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OPTIMAL QUADRATURE FORMULAS FOR FOURIER COEFFICIENTS IN $W_2^{(m,m-1)}$ SPACE

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Abstract This paper studies the problem of construction of optimal quadrature formulas in the sense of Sard in the $W_2^{(m,m-1)}[0,1]$ space for calculating Fourier coefficients. Using S. L. Sobolev's method we obtain new optimal quadrature formulas of such type for $N+1 \geq m$, where $N+1$ is the number of the nodes. Moreover, explicit formulas for the optimal coefficients are obtained. We investigate the order of convergence of the optimal formula for $m=1$. The obtained optimal quadrature formula in the $W_2^{(m,m-1)}[0,1]$ space is exact for $\exp(-x)$ and $P_{m-2}(x)$, where $P_{m-2}(x)$ is a polynomial of degree $m-2$. Furthermore, we present some numerical results, which confirm the obtained theoretical results.

Keywords Fourier coefficients, optimal quadrature formulas, the error functional, extremal function, Hilbert space.

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1. Introduction

Numerical calculation of integrals of highly oscillating functions is one of the more important problems of numerical analysis, because such integrals are encountered in applications in many branches of mathematics as well as in other science such as quantum physics, flow mechanics and electromagnetism. Main examples of strongly oscillating integrands are encountered in different transformation, for example, the Fourier transformation and Fourier-Bessel transformation. Standard methods of numerical integration frequently require more computational works and they cannot be successfully applied. The earliest formulas for numerical integration of highly oscillatory functions were given by Filon [11] in 1928. Filon's approach for Fourier integrals

$$I[f;\omega] = \int_a^b \mathrm{e}^{\mathrm{i}\omega x} f(x)\,\mathrm{d}x$$

is based on piecewise approximation of $f(x)$ by arcs of the parabola on the integration interval. Then finite integrals on the subintervals are exactly integrated.

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Afterwards for integrals with different type highly oscillating functions many special effective methods such as Filon-type method, Clenshaw-Curtis-Filon type method, Levin type methods, modified Clenshaw-Curtis method, generalized quadrature rule, Gauss-Laguerre quadrature are worked out (see, for example, [4, 15, 20, 40], for more review see, for instance, [21, 23] and references therein).

In [22] the authors studied approximate computation of univariate oscillatory integrals (Fourier coefficients) for the standard Sobolev spaces H^s of periodic and non-periodic functions with an arbitrary integer $s \geq 1$. They found matching lower and upper bounds on the minimal worst case error of algorithms that use n function or derivative values. They also found sharp bounds on the information complexity which is the minimal n for which the absolute or normalized error is at most ε.

In the work [29] the weight lattice optimal cubature formulas in the periodic Sobolev's space $\tilde{L}_2^{(m)}(\Omega)$ were constructed. In particular, from the result of the work [29], in univariate case when the weight is the function $\exp(i\sigma x)$ (where $x \in [0, 2\pi]$ and σ is an integer), the Babuška optimal quadrature formula for Fourier coefficients was obtained [3].

Recently, some optimal quadrature formulas for Fourier coefficients in the Sobolev space $L_2^{(m)}(0,1)$ of non-periodic functions have been constructed in [6].

This paper is devoted to construction of optimal quadrature formulas for approximate calculation of Fourier integrals in a Hilbert space of non-periodic functions. Precisely, we study the problem of construction such optimal formulas in the sense of Sard in the $W_2^{(m,m-1)}[0,1]$ space.

We consider the following quadrature formula

$$\int_0^1 \mathrm{e}^{2\pi \mathrm{i} \omega x} \varphi(x)\, \mathrm{d}x \cong \sum_{\beta=0}^{N} C_\beta \varphi(h\beta) \tag{1.1}$$

with the error functional

$$\ell(x) = \mathrm{e}^{2\pi \mathrm{i} \omega x} \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C_\beta \delta(x - h\beta), \tag{1.2}$$

where C_β are the coefficients of formula (1.1), $h = 1/N$, $N \in \mathbb{N}$, $\mathrm{i}^2 = -1$, $\omega \in \mathbb{Z}$, $\varepsilon_{[0,1]}(x)$ is the indicator of the interval $[0,1]$ and $\delta(x)$ is the Dirac delta-function. Functions φ belong to the space $W_2^{(m,m-1)}[0,1]$, where

$$W_2^{(m,m-1)}[0,1] = \left\{ \varphi : [0,1] \to \mathbb{C} \;\middle|\; \varphi^{(m-1)} \in AC[0,1] \text{ and } \varphi^{(m)} \in L_2[0,1] \right\}$$

is the Hilbert space of complex valued functions and in this space the inner product is defined by the equality

$$\langle \varphi, \psi \rangle = \int_0^1 \left(\varphi^{(m)}(x) + \varphi^{(m-1)}(x) \right) \left(\overline{\psi}^{(m)}(x) + \overline{\psi}^{(m-1)}(x) \right) \mathrm{d}x, \tag{1.3}$$

where $\overline{\psi}$ is the conjugate function to the function ψ and the norm of the function φ is correspondingly defined by the formula

$$\|\varphi | W_2^{(m,m-1)}[0,1]\| = \langle \varphi, \varphi \rangle^{1/2}$$

and

$$\int_0^1 \left(\varphi^{(m)}(x)+\varphi^{(m-1)}(x)\right)\left(\overline{\varphi}^{(m)}(x)+\overline{\varphi}^{(m-1)}(x)\right)\mathrm{d}x<\infty.$$

We note that the coefficients C_β depend on ω, N and m, i.e., $C_\beta = C_\beta(\omega, N, m)$.

It should be noted that for a linear differential operator of order m, $L \equiv P_m(\mathrm{d}/\mathrm{d}x)$, Ahlberg, Nilson, and Walsh in the book [1, Chapter 6] investigated the Hilbert spaces $K_2(P_m)$ in the context of generalized splines. Namely, with the inner product

$$\langle\varphi,\psi\rangle=\int_0^1 L\varphi(x)\cdot L\psi(x)\,\mathrm{d}x,$$

$K_2(P_m)$ is a Hilbert space if we identify functions that differ by a solution of $L\varphi = 0$. Also, such a type of spaces of periodic functions and optimal quadrature formulas were discussed in [8].

The difference

$$(\ell,\varphi)=\int_0^1 \mathrm{e}^{2\pi\mathrm{i}\omega x}\varphi(x)\,\mathrm{d}x-\sum_{\beta=0}^{N}C_\beta\varphi(x_\beta)=\int_{-\infty}^{\infty}\ell(x)\varphi(x)\,\mathrm{d}x \tag{1.4}$$

is called *the error* of the quadrature formula (1.1). The error of the formula (1.1) is a linear functional in $W_2^{(m,m-1)*}[0,1]$, where $W_2^{(m,m-1)*}[0,1]$ is the conjugate space to the space $W_2^{(m,m-1)}[0,1]$.

By the Cauchy-Schwarz inequality

$$|(\ell,\varphi)|\le\|\varphi|W_2^{(m,m-1)}[0,1]\|\cdot\|\ell|W_2^{(m,m-1)*}[0,1]\|.$$

So, the error (1.4) of formula (1.1) is estimated by the norm

$$\left\|\ell|W_2^{(m,m-1)*}[0,1]\right\|=\sup_{\|\varphi|W_2^{(m,m-1)}[0,1]\|=1}|(\ell,\varphi)|$$

of the error functional (1.2).

Thus, the estimation of the error of the quadrature formula (1.1) over functions of the space $W_2^{(m,m-1)}$ is reduced to finding the norm of the error functional ℓ in the conjugate space $W_2^{(m,m-1)*}$.

Clearly the norm of the error functional ℓ depends on the coefficients C_β. The problem of finding the minimum of the norm of the error functional ℓ by coefficients C_β when the nodes are fixed (in our case distances between neighbor nodes of formula (1.1) are equal, i.e., $x_\beta = h\beta$, $\beta = 0, 1, \ldots, N$, $h = 1/N$) is called *Sard's problem*. And the obtained formula is called *the optimal quadrature formula in the sense of Sard*. This problem was first investigated by A. Sard [24] in the space $L_2^{(m)}$ for some m. Here $L_2^{(m)}$ is the Sobolev space of functions which $(m-1)$-st derivative is absolutely continuous and m-th derivative is square integrable.

There are several methods for constructing of optimal quadrature formulas in the sense of Sard such as the spline method, the ϕ-function method (cf. [5], [25]) and Sobolev's method. Note that Sobolev's method is based on the construction of a discrete analogue to a linear differential operator (cf. [37–39]). In different spaces based on these methods, the Sard problem was investigated by many authors (see, for example, [2, 5, 7, 9, 10, 14, 16–19, 24–28, 30, 31, 33, 36–39, 41, 42] and references therein).

The main aim of the present paper is to solve the Sard problem for quadrature formulas (1.1) in the space $W_2^{(m,m-1)}[0,1]$ using S. L. Sobolev's method with $N+1 \geq m$, i.e., to look for the coefficients C_β that satisfy the following equality

$$\big\|\mathring{\ell}|W_2^{(m,m-1)*}[0,1]\big\| = \inf_{C_\beta}\big\|\ell\,|W_2^{(m,m-1)*}[0,1]\big\|. \tag{1.5}$$

Thus, to construct Sard's optimal quadrature formula of the form (1.1) in the space $W_2^{(m,m-1)}[0,1]$, we need to solve the following problems.

Problem 1. Find the norm of the error functional ℓ of quadrature formulas (1.1) in the space $W_2^{(m,m-1)*}[0,1]$.

Problem 2. Find the coefficients C_β that satisfy equality (1.5).

It should be noted that Problems 1 and 2 were solved in [34] for the case $\omega = 0$, i.e., in the work [34] the optimal quadrature formulas of the form

$$\int_0^1 \varphi(x)\,\mathrm{d}x \cong \sum_{\beta=0}^{N} C_\beta \varphi(h\beta)$$

in the sense of Sard were constructed. In the sequel we will solve Problems 1 and 2 in the cases when $\omega \in \mathbb{Z}$ and $\omega \neq 0$.

The paper is organized as follows. In the second section the extremal function, which corresponds to the error functional ℓ, is given and, with its help, a representation of the norm of the error functional (1.2) is calculated, i.e., Problem 1 is solved. In Section 3 we obtain the system of linear equations for coefficients of the optimal quadrature formulas in the space $W_2^{(m,m-1)}[0,1]$. Moreover, the existence and uniqueness of the solution of this system are discussed. In Section 4, in the cases $m \geq 2$, the explicit formulas for the coefficients of the optimal quadrature formulas of the form (1.1) are found, i.e., Problem 2 is solved in the cases $m \geq 2$. The obtained optimal quadrature formulas are exact for any polynomial of order $\leq m-2$ and for the exponential function $\exp(-x)$. In Section 5 we solve Problem 2 in the case $m = 1$ and we calculate the norm of the error functional of the optimal quadrature formula in the $W_2^{(1,0)}[0,1]$ space. The obtained explicit formula for the norm of the error functional shows dependence on ω and h of the error of the optimal quadrature formula of the form (1.1) in $W_2^{(1,0)}[0,1]$ space. Finally, in Section 6 we present some numerical results which are confirm the obtained theoretical results of the present work.

2. Extremal function and norm of the error functional

To solve Problem 1, i.e., to get the explicit expression for the norm of the error functional (1.2) in the space $W_2^{(m,m-1)*}[0,1]$, we use the concept of the extremal function. The function ψ_ℓ is called *the extremal function* for the functional ℓ (see, [37]), if the following equality holds

$$(\ell, \psi_\ell) = \big\|\ell\,|\,W_2^{(m,m-1)*}[0,1]\big\| \cdot \big\|\psi_\ell\,|\,W_2^{(m,m-1)}[0,1]\big\|. \tag{2.1}$$

Since $W_2^{(m,m-1)}[0,1]$ is a Hilbert space, then the extremal function ψ_ℓ in this space, is found with the help of the general form of a linear continuous functional on Hilbert spaces given by the Riesz theorem. Then for the functional ℓ and for any $\varphi \in W_2^{(m,m-1)}[0,1]$ there exists the function $\psi_\ell \in W_2^{(m,m-1)}[0,1]$ for which the following equation holds

$$(\ell,\varphi) = \langle \psi_\ell, \varphi \rangle\,, \tag{2.2}$$

where

$$\langle \psi_\ell, \varphi \rangle = \int_0^1 \left(\overline{\psi}_\ell^{(m)}(x) + \overline{\psi}_\ell^{(m-1)}(x) \right) \left(\varphi^{(m)}(x) + \varphi^{(m-1)}(x) \right) \mathrm{d}x \tag{2.3}$$

is the inner product defined in the space $W_2^{(m,m-1)}[0,1]$.

From (2.2) taking into account (2.3) for the extremal function ψ_ℓ we get the following boundary value problem

$$\psi_\ell^{(2m)}(x) - \psi_\ell^{(2m-2)}(x) = (-1)^m \bar{\ell}(x), \tag{2.4}$$

$$\left(\psi_\ell^{(m+s)}(x) - \psi_\ell^{(m+s-2)}(x) \right) \Big|_{x=0}^{x=1} = 0, \quad s = \overline{1, m-1}, \tag{2.5}$$

$$\left(\psi_\ell^{(m)}(x) + \psi_\ell^{(m-1)}(x) \right) \Big|_{x=0}^{x=1} = 0, \tag{2.6}$$

where $\bar{\ell}$ is the conjugate to ℓ.

Theorem 2.1. *The solution of the boundary value problem* (2.4)–(2.6) *is the extremal function* ψ_ℓ *of the error functional* ℓ *and has the following form*

$$\psi_\ell(x) = (-1)^m \bar{\ell}(x) * G_m(x) + P_{m-2}(x) + d\,\mathrm{e}^{-x},$$

where

$$G_m(x) = \frac{\mathrm{sgn} x}{2} \left(\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} - \sum_{k=1}^{m-1} \frac{x^{2k-1}}{(2k-1)!} \right) \tag{2.7}$$

is a solution of the equation

$$G_m^{(2m)}(x) - G_m^{(2m-2)}(x) = \delta(x), \tag{2.8}$$

d *is any complex number and* $P_{m-2}(x)$ *is a polynomial of degree* $m-2$ *with complex coefficients, and* $*$ *is the operation of convolution.*

Theorem 2.1 can be proved as Theorem 2.1 in [34].

For the error functional (1.2) to be defined on the space $W_2^{(m,m-1)}(0,1)$ it is necessary to impose the following conditions

$$(\ell, x^\alpha) = 0, \quad \alpha = 0,1,2,\dots,m-2, \qquad (\ell, \mathrm{e}^{-x}) = 0. \tag{2.9}$$

Hence, it is clear that for existence of the quadrature formulas of the form (1.1) the condition $N+1 \geq m$ has to be met.

The equalities (2.9) mean that our quadrature formula is exact for the function e^{-x} and for any polynomial of degree $\leq m-2$.

Now, using Theorem 2.1 we will get the representation of the square of the norm of the error functional (1.2).

