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A SUBDIVISION COLLOCATION METHOD FOR SOLVING TWO POINT BOUNDARY VALUE PROBLEMS OF ORDER THREE*

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Abstract In this paper, we propose a method for the numerical solution of self adjoint singularly perturbed third order boundary value problems in which the highest order derivative is multiplied by a small parameter ε. In this method, first we introduce the derivatives of two scale relations satisfied by the subdivision schemes. After that we use these derivatives to construct the subdivision collocation method for the numerical solution of singularly perturbed boundary value problems. Convergence of the subdivision collocation method is also discussed. Numerical examples are presented to illustrate the proposed method.

Keywords Singularly perturbed, boundary value problem, ordinary differential equation, subdivision schemes, collocation method.

MSC(2010) 65L11, 65L10, 65L99, 65D07, 65M70.

1. Introduction

In modern development of mathematics there are so many research problems occurs and our researchers work on them day by day. Many practical problem such as the mathematical boundary layer theory, numerical solutions of various problems described by differential equations involving parameters have become increasingly complex. Therefore we require the use of asymptotic methods. The numerical treatment of singularly perturbed problems is currently a field, in which active research is going on these days. Singularly perturbed problems in which the term containing the highest order derivative is multiplied by a small parameter ε occur in a number of areas of applied mathematics, science and engineering.

The solution of a singularly perturbed boundary value problem (bvp) act like a multi-scale character. The solution varies quickly near at thin transition layer while away from the layer the solution performs regularly and varies slowly. Therefore many obstacles are met in solving singularly perturbed boundary value problems using standard numerical methods. In recent development a large number of methods for different purpose have been established to provide accurate results. Major techniques that are used to solve such type of problems are finite difference techniques, finite element techniques and spline approximation techniques. Most

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of the research papers regarding computational aspects are restricted to second order equations. For example spline based methods [2, 4, 7, 9, 10], finite difference methods [12, 15–17] and haar wavelet methods [8, 18] etc. Only a few results are available for higher order equations, the class of third order singularly perturbed boundary value problems has been solved [8] by using fourth order finite difference scheme based on nonuniform mesh for a class of singular two point boundary value problems. Akram [1] solved third order self-adjoint singulary perturbed boundary problem by using fourth degree spline. Cui and Geng [3] presented a new numerical method for the class of third order boundary value problems with a boundary layer at the left of the underlying interval. Boundary value technique for the solution of class of third order singulary perturbed boundary value problems is presented in Valarmatht and Ramanujam [23]. Su-rang et al. [22] presented a method to solved singularly perturbed boundary value problem for quasi linear third order ordinary differential equations involving two parameters.

Subdivision based methods for the numerical solution of boundary value problems have also been used in the literature by Qu and Agarwal [19–21]. They developed subdivision based methods only for second order boundary value problems. Later on, Mustafa and Ejaz [14] solved linear third order boundary value problems by using the subdivision method. Numerical solutions of linear fourth order boundary value problems by the subdivision method have been presented by Ejaz et al. [6]. Further, Ejaz and Mustafa [5] presented an algorithm for the solution of nonlinear third order boundary value problems. An iterative collocation numerical approach based on interpolating subdivision scheme for the solution of nonlinear fourth order boundary value problems has been presented by Mustafa et al. [13]. Nowadays, the singularly perturbed boundary value problems become very attractive field and researchers are working to developed new techniques for their solution. Since the third order singularly perturbed boundary value problems have not been solved by subdivision methods. This motivates us to solve this type of problems.

This paper is divided into five sections. Self adjoint singularly perturbed bvp, subdivision scheme and their derivative are presented in Section 2. Numerical method for the solution of third order singularly perturbed bvp is discusses in Section 3. Convergence analysis of the numerical algorithm is given in Section 4. Numerical examples and their discussion for the illustration of the method is given in Section 5. Conclusion about the proposed method is given in Section 6.

2. Preliminaries

We present third order singularly perturbed bvp, binary subdivision scheme and their derivatives in this section.

2.1. Third order singularly perturbed bvp

We consider the general expression for third order singulary perturbed problem as

$$\begin{aligned} &-\varepsilon Y^{(3)}(x) + p(x)Y(x) = f(x), \;\; p(x) \geqslant 0, \\ &Y(0) = \beta_0, \;\; Y(1) = \beta_1, \;\; Y^{(1)}(0) = \beta_2, \end{aligned} \tag{2.1}$$

or

$$\begin{aligned} &-\varepsilon Y^{(3)}(x) + p(x)Y(x) = f(x), \;\; p(x) \geqslant 0, \\ &Y(0) = \beta_0, \;\; Y(1) = \beta_1, \;\; Y^{(2)}(0) = \beta_3, \end{aligned} \tag{2.2}$$

where $\beta_0, \beta_1, \beta_2$ and β_3 are constant, $p(x)$, $f(x)$ are smooth functions and ε is a small positive parameter with $\varepsilon \ll 1$. These type of problems usually occur in quantum mechanics, fluid mechanics, optical control, chemical reactions etc.

