convention is used. $\dfrac{\delta}{\delta u}$ is the variational derivative operator and given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1}^{\infty} (-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u_{i_1 \ldots i_s}^\alpha}, \quad \alpha = 1, ..., m. \tag{4.3}$$

Definition 4.1. The conserved vector of (4.1), where each $T^i \epsilon A,\ A$ is the space of all differential functions, satisfies the equation

$$D_i T^i_{|(4.1)} = 0 \tag{4.4}$$

along the solution of (4.1).

A multiplier $\Lambda_\alpha(x, u, u_1, ...)$ has the property that

$$\Lambda_\alpha E^\alpha = D_i T^i \tag{4.5}$$

satisfies identically. Here we will consider multipliers of zeroth, that is $\Lambda_\alpha = \Lambda_\alpha(x, t, u)$. The right side of (4.5) is a divergence condition. To find multipliers, we construct determining equation for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E^\alpha)}{\delta u^\alpha} = 0. \tag{4.6}$$

All the corresponding multipliers can be found with the aid of (4.6) for which the equation can be expressed as a local conserved vector. [3, 6, 23, 24].

The determining equation for the zeroth order multiplier $\Lambda(x, t, u)$ is

$$\frac{\delta}{\delta u}\left[\Lambda\left(u_{tt} + \beta(u^2)_{xx} + \gamma u_{xxxx}\right)\right] = 0. \tag{4.7}$$

Expanding and then separating (4.7) with respect to different combinations of derivatives of u yields the following overdetermined system for the multipliers:

$$\Lambda_{tt} = 0, \quad \Lambda_{xx} = 0, \quad \Lambda_u = 0.$$

After solving this system we get the multipliers

$$\Lambda = (c_3 x + c_2)t + c_1 x + c_4, \tag{4.8}$$

where $c_1,\ c_2,\ c_3$ and c_4 are constants. Corresponding to the above multipliers, we have the following conserved vectors of (1.1):

$$\begin{aligned}
\Lambda_1 &= x, \\
T^t &= xu_t, \\
T^x &= 2\beta x u_x u - \beta u^2 - \gamma u_{xx} + \gamma x u_{xxx}, \\
\Lambda_2 &= t, \\
T^t &= -u + tu_t, \\
T^x &= 2u\beta t u_x + \gamma t u_{xxx}, \\
\Lambda_3 &= xt, \\
T^t &= -xu + xtu_t, \\
T^x &= 2\beta t x u_x u - \beta t u^2 - \gamma t u_{xx} + \gamma t x u_{xxx}, \\
\Lambda_4 &= 1, \\
T^t &= u_t, \\
T^x &= 2\beta u_x u + \gamma u_{xxx}.
\end{aligned}$$

5. Conclusion

In this work, we gave the decription of the the $\exp(-\Phi(\xi))$ method. Then we gave an implementation of this method on the Benjamin-Ono equation. We have obtained the exact solutions of this equation in terms of hyperbolic functions, trigonometric functions and rational functions. Also via the multiplier approach, we have found the three nontrivial and one trivial conservation laws of this equation. We foresee that, the obtained results can be found potentially advantageous for applications in mathematical physics and engineering. All results in this paper found and checked by putting them in to the original equations with the help of Maple software.

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