We recall that a convolution of two functions is defined by the formula

$$\varphi(x) * \psi(x) = \int_{-\infty}^{\infty} \varphi(x-y)\psi(y)\,\mathrm{d}y = \int_{-\infty}^{\infty} \varphi(y)\psi(x-y)\,\mathrm{d}y.$$

Taking into account the definition of convolution and equality (1.2) we calculate the convolution $\bar{\ell}(x) * G_m(x)$, i.e.,

$$\bar{\ell}(x)*G_m(x) = \int_{-\infty}^{\infty} \bar{\ell}(y)G_m(x-y)\,\mathrm{d}y = \int_0^1 \mathrm{e}^{-2\pi\mathrm{i}\omega y}G_m(x-y)\,\mathrm{d}y - \sum_{\beta=0}^{N} \bar{C}_\beta G_m(x-h\beta),$$

where $\bar{\ell}$ and $\bar{C}_\beta$ are conjugates to ℓ and C_β, respectively. Then keeping in mind (2.2), (2.3) and Theorem 2.1, we have

$$\|\ell\|^2 = (\ell, \psi_\ell) = \langle \psi_\ell, \psi_\ell \rangle = \int_{-\infty}^{\infty} \ell(x)\psi_\ell(x)\,\mathrm{d}x = (-1)^m \int_{-\infty}^{\infty} \ell(x)\cdot\big(\bar{\ell}(x) * G_m(x)\big)\,\mathrm{d}x,$$

i.e.,

$$\|\ell\|^2 = (-1)^m \int_{-\infty}^{\infty} \left(\mathrm{e}^{2\pi\mathrm{i}\omega x}\varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C_\beta \delta(x-h\beta)\right) \times\left(\int_0^1 \mathrm{e}^{-2\pi\mathrm{i}\omega y}G_m(x-y)\,\mathrm{d}y - \sum_{\gamma=0}^{N} \bar{C}_\gamma G_m(x-h\gamma)\right)\mathrm{d}x.$$

Hence we obtain

$$\|\ell\|^2 = (-1)^m \Bigg\{ \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta \overline{C}_\gamma G_m(h\beta - h\gamma) - \sum_{\beta=0}^{N} \int_0^1 \left(\overline{C}_\beta \mathrm{e}^{2\pi\mathrm{i}\omega x} + C_\beta \mathrm{e}^{-2\pi\mathrm{i}\omega x}\right) G_m(x-h\beta)\,\mathrm{d}x + \int_0^1\int_0^1 \mathrm{e}^{2\pi\mathrm{i}\omega x}\mathrm{e}^{-2\pi\mathrm{i}\omega y} G_m(x-y)\,\mathrm{d}x\,\mathrm{d}y \Bigg\}. \tag{2.10}$$

Now we show that the right hand side of (2.10) is real. Really, let $C_\beta = C_\beta^R + \mathrm{i}C_\beta^I$, $\mathrm{i}^2 = -1$, where C_β^R and C_β^I are real. Using Euler's formula $\mathrm{e}^{2\pi\mathrm{i}\omega x} = \cos 2\pi\omega x + \mathrm{i}\sin 2\pi\omega x$, we get the following equalities

$$\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta \bar{C}_\gamma G_m(h\beta - h\gamma) = \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} \left(C_\beta^R C_\gamma^R + C_\beta^I C_\gamma^I\right) G_m(h\beta - h\gamma),$$

$$\bar{C}_\beta \mathrm{e}^{2\pi\mathrm{i}\omega x} + C_\beta \mathrm{e}^{-2\pi\mathrm{i}\omega x} = 2C_\beta^R \cos 2\pi\omega x + 2C_\beta^I \sin 2\pi\omega x,$$

$$\int_0^1\int_0^1 \mathrm{e}^{2\pi\mathrm{i}\omega x}\mathrm{e}^{-2\pi\mathrm{i}\omega y} G_m(x-y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\int_0^1 \cos[2\pi\omega(x-y)] G_m(x-y)\,\mathrm{d}x\,\mathrm{d}y.$$

Keeping in mind the last three equalities, from (2.10) for the norm of the error functional we have

$$\|\ell\|^2 = (-1)^m \Bigg[\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N}(C_\beta^R C_\gamma^R + C_\beta^I C_\gamma^I)\ G_m(h\beta - h\gamma)$$
$$-2\sum_{\beta=0}^{N} C_\beta^R \int_0^1 \cos 2\pi\omega x\ G_m(x-h\beta)\,\mathrm{d}x$$
$$-2\sum_{\beta=0}^{N} C_\beta^I \int_0^1 \sin 2\pi\omega x\ G_m(x-h\beta)\,\mathrm{d}x$$
$$+\int_0^1\int_0^1 \cos[2\pi\omega(x-y)]\ G_m(x-y)\,\mathrm{d}x\,\mathrm{d}y \Bigg], \tag{2.11}$$

and from (2.9), we have the following equalities

$$\sum_{\beta=0}^{N} C_\beta^R (h\beta)^\alpha = \int_0^1 x^\alpha \cos 2\pi\omega x\,\mathrm{d}x, \quad \alpha = 0,1,2,\dots,m-2, \tag{2.12}$$

$$\sum_{\beta=0}^{N} C_\beta^R \mathrm{e}^{-h\beta} = \int_0^1 \mathrm{e}^{-x}\cos 2\pi\omega x\,\mathrm{d}x, \tag{2.13}$$

$$\sum_{\beta=0}^{N} C_\beta^I (h\beta)^\alpha = \int_0^1 x^\alpha \sin 2\pi\omega x\,\mathrm{d}x, \quad \alpha = 0,1,2,\dots,m-2, \tag{2.14}$$

$$\sum_{\beta=0}^{N} C_\beta^I \mathrm{e}^{-h\beta} = \int_0^1 \mathrm{e}^{-x}\sin 2\pi\omega x\,\mathrm{d}x, \tag{2.15}$$

Thus, Problem 1 is solved. Further in Sections 3 and 4 we solve Problem 2.

3. The system for coefficients of optimal quadrature formulas (1.1) in the space $W_2^{(m,m-1)}[0,1]$

To find the minimum of the expression (2.11) under the conditions (2.12)–(2.15) we apply the Lagrange method.

Consider the function

$$\Psi(C_0^R,\dots,C_N^R,C_0^I,\dots,C_N^I,a_0^R,\dots,a_{m-2}^R,a_0^I,\dots,a_{m-2}^I,d^R,d^I)$$
$$= \|\ell\|^2 - 2(-1)^m \sum_{\alpha=0}^{m-2} a_\alpha^R \left(\int_0^1 x^\alpha \cos 2\pi\omega x\,\mathrm{d}x - \sum_{\beta=0}^{N} C_\beta^R (h\beta)^\alpha\right)$$
$$-2(-1)^m \sum_{\alpha=0}^{m-2} a_\alpha^I \left(\int_0^1 x^\alpha \sin 2\pi\omega x\,\mathrm{d}x - \sum_{\beta=0}^{N} C_\beta^I (h\beta)^\alpha\right)$$
$$-2(-1)^m d^R \left(\int_0^1 \mathrm{e}^{-x}\cos 2\pi\omega x\,\mathrm{d}x - \sum_{\beta=0}^{N} C_\beta^R \mathrm{e}^{-h\beta}\right)$$

$$-2(-1)^m d^I \left(\int_0^1 \mathrm{e}^{-x} \sin 2\pi\omega x \, \mathrm{d}x - \sum_{\beta=0}^{N} C_\beta^I \mathrm{e}^{-h\beta} \right).$$

Equating to 0 the partial derivatives of Ψ with respect to C_β^R, C_β^I, $(\beta = \overline{0,N})$, a_α^R, a_α^I, $(\alpha = \overline{0,m-2})$, d^R, and d^I, we get the following system of linear equations, for $\alpha = 0,1,\ldots,m-2$ and $\beta = 0,1,\ldots,N$,

$$\sum_{\gamma=0}^{N} C_\gamma^R G_m(h\beta - h\gamma) + \sum_{\alpha=0}^{m-2} a_\alpha^R (h\beta)^\alpha + d^R \mathrm{e}^{-h\beta} = \int_0^1 \cos 2\pi\omega x G_m(x - h\beta)\,\mathrm{d}x, \quad (3.1)$$

$$\sum_{\gamma=0}^{N} C_\gamma^R (h\gamma)^\alpha = \int_0^1 x^\alpha \cos 2\pi\omega x \, \mathrm{d}x, \quad (3.2)$$

$$\sum_{\gamma=0}^{N} C_\gamma^R \mathrm{e}^{-h\gamma} = \int_0^1 \mathrm{e}^{-x} \cos 2\pi\omega x \, \mathrm{d}x, \quad (3.3)$$

$$\sum_{\gamma=0}^{N} C_\gamma^I G_m(h\beta - h\gamma) + \sum_{\alpha=0}^{m-2} a_\alpha^I (h\beta)^\alpha + d^I \mathrm{e}^{-h\beta} = \int_0^1 \sin 2\pi\omega x G_m(x - h\beta)\,\mathrm{d}x, \quad (3.4)$$

$$\sum_{\gamma=0}^{N} C_\gamma^I (h\gamma)^\alpha = \int_0^1 x^\alpha \sin 2\pi\omega x \, \mathrm{d}x, \quad (3.5)$$

$$\sum_{\gamma=0}^{N} C_\gamma^I \mathrm{e}^{-h\gamma} = \int_0^1 \mathrm{e}^{-x} \sin 2\pi\omega x \, \mathrm{d}x. \quad (3.6)$$

Now, multiplying both sides of (3.4), (3.5), and (3.6) by i and adding to both sides of (3.1), (3.2), and (3.3), respectively, using notations $C_\beta = C_\beta^R + \mathrm{i}C_\beta^I$ $(\beta = \overline{0,N})$, $a_\alpha = a_\alpha^R + \mathrm{i}a_\alpha^I$ $(\alpha = \overline{0,m-2})$, and $d = d^R + \mathrm{i}d^I$, for the coefficients of the optimal quadrature formulas of the form (1.1) we get the following system of $N+m+1$ linear equations, for $\alpha = 0,1,\ldots,m-2$ and $\beta = 0,1,\ldots,N$,

$$\sum_{\gamma=0}^{N} C_\gamma G_m(h\beta - h\gamma) + \sum_{\alpha=0}^{m-2} a_\alpha (h\beta)^\alpha + d\,\mathrm{e}^{-h\beta} = f_m(h\beta), \quad (3.7)$$

$$\sum_{\gamma=0}^{N} C_\gamma (h\gamma)^\alpha = \int_0^1 \mathrm{e}^{2\pi\mathrm{i}\omega x} x^\alpha \, \mathrm{d}x, \quad (3.8)$$

$$\sum_{\gamma=0}^{N} C_\gamma \mathrm{e}^{-h\gamma} = \int_0^1 \mathrm{e}^{2\pi\mathrm{i}\omega x} \mathrm{e}^{-x} \, \mathrm{d}x, \quad (3.9)$$

where $G_m(x)$ is defined by equality (2.7),

$$f_m(h\beta) = \int_0^1 \mathrm{e}^{2\pi\mathrm{i}\omega x} G_m(x - h\beta)\,\mathrm{d}x. \quad (3.10)$$

We note that the system (3.7)–(3.9) has a unique solution when $N+1 \geq m$ and this solution gives the minimum to $\|\ell\|^2$ under the conditions (3.8) and (3.9). The uniqueness of the solution of this system is obtained from Theorems 3.1 and 3.2 of [34].

From (2.11) and these theorems from [34], it follows that the square of the norm of the error functional ℓ, being a quadratic functions of the coefficients C_β has a unique minimum in some concrete value of $C_\beta = \mathring{C}_\beta$.

As it was said in the first section, the quadrature formula with the coefficients $\mathring{C}_\beta$ ($\beta = \overline{0,N}$), corresponding to this minimum, is called *the optimal quadrature formula in the sense of Sard*, and $\mathring{C}_\beta$ ($\beta = \overline{0,N}$) are called *the optimal coefficients*.

Below, for the purposes of convenience, the optimal coefficients $\mathring{C}_\beta$ will be denoted as C_β.

4. Coefficients of optimal quadrature formulas (1.1)

In the present section we solve the system (3.7)–(3.9) and we find the explicit formulas for the optimal coefficients C_β. Here we use a similar method to the one suggested by S. L. Sobolev [38] for finding the coefficients of optimal quadrature formulas in the space $L_2^{(m)}(0,1)$. Here the main concept used is that of functions of discrete argument and operations on them. Theory of discrete argument functions is given in [37,39]. For the purposes of completeness we give some definitions about functions of discrete argument.

Suppose that $\varphi(x)$ and $\psi(x)$ are real-valued functions of real variable and are defined in real line $\mathbb{R}$.

Definition 4.1. A function $\varphi(h\beta)$ is called *function of discrete argument* if it is defined on some set of integer values of β.

Definition 4.2. We define *the inner product* of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ as the following number

$$[\varphi(h\beta), \psi(h\beta)] = \sum_{\beta=-\infty}^{\infty} \varphi(h\beta) \cdot \psi(h\beta),$$

if the series on the right hand side of the last equality converges absolutely.

Definition 4.3. We define *convolution* of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ as the inner product

$$\varphi(h\beta) * \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma).$$

Now, we return to our problem.

Suppose that $C_\beta = 0$ when $\beta < 0$ and $\beta > N$. Using the above mentioned definitions, we rewrite the system (3.7)–(3.9) in the following convolution form

$$G_m(h\beta) * C_\beta + P_{m-2}(h\beta) + d\,\mathrm{e}^{-h\beta} = f_m(h\beta), \quad \beta = 0, 1, \ldots, N, \tag{4.1}$$

$$\sum_{\beta=0}^{N} C_\beta \cdot (h\beta)^\alpha = g_\alpha, \quad \alpha = 0, 1, \ldots, m-2, \tag{4.2}$$

$$\sum_{\beta=0}^{N} C_\beta \cdot \mathrm{e}^{-h\beta} = \frac{\mathrm{e}^{-1} - 1}{2\pi\mathrm{i}\omega - 1}, \tag{4.3}$$

where $P_{m-2}(h\beta) = \sum_{\alpha=0}^{m-2} a_\alpha (h\beta)^\alpha$ is a polynomial of degree $m-2$,

$$f_m(h\beta) = \int_0^1 \mathrm{e}^{2\pi \mathrm{i}\omega x} G_m(x - h\beta)\,\mathrm{d}x, \tag{4.4}$$

$$g_\alpha = \int_0^1 \mathrm{e}^{2\pi \mathrm{i}\omega x} x^\alpha \,\mathrm{d}x = \frac{1}{2\pi \mathrm{i}\omega} + \sum_{k=1}^{\alpha-1} (-1)^k \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{(2\pi\mathrm{i}\omega)^{k+1}} \tag{4.5}$$

for $\alpha = 1, 2, \ldots, m-2$, $g_0 = 0$, d is a constant, and $G_m(x)$ is defined by (2.7).

Consider the following problem:

Problem 3. Find a discrete function C_β, a polynomial $P_{m-2}(h\beta)$ of degree $m-2$ and a constant d which satisfy the system (4.1)–(4.3) for the given $f_m(h\beta)$.

Further we investigate Problem 3 and instead of C_β we introduce the functions

$$v(h\beta) = G_m(h\beta) * C_\beta \quad \text{and} \quad u(h\beta) = v\,(h\beta) + P_{m-2}(h\beta) + d\,\mathrm{e}^{-h\beta}. \tag{4.6}$$

In this statement it is necessary to express the coefficients C_β by the function $u(h\beta)$. For this, we need such an operator $D_m(h\beta)$ which satisfies the equality

$$D_m(h\beta) * G_m(h\beta) = \delta_{\mathrm{d}}(h\beta), \tag{4.7}$$

where $\delta_{\mathrm{d}}(h\beta)$ is equal to 0 when $\beta \neq 0$ and is equal to 1 when $\beta = 0$, i.e., $\delta_{\mathrm{d}}(h\beta)$ is the discrete delta-function.