2.2. Subdivision scheme and derivatives

The subdivision schemes have been considered esteemed in many arenas of computational sciences. Such as computer animation, computer graphics and computer aided geometric design due to its efficient and simple characteristics. The subdivision scheme defines a curve out of an initial control polygon by subdividing them according to some refining rules recursively. Consider eight point binary interpolating scheme presented by [11]

$$\begin{cases} p_{2i}^{k+1} = p_i^k, \\ p_{2i+1}^{k+1} = -\omega\left(p_{i-3}^k + p_{i+4}^k\right) + \left(5\omega + \frac{3}{256}\right)\left(p_{i-2}^k + p_{i+3}^k\right) \\ \qquad\quad - \left(9\omega + \frac{25}{256}\right)\left(p_{i-1}^k + p_{i+2}^k\right) + \left(5\omega + \frac{75}{128}\right)\left(p_i^k + p_{i+1}^k\right). \end{cases} \tag{2.3}$$

The scheme (2.3) is C^3 derivable continuous for $0.0016 < \omega < 0.0084$, the support width for the mask of the scheme is $[-6, 6]$, the approximation order is six and satisfies following two scale relation

$$\begin{aligned} \Psi(x) = \Psi(2x) + \Big[&-\omega\left\{\Psi(2x-1) + \Psi(2x+1)\right\} + \left(5\omega + \tfrac{3}{256}\right)\left\{\Psi(2x-3)\right. \\ &\left.+\Psi(2x+3)\right\} - \left(9\omega + \tfrac{25}{256}\right)\left\{\Psi(2x-5) + \Psi(2x+5)\right\} + \left(5\omega + \tfrac{75}{128}\right) \\ &\left\{\Psi(2x-7) + \Psi(2x+7)\right\}\Big], \quad x \in \mathbb{R}, \end{aligned} \tag{2.4}$$

where

$$\Psi(x) = \begin{cases} 1 & \text{for} \quad x = 0, \\ 0 & \text{for} \quad x \neq 0. \end{cases} \tag{2.5}$$

As the function $\Psi(x) \in C^3$, then the first, second and third derivatives can be obtained by using the similar approach as in [14]. The third derivatives of (2.4) for the parametric value $\omega = 0.0032$ are given below:

$$\begin{cases} \Psi^{(3)}(0) = 0, & \\ \Psi^{(3)}(\pm 1) = \pm\frac{1122400355000}{418234124847}, & \Psi^{(3)}(\pm 2) = \mp\frac{1502922273911}{836468249694}, \\ \Psi^{(3)}(\pm 3) = \pm\frac{11666675000}{38021284077}, & \Psi^{(3)}(\pm 4) = \pm\frac{15980430727}{1672936499388}, \\ \Psi^{(3)}(\pm 5) = \mp\frac{4221440000}{418234124847}, & \Psi^{(3)}(\pm 6) = \pm\frac{108068864}{418234124847}. \end{cases} \tag{2.6}$$

3. Subdivision Collocation Method

This section describe the method for the numerical solutions of singularly perturbed linear third order boundary value problems with non-homogeneous boundary conditions. The method is described as follows:

Let N be a positive integer ($N \geqslant 6$), $h = 1/N$ and $x_i = i/N = ih$, $i = 0, 1, 2, \cdots N$, and

$$U(x) = \sum_{i=-6}^{N+6} u_i \Psi\left(\frac{x - x_i}{h}\right), \quad 0 \leq x \leq 1, \tag{3.1}$$

be the approximate solution to (2.1) or (2.2) where $\{u_i\}$ are the unknown to be determined. Then

$$-\varepsilon U^{(3)}(x_j) + p(x_j)U(x_j) = f(x_j), \quad j = 0, 1, 2, \cdots, N, \tag{3.2}$$

with the following given boundary conditions

$$\begin{aligned} & U(0) = \beta_0, \;\; U(1) = \beta_1, \;\; U^{(1)}(0) = \beta_2, \\ \text{or} \quad & U(0) = \beta_0, \;\; U(1) = \beta_1, \;\; U^{(2)}(0) = \beta_3. \end{aligned} \tag{3.3}$$

Let we define $p(x_j) = p_j$, and $f(x_j) = f_j$, then above equation can be written as

$$-\varepsilon U^{(3)}(x_j) + p_j U(x_j) = f_j, \quad j = 0, 1, 2, \cdots, N. \tag{3.4}$$

From (3.1) we have

$$U^{(3)}(x_j) = \frac{1}{h^3} \sum_{i=-6}^{N+6} u_i \Psi^{(3)}\left(\frac{x_j - x_i}{h}\right). \tag{3.5}$$