In [32,35] the discrete analogue $D_m(h\beta)$ of the operator $\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}} - \frac{\mathrm{d}^{2m-2}}{\mathrm{d}x^{2m-2}}$, which satisfies equation (4.7) is constructed and its some properties are investigated.

The following results are proved in [32,35].

Theorem 4.1. *The discrete analogues to the differential operator* $\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}} - \frac{\mathrm{d}^{2m-2}}{\mathrm{d}x^{2m-2}}$ *satisfying the equation* (4.7) *has the form*

$$D_m(h\beta) = \frac{1}{p_{2m-2}^{(2m-2)}} \begin{cases} \sum\limits_{k=1}^{m-1} A_k \lambda_k^{|\beta|-1}, & |\beta| \geq 2, \\ -2\mathrm{e}^h + \sum\limits_{k=1}^{m-1} A_k, & |\beta| = 1, \\ 2C + \sum\limits_{k=1}^{m-1} \frac{A_k}{\lambda_k}, & \beta = 0, \end{cases} \tag{4.8}$$

where

$$C = 1 + (2m-2)\mathrm{e}^h + \mathrm{e}^{2h} + \frac{\mathrm{e}^h p_{2m-3}^{(2m-2)}}{p_{2m-2}^{(2m-2)}},$$

$$A_k = \frac{2(1-\lambda_k)^{2m-2}[\lambda_k(\mathrm{e}^{2h}+1) - \mathrm{e}^h(\lambda_k^2+1)]p_{2m-2}^{(2m-2)}}{\lambda_k\ \mathcal{P}'_{2m-2}(\lambda_k)},$$

and

$$\mathcal{P}_{2m-2}(\lambda) = \sum_{s=0}^{2m-2} p_s^{(2m-2)}\lambda^s \tag{4.9}$$
$$= (1-\mathrm{e}^{2h})(1-\lambda)^{2m-2} - 2[\lambda(\mathrm{e}^{2h}+1) - \mathrm{e}^{h}(\lambda^2+1)]$$
$$\times \Big[h(1-\lambda)^{2m-4} + \frac{h^3(1-\lambda)^{2m-6}E_2(\lambda)}{3!} + \cdots + \frac{h^{2m-3}E_{2m-4}(\lambda)}{(2m-3)!}\Big].$$

Here, $p_{2m-2}^{(2m-2)}$ *and* $p_{2m-3}^{(2m-2)}$ *are the coefficients of the polynomial* $\mathcal{P}_{2m-2}(\lambda)$ *defined by equality* (4.9)*,* λ_k *are roots of the polynomial* $\mathcal{P}_{2m-2}(\lambda)$*,* $|\lambda_k| < 1$*, and* $E_k(\lambda)$ *is the Euler-Frobenius polynomial of degree* k (see [39]).

Theorem 4.2. *The discrete analogue* $D_m(h\beta)$ *of the differential operator*

$$\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}} - \frac{\mathrm{d}^{2m-2}}{\mathrm{d}x^{2m-2}}$$

satisfies the following equalities

1) $D_m(h\beta) * \mathrm{e}^{h\beta} = 0$,
2) $D_m(h\beta) * \mathrm{e}^{-h\beta} = 0$,
3) $D_m(h\beta) * (h\beta)^n = 0$, $n \le 2m-3$,
4) $D_m(h\beta) * G_m(h\beta) = \delta_{\mathrm{d}}(h\beta)$.

Here, $G_m(h\beta)$ *is the function of discrete argument corresponding to the function* $G_m(x)$*, defined by equality* (2.7) *and* $\delta_{\mathrm{d}}(h\beta)$ *is the discrete delta function.*

Then taking into account (4.6), (4.7) and Theorems 4.1 and 4.2, for the optimal coefficients we have

$$C_\beta = D_m(h\beta) * u(h\beta). \tag{4.10}$$

Thus, if we find the function $u(h\beta)$, then the optimal coefficients can be obtained from equality (4.10).

To calculate this convolution, it is required to find the representation of the function $u(h\beta)$ for all integer values of β. From equality (4.1), we get that $u(h\beta) = f_m(h\beta)$ when $h\beta \in [0,1]$. Now we need to find the representation of the function $u(h\beta)$ when $\beta < 0$ and $\beta > N$.

Since $C_\beta = 0$ when $h\beta \notin [0,1]$ then $C_\beta = D_m(h\beta) * u(h\beta) = 0$, $h\beta \notin [0,1]$.

Now, we calculate the convolution $v(h\beta) = G_m(h\beta) * C_\beta$ when $h\beta \notin [0,1]$.

Suppose $\beta < 0$ then, taking into account equalities (2.7), (4.2), (4.3), we have

$$v(h\beta) = G_m(h\beta) * C_\beta$$
$$= -\frac{1}{2}\sum_{\gamma=0}^{N} C_\gamma\left(\frac{\mathrm{e}^{h\beta-h\gamma} - \mathrm{e}^{-h\beta+h\gamma}}{2} - \sum_{k=1}^{m-1}\frac{(h\beta-h\gamma)^{2k-1}}{(2k-1)!}\right)$$
$$= -\frac{\mathrm{e}^{h\beta}}{4}\frac{\mathrm{e}^{-1}-1}{2\pi\mathrm{i}\omega - 1} + De^{-h\beta} + R_{2m-3}(h\beta) + Q_{m-2}(h\beta), \tag{4.11}$$

where

$$R_{2m-3}(h\beta)=\frac{1}{2}\left(\sum_{k=1}^{\left[\frac{m+1}{2}\right]-1}\sum_{\alpha=0}^{2k-1}\frac{(h\beta)^{2k-1-\alpha}(-1)^{\alpha}}{(2k-1-\alpha)!\alpha!}g_{\alpha}\right.$$

$$\left.+\sum_{k=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{\alpha=0}^{m-2}\frac{(h\beta)^{2k-1-\alpha}(-1)^{\alpha}}{(2k-1-\alpha)!\alpha!}g_{\alpha}\right) \tag{4.12}$$

is a polynomial of degree $2m-3$ in $(h\beta)$,

$$Q_{m-2}(h\beta)=\frac{1}{2}\sum_{k=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{\alpha=m-1}^{2k-1}\frac{(h\beta)^{2k-1-\alpha}(-1)^{\alpha}}{(2k-1-\alpha)!\alpha!}\sum_{\gamma=0}^{N}C_{\gamma}(h\gamma)^{\alpha} \tag{4.13}$$

is an unknown polynomial of degree $m-2$ also in $(h\beta)$, and

$$D=\frac{1}{4}\sum_{\gamma=0}^{N}C_{\gamma}\mathrm{e}^{h\gamma}. \tag{4.14}$$

Similarly, in the case $\beta>N$, for the convolution $v(h\beta)=G_m(h\beta)*C_{\beta}$, we obtain

$$v(h\beta)=\frac{\mathrm{e}^{h\beta}}{4}\frac{\mathrm{e}^{-1}-1}{2\pi\mathrm{i}\omega-1}-De^{-h\beta}-R_{2m-3}(h\beta)-Q_{m-2}(h\beta). \tag{4.15}$$

We denote

$$Q_{m-2}^{(-)}(h\beta)=P_{m-2}(h\beta)+Q_{m-2}(h\beta),\quad a^{-}=d+D, \tag{4.16}$$

$$Q_{m-2}^{(+)}(h\beta)=P_{m-2}(h\beta)-Q_{m-2}(h\beta),\quad a^{+}=d-D, \tag{4.17}$$

and, taking into account (4.11), (4.15), (4.6), we get the following problem.

Problem 4. Find the solution of the equation

$$D_m(h\beta)*u(h\beta)=0,\quad h\beta\notin[0,1], \tag{4.18}$$

having the form

$$u(h\beta)=\begin{cases}-\dfrac{\mathrm{e}^{h\beta}}{4}\dfrac{\mathrm{e}^{-1}-1}{2\pi\mathrm{i}\omega-1}+a^{-}\mathrm{e}^{-h\beta}+R_{2m-3}(h\beta)+Q_{m-2}^{(-)}(h\beta), & \beta<0,\\ f_m(h\beta), & 0\le\beta\le N,\\ \dfrac{\mathrm{e}^{h\beta}}{4}\dfrac{\mathrm{e}^{-1}-1}{2\pi\mathrm{i}\omega-1}+a^{+}\mathrm{e}^{-h\beta}-R_{2m-3}(h\beta)+Q_{m-2}^{(+)}(h\beta), & \beta>N.\end{cases}$$

Here, $Q_{m-2}^{(-)}(h\beta)$ and $Q_{m-2}^{(+)}(h\beta)$ are unknown polynomials of degree $m-2$ with respect to $(h\beta)$, a^{-} and a^{+} are unknown constants.

If we find $Q_{m-2}^{(-)}(h\beta)$, $Q_{m-2}^{(+)}(h\beta)$, a^{-} and a^{+}, then from (4.16), (4.17) we have

$$P_{m-2}(h\beta)=\frac{1}{2}\left(Q_{m-2}^{(-)}(h\beta)+Q_{m-2}^{(+)}(h\beta)\right),\quad d=\frac{1}{2}(a^{-}+a^{+}),$$

$$Q_{m-2}(h\beta) = \frac{1}{2}\left(Q^{(-)}_{m-2}(h\beta) - Q^{(+)}_{m-2}(h\beta)\right), \quad D = \frac{1}{2}(a^- - a^+).$$

Unknowns $Q^{(-)}_{m-2}(h\beta)$, $Q^{(+)}_{m-2}(h\beta)$, a^- and a^+ can be found from the equation (4.18), using the function $D_m(h\beta)$. Then we can obtain the explicit form of the function $u(h\beta)$ and find the optimal coefficients C_β. Thus, Problem 4 and, respectively, Problem 3 can be solved.

But here we will not find $Q^{(-)}_{m-2}(h\beta)$, $Q^{(+)}_{m-2}(h\beta)$, a^- and a^+. Instead of them, using $D_m(h\beta)$ and $u(h\beta)$, taking into account (4.10), we find now the expressions for the optimal coefficients C_β when $\beta = 1, \ldots, N-1$.

We denote

$$a_k = \frac{A_k}{\lambda_k p}\sum_{\gamma=1}^{\infty}\lambda_k^\gamma\Bigg(-\frac{\mathrm{e}^{-h\gamma}}{4}\frac{\mathrm{e}^{-1}-1}{2\pi\mathrm{i}\omega-1} + R_{2m-3}(-h\gamma) + Q^{(-)}_{m-2}(-h\gamma) + a^-\mathrm{e}^{h\gamma} - f_m(-h\gamma)\Bigg), \tag{4.19}$$

$$b_k = \frac{A_k}{\lambda_k p}\sum_{\gamma=1}^{\infty}\lambda_k^\gamma\Bigg(\frac{\mathrm{e}^{h\gamma+1}}{4}\frac{\mathrm{e}^{-1}-1}{2\pi\mathrm{i}\omega-1} - R_{2m-3}(1+h\gamma) + Q^{(+)}_{m-2}(1+h\gamma) + a^+\mathrm{e}^{-1-h\gamma} - f_m(1+h\gamma)\Bigg), \tag{4.20}$$

where λ_k are roots and p is the leading coefficient of the polynomial $\mathcal{P}_{2m-2}(\lambda)$ of degree $2m-2$ defined by (4.9) and $|\lambda_k| < 1$. The series in the notations (4.19), (4.20) are convergent.

The following statement holds:

Theorem 4.3 (Theorem 3, [31]). *The coefficients of optimal quadrature formulas in the sense of Sard of the form* (1.1) *in the space* $W_2^{(m,m-1)}[0,1]$ *have the following form*

$$C_\beta = D_m(h\beta) * f_m(h\beta) + \sum_{k=1}^{m-1}\left(a_k\lambda_k^\beta + b_k\lambda_k^{N-\beta}\right), \quad \beta = 1, 2, \ldots, N-1, \tag{4.21}$$

where a_k *and* b_k *are unknowns and have the form* (4.19) *and* (4.20) *respectively,* λ_k *are the roots of the polynomial* $\mathcal{P}_{2m-2}(\lambda)$ *which is defined by equality* (4.9) *and* $|\lambda_k| < 1$.

From Theorem 4.3, it is clear that to obtain the explicit forms of the optimal coefficients C_β in the space $W_2^{(m,m-1)}[0,1]$ it is sufficient to find a_k and b_k ($k = \overline{1, m-1}$). But here we will not calculate series (4.19) and (4.20). Instead of that substituting equality (4.21) into (4.1) we obtain the identity with respect to $(h\beta)$. Whence, equating the corresponding coefficients in the left and the right hand sides of equation (4.1) and using (4.2) when $\alpha = 1, 2, \ldots, m-2$, we find a_k and b_k. The coefficients C_0 and C_N can be found from (4.2) when $\alpha = 0$ and (4.3), respectively. Below we do it.

In the present section we solve the system (4.1)–(4.3) for any $m \geq 2$ and for natural N that $N+1 \geq m$. As it was mentioned above, it is sufficient to find a_k and b_k ($k = \overline{1, m-1}$) in (4.21).