Using (3.1) and (3.5) in (3.4), we get following $(N+1)$ system of equations

$$-\varepsilon \sum_{i=-6}^{N+6} u_i \Psi^{(3)}\left(\frac{x_j - x_i}{h}\right) + h^3 p_j \sum_{i=-6}^{N+6} u_i \Psi\left(\frac{x_j - x_i}{h}\right) = h^3 f_j.$$

This implies

$$\sum_{i=-6}^{N+6} u_i \left\{-\varepsilon \Psi^{(3)}\left(\frac{x_j - x_i}{h}\right) + h^3 p_j \Psi\left(\frac{x_j - x_i}{h}\right)\right\} = h^3 f_j.$$

Further implies

$$\sum_{i=-6}^{N+6} u_i \left\{-\varepsilon \Psi^{(3)}(j-i) + h^3 p_j \Psi(j-i)\right\} = h^3 f_j, \tag{3.6}$$

where $j = 0, 1, 2, \cdots, N$ and $x_i = ih$ or $x_j = jh$. By using $\Psi(i) = \Psi_i$, (3.6) can be written as

$$\sum_{i=-6}^{N+6} u_i \left\{-\varepsilon \Psi^{(3)}_{j-i} + h^3 p_j \Psi_{j-i}\right\} = h^3 f_j, \;\; j = 0, 1, 2, \cdots, N. \tag{3.7}$$

As we observe from (2.6), $\Psi_{-i}^{(3)} = -\Psi_i^{(3)}$, then (3.7) becomes

$$\sum_{i=-6}^{N+6} u_i \left\{ \varepsilon \Psi_{i-j}^{(3)} + h^3 p_j \Psi_{i-j} \right\} = h^3 f_j, \;\; j = 0, 1, 2, \cdots, N. \tag{3.8}$$

Remark 3.1. The system (3.8) is equivalent to

$$\sum_{i=-6}^{6} u_{j+i} Q_i^j = h^3 f_j, \quad j = 0, 1, 2, \cdots, N, \tag{3.9}$$

where

$$Q_i^j = \begin{cases} \varepsilon \Psi_0^{(3)} + h^3 p_j, & i = 0, \\ \varepsilon \Psi_i^{(3)}, & i \neq 0. \end{cases} \tag{3.10}$$

3.1. Singularly perturbed system

The system of equations (3.9) are the singularly perturbed linear equations. These equations can be written in matrix form as

$$\mathbb{A}\mathbb{U} = \mathbb{F}_1, \tag{3.11}$$

where

$$\mathbb{A} = (q_s^{r-1})_{(N+1)\times(N+13)}, \tag{3.12}$$

$$\mathbb{U} = (u_s)_{1\times(N+13)}, \tag{3.13}$$

$$\mathbb{F}_1 = h^3 \times (u_\ell)_{1\times(N+13)}, \tag{3.14}$$

"r" and "s" represent rows and columns respectively. Where

$$q_s^{r-1} = \begin{cases} Q_i^{r-1}, & \text{for} \quad -6 \leqslant i \leqslant 6, \\ 0, & \text{for} \quad \text{otherwise}, \end{cases}$$

$$\ell = 0, 1, \cdots, N, \;\; r = 1, 2, \cdots, N+2, \;\; s = -6, -5, \cdots, N+5, N+6,$$

and Q_i^{r-1} is defined in (3.10).

To find the unique solution of the system (3.11), we need twelve more conditions. Three conditions are given in (3.3) i.e. $U(0)$, $U(1)$ and $U^{(1)}(0)$ or $U^{(2)}(0)$. As in given conditions first or second derivative is involved so first we replace first or second derivative conditions by their approximation. The approximation of these derivatives is given as follows:

3.2. Approximation of derivative conditions

We approximate first and second derivatives of the function $U(x)$ by finite differences method. Given a non-zero value of h, the l^{th} order derivative satisfies the following equation where the integer order of error $p > 0$ may be selected as desired

$$U^{(l)}(x) = \frac{l!}{h^l} \sum_{i=i_{\min}}^{i_{\max}} c_i U(x + ih) + O(h^p). \tag{3.15}$$

A forward difference approximation occurs if we set $i_{\min} = 0$ and $i_{\max} = l + p - 1$. The vector $C = (c_{i_{\min}}, \cdots, c_{i_{\max}})$ is called the convolution mask for the approximation. In order for equation (3.15) satisfied, it is necessary that

$$\sum_{i=i_{\min}}^{i_{\max}} i^n c_i = \begin{cases} 0, & \text{for} \quad 0 \leqslant n \leqslant l+p-1 \text{ and } n \neq l, \\ 1, & \text{for} \qquad n = l. \end{cases} \tag{3.16}$$