The case $m = 1$ we consider in the next section. In the case $m \geq 2$ the following results hold:

Theorem 4.4. *The coefficients of optimal quadrature formulas of the form* (1.1) *with the error functional* (1.2) *and with equal spaced nodes in the space* $W_2^{(m,m-1)}[0,1]$ *when* $m \geq 2$, $N+1 \geq m$ *and* $\omega h \notin \mathbb{Z}$ *are expressed by formulas*

$$C_0 = \frac{K e^{4\pi i \omega h}}{(e^{2\pi i\omega h} - e^h)(e^{2\pi i \omega h} - 1)} + \frac{2\pi i\omega(1-e^h) - 1}{2\pi i \omega(1 - 2\pi i\omega)(1-e^h)} + \sum_{k=1}^{m-1}\left(\frac{a_k\lambda_k^2}{(1-\lambda_k)(e^h-\lambda_k)} + \frac{b_k \lambda_k^N}{(1-\lambda_k)(1-\lambda_k e^h)}\right),$$

$$C_\beta = e^{2\pi i\omega h\beta} K + \sum_{k=1}^{m-1}\left(a_k\lambda_k^\beta + b_k\lambda_k^{N-\beta}\right), \qquad \beta = \overline{1, N-1},$$

$$C_N = \frac{K e^h}{(e^{2\pi i\omega h} - e^h)(e^{2\pi i \omega h} - 1)} + \frac{2\pi i\omega(e^h - 1) - e^h}{2\pi i \omega(1 - 2\pi i\omega)(1-e^h)} + e^h \sum_{k=1}^{m-1}\left(\frac{a_k\lambda_k^N}{(1-\lambda_k)(e^h-\lambda_k)} + \frac{b_k \lambda_k^2}{(1-\lambda_k)(1-\lambda_k e^h)}\right),$$

where a_k *and* b_k $(k = \overline{1, m-1})$ *are defined by the following system of* $2m-2$ *linear equations*

$$\sum_{k=1}^{m-1}\frac{a_k\lambda_k}{(\lambda_k - 1)(\lambda_k - e^h)} + \sum_{k=1}^{m-1}\frac{b_k\lambda_k^{N+1}}{(\lambda_k-1)(\lambda_k e^h - 1)} = \frac{1}{2\pi i\omega(1-2\pi i\omega)(1-e^h)} + \frac{K e^{2\pi i\omega h}}{(e^{2\pi i\omega h} - e^h)(1 - e^{2\pi i \omega h})},$$

$$\sum_{k=1}^{m-1}\frac{a_k\lambda_k^{N+1}}{(\lambda_k - 1)(\lambda_k - e^h)} + \sum_{k=1}^{m-1}\frac{b_k\lambda_k}{(\lambda_k-1)(\lambda_k e^h - 1)} = \frac{1}{2\pi i\omega(1-2\pi i\omega)(1-e^h)} + \frac{K e^{2\pi i\omega h}}{(e^{2\pi i\omega h} - e^h)(1 - e^{2\pi i \omega h})},$$

$$\sum_{k=1}^{m-1} a_k \sum_{t=1}^{j}\frac{\lambda_k \Delta^t 0^j}{(\lambda_k - 1)^{t+1}} + \sum_{k=1}^{m-1} b_k \sum_{t=1}^{j}\frac{\lambda_k^{N+t}\Delta^t 0^j}{(1-\lambda_k)^{t+1}} = \frac{j!h}{(2\pi i\omega h)^{j+1}} - \sum_{t=1}^{j}\frac{K e^{2\pi i\omega h}\Delta^t 0^j}{(e^{2\pi i\omega h} - 1)^{t+1}}, \qquad j = \overline{1, m-2},$$

and

$$\sum_{k=1}^{m-1} a_k\left[h^j\sum_{t=1}^{j}\frac{\lambda_k^t\Delta^t 0^j}{(1-\lambda_k)^{t+1}} - \sum_{i=1}^{j} h^i C_j^i \sum_{t=1}^{i}\frac{\lambda_k^{N+t}\Delta^t 0^i}{(1-\lambda_k)^{t+1}}\right] + \sum_{k=1}^{m-1} b_k\left[h^j\sum_{t=1}^{j}\frac{\lambda_k^{N+1}\Delta^t 0^j}{(\lambda_k - 1)^{t+1}} - \sum_{i=1}^{j} h^i C_j^i \sum_{t=1}^{i}\frac{\lambda_k\Delta^t 0^i}{(\lambda_k-1)^{t+1}}\right]$$

$$= \sum_{k=1}^{j-1}(-1)^k \frac{j(j-1)\cdots(j-k+1)}{(2\pi \mathrm{i}\omega)^{k+1}} + K\sum_{i=1}^{j-1} h^i C_j^i \sum_{t=1}^{i} \frac{\mathrm{e}^{2\pi \mathrm{i}\omega h t}\Delta^t 0^i}{(1-\mathrm{e}^{2\pi \mathrm{i}\omega h})^{t+1}},$$

for $j=\overline{1,m-2}$, *where*

$$K = \frac{L}{p_{2m-2}^{(2m-2)}}\left\{\sum_{k=1}^{m-1}\left[\frac{2A_k}{\lambda_k}\cdot\frac{1-\lambda_k\cos(2\pi\omega h)}{\lambda_k^2+1-2\lambda_k\cos(2\pi\omega h)} - \frac{A_k}{\lambda_k}\right] - 4\mathrm{e}^h\cos(2\pi\omega h) + 2C\right\},$$

$$L = \frac{1}{(2\pi \mathrm{i}\omega)^2-1} - \sum_{k=1}^{m-1}\frac{1}{(2\pi \mathrm{i}\omega)^{2k}},$$

λ_k *are the roots of the polynomial* (4.9), $|\lambda_k|<1$, *and* $p_{2m-2}^{(2m-2)}$, A_k *and* C *are defined in Theorem* 4.1.

Theorem 4.5. *The coefficients of optimal quadrature formulas of the form* (1.1) *with the error functional* (1.2) *and with equal spaced nodes in the space* $W_2^{(m,m-1)}[0,1]$ *when* $m\ge 2$, $N+1\ge m$ *and* $\omega h\in\mathbb{Z}$, $\omega\ne 0$, *are expressed by formulas*

$$C_0 = \frac{2\pi \mathrm{i}\omega(1-\mathrm{e}^h)-1}{2\pi \mathrm{i}\omega(1-2\pi \mathrm{i}\omega)(1-\mathrm{e}^h)} + \sum_{k=1}^{m-1}\left[\frac{a_k\lambda_k^2}{(1-\lambda_k)(\mathrm{e}^h-\lambda_k)} + \frac{b_k\lambda_k^N}{(1-\lambda_k)(1-\lambda_k\mathrm{e}^h)}\right],$$

$$C_\beta = \sum_{k=1}^{m-1}\left(a_k\lambda_k^\beta + b_k\lambda_k^{N-\beta}\right), \qquad \beta=\overline{1,N-1},$$

$$C_N = \frac{2\pi \mathrm{i}\omega(\mathrm{e}^h-1)-\mathrm{e}^h}{2\pi \mathrm{i}\omega(1-2\pi \mathrm{i}\omega)(1-\mathrm{e}^h)} + \mathrm{e}^h\sum_{k=1}^{m-1}\left[\frac{a_k\lambda_k^N}{(1-\lambda_k)(\mathrm{e}^h-\lambda_k)} + \frac{b_k\lambda_k^2}{(1-\lambda_k)(1-\lambda_k\mathrm{e}^h)}\right],$$

where a_k *and* b_k, $k=\overline{1,m-1}$, *are defined by the following system of* $2m-2$ *linear equations*

$$\sum_{k=1}^{m-1}\frac{a_k\lambda_k}{(\lambda_k-1)(\lambda_k-\mathrm{e}^h)} + \sum_{k=1}^{m-1}\frac{b_k\lambda_k^{N+1}}{(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)} = \frac{1}{2\pi \mathrm{i}\omega(1-2\pi \mathrm{i}\omega)(1-\mathrm{e}^h)},$$

$$\sum_{k=1}^{m-1}\frac{a_k\lambda_k^{N+1}}{(\lambda_k-1)(\lambda_k-\mathrm{e}^h)} + \sum_{k=1}^{m-1}\frac{b_k\lambda_k}{(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)} = \frac{1}{2\pi \mathrm{i}\omega(1-2\pi \mathrm{i}\omega)(1-\mathrm{e}^h)},$$

$$\sum_{k=1}^{m-1}a_k\sum_{t=1}^{j}\frac{\lambda_k\Delta^t 0^j}{(\lambda_k-1)^{t+1}} + \sum_{k=1}^{m-1}b_k\sum_{t=1}^{j}\frac{\lambda_k^{N+t}\Delta^t 0^j}{(1-\lambda_k)^{t+1}} = \frac{j!h}{(2\pi \mathrm{i}\omega h)^{j+1}}, \quad j=\overline{1,m-2},$$

and

$$\sum_{k=1}^{m-1}a_k\left[h^j\sum_{t=1}^{j}\frac{\lambda_k^t\Delta^t 0^j}{(1-\lambda_k)^{t+1}} - \sum_{i=1}^{j}h^iC_j^i\sum_{t=1}^{i}\frac{\lambda_k^{N+t}\Delta^t 0^i}{(1-\lambda_k)^{t+1}}\right]$$

$$+\sum_{k=1}^{m-1}b_k\left[h^j\sum_{t=1}^{j}\frac{\lambda_k^{N+1}\Delta^t 0^j}{(\lambda_k-1)^{t+1}} - \sum_{i=1}^{j}h^iC_j^i\sum_{t=1}^{i}\frac{\lambda_k\Delta^t 0^i}{(\lambda_k-1)^{t+1}}\right]$$

$$= \sum_{k=1}^{j-1}(-1)^k\frac{j(j-1)\cdots(j-k+1)}{(2\pi \mathrm{i}\omega)^{k+1}}, \qquad j=\overline{1,m-2}.$$

Here, λ_k are the roots of the polynomial (4.9) *and $|\lambda_k|<1$, A_k and C are defined in Theorem* 4.1.

In order to prove Theorem 4.4 we use the following formulas (cf. [13], [12])

$$\begin{aligned}\sum_{\gamma=0}^{n-1} q^\gamma \gamma^k &= \frac{1}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^i \Delta^i 0^k - \frac{q^n}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^i \Delta^i\gamma^k\big|_{\gamma=n},\\ \sum_{\gamma=0}^{\beta-1}\gamma^k &= \sum_{j=1}^{k+1}\frac{k!\,B_{k+1-j}}{j!\,(k+1-j)!}\,\beta^j,\end{aligned} \tag{4.22}$$

where $\Delta^i 0^k = \sum\limits_{\ell=1}^{i}(-1)^{i-\ell}C_i^\ell \ell^k$, $\Delta^i\gamma^k$ is the finite difference of order i of γ^k, and B_{k+1-j} are the Bernoulli numbers, as well as

$$\Delta^\alpha x^\nu = \sum_{p=0}^{\nu} C_\nu^p \Delta^\alpha 0^p x^{\nu-p}. \tag{4.23}$$

Proof of Theorem 4.4. Using the binomial formula in equality (4.4), for $f_m(h\beta)$ we deduce

$$\begin{aligned} f_m(h\beta) = {}& \frac{(\mathrm{e}+1)\mathrm{e}^{-h\beta}}{4(2\pi \mathrm{i}\omega+1)} - \frac{(1+\mathrm{e}^{-1})\mathrm{e}^{h\beta}}{4(2\pi\mathrm{i}\omega-1)} + \mathrm{e}^{2\pi\mathrm{i}\omega h\beta}\left[\frac{1}{(2\pi\mathrm{i}\omega)^2-1} - \sum_{k=1}^{m-1}\frac{1}{(2\pi\mathrm{i}\omega)^{2k}}\right]\\ &+ \sum_{k=1}^{\left[\frac{m+1}{2}\right]-1}\sum_{\alpha=0}^{2k-1}\frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha}{2(2k-1-\alpha)!\alpha!}g_\alpha + \sum_{k=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{\alpha=0}^{m-2}\frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha}{2(2k-1-\alpha)!\alpha!}g_\alpha\\ &+ \sum_{k=\left[\frac{m+1}{2}\right]}^{m-1}\sum_{\alpha=m-1}^{m-2}\frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha}{2(2k-1-\alpha)!\alpha!}g_\alpha. \end{aligned} \tag{4.24}$$

Then, using (4.24), Definition 4.3 and Theorems 4.1 and 4.2, after certain calculations for the convolution $D_m(h\beta)*f_m(h\beta)$ we get

$$\begin{aligned} D_m(h\beta)*f_m(h\beta) &= D_m(h\beta)*\left[\mathrm{e}^{2\pi\mathrm{i}\omega h\beta}\left(\frac{1}{(2\pi i\omega)^2-1}-\sum_{k=1}^{m-1}\frac{1}{(2\pi i\omega)^{2k}}\right)\right]\\ &= \mathrm{e}^{2\pi\mathrm{i}\omega h\beta}\left[\frac{1}{(2\pi i\omega)^2-1}-\sum_{k=1}^{m-1}\frac{1}{(2\pi i\omega)^{2k}}\right]\sum_{\gamma=-\infty}^{\infty}D_m(h\gamma)\mathrm{e}^{2\pi\mathrm{i}\omega h\gamma}\\ &= K\mathrm{e}^{2\pi\mathrm{i}\omega h\beta}, \end{aligned}$$

where K is given in Theorem 4.4.

Therefore, from Theorem 4.3, taking into account the last equality, for coefficients C_β, $\beta=\overline{1,N-1}$, we have

$$C_\beta = K\mathrm{e}^{2\pi\mathrm{i}\omega h\beta} + \sum_{k=1}^{m-1}\left(a_k\lambda_k^\beta + b_k\lambda_k^{N-\beta}\right), \quad \beta=1,2,\ldots,N-1. \tag{4.25}$$

For the convolution $G_m(h\beta) * C_\beta$ of equality (4.1) we have

$$S(h\beta) = C_0 \left(\frac{\mathrm{e}^{h\beta} - \mathrm{e}^{-h\beta}}{2} - \sum_{k=1}^{m-1} \frac{(h\beta)^{2k-1}}{(2k-1)!} \right) + S_1(h\beta) + S_2(h\beta), \tag{4.26}$$

where

$$S_1(h\beta) = \sum_{\gamma=1}^{\beta-1} C_\gamma \left(\frac{\mathrm{e}^{h\beta-h\gamma} - \mathrm{e}^{h\gamma-h\beta}}{2} - \sum_{k=1}^{m-1} \frac{(h\beta-h\gamma)^{2k-1}}{(2k-1)!} \right),$$

$$S_2(h\beta) = -\frac{1}{2} \sum_{\gamma=0}^{N} C_\gamma \left(\frac{\mathrm{e}^{h\beta-h\gamma} - \mathrm{e}^{h\gamma-h\beta}}{2} - \sum_{k=1}^{m-1} \frac{(h\beta-h\gamma)^{2k-1}}{(2k-1)!} \right).$$

Then, using (4.25), (4.22), (4.23) and taking into account that λ_k are roots of the polynomial (4.9), after some simplifications, we get

$$\begin{aligned}
S_1(h\beta) =& \mathrm{e}^{2\pi \mathrm{i}\omega h\beta} \Bigg[\frac{K\mathrm{e}^{h}}{2(\mathrm{e}^{2\pi\mathrm{i}\omega h} - \mathrm{e}^{h})} - \frac{K}{2(\mathrm{e}^{h+2\pi\mathrm{i}\omega h} - 1)} \\
& - \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{\mathrm{e}^{2\pi\mathrm{i}\omega h} - 1} \sum_{k=1}^{m-1} \frac{h^{2l-1}}{(2l-1)!} \sum_{t=0}^{2l-1} \frac{\Delta^t 0^{2l-1}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h} - 1)^t} \Bigg] \\
& - \frac{\mathrm{e}^{h\beta}}{2} \left[\frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{\mathrm{e}^{2\pi\mathrm{i}\omega h} - \mathrm{e}^{h}} + \sum_{k=1}^{m-1} \left(\frac{a_k\lambda_k}{\lambda_k - \mathrm{e}^{h}} + \frac{b_k\lambda_k^N}{1-\lambda_k\mathrm{e}^{h}} \right) \right] \\
& + \frac{\mathrm{e}^{-h\beta}}{2} \left[\frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h+h}}{\mathrm{e}^{h+2\pi\mathrm{i}\omega h} - 1} + \sum_{k=1}^{m-1} \left(\frac{a_k\lambda_k\mathrm{e}^{h}}{\lambda_k\mathrm{e}^{h} - 1} + \frac{b_k\lambda_k^N\mathrm{e}^{h}}{\mathrm{e}^{h} - \lambda_k} \right) \right] \\
& + \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{\mathrm{e}^{2\pi\mathrm{i}\omega h} - 1} \sum_{l=1}^{m-1} \frac{h^{2l-1}}{(2l-1)!} \sum_{j=0}^{2l-1} C_{2l-1}^j \beta^j \sum_{t=0}^{2l-1} \frac{\Delta^t 0^{2l-1-j}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h} - 1)^t} \\
& + \sum_{\ell=1}^{m-1} \frac{h^{2\ell-1}}{(2\ell-1)!} \sum_{j=0}^{2\ell-1} C_{2\ell-1}^j \beta^j \sum_{k=1}^{m-1} \frac{a_k\lambda_k}{\lambda_k - 1} \sum_{t=0}^{2\ell-1} \frac{\Delta^t 0^{2\ell-1-j}}{(\lambda_k - 1)^t} \\
& + \sum_{\ell=1}^{m-1} \frac{h^{2\ell-1}}{(2\ell-1)!} \sum_{j=0}^{2\ell-1} C_{2\ell-1}^j \beta^j \sum_{k=1}^{m-1} \frac{b_k\lambda_k^N}{1-\lambda_k} \sum_{t=0}^{2\ell-1} \left(\frac{\lambda_k}{1-\lambda_k} \right)^t \Delta^t 0^{2\ell-1-j}.
\end{aligned} \tag{4.27}$$