Approximation of $U^{(1)}(x)$ with error $O(h^7)$, we have $i_{\min} = 0$ and $i_{\max} = 7$. The convolution matrix $(c_0, c_1, \cdots, c_7)$ is obtained by solving the linear system

$$\sum_{i=0}^{7} i^n c_i = \begin{cases} 0, & \text{for} \quad 0 \leqslant n \leqslant 7 \text{ and } n \neq 1, \\ 1, & \text{for} \qquad n = 1. \end{cases} \tag{3.17}$$

After solving (3.17) substituting the values of c_i in (3.15), we obtain first derivative approximation as

$$U^{(1)}(0) = \left(\frac{N}{60}\right)\left[-147u_0 + 360u_1 - 450u_2 + 400u_3 - 225u_4 + 72u_5 - 10u_6\right]. \tag{3.18}$$

Similarly approximation of $U^{(2)}(x)$ with error $O(h^7)$, we have $i_{\min} = 0$ and $i_{\max} = 8$. The convolution matrix $(c_0, c_1, \cdots, c_8)$ is obtained by solving the linear system

$$\sum_{i=0}^{8} i^n c_i = \begin{cases} 0, & \text{for} \quad 0 \leqslant n \leqslant 8 \text{ and } n \neq 2, \\ 1, & \text{for} \qquad n = 2. \end{cases} \tag{3.19}$$

After solving (3.19) substituting the values of c_i in (3.15), we obtain second derivative approximation as

$$\begin{aligned} U^{(2)}(0) = \left(\frac{N}{360}\right) & \left[938u_0 - 4014u_1 + 7911u_2 - 9490u_3 + 7380u_4 - 3618u_5\right. \\ & \left. + 1019u_6 - 126u_7\right]. \end{aligned} \tag{3.20}$$

The remaining nine conditions are discussed in the next section.

3.3. Necessitated conditions

To find the unique solution of (3.11) with (3.3), we require nine more conditions. For this purpose we define these conditions, named necessitated conditions, in this section. These conditions can be defined as follows:

The values $u_{-5}, u_{-4}, u_{-3}, u_{-2}, u_{-1}$ can be determined by the sixth order polynomial $S_1(x)$ interpolating (x_i, u_i), $0 \leq i \leq 5$. Precisely, we have

$$u_{-i} - S_1(-x_i) = 0, \; i = 1, 2, 3, 4, 5,$$

where

$$S_1(x_i) = \sum_{j=1}^{6} \binom{6}{j} (-1)^{j+1} U(x_{i-j}).$$

Since by (3.1), $U(x_i) = u_i$ for $i = 1, 2, 3, 4, 5$ then by replacing x_i by $-x_i$, we have

$$S_1(-x_i) = \sum_{j=1}^{6} \begin{pmatrix} 6 \\ j \end{pmatrix} (-1)^{j+1} u_{-i+j}.$$

Hence the following necessitated conditions can be employed at the left end

$$\sum_{j=0}^{6} \begin{pmatrix} 6 \\ j \end{pmatrix} (-1)^{j} u_{-i+j} = 0, \ \ i = 1, 2, 3, 4, 5. \tag{3.21}$$

Similarly for the right end, we can define $u_i - S_1(x_i) = 0$, $i = N+1$, $N+2$, $N+3$, $N+4$. So we have the following necessitated boundary conditions at the right end

$$\sum_{j=0}^{6} \begin{pmatrix} 6 \\ j \end{pmatrix} (-1)^{j} u_{i-j} = 0, \ \ i = N+1, N+2, N+3, N+4. \tag{3.22}$$

3.4. Stable linear system of equations

By using above necessitated conditions, we get a following new system of $(N+13)$ singularly perturbed linear equations with $(N+13)$ unknowns $\{u_i\}$, in which $N+1$ equations are obtained from (3.9) and three equations obtained from given boundary conditions (3.3). Further nine equations are obtained from necessitated conditions defined in (3.21) and (3.22).

If we use necessitated conditions then stable singularly perturbed linear system of equations becomes

$$\mathbb{B}\mathbb{U} = \mathbb{F}, \tag{3.23}$$

where the coefficients matrix $\mathbb{B} = (\mathbb{L}^T, \mathbb{A}^T, \mathbb{R}^T)^T$, $\mathbb{A}$ is defined by (3.12).

The matrix $\mathbb{L}$ of order $7 \times (N+13)$ for left end boundary conditions is defined as: First five rows are obtained from (3.21), second last row is obtained from (3.3) either $U^{(1)}(0)$ or $U^{(2)}(0)$ and last row is also obtained from (3.3) i.e. $U(0)$.