Now, using the binomial formula and equalities (4.2) and (4.3), we obtain

$$\begin{aligned}
S_2(h\beta) = \frac{1}{2} \Bigg\{ & \frac{(\mathrm{e}^{-1} - 1)\mathrm{e}^{h\beta}}{2(2\pi\mathrm{i}\omega - 1)} + \frac{\mathrm{e}^{-h\beta}}{2} \sum_{\gamma=0}^{N} C_\gamma \mathrm{e}^{h\gamma} \\
& + \Bigg[\sum_{k=1}^{\left[\frac{m+1}{2}\right]-1} \sum_{\alpha=0}^{2k-1} \frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha}{(2k-1-\alpha)!\alpha!} g_\alpha + \sum_{k=\left[\frac{m+1}{2}\right]}^{m-1} \sum_{\alpha=0}^{m-2} \frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha}{(2k-1-\alpha)!\alpha!} g_\alpha \\
& + \sum_{k=\left[\frac{m+1}{2}\right]}^{m-1} \sum_{\alpha=m-1}^{2k-1} \frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha}{(2k-1-\alpha)!\alpha!} \sum_{\gamma=0}^{N} C_\gamma (h\gamma)^\alpha \Bigg] \Bigg\}.
\end{aligned} \tag{4.28}$$

Taking into account (4.27), (4.28) and putting (4.26), (4.24) into (4.1), we get the following identity with respect to $(h\beta)$:

$$S(h\beta) + P_{m-2}(h\beta) + d\,\mathrm{e}^{-h\beta} = f_m(h\beta). \tag{4.29}$$

As it was said above, equality (4.29) is the identity with respect to $(h\beta)$. Keeping in mind (4.27), (4.28), (4.24), equating the coefficients of $\mathrm{e}^{h\beta}$ and the terms which consist of $(h\beta)^\alpha$, $\alpha = \overline{m-1, 2m-3}$ in both sides of (4.29), we get the following equations for a_k and b_k

$$\sum_{k=1}^{m-1}\left\{a_k \frac{\lambda_k - \lambda_k^{N+1}}{(e^h-\lambda_k)(1-\lambda_k)} + b_k \frac{\lambda_k - \lambda_k^{N+1}}{(\lambda_k e^h - 1)(1-\lambda_k)}\right\} = 0, \tag{4.30}$$

$$\begin{aligned}\sum_{\ell=[\frac{m+1}{2}]}^{m-1}\Bigg\{&-C_0\frac{(h\beta)^{2\ell-1}}{(2\ell-1)!} - \sum_{j=m-1}^{2\ell-1}\frac{(h\beta)^j h^{2\ell-j-1}}{j!\,(2\ell-j-1)!}\sum_{t=0}^{2l-1-j}\frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}\Delta^t 0^{2l-1-j}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)^{t+1}}\\ &+\sum_{j=m-1}^{2\ell-1}\frac{(h\beta)^j h^{2\ell-j-1}}{(2\ell-1-j)!j!}\sum_{k=1}^{m-1}a_k\sum_{t=0}^{2\ell-1}\frac{\lambda_k\Delta^t 0^{2\ell-1-j}}{(\lambda_k-1)^{t+1}}\\ &+\sum_{j=m-1}^{2\ell-1}(h\beta)^j\frac{h^{2\ell-j-1}}{(2\ell-1-j)!j!}\sum_{k=1}^{m-1}b_k\sum_{t=0}^{2\ell-1}\frac{\lambda_k^{N+i}\Delta^t 0^{2\ell-1-j}}{(1-\lambda_k)^{t+1}}\Bigg\} = 0.\end{aligned} \tag{4.31}$$

Unknown polynomial $P_{m-2}(h\beta)$ and the coefficient d can be found from (4.29) by equating the corresponding coefficients of $(h\beta)^\alpha$ when $\alpha = 0, 1, \ldots, m-2$ and $\mathrm{e}^{-h\beta}$, respectively.

Now, from equations (4.2) when $\alpha = 0$ and (4.3), taking into account (4.25), using identities (4.22) and (4.23), after some simplifications for the coefficients C_0 and C_N, we get the following expressions

$$\begin{aligned}C_0 = {}&\frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h} - \frac{1}{2\pi\mathrm{i}\omega - 1}\\ &+\sum_{k=1}^{m-1}\Bigg\{a_k\frac{\lambda_k(\mathrm{e}^h-\mathrm{e}) + \lambda_k^2(\mathrm{e}-1) + \lambda_k^{N+1}(1-\mathrm{e}^h)}{(\mathrm{e}-1)(1-\lambda_k)(\mathrm{e}^h-\lambda_k)}\\ &\qquad + b_k\frac{\lambda_k^{N+1}(\mathrm{e}^h-\mathrm{e}) + \lambda_k^N(\mathrm{e}-1) + \lambda_k(1-\mathrm{e}^h)}{(\mathrm{e}-1)(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)}\Bigg\}\end{aligned} \tag{4.32}$$

and

$$\begin{aligned}C_N = {}&\frac{K\mathrm{e}^h}{\mathrm{e}^h - \mathrm{e}^{2\pi i\omega h}} + \frac{1}{2\pi\mathrm{i}\omega-1}\\ &+\sum_{k=1}^{m-1}\Bigg\{a_k\frac{\lambda_k(\mathrm{e}-\mathrm{e}^{h+1}) + \lambda_k^N(\mathrm{e}^{h+1}-\mathrm{e}^h) + \lambda_k^{N+1}(\mathrm{e}^h-\mathrm{e})}{(\mathrm{e}-1)(1-\lambda_k)(\mathrm{e}^h-\lambda_k)}\\ &\qquad + b_k\frac{\lambda_k^{N+1}(\mathrm{e}-\mathrm{e}^{h+1}) + \lambda_k^2(\mathrm{e}^{h+1}-\mathrm{e}^h) + \lambda_k(\mathrm{e}^h-\mathrm{e})}{(\mathrm{e}-1)(1-\lambda_k)(1-\lambda_k\mathrm{e}^h)}\Bigg\},\end{aligned} \tag{4.33}$$

respectively. Then, from (4.31), using (4.32), grouping the coefficients of same degrees of $(h\beta)$ and equating to zero, for a_k and b_k we obtain the following $m-1$

linear equations:

$$\sum_{k=1}^{m-1} a_k \left[\sum_{l=1}^{j} \frac{h^{2l-2}}{(2l-2)!} \sum_{t=0}^{2l-2} \frac{\lambda_k \Delta^t 0^{2l-2}}{(\lambda_k-1)^{t+1}} - \frac{\lambda_k(\mathrm{e}^h-\mathrm{e})+\lambda_k^2(\mathrm{e}-1)+\lambda_k^{N+1}(1-\mathrm{e}^h)}{(\mathrm{e}-1)(\lambda_k-1)(\lambda_k-\mathrm{e}^h)} \right]$$

$$+\sum_{k=1}^{m-1} b_k \left[\sum_{l=1}^{j} \frac{h^{2l-2}}{(2l-2)!} \sum_{t=0}^{2l-2} \frac{\lambda_k^{N+t} \Delta^t 0^{2l-2}}{(1-\lambda_k)^{t+1}} - \frac{\lambda_k(1-\mathrm{e}^h)+\lambda_k^N(\mathrm{e}-1)+\lambda_k^{N+1}(\mathrm{e}^h-\mathrm{e})}{(\mathrm{e}-1)(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)} \right]$$

$$=\sum_{l=1}^{j} \left[\frac{1}{(2\pi\mathrm{i}\omega)^{2l-1}} - \frac{h^{2l-2}}{(2l-2)!} \sum_{t=0}^{2l-2} \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}\Delta^t 0^{2l-2}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)^{t+1}} \right] - \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h} - \frac{1}{2\pi\mathrm{i}\omega h-1},$$

for $j=\overline{1,[m/2]}$, and

$$\sum_{k=1}^{m-1} a_k \left[\sum_{l=1}^{j} \frac{h^{2l-1}}{(2l-1)!} \sum_{t=0}^{2l-1} \frac{\lambda_k \Delta^t 0^{2l-1}}{(\lambda_k-1)^{t+1}} \right] + \sum_{k=1}^{m-1} b_k \left[\sum_{l=1}^{j} \frac{h^{2l-1}}{(2l-1)!} \sum_{t=0}^{2l-1} \frac{\lambda_k^{N+t} \Delta^i 0^{2l-1}}{(1-\lambda_k)^{t+1}} \right]$$

$$=\sum_{l=1}^{j} \left[\frac{1}{(2\pi\mathrm{i}\omega)^{2l}} - \frac{h^{2l-1}}{(2l-1)!} \sum_{t=0}^{2l-1} \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}\Delta^t 0^{2l-1}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)^{t+1}} \right],$$

for $j=\overline{1,[(m-1)/2]}$.

Further, from (4.2) when $\alpha=1,\dots,m-2$, using equalities (4.25), (4.33) and identities (4.22) and (4.23) for a_k and b_k we have the following $m-2$ linear equations:

$$\sum_{k=1}^{m-1} a_k \Bigg\{ h^j \sum_{i=0}^{j} \frac{\lambda_k^i - \lambda_k^{N+i}}{(1-\lambda_k)^{i+1}} \Delta^i 0^j - \sum_{l=0}^{j-1} h^l C_j^l \sum_{i=0}^{l} \frac{\lambda_k^{N+i}\Delta^i 0^l}{(1-\lambda_k)^{i+1}}$$

$$+\frac{\lambda_k(\mathrm{e}-\mathrm{e}^{h+1})+\lambda_k^N(\mathrm{e}^{h+1}-\mathrm{e}^h)+\lambda_k^{N+1}(\mathrm{e}^h-\mathrm{e})}{(\mathrm{e}-1)(\lambda_k-1)(\lambda_k-\mathrm{e}^h)} \Bigg\}$$

$$+\sum_{k=1}^{m-1} b_k \Bigg\{ h^j \sum_{i=0}^{j} \frac{\lambda_k^{N+1}-\lambda_k}{(\lambda_k-1)^{i+1}} \Delta^i 0^j - \sum_{l=0}^{j-1} h^l C_j^l \sum_{i=0}^{l} \frac{\lambda_k\Delta^i 0^l}{(\lambda_k-1)^{i+1}}$$

$$+\frac{\lambda_k^{N+1}(\mathrm{e}-\mathrm{e}^{h+1})+\lambda_k(\mathrm{e}^h-\mathrm{e})+\lambda_k^2(\mathrm{e}^{h+1}-\mathrm{e}^h)}{(\mathrm{e}-1)(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)} \Bigg\}$$

$$= \frac{1}{2\pi\mathrm{i}\omega} + \sum_{l=1}^{j-1}(-1)^l \frac{j(j-1)(j-2)\cdots(j-l+1)}{(2\pi\mathrm{i}\omega)^{l+1}} - \frac{K\mathrm{e}^h}{\mathrm{e}^h-\mathrm{e}^{2\pi\mathrm{i}\omega h}}$$

$$-\frac{1}{2\pi\mathrm{i}\omega-1} + K\sum_{l=0}^{j-1} h^l C_j^l \sum_{t=0}^{l} \frac{\mathrm{e}^{2\pi\mathrm{i}\omega t h}\Delta^t 0^l}{(1-\mathrm{e}^{2\pi\mathrm{i}\omega h})^{t+1}},$$

where $j=\overline{1,m-2}$.

Finally, after some simplifications in (4.30) and the previous systems of equations for a_k and b_k, we get the system which is given in the assertion of this theorem. □

The proof of Theorem 4.5 is similar to one of Theorem 4.4. Only one difference is that $D_m(h\beta) * f_m(h\beta) = K = 0$.

For $m=2,\ m=3$ and $m=4$, from Theorem 4.4 we have the following results:

Corollary 4.1. *The coefficients of optimal quadrature formulas of the form* (1.1), *with the error functional* (1.2), *and with equal spaced nodes when $\omega h \notin \mathbb{Z}$ in the space $W_2^{(2,1)}[0,1]$, are expressed by formulas*

$$\begin{aligned} C_0 &= \frac{K\mathrm{e}^{4\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h)(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)} + \frac{2\pi\mathrm{i}\omega(1-\mathrm{e}^h)-1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} \\ &\qquad + \frac{1}{1-\lambda_1}\left\{\frac{a_1\lambda_1^2}{\mathrm{e}^h-\lambda_1}+\frac{b_1\lambda_1^N}{1-\lambda_1\mathrm{e}^h}\right\}, \\ C_\beta &= \mathrm{e}^{2\pi\mathrm{i}\omega h\beta}K + a_1\lambda_1^\beta + b_1\lambda_1^{N-\beta}, \qquad \beta=\overline{1,N-1}, \\ C_N &= \frac{K\mathrm{e}^{h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h)(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)} + \frac{2\pi\mathrm{i}\omega(\mathrm{e}^h-1)-\mathrm{e}^h}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} \\ &\qquad + \frac{\mathrm{e}^h}{1-\lambda_1}\left\{\frac{a_1\lambda_1^N}{\mathrm{e}^h-\lambda_1}+\frac{b_1\lambda_1^2}{1-\lambda_1\mathrm{e}^h}\right\}, \end{aligned}$$

where

$$\begin{aligned} a_1 &= \frac{(\mathrm{e}^h-\lambda_1)(1-\lambda_1)}{\lambda_1(\mathrm{e}^h-1)(\lambda_1^N+1)}\left[\frac{1}{2\pi i\omega(2\pi i\omega-1)}+\frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}(1-\mathrm{e}^h)}{(\mathrm{e}^h-\mathrm{e}^{2\pi\mathrm{i}\omega h})(1-\mathrm{e}^{2\pi\mathrm{i}\omega h})}\right], \\ b_1 &= \frac{(1-\mathrm{e}^h\lambda_1)(1-\lambda_1)}{\lambda_1(\mathrm{e}^h-1)(\lambda_1^N+1)}\left[\frac{1}{2\pi i\omega(2\pi i\omega-1)}+\frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}(1-\mathrm{e}^h)}{(\mathrm{e}^h-\mathrm{e}^{2\pi\mathrm{i}\omega h})(1-\mathrm{e}^{2\pi\mathrm{i}\omega h})}\right], \\ \lambda_1 &= \frac{h(\mathrm{e}^{2h}+1)-\mathrm{e}^{2h}+1-(\mathrm{e}^h-1)\sqrt{h^2(\mathrm{e}^h+1)^2+2h(1-\mathrm{e}^{2h})}}{1-\mathrm{e}^{2h}+2h\mathrm{e}^h} \\ K &= \frac{L}{p_2^{(2)}}\left[\frac{2A_1}{\lambda_1}\cdot\frac{1-\lambda_1\cos(2\pi\omega h)}{\lambda_1^2+1-2\lambda_1\cos(2\pi\omega h)}-\frac{A_1}{\lambda_1}-4\mathrm{e}^h\cos(2\pi\omega h)+2C\right], \\ L &= \frac{1}{(2\pi\omega)^2((2\pi\omega)^2+1)}, \quad A_1=\frac{2(\lambda_1-1)(\lambda_1\mathrm{e}^h-1)(\mathrm{e}^h-\lambda_1)}{\lambda_1+1}, \\ p_2^{(2)} &= 1-\mathrm{e}^{2h}+2h\mathrm{e}^h, \quad C=\left(1+\mathrm{e}^h\right)^2-\mathrm{e}^h\frac{\lambda_1^2+1}{\lambda_1}. \end{aligned}$$

Remark 4.1. For λ_1 in Corollary 4.1 the following expansion

$$\lambda_1=\sqrt{3}-2+\frac{2\sqrt{3}-3}{30}h^2-\frac{3\sqrt{3}-1}{4200}h^4+O\left(h^6\right)$$

holds.

Corollary 4.2. *The coefficients of optimal quadrature formulas of the form* (1.1), *with the error functional* (1.2), *and with equal spaced nodes when $\omega h \notin \mathbb{Z}$ in the space $W_2^{(3,2)}[0,1]$, are expressed by formulas*

$$\begin{aligned} C_0 &= \frac{K\mathrm{e}^{4\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h)(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)} + \frac{2\pi\mathrm{i}\omega(1-\mathrm{e}^h)-1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} \\ &\qquad + \sum_{k=1}^{2}\frac{1}{1-\lambda_k}\left\{\frac{a_k\lambda_k^2}{\mathrm{e}^h-\lambda_k}+\frac{b_k\lambda_k^N}{1-\lambda_k\mathrm{e}^h}\right\}, \end{aligned}$$

$$C_\beta = K\mathrm{e}^{2\pi \mathrm{i}\omega h\beta} + \sum_{k=1}^{2}\left(a_k\lambda_k^{\beta} + b_k\lambda_k^{N-\beta}\right), \qquad \beta = \overline{1, N-1},$$

$$C_N = \frac{K\mathrm{e}^h}{(\mathrm{e}^{2\pi\mathrm{i}\omega h} - \mathrm{e}^h)(\mathrm{e}^{2\pi\mathrm{i}\omega h} - 1)} + \frac{2\pi\mathrm{i}\omega(\mathrm{e}^h - 1) - \mathrm{e}^h}{2\pi\mathrm{i}\omega(1 - 2\pi\mathrm{i}\omega)(1 - \mathrm{e}^h)} + e^h \sum_{k=1}^{2} \frac{1}{1-\lambda_k}\left\{\frac{a_k\lambda_k^N}{\mathrm{e}^h - \lambda_k} + \frac{b_k\lambda_k^2}{1 - \lambda_k\mathrm{e}^h}\right\},$$

where a_k and b_k $(k = \overline{1,2})$ are defined by the following system of linear equations

$$\sum_{k=1}^{2}\frac{a_k\lambda_k}{(\lambda_k - 1)(\lambda_k - \mathrm{e}^h)} + \sum_{k=1}^{2}\frac{b_k\lambda_k^{N+1}}{(\lambda_k - 1)(\lambda_k\mathrm{e}^h - 1)} = \frac{1}{2\pi i\omega(1 - 2\pi i\omega)(1 - \mathrm{e}^h)} + \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h} - \mathrm{e}^h)(1 - \mathrm{e}^{2\pi\mathrm{i}\omega h})};$$

$$\sum_{k=1}^{2}\frac{a_k\lambda_k^{N+1}}{(\lambda_k - 1)(\lambda_k - \mathrm{e}^h)} + \sum_{k=1}^{2}\frac{b_k\lambda_k}{(\lambda_k - 1)(\lambda_k\mathrm{e}^h - 1)} = \frac{1}{2\pi i\omega(1 - 2\pi i\omega)(1 - \mathrm{e}^h)} + \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h} - \mathrm{e}^h)(1 - \mathrm{e}^{2\pi\mathrm{i}\omega h})};$$

$$\sum_{k=1}^{2}\frac{a_k\lambda_k}{(\lambda_k - 1)^2} + \sum_{k=1}^{2}\frac{b_k\lambda_k^{N+1}}{(\lambda_k - 1)^2} = \frac{h}{(2\pi\mathrm{i}\omega h)^2} - \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h} - 1)^2};$$

$$\sum_{k=1}^{2}\frac{a_k\lambda_k^{N+1}}{(\lambda_k - 1)^2} + \sum_{k=1}^{2}\frac{b_k\lambda_k}{(\lambda_k - 1)^2} = \frac{h}{(2\pi\mathrm{i}\omega h)^2} - \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h} - 1)^2}.$$

Here λ_k, $k = 1, 2$, are roots of the polynomial

$$\mathcal{P}_4(\lambda) = (1 - \mathrm{e}^{2h})(1-\lambda)^4 - 2\big[\lambda(\mathrm{e}^{2h} + 1) - \mathrm{e}^h(\lambda^2 + 1)\big]\big[h(1-\lambda)^2 + \frac{h^3}{6}(1 + 4\lambda + \lambda^2)\big],$$

for which $|\lambda_k| < 1$,

$$K = L\left\{\sum_{k=1}^{2}\left(\frac{2A_k}{\lambda_k}\cdot\frac{1 - \lambda_k\cos(2\pi\omega h)}{\lambda_k^2 + 1 - 2\lambda_k\cos(2\pi\omega h)} - \frac{A_k}{\lambda_k}\right) - 4\mathrm{e}^h\cos(2\pi\omega h) + 2C\right\},$$

$$L = \frac{1}{p_4^{(4)}}\left\{\frac{1}{(2\pi\mathrm{i}\omega)^2 - 1} - \sum_{k=1}^{2}\frac{1}{(2\pi\mathrm{i}\omega)^{2k}}\right\},$$

and A_k and C are defined in Theorem 4.1.

Corollary 4.3. *The coefficients of optimal quadrature formulas of the form* (1.1), *with the error functional* (1.2), *and with equal spaced nodes when $\omega h \notin \mathbb{Z}$ in the* *space* $W_2^{(4,3)}[0,1]$, *are expressed by formulas*

$$C_0 = \frac{K\mathrm{e}^{4\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h)(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)} + \frac{2\pi\mathrm{i}\omega(1-\mathrm{e}^h)-1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} + \sum_{k=1}^{3}\frac{1}{1-\lambda_k}\left\{\frac{a_k\lambda_k^2}{\mathrm{e}^h-\lambda_k}+\frac{b_k\lambda_k^N}{1-\lambda_k\mathrm{e}^h}\right\},$$

$$C_\beta = \mathrm{e}^{2\pi\mathrm{i}\omega h\beta}K + \sum_{k=1}^{3}\left(a_k\lambda_k^{\beta}+b_k\lambda_k^{N-\beta}\right), \qquad \beta=\overline{1,N-1},$$

$$C_N = \frac{K\mathrm{e}^{h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h)(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)} + \frac{2\pi\mathrm{i}\omega(\mathrm{e}^h-1)-\mathrm{e}^h}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} + \mathrm{e}^h\sum_{k=1}^{3}\frac{1}{1-\lambda_k}\left\{\frac{a_k\lambda_k^N}{\mathrm{e}^h-\lambda_k}+\frac{b_k\lambda_k^2}{1-\lambda_k\mathrm{e}^h}\right\},$$

where a_k *and* b_k $(k=\overline{1,3})$ *are defined by the following system of linear equations*

$$\sum_{k=1}^{3}\frac{a_k\lambda_k}{(\lambda_k-1)(\lambda_k-\mathrm{e}^h)} + \sum_{k=1}^{3}\frac{b_k\lambda_k^{N+1}}{(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)} = \frac{1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} + \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h)(1-\mathrm{e}^{2\pi\mathrm{i}\omega h})};$$

$$\sum_{k=1}^{3}\frac{a_k\lambda_k^{N+1}}{(\lambda_k-1)(\lambda_k-\mathrm{e}^h)} + \sum_{k=1}^{3}\frac{b_k\lambda_k}{(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)} = \frac{1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} + \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-\mathrm{e}^h)(1-\mathrm{e}^{2\pi\mathrm{i}\omega h})};$$

$$\sum_{k=1}^{3}\frac{a_k\lambda_k}{(\lambda_k-1)^2} + \sum_{k=1}^{3}\frac{b_k\lambda_k^{N+1}}{(\lambda_k-1)^2} = \frac{h}{(2\pi\mathrm{i}\omega h)^2} - \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)^2};$$

$$\sum_{k=1}^{3}\frac{a_k\lambda_k^{N+1}}{(\lambda_k-1)^2} + \sum_{k=1}^{3}\frac{b_k\lambda_k}{(\lambda_k-1)^2} = \frac{h}{(2\pi\mathrm{i}\omega h)^2} - \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)^2};$$

$$\sum_{k=1}^{3}\frac{a_k\lambda_k}{(\lambda_k-1)^3} + \sum_{k=1}^{3}\frac{b_k\lambda_k^{N+2}}{(1-\lambda_k)^3} - \frac{h}{(2\pi\mathrm{i}\omega h)^3} = -\frac{h}{2(2\pi\mathrm{i}\omega h)^2} - \frac{K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)^3};$$

$$\sum_{k=1}^{3}\frac{a_k\lambda_k^2(1-\lambda_k^N)}{(1-\lambda_k)^3} + \sum_{k=1}^{3}\frac{b_k\lambda_k(\lambda_k^N-1)}{(\lambda_k-1)^3} = \frac{(1-2h)K\mathrm{e}^{2\pi\mathrm{i}\omega h}}{2h^2(\mathrm{e}^{2\pi\mathrm{i}\omega h}-1)^2}.$$

Here λ_k, $k=1,2,3$ *are the roots of the polynomial*

$$\mathcal{P}_6(\lambda) = (1-\mathrm{e}^{2h})(1-\lambda)^6 - 2[\lambda(\mathrm{e}^{2h}+1)-\mathrm{e}^h(\lambda^2+1)]\Big[h(1-\lambda)^4 + \frac{h^3}{6}(1-\lambda)^2(1+4\lambda+\lambda^2) + \frac{h^5}{120}(1+26\lambda+66\lambda^2+26\lambda^3+\lambda^4)\Big],$$

for which $|\lambda_k| < 1$,

$$K = \frac{L}{p_6^{(6)}} \left\{ \sum_{k=1}^{3} \left(\frac{2A_k}{\lambda_k} \cdot \frac{1 - \lambda_k \cos(2\pi\omega h)}{\lambda_k^2 + 1 - 2\lambda_k \cos(2\pi\omega h)} - \frac{A_k}{\lambda_k} \right) - 4\mathrm{e}^h \cos(2\pi\omega h) + 2C \right\},$$

$$L = \frac{1}{(2\pi \mathrm{i}\omega)^2 - 1} - \sum_{k=1}^{3} \frac{1}{(2\pi \mathrm{i}\omega)^{2k}},$$

$p_6^{(6)}$ *is the leading coefficient of the polynomial* $\mathcal{P}_6(\lambda)$, *and* A_k *and* C *are defined in Theorem* 4.1 *for* $m = 4$.

Now from Theorem 4.5 for $m = 2$, $m = 3$ and $m = 4$, we have the following corollaries:

Corollary 4.4. *The coefficients of optimal quadrature formulas of the form* (1.1), *with the error functional* (1.2), *and with equal spaced nodes when* $\omega h \in \mathbb{Z}$ *and* $\omega \neq 0$ *in the space* $W_2^{(2,1)}[0,1]$, *are expressed by formulas*

$$C_0 = \frac{2\pi \mathrm{i}\omega(1 - \mathrm{e}^h) - 1}{2\pi \mathrm{i}\omega(1 - 2\pi \mathrm{i}\omega)(1 - \mathrm{e}^h)} + \frac{1}{1 - \lambda_1} \left\{ \frac{a_1 \lambda_1^2}{\mathrm{e}^h - \lambda_1} + \frac{b_1 \lambda_1^N}{1 - \lambda_1 \mathrm{e}^h} \right\},$$

$$C_\beta = a_1 \lambda_1^\beta + b_1 \lambda_1^{N-\beta}, \quad \beta = \overline{1, N-1},$$

$$C_N = \frac{2\pi \mathrm{i}\omega(\mathrm{e}^h - 1) - \mathrm{e}^h}{2\pi \mathrm{i}\omega(1 - 2\pi \mathrm{i}\omega)(1 - \mathrm{e}^h)} + \frac{\mathrm{e}^h}{1 - \lambda_1} \left\{ \frac{a_1 \lambda_1^N}{\mathrm{e}^h - \lambda_1} + \frac{b_1 \lambda_1^2}{1 - \lambda_1 \mathrm{e}^h} \right\},$$

where

$$a_1 = \frac{(\mathrm{e}^h - \lambda_1)(1 - \lambda_1)}{2\pi \mathrm{i}\omega(2\pi \mathrm{i}\omega - 1)\lambda_1(\mathrm{e}^h - 1)(\lambda_1^N + 1)},$$

$$b_1 = \frac{(1 - \mathrm{e}^h \lambda_1)(1 - \lambda_1)}{2\pi \mathrm{i}\omega(2\pi \mathrm{i}\omega - 1)\lambda_1(\mathrm{e}^h - 1)(\lambda_1^N + 1)},$$

and

$$\lambda_1 = \frac{h(\mathrm{e}^{2h} + 1) - \mathrm{e}^{2h} + 1 - (\mathrm{e}^h - 1)\sqrt{h^2(\mathrm{e}^h + 1)^2 + 2h(1 - \mathrm{e}^{2h})}}{1 - \mathrm{e}^{2h} + 2h\mathrm{e}^h}.$$

Corollary 4.5. *The coefficients of optimal quadrature formulas of the form* (1.1), *with the error functional* (1.2), *and with equal spaced nodes when* $\omega h \in \mathbb{Z}$ *and* $\omega \neq 0$ *in the space* $W_2^{(3,2)}[0,1]$, *are expressed by formulas*

$$C_0 = \frac{2\pi \mathrm{i}\omega(1 - \mathrm{e}^h) - 1}{2\pi \mathrm{i}\omega(1 - 2\pi \mathrm{i}\omega)(1 - \mathrm{e}^h)} + \sum_{k=1}^{2} \frac{1}{1 - \lambda_k} \left\{ \frac{a_k \lambda_k^2}{\mathrm{e}^h - \lambda_k} + \frac{b_k \lambda_k^N}{1 - \lambda_k \mathrm{e}^h} \right\},$$

$$C_\beta = \sum_{k=1}^{2} \left(a_k \lambda_k^\beta + b_k \lambda_k^{N-\beta} \right), \quad \beta = \overline{1, N-1},$$

$$C_N = \frac{2\pi \mathrm{i}\omega(\mathrm{e}^h - 1) - \mathrm{e}^h}{2\pi \mathrm{i}\omega(1 - 2\pi \mathrm{i}\omega)(1 - \mathrm{e}^h)} + \mathrm{e}^h \sum_{k=1}^{2} \frac{1}{1 - \lambda_k} \left\{ \frac{a_k \lambda_k^N}{\mathrm{e}^h - \lambda_k} + \frac{b_k \lambda_k^2}{1 - \lambda_k \mathrm{e}^h} \right\},$$

where a_k *and* b_k $(k=\overline{1,2})$ *are defined by the following system of linear equations*

$$\sum_{k=1}^{2} a_k \frac{\lambda_k}{(\lambda_k-1)(\lambda_k-\mathrm{e}^h)} + \sum_{k=1}^{2} b_k \frac{\lambda_k^{N+1}}{(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)} = \frac{1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)},$$

$$\sum_{k=1}^{2} a_k \frac{\lambda_k^{N+1}}{(\lambda_k-1)(\lambda_k-\mathrm{e}^h)} + \sum_{k=1}^{2} b_k \frac{\lambda_k}{(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)} = \frac{1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)},$$

$$\sum_{k=1}^{2} a_k \frac{\lambda_k}{(\lambda_k-1)^2} + \sum_{k=1}^{2} b_k \frac{\lambda_k^{N+1}}{(\lambda_k-1)^2} = \frac{h}{(2\pi\mathrm{i}\omega h)^2},$$

$$\sum_{k=1}^{2} a_k \frac{\lambda_k^{N+1}}{(\lambda_k-1)^2} + \sum_{k=1}^{2} b_k \frac{\lambda_k}{(\lambda_k-1)^2} = \frac{h}{(2\pi\mathrm{i}\omega h)^2}.$$

Here λ_k, $k=1,2$, *are roots of the polynomial*

$$\mathcal{P}_4(\lambda) = (1-\mathrm{e}^{2h})(1-\lambda)^4 - 2\big[\lambda(\mathrm{e}^{2h}+1)-\mathrm{e}^h(\lambda^2+1)\big]\Big[h(1-\lambda)^2+\frac{h^3}{6}(1+4\lambda+\lambda^2)\Big]$$

for which $|\lambda_k|<1$.