The matrix $\mathbb{R}$ of order $6 \times (N+13)$ for right end boundary conditions is constructed as: First row of above matrix is obtained from given condition (3.3) i.e. $U(1)$ and the remaining five conditions are obtained from the conditions (3.22). The vector $\mathbb{U}$ is defined in (3.13) and $\mathbb{F}$ is define as

$$\begin{aligned} &\mathbb{F} = (0, 0, 0, 0, 0, \beta_2, \beta_0, \mathbb{F}_1^T, \beta_1, 0, 0, 0, 0, 0)^T, \\ \text{or} \quad &\mathbb{F} = (0, 0, 0, 0, 0, \beta_3, \beta_0, \mathbb{F}_1^T, \beta_1, 0, 0, 0, 0, 0)^T, \end{aligned} \tag{3.24}$$

where $\mathbb{F}_1$ is defined in (3.14) and β_0, β_1, β_2, β_3, given in (3.3). For the existences of the unique solution, first we check the non-singularity of the coefficient $\mathbb{B}$. We observed that $\mathbb{B}$ remains non-singulary for $N \leqslant 500$ and for large N it may or may not be non-singular.

4. Convergence of the Method

In this section, we discuss convergence of the method described in Section 3.
Let $Y(x)$ be the analytic solution of the problem (2.1) or (2.2) then for $j = 0, 1, \cdots N$, we have

$$-\varepsilon Y^{(3)}(x_j) + p(x_j)Y(x_j) = f(x_j). \tag{4.1}$$

Let the vector $Y(x)$ be defined as

$$Y(x) = (y(x_0), y(x_1), \cdots , y(x_N))^T.$$

By Taylor's series

$$\begin{aligned}Y^{(3)}(x_j) = &-\frac{1}{1672936499388h^3}\left[-432275456y_{(j-6)h} + 16885760000y_{(j-5)h}\right.\\ &-15980430727y_{(j-4)h} - 513333700000y_{(j-3)h} + 3005844547822y_{(j-2)h}\\ &-4489601420000y_{(j-1)h} + 4489601420000y_{(j+1)h} - 3005844547822y_{(j+2)h}\\ &+513333700000y_{(j+3)h} + 15980430727y_{(j+4)h} - 16885760000y_{(j+5)h}\\ &\left.+432275456y_{(j+6)h}\right] + o(h^7),\end{aligned}$$

where $y(x_j - th) = y_{(j-t)h}$ for $t = -6, -5, \cdots , N + 6$. The system of equations (3.23) provides the required subdivision based approximate solution $U(x)$ of (2.1) or (2.2) then by (3.3), for $j = 0, 1, \cdots , N$

$$-\varepsilon U^{(3)}(x_j) + p(x_j)U(x_j) = f(x_j), \tag{4.2}$$

where $U^{(3)}(x_j)$ is defined as

$$\begin{aligned}U^{(3)}(x_j) = &-\frac{1}{1672936499388h^3}\left[-432275456u_{(j-6)h} + 16885760000u_{(j-5)h}\right.\\ &-15980430727u_{(j-4)h} - 513333700000u_{(j-3)h} + 3005844547822u_{(j-2)h}\\ &-4489601420000u_{(j-1)h} + 4489601420000u_{(j+1)h} - 3005844547822u_{(j+2)h}\\ &+513333700000u_{(j+3)h} + 15980430727u_{(j+4)h} - 16885760000u_{(j+5)h}\\ &\left.+432275456u_{(j+6)h}\right] + o(h^7)\end{aligned}$$

and $u(x_j - th) = u_{(j-t)h}$ for $t = -6, -5, \cdots , N + 6$. Let the error function $\mathbb{E}$ is defined as $\mathbb{E}(x) = Y(x) - U(x)$ and

$$\mathbb{E} = (E_{-6}, E_{-5}, \cdots , E_{N+5}, E_{N+6}).$$

Then error vector at the node points is

$$\mathbb{E}(x_j) = Y(x_j) - U(x_j), \;\; -6 \leqslant j \leqslant N + 6.$$

This implies

$$\mathbb{E}^{(3)}(x_j) = Y^{(3)}(x_j) - U^{(3)}(x_j), \;\; -6 \leqslant j \leqslant N + 6.$$

By subtracting (4.2) from (4.1), we get

$$-\varepsilon\left[Y^{(3)}(x_j) - U^{(3)}(x_j)\right] + p(x_j)\left[Y(x_j) - U(x_j)\right] = 0.$$

By definition of error vector

$$-\varepsilon\mathbb{E}^{(3)}(x_j)+p(x_j)\mathbb{E}(x_j)=0,\ \ 0\leqslant j\leqslant N.$$

This implies

$$-\varepsilon\mathbb{E}^{(3)}(x_j)+p(x_j)\mathbb{E}(x_j)=0,\qquad 0\leqslant j\leqslant N, \tag{4.3}$$

where for $0\leqslant j\leqslant N$,

$$\begin{aligned}\mathbb{E}^{(3)}(x_j)=&\frac{1}{1672936499388h^3}\big[-432275456E_{(j-6)h}+16885760000E_{(j-5)h}\\&-15980430727E_{(j-4)h}-513333700000E_{(j-3)h}+3005844547822E_{(j-2)h}\\&4489601420000E_{(j-1)h}+4489601420000E_{(j+1)h}-3005844547822E_{(j+2)h}\\&+513333700000E_{(j+3)h}+15980430727E_{(j+4)h}-16885760000E_{(j+5)h}\\&+432275456E_{(j+6)h}\big]+o(h^7).\end{aligned}$$

As $0\leq x\leq 1$ and $x_j=jh$, $j=0,1,2,\cdots,N$ so $E_0,E_1,\cdots,E_N$ are non zero while $E_{-6},\cdots,E_{-1}$ and $E_{N+1},\cdots,E_{N+6}$ are zero because they lie outside the interval $[0,1]$. Let us define error values at the end points as

$$E_j=\begin{cases}\max\limits_{0\leq k\leq 4}\{|E_k|\}O(h^6),\ -6\leq j<0,\\ \max\limits_{N-4\leq k\leq N}\{|E_k|\}O(h^6),\ N<j\leq N+6.\end{cases} \tag{4.4}$$

By expanding (4.3), we get

$$(\mathbb{B}+O(h^5))\mathbb{E}=0.$$

These are equivalent to

$$(\mathbb{B}+O(h^5))\mathbb{E}=O(h^5)\parallel\mathbb{E}\parallel=O(h^5).$$

Since for small h, the coefficient matrix $\mathbb{B}+O(h^5)$ will be invertible and thus using the standard result from linear algebra, we have

$$||\mathbb{E}||\leq\left(\tfrac{||\mathbb{B}^{-1}||}{1-O(h^6)}O(h^5)\right)=O(h^3).$$

Hence $\parallel\mathbb{E}\parallel=O(h^3)$. The result is summarized in the following theorem.

Theorem 4.1. *Let Y be the exact solution of the system (2.1) and U, be the approximate solution of (2.1) then $\parallel\mathbb{E}\parallel_\infty=\parallel Y-U\parallel_\infty=O(h^3)$.*

5. Numerical Results and Discussion

In this section, we have solved four examples by using subdivision based numerical algorithm to show the accuracy of our method. Numerical results of these examples are calculated by using MATLAB. We observe that accuracy between the analytic and approximate solutions is good.

Example 5.1. Consider the following singularly perturbed boundary value problem:

$$
\begin{aligned}
&-\varepsilon Y^{(3)} + Y(x) = f(x), \quad x \in [0,1], \\
&Y(0) = 0, \quad Y(1) = 0, \quad Y^{(1)}(0) = 0,
\end{aligned}
\tag{5.1}
$$

where

$$f(x) = 6\varepsilon(1-x)^5 x^3 - 6\varepsilon^2[6(1-x)^5 - 90x(1-x)^4 + 180x^2(1-x)^3 - 60x^3(1-x)^2].$$

The analytic solution of (5.1) is

$$Y(x) = 6x^3\varepsilon(1-x)^5.$$

The numerical results of (5.1) for different values of N and ε are given in Tables 1 and 2. Graphical representation of these numerical results is shown in Figures 1 and 2.

Table 1. Maximum absolute errors for $N = 10$ of Example 5.1

ε	Our method	By [1]
$\frac{1}{16}$	9.5454E-04	2.9E-03
$\frac{1}{32}$	4.2571E-04	9.2E-04
$\frac{1}{64}$	1.7964E-04	1.4E-04

Table 2. Maximum absolute errors of Example 5.1

ε	$N = 10$	$N = 50$	$N = 100$	$N = 150$	$N = 200$	$N = 250$
0.1	1.6053E-03	7.4832E-04	7.1446E-04	7.0910E-04	7.0744E-04	7.0694E-04
0.01	1.0097E-04	4.7393E-05	4.6896E-05	4.7464E-05	4.7877E-05	4.8167E-05
0.001	9.8678E-06	1.5662E-06	1.1965E-06	1.3800E-06	1.4973E-06	1.5754E-06
0.0001	1.0634E-04	2.3926E-07	1.0401E-07	6.1803E-08	4.3221E-08	3.3448E-08

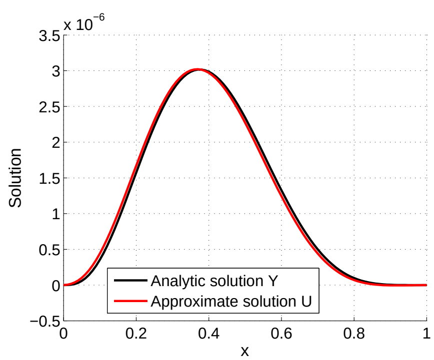

Figure 1. Comparability of exact and approximate solutions of Example 5.1 for $N = 100$ with $\varepsilon = 10^{-4}$.