Corollary 4.6. *The coefficients of optimal quadrature formulas of the form* (1.1), *with the error functional* (1.2), *and with equal spaced nodes when* $\omega h\in\mathbb{Z}$ *and* $\omega\neq 0$ *in the space* $W_2^{(4,3)}[0,1]$, *are expressed by formulas*

$$C_0 = \frac{2\pi\mathrm{i}\omega(1-\mathrm{e}^h)-1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} + \sum_{k=1}^{3}\frac{1}{1-\lambda_k}\left\{\frac{a_k\lambda_k^2}{\mathrm{e}^h-\lambda_k}+\frac{b_k\lambda_k^N}{1-\lambda_k\mathrm{e}^h}\right\},$$

$$C_\beta = \sum_{k=1}^{3}\left(a_k\lambda_k^{\beta}+b_k\lambda_k^{N-\beta}\right), \quad \beta=\overline{1,N-1},$$

$$C_N = \frac{2\pi\mathrm{i}\omega(\mathrm{e}^h-1)-\mathrm{e}^h}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)} + \mathrm{e}^h\sum_{k=1}^{3}\frac{1}{1-\lambda_k}\left\{\frac{a_k\lambda_k^N}{\mathrm{e}^h-\lambda_k}+\frac{b_k\lambda_k^2}{1-\lambda_k\mathrm{e}^h}\right\},$$

where a_k *and* b_k $(k=\overline{1,3})$ *are defined by the following system of linear equations*

$$\sum_{k=1}^{3} a_k \frac{\lambda_k}{(\lambda_k-1)(\lambda_k-\mathrm{e}^h)} + \sum_{k=1}^{3} b_k \frac{\lambda_k^{N+1}}{(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)} = \frac{1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)},$$

$$\sum_{k=1}^{3} a_k \frac{\lambda_k^{N+1}}{(\lambda_k-1)(\lambda_k-\mathrm{e}^h)} + \sum_{k=1}^{3} b_k \frac{\lambda_k}{(\lambda_k-1)(\lambda_k\mathrm{e}^h-1)} = \frac{1}{2\pi\mathrm{i}\omega(1-2\pi\mathrm{i}\omega)(1-\mathrm{e}^h)},$$

$$\sum_{k=1}^{3} a_k \frac{\lambda_k}{(\lambda_k-1)^2} + \sum_{k=1}^{3} b_k \frac{\lambda_k^{N+1}}{(\lambda_k-1)^2} = \frac{h}{(2\pi\mathrm{i}\omega h)^2},$$

$$\sum_{k=1}^{3} a_k \frac{\lambda_k^{N+1}}{(\lambda_k-1)^2} + \sum_{k=1}^{3} b_k \frac{\lambda_k}{(\lambda_k-1)^2} = \frac{h}{(2\pi\mathrm{i}\omega h)^2},$$

$$\sum_{k=1}^{3} a_k \frac{\lambda_k}{(\lambda_k-1)^3} + \sum_{k=1}^{3} b_k \frac{\lambda_k^{N+2}}{(1-\lambda_k)^3} = \frac{h}{(2\pi\mathrm{i}\omega h)^3} - \frac{h}{2(2\pi\mathrm{i}\omega h)^2},$$

$$\sum_{k=1}^{3} a_k \frac{\lambda_k^2 - \lambda_k^{N+2}}{(1-\lambda_k)^3} + \sum_{k=1}^{3} b_k \frac{\lambda_k^{N+1} - \lambda_k}{(\lambda_k - 1)^3} = 0.$$

Here λ_k, $k = 1, 2, 3$, *are roots of the polynomial*

$$\mathcal{P}_6(\lambda) = (1 - \mathrm{e}^{2h})(1-\lambda)^6 - 2\big[\lambda(\mathrm{e}^{2h}+1) - \mathrm{e}^{h}(\lambda^2+1)\big]\Big[h(1-\lambda)^4 + \frac{h^3}{6}(1-\lambda)^2(1+4\lambda+\lambda^2) + \frac{h^5}{120}(1+26\lambda+66\lambda^2+26\lambda^3+\lambda^4)\Big],$$

for which $|\lambda_k| < 1$.

5. Coefficients and norm of the error functional of optimal quadrature formulas (1.1) in $W_2^{(1,0)}[0,1]$

Here we get the explicit expressions for coefficients and calculate the square of the norm of the error functional (1.2), of the optimal quadrature formula (1.1), on the space $W_2^{(1,0)}[0,1]$.

For $m = 1$ the system (4.1)–(4.3) takes the form

$$\sum_{\gamma=0}^{N} C_\gamma G_1(h\beta - h\gamma) + d\,\mathrm{e}^{-h\beta} = f_1(h\beta), \quad \beta = 0, 1, \ldots, N, \tag{5.1}$$

$$\sum_{\beta=0}^{N} C_\beta \mathrm{e}^{-h\beta} = \frac{e^{-1}-1}{2\pi\mathrm{i}\omega - 1}, \tag{5.2}$$

where

$$G_1(x) = \frac{\mathrm{sign}\,(x)}{4}\big(\mathrm{e}^{x} - \mathrm{e}^{-x}\big),$$

$$f_1(h\beta) = -\frac{\mathrm{e}^{h\beta}(\mathrm{e}^{2\pi\mathrm{i}\omega-1}+1)}{4(2\pi\mathrm{i}\omega-1)} + \frac{\mathrm{e}^{-h\beta}(\mathrm{e}^{2\pi\mathrm{i}\omega+1}+1)}{4(2\pi\mathrm{i}\omega+1)} + \frac{\mathrm{e}^{2\pi\mathrm{i}\omega h\beta}}{(2\pi\mathrm{i}\omega+1)(2\pi i\omega-1)}, \tag{5.3}$$

and C_β $(\beta = 0, 1, \ldots, N)$ and d are unknowns.

In this case Problem 4 is expressed as follows:

Problem 5. Find the solution of the equation

$$D_1(h\beta) * u(h\beta) = 0, \quad h\beta \notin [0,1], \tag{5.4}$$

having the form

$$u(h\beta) = \begin{cases} -\dfrac{\mathrm{e}^{h\beta}}{4}\dfrac{\mathrm{e}^{-1}-1}{2\pi\mathrm{i}\omega-1} + a^{-}\mathrm{e}^{-h\beta}, & \beta < 0, \\ f_1(h\beta), & 0 \le \beta \le N, \\ \dfrac{\mathrm{e}^{h\beta}}{4}\dfrac{\mathrm{e}^{-1}-1}{2\pi\mathrm{i}\omega-1} + a^{+}\mathrm{e}^{-h\beta}, & \beta > N, \end{cases} \tag{5.5}$$

where $f_1(h\beta)$ is defined by (5.3), a^- and a^+ are unknowns.

For $m = 1$, from Theorem 4.1 for $D_1(h\beta)$, we obtain

$$D_1(h\beta) = \frac{1}{1-\mathrm{e}^{2h}} \begin{cases} 0, & |\beta| \geq 2, \\ -2\mathrm{e}^{h}, & |\beta| = 1, \\ 2(1+\mathrm{e}^{2h}), & \beta = 0. \end{cases} \tag{5.6}$$

Now, taking into account (5.6), for the convolution $C_\beta = D_1(h\beta) * u(h\beta)$, we have

$$D_1(h\beta) * u(h\beta) = D_1(h)(u(h\beta - h) + u(h\beta + h)) + D_1(0)u(h\beta).$$

Hence, keeping in mind (5.4) for $\beta = -1$ and $\beta = N+1$, we get the following system

$$D_1(h)(u(-2h) + u(0)) + D_1(0)u(-h) = 0,$$
$$D_1(h)(u(Nh) + u(Nh + 2h)) + D_1(0)u(Nh + h) = 0.$$

Whence, taking into account (5.5), (5.6) for a^- and a^+, we have

$$a^- = \frac{\mathrm{e}-1}{4(2\pi\mathrm{i}\omega + 1)}, \quad a^+ = -\frac{\mathrm{e}-1}{4(2\pi\mathrm{i}\omega + 1)}. \tag{5.7}$$

Then, using (5.7), from (4.16) and (4.17) we obtain

$$d = \frac{1}{2}(a^- + a^+) = 0, \quad D = \frac{1}{2}(a^- - a^+) = \frac{\mathrm{e}-1}{4(2\pi\mathrm{i}\omega + 1)}. \tag{5.8}$$

Substituting (5.7) into (5.5) for $u(h\beta)$ we have the following expression

$$u(h\beta) = \begin{cases} -\dfrac{\mathrm{e}^{h\beta}}{4}\dfrac{\mathrm{e}^{-1}-1}{2\pi\mathrm{i}\omega - 1} + \dfrac{\mathrm{e}^{-h\beta}}{4}\dfrac{\mathrm{e}-1}{2\pi\mathrm{i}\omega + 1}, & \beta < 0, \\ f_1(h\beta), & 0 \leq \beta \leq N, \\ \dfrac{\mathrm{e}^{h\beta}}{4}\dfrac{\mathrm{e}^{-1}-1}{2\pi\mathrm{i}\omega - 1} - \dfrac{\mathrm{e}^{-h\beta}}{4}\dfrac{\mathrm{e}-1}{2\pi\mathrm{i}\omega + 1}, & \beta > N. \end{cases} \tag{5.9}$$

Using (5.9) and (5.6), taking into account (5.3), by direct calculations for optimal coefficients $C_\beta = D_1(h\beta) * u(h\beta)$ $(\beta = 0, 1, \ldots, N)$ we obtain the following result:

Theorem 5.1. *Coefficients of the optimal quadrature formulas of the form* (1.1) *in the sense of Sard in the space* $W_2^{(1,0)}[0,1]$ *have the form*

$$C_0 = \frac{1 + \mathrm{e}^{2h} - 2\mathrm{e}^{2\pi\mathrm{i}\omega h + h} - 2\pi\mathrm{i}\omega(1 - \mathrm{e}^{2h})}{(\mathrm{e}^{2h} - 1)(4\pi^2\omega^2 + 1)},$$

$$C_\beta = \frac{2(1 + \mathrm{e}^{2h} - 2\mathrm{e}^{h}\cos 2\pi\omega h)}{(\mathrm{e}^{2h} - 1)(4\pi^2\omega^2 + 1)}\,\mathrm{e}^{2\pi\mathrm{i}\omega h\beta}, \quad \beta = 1, 2, \ldots, N-1,$$

$$C_N = \frac{1 + \mathrm{e}^{2h} - 2\mathrm{e}^{h - 2\pi\mathrm{i}\omega h} + 2\pi\mathrm{i}\omega(1 - \mathrm{e}^{2h})}{(\mathrm{e}^{2h} - 1)(4\pi^2\omega^2 + 1)}.$$

Note that, in Theorem 5.1, the formulas for the optimal coefficients C_β are decomposed into two parts – real and imaginary parts. Therefore, from Theorem 5.1 we get the following results:

Corollary 5.1. *Coefficients for the optimal quadrature formulas of the form*

$$\int_0^1 \varphi(x)\cos 2\pi\omega x\,\mathrm{d}x \cong \sum_{\beta=0}^{N} C_\beta^R \varphi(h\beta)$$

in the sense of Sard in the space $W_2^{(1,0)}[0,1]$ *have the form*

$$C_0^R = \frac{1+\mathrm{e}^{2h}-2\mathrm{e}^h\cos 2\pi\omega h}{(\mathrm{e}^{2h}-1)(4\pi^2\omega^2+1)},\quad C_N^R = \frac{1+\mathrm{e}^{2h}-2\mathrm{e}^h\cos 2\pi\omega h}{(\mathrm{e}^{2h}-1)(4\pi^2\omega^2+1)},$$
$$C_\beta^R = \frac{2(1+\mathrm{e}^{2h}-2\mathrm{e}^h\cos 2\pi\omega h)}{(\mathrm{e}^{2h}-1)(4\pi^2\omega^2+1)}\cos 2\pi\omega h\beta,\quad \beta=1,2,\ldots,N-1.$$

Corollary 5.2. *Coefficients for the optimal quadrature formulas of the form*

$$\int_0^1 \varphi(x)\sin 2\pi\omega x\,\mathrm{d}x \cong \sum_{\beta=0}^{N} C_\beta^I \varphi(h\beta)$$

in the sense of Sard in the space $W_2^{(1,0)}[0,1]$ *have the form*

$$C_0^I = \frac{2\pi\omega(\mathrm{e}^{2h}-1)-2\mathrm{e}^h\sin 2\pi\omega h}{(\mathrm{e}^{2h}-1)(4\pi^2\omega^2+1)},\quad C_N^I = -\frac{2\pi\omega(\mathrm{e}^{2h}-1)-2\mathrm{e}^h\sin 2\pi\omega h}{(\mathrm{e}^{2h}-1)(4\pi^2\omega^2+1)},$$
$$C_\beta^I = \frac{2(1+\mathrm{e}^{2h}-2\mathrm{e}^h\cos 2\pi\omega h)}{(\mathrm{e}^{2h}-1)(4\pi^2\omega^2+1)}\sin 2\pi\omega h\beta,\quad \beta=1,2,\ldots,N-1.$$

Remark 5.1. When $\omega=0$, Theorem 5.1 reduces to Theorem 4.4 from [34].

Theorem 5.2. *The square of the norm of the error functional* (1.2)*, of the optimal quadrature formula* (1.1)*, on the space* $W_2^{(1,0)}[0,1]$*, has the form*

$$\|\mathring{\ell}\|^2 = \frac{1}{(4\pi^2\omega^2+1)^2}\left(4\pi^2\omega^2+1-\frac{2(\mathrm{e}^{2h}+1-2\mathrm{e}^h\cos 2\pi\omega h)}{h(\mathrm{e}^{2h}-1)}\right). \tag{5.10}$$

Proof. For $m=1$ we rewrite the equality (2.11) in the following form

$$\begin{aligned}\|\ell\|^2 = -\Bigg[&\sum_{\beta=0}^{N} C_\beta^R\left(\sum_{\gamma=0}^{N} C_\gamma^R G_1(h\beta-h\gamma)-\int_0^1 \cos 2\pi\omega x\ G_1(x-h\beta)\,\mathrm{d}x\right)\\
&+\sum_{\beta=0}^{N} C_\beta^I\left(\sum_{\gamma=0}^{N} C_\gamma^I G_1(h\beta-h\gamma)-\int_0^1 \sin 2\pi\omega x\ G_1(x-h\beta)\,\mathrm{d}x\right)\\
&-\sum_{\beta=0}^{N} C_\beta^R\int_0^1 \cos 2\pi\omega x\ G_1(x-h\beta)\,\mathrm{d}x-\sum_{\beta=0}^{N} C_\beta^I\int_0^1 \sin 2\pi\omega x\ G_1(x-h\beta)\,\mathrm{d}x\\
&+\int_0^1\int_0^1 \cos[2\pi\omega(x-y)]G_1(x-y)\,\mathrm{d}x\,\mathrm{d}y\Bigg],\end{aligned}$$

where $G_1(x)$ is defined by (5.3).

Taking into account (5.8), from (5.1) we get

$$\sum_{\gamma=0}^{N} C_\gamma^R G_1(h\beta - h\gamma) - \int_0^1 \cos 2\pi\omega x \; G_1(x - h\beta)\,\mathrm{d}x = 0$$

and

$$\sum_{\gamma=0}^{N} C_\gamma^I G_1(h\beta - h\gamma) - \int_0^1 \sin 2\pi\omega x \; G_1(x - h\beta)\,\mathrm{d}x = 0.$$

Then, using the last two equalities, for $\|\ell\|^2$ we obtain

$$\|\ell\|^2 = \sum_{\beta=0}^{N} C_\beta^R \int_0^1 \cos 2\pi\omega x \; G_1(x-h\beta)\,\mathrm{d}x + \sum_{\beta=0}^{N} C_\beta^I \int_0^1 \sin 2\pi\omega x \; G_1(x-h\beta)\,\mathrm{d}x$$
$$- \int_0^1 \int_0^1 \cos[2\pi\omega(x-y)] G_1(x-y)\,\mathrm{d}x\,\mathrm{d}y.$$

Finally, calculating these integrals and using Corollaries 5.1 and 5.2, after some simplifications, we get (5.10). □

Remark 5.2. When $\omega = 0$, Theorem 5.2 reduces to Theorem 5.1 from [34].

6. Numerical results

In this section we give some numerical results of the upper bounds for the errors in the optimal quadrature formulas of the form (1.1), as well their analysis in the cases $m = 1$ and $m = 2$.

According to the Cauchy-Schwarz inequality, in the space $W_2^{(m,m-1)}[0,1]$ for the absolute value of the difference (1.4) we get

$$|(\mathring{\ell}, \varphi)| \le \|\varphi\| \cdot \|\mathring{\ell}\|,$$

where $\|\mathring{\ell}\|$ is the norm of *the optimal error functional* which corresponds to the optimal quadrature formulas (1.1).

1° *First we consider the case* $m = 1$.

Using Theorem 5.2, for $\|\mathring{\ell}|W_2^{(1,0)*}[0,1]\|$, when $N = 1, 10, 10^2, 10^3, 10^4$ and $\omega = 1, 11, 101, 1001, 10001$, we get numerical results which are presented in Table 1. Numbers in parenthesis indicate the decimal exponents. From the first column of this table we see that order of convergence of our optimal quadrature formula is $O(N^{-1})$ and from the first row of Table 1 it is clear that the quantity $\|\mathring{\ell}\|$ converges as $O(|\omega|^{-1})$. From other columns and rows of Table 1 we conclude that order of convergence of our optimal quadrature formula in the case $m = 1$ is $O((N+|\omega|)^{-1})$.

Table 1. The numerical results for $\|\mathring{\ell}\|$ in the case $m = 1$ when $N = 10^k$, $k = 0, 1, 2, 3, 4$, and $\omega = 1, 11, 101, 1001, 10001$.

N	$\omega = 1$	$\omega = 11$	$\omega = 101$	$\omega = 1001$	$\omega = 10001$
1	1.5537(−1)	1.44657(−2)	1.5757878(−3)	1.589959433(−4)	1.5913902915020(−5)
10	2.8664(−2)	1.44078(−2)	1.5757130(−3)	1.589958665(−4)	1.5913902838027(−5)
10^2	2.8865(−3)	2.86386(−3)	1.5757104(−3)	1.589958638(−4)	1.5913902835341(−5)
10^3	2.8867(−4)	2.88652(−4)	2.8674495(−4)	1.589958638(−4)	1.5913902835314(−5)
10^4	2.8868(−5)	2.88675(−5)	2.8865576(−5)	2.867790858(−5)	1.5913902835314(−5)

Now, as an integrand we take the function $\varphi(x) = \mathrm{e}^{2x}$. Then for the actual error $R_N(\omega)$ of the optimal quadrature formula (1.1) we have the following estimate

$$\begin{aligned} R_N(\omega) = |(\mathring{\ell}, \mathrm{e}^{2x})| &= \left| \int_0^1 \mathrm{e}^{2\pi \mathrm{i}\omega x} \mathrm{e}^{2x}\,\mathrm{d}x - \sum_{\beta=0}^{N} C_\beta \mathrm{e}^{2h\beta} \right| \\ &\le \|\mathrm{e}^{2x}|W_2^{(1,0)}[0,1]\| \cdot \|\mathring{\ell}\,|W_2^{(1,0)*}[0,1]\| \\ &= \frac{3}{2}\sqrt{\mathrm{e}^4 - 1}\ \|\mathring{\ell}\,|W_2^{(1,0)*}[0,1]\|. \end{aligned}$$

For the same values of N and ω, using formulas for the optimal coefficients C_β from Theorem 5.1 and formula (5.10), we get the numerical values for the actual error $R_N(\omega)$ and for the bound $B_N(\omega)$ on the right hand side in the previous inequality. These results are presented in Table 2.

Table 2. Numerical values of $R_N(\omega) = |(\mathring{\ell}, \mathrm{e}^{2x})|$ and $B_N(\omega) = \|\mathrm{e}^{2x}\|\|\mathring{\ell}\|$ in the case $m = 1$ for some selected values of N and ω.

N	$\omega = 1$		$\omega = 11$		$\omega = 101$		$\omega = 1001$		$\omega = 10001$	
	$R_N(\omega)$	$B_N(\omega)$	$R_N(\omega)$	$B_N(\omega)$	$R_N(\omega)$	$B_N(\omega)$	$R_N(\omega)$	$B_N(\omega)$	$R_N(\omega)$	$B_N(\omega)$
1	2.1(−1)	1.7(0)	1.9(−3)	1.6(−1)	2.2(−5)	1.7(−2)	2.3(−7)	1.7(−3)	2.3(−9)	1.7(−4)
10	2.4(−3)	3.1(−1)	5.3(−4)	1.6(−1)	6.9(−6)	1.7(−2)	7.1(−8)	1.7(−3)	7.1(−10)	1.7(−4)
10^2	2.4(−5)	3.2(−2)	2.3(−6)	3.1(−2)	7.1(−6)	1.7(−2)	7.3(−8)	1.7(−3)	7.4(−10)	1.7(−4)
10^3	2.4(−7)	3.2(−3)	2.3(−8)	3.2(−3)	2.5(−9)	3.1(−3)	7.3(−8)	1.7(−3)	7.4(−10)	1.7(−4)
10^4	2.4(−9)	3.2(−4)	2.3(−10)	3.2(−4)	2.5(−11)	3.2(−4)	2.6(−12)	3.1(−4)	7.4(−10)	1.7(−4)

These numerical results confirm our theoretical results obtained in the previous sections.

2° *Now we consider the case* $m = 2$.

From (2.11), taking into account (2.7), after some calculations for the norm of the error functional of the optimal quadrature formula (1.1) we get the following expression

$$\begin{aligned} \|\mathring{\ell}\|^2 = &\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} (C_\beta^R C_\gamma^R + C_\beta^I C_\gamma^I) \frac{\mathrm{sgn}(h\beta - h\gamma)}{2} \left[\sinh(h\beta - h\gamma) - h\beta + h\gamma\right] \\ &+ \frac{4\pi^2\omega^2 + 2}{4\pi^2\omega^2(4\pi^2\omega^2 + 1)} - \sum_{\beta=0}^{N} C_\beta^R \left[\frac{(\mathrm{e}^{-1} + 1)\mathrm{e}^{h\beta}}{2(4\pi^2\omega^2 + 1)} + \frac{\cos(2\pi\omega h\beta)}{2\pi^2\omega^2(4\pi^2\omega^2 + 1)} \right] \\ &- \sum_{\beta=0}^{N} C_\beta^I \left[\frac{\pi\omega(\mathrm{e}^{-1} + 1)\mathrm{e}^{h\beta}}{4\pi^2\omega^2 + 1} + \frac{\sin(2\pi\omega h\beta)}{2\pi^2\omega^2(4\pi^2\omega^2 + 1)} - \frac{h\beta}{\pi\omega} \right]. \end{aligned} \tag{6.1}$$

Hence, using the formulas for the optimal coefficients C_β which are given in Corollary 4.4 when $N = 1$ and $\omega = 1, 11, 101, 1001, 10001$, we get the results which are presented in the first row of Table 3. Using the formulas of the optimal coefficients C_β, which are given in Corollary 4.1, when $N = 10, 100, 1000, 10000$ and $\omega = 1, 11, 101, 1001, 10001$, we obtain the numerical results presented in other rows of Table 3. From the numerical results of the first column of Table 3 we see that order of convergence of the optimal quadrature formula (1.1) is $O(N^{-2})$. And from the results presented in the first row of Table 3 we conclude that order of convergence is $O(|\omega|^{-2})$. From the results which are given in other columns and rows of Table 3 we have that order of our optimal quadrature formula in this case is $O((N + |\omega|)^{-2})$.

Table 3. The numerical results for $\|\mathring{\ell}\|$ in the case $m = 2$ when $N = 10^k$, $k = 0, 1, 2, 3, 4$, and $\omega = 1, 11, 101, 1001, 10001$.

N	$\omega = 1$	$\omega = 11$	$\omega = 101$	$\omega = 1001$	$\omega = 10001$
1	3.5377(−2)	2.96022(−4)	3.5116561(−6)	3.575090999(−8)	3.581528459606(−10)
10	4.3982(−4)	2.15172(−4)	2.5538002(−6)	2.599946583(−8)	2.604628302387(−10)
10^2	3.7819(−6)	3.99301(−6)	2.4902722(−6)	2.535258220(−8)	2.539823327481(−10)
10^3	3.7322(−8)	3.73427(−8)	3.9122983(−8)	2.528700745(−8)	2.533254031754(−10)
10^4	3.7273(−10)	3.72734(−10)	3.7291046(−10)	3.904339277(−10)	2.532596167387(−10)

Now we consider the function $\varphi(x) = x^2$ as an integrand. Then for the error of the optimal quadrature formula (1.1) we have

$$|(\mathring{\ell}, x^2)| = \left| \int_0^1 \mathrm{e}^{2\pi \mathrm{i}\omega x} x^2 \,\mathrm{d}x - \sum_{\beta=0}^{N} C_\beta (h\beta)^2 \right| \le \|x^2 | W_2^{(2,1)}[0,1]\| \cdot \|\mathring{\ell}\, | W_2^{(2,1)*}[0,1]\|$$
$$\le \frac{2}{3}\sqrt{21}\; \|\mathring{\ell}\, | W_2^{(2,1)*}[0,1]\|.$$

Using formulas for the optimal coefficients C_β which are given in Corollary 4.4 and formula (6.1) for the left and the right hand sides of the last inequality, respectively, when $N = 1$ and $\omega = 1, 11, 101, 1001, 10001$, we get the numerical results given in the first row of Table 4. The numerical results which are presented in other rows of Table 4 are obtained by using Corollary 4.1 and formula (6.1).

Table 4. Numerical values of $R_N(\omega) = |(\mathring{\ell}, x^2)|$ and $B_N(\omega) = \|x^2\|\|\mathring{\ell}\|$ in the case $m = 2$ for some selected values of N and ω.

N	$\omega = 1$		$\omega = 11$		$\omega = 101$		$\omega = 1001$		$\omega = 10001$	
	$R_N(\omega)$	$B_N(\omega)$	$R_N(\omega)$	$B_N(\omega)$	$R_N(\omega)$	$B_N(\omega)$	$R_N(\omega)$	$B_N(\omega)$	$R_N(\omega)$	$B_N(\omega)$
1	7.5(−2)	1.1(−1)	6.3(−4)	9.0(−4)	7.4(−6)	1.1(−5)	7.6(−8)	1.1(−7)	7.6(−10)	1.1(−9)
10	1.5(−4)	1.3(−3)	3.6(−5)	6.6(−4)	4.3(−7)	7.8(−6)	4.3(−9)	7.9(−8)	4.3(−11)	8.0(−10)
10^2	1.4(−7)	1.2(−5)	1.5(−7)	1.2(−5)	4.4(−8)	7.6(−6)	4.4(−10)	7.7(−8)	4.4(−12)	7.8(−10)
10^3	1.4(−10)	1.1(−7)	1.4(−10)	1.1(−7)	1.5(−10)	1.2(−7)	4.5(−11)	7.7(−8)	4.4(−13)	7.7(−10)
10^4	1.4(−13)	1.1(−9)	1.4(−13)	1.1(−9)	1.4(−13)	1.1(−9)	1.5(−13)	1.2(−9)	4.5(−14)	7.7(−10)

Finally, for the function $x \mapsto \varphi(x) = x^2 \mathrm{e}^{-x}$, we consider an example of calculating Fourier coefficients $\int_0^1 \mathrm{e}^{2\pi \mathrm{i}\omega x}\varphi(x)\,\mathrm{d}x$ using the optimal quadrature formula in the space $W_2^{(2,1)}$. The real part of this integrand, $\cos(2\pi\omega x)\varphi(x)$, for $\omega = 80$ is presented in Figure 1 (left).

The exact value of the corresponding Fourier integral can be obtained in an

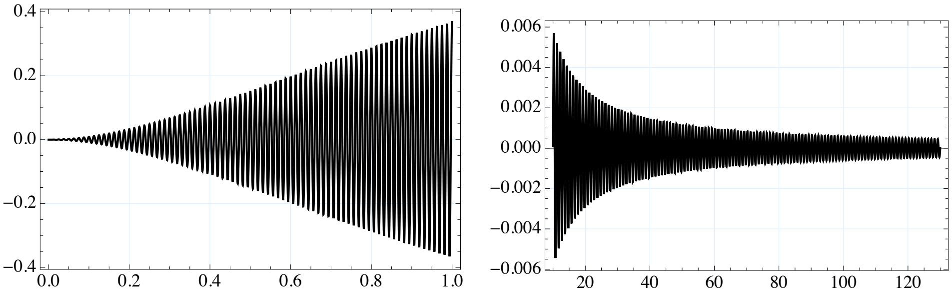