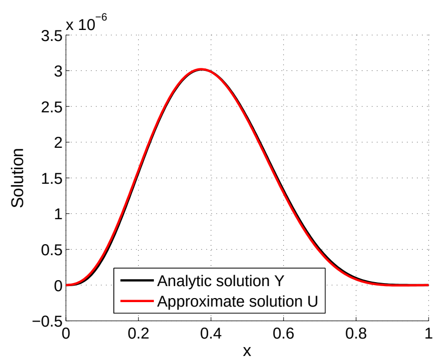

Figure 2. Comparability of exact and approximate solutions of Example 5.1 for $N = 200$ with $\varepsilon = 10^{-4}$.

Example 5.2. Consider the following singularly perturbed boundary value problem:

$$\begin{aligned} -\varepsilon Y^{(3)} + Y(x) &= f(x), \quad x \in [0,1], \\ Y(0) = 0, \quad Y(1) &= 0, \quad Y^{(2)}(0) = 0, \end{aligned} \tag{5.2}$$

where

$$f(x) = 6\varepsilon(1-x)^5 x^3 - 6\varepsilon^2[6(1-x)^5 - 90x(1-x)^4 + 180x^2(1-x)^3 - 60x^3(1-x)^2].$$

The analytic solution of (5.2) is $Y(x) = 6x^3\varepsilon(1-x)^5$. Numerical results of this example is shown in Tables 3 and 4 for different values of N and ε. Graphical representation of these numerical results is shown in Figures 3 and 4.

Table 3. Maximum absolute errors for $N = 10$ of Example 5.2

ε	Our method	By [1]
$\frac{1}{16}$	6.2854E-03	1.3E-02
$\frac{1}{32}$	1.9707E-03	3.2E-03
$\frac{1}{64}$	3.9065E-04	3.4E-04

Table 4. Maximum absolute errors of Example 5.2

ε	$N = 10$	$N = 50$	$N = 100$	$N = 150$	$N = 200$	$N = 250$
0.1	1.6190E-02	7.3371E-04	6.4463E-04	6.3671E-04	6.3496E-04	6.3446E-04
0.01	5.4777E-04	3.5302E-05	3.2708E-05	3.3005E-05	3.3331E-05	3.3579E-05
0.001	4.3814E-05	2.4150E-06	1.3966E-06	1.1544E-06	1.2348E-06	1.2913E-06
0.0001	7.5623E-06	2.4329E-07	1.1223E-07	7.6323E-08	6.1521E-08	5.3811E-08

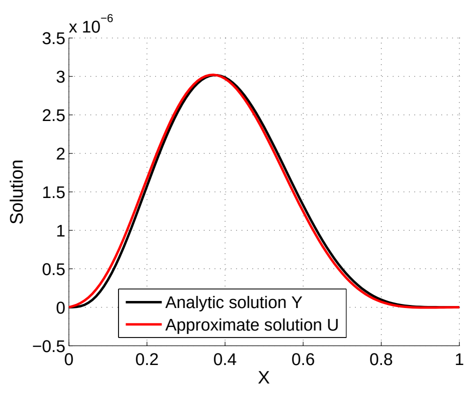

Figure 3. Comparability of exact and approximate solutions of Example 5.2 for $N = 100$ with $\varepsilon = 10^{-4}$.

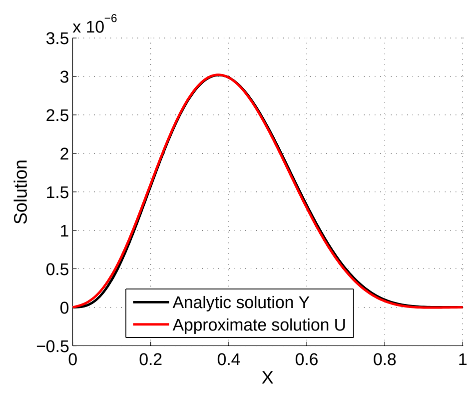

Figure 4. Comparability of exact and approximate solutions of Example 5.2 for $N = 200$ with $\varepsilon = 10^{-4}$.

Example 5.3. Consider the following boundary value problems

$$\begin{aligned} -\varepsilon Y^3(x)+Y(x) &= 81\varepsilon^2\cos 3x+3\varepsilon\sin 3x, \qquad x\in[0,1], \\ Y(0)=0,\ \ Y(1) &= 3\varepsilon\sin 3,\ \ Y^{(1)}(0)=9\varepsilon. \end{aligned} \tag{5.3}$$

The analytic solution of the system (5.3)

$$Y(x)=3\varepsilon\sin 3x.$$

The numerical results for (5.3) is given in Table 5. Graphical representation of these numerical results is shown in Figures 5 and 6.

Table 5. Maximum absolute errors of Example 5.3

ε	$N=10$	$N=50$	$N=100$	$N=150$	$N=200$	$N=250$
0.001	7.4049E-04	1.1109E-04	4.4739E-05	2.3062E-05	1.2307E-05	6.0161E-06
0.0001	3.0209E-02	1.8618E-05	9.0607E-06	5.9279E-06	4.3685E-06	3.4355E-06
0.00001	$\cdots$	2.1405E-06	1.0275E-06	6.7970E-07	5.0807E-07	4.0537E-07

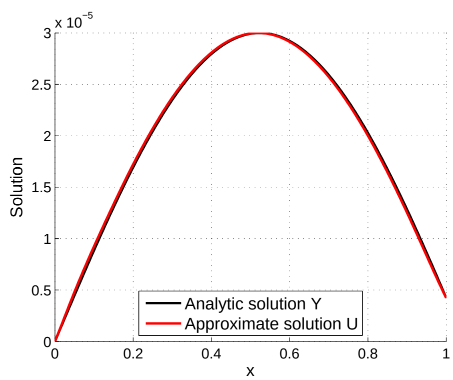

Figure 5. Comparability of exact and approximate solutions of Example 5.3 for $N = 250$ with $\varepsilon = 10^{-5}$.

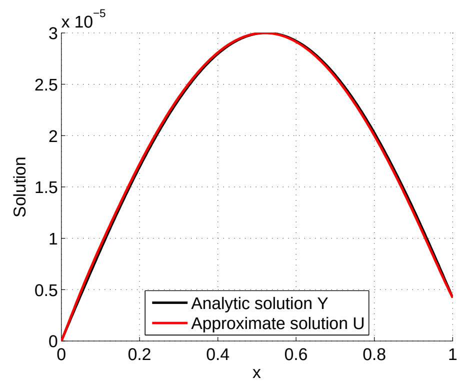

Figure 6. Comparability of exact and approximate solutions of Example 5.3 for $N = 300$ with $\varepsilon = 10^{-5}$.

Example 5.4. Consider the following boundary value problems

$$\begin{aligned} -\varepsilon Y^3(x)+Y(x) &= 81\varepsilon^2\cos 3x+3\varepsilon\sin 3x, \qquad x\in[0,1], \\ Y(0)=0,\ \ Y(1) &= 3\varepsilon\sin 3,\ \ Y^{(2)}(0)=0. \end{aligned} \tag{5.4}$$

The analytic solution of the system (5.4)

$$Y(x)=3\varepsilon\sin 3x.$$

Numerical results of (5.4) is shown in Table 6. Graphical representation of these numerical results is shown in Figures 7 and 8.

Table 6. Maximum absolute errors of Example 5.4

ε	$N = 10$	$N = 50$	$N = 100$	$N = 150$	$N = 200$	$N = 250$
0.001	2.5276E-03	1.7508E-04	7.0100E-05	3.6747E-05	2.0355E-05	1.0633E-05
0.0001	1.9938E-03	2.5243E-05	1.1440E-05	7.3469E-06	5.3722E-06	4.2054E-06
0.00001	$\cdots$	2.0415E-05	1.3333E-06	8.3294E-07	6.0941E-07	4.8045E-07

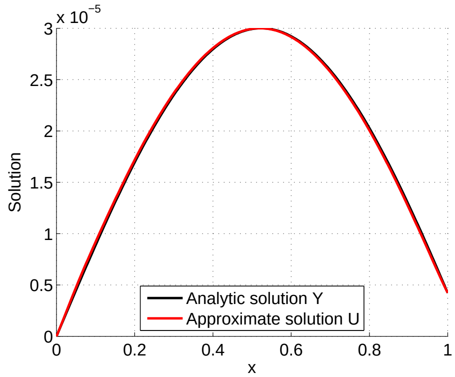

Figure 7. Comparability of exact and approximate solutions of Example 5.4 for $N = 250$ with $\varepsilon = 10^{-5}$.

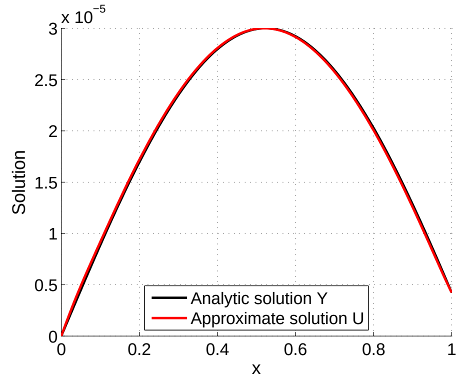

Figure 8. Comparability of exact and approximate solutions of Example 5.4 for $N = 300$ with $\varepsilon = 10^{-5}$.

6. Concluding Remark

A binary interpolating subdivision scheme is used to construct a numerical method for solving third order singularly perturbed boundary value problems. The method is third order convergent. The numerical illustration shows that the developed method maintains a very remarkable high accuracy that makes it very encouraging for dealing with the solution of singularly perturbed boundary value problems. We have compared the numerical results with the method of [1] and have observed that our results are better.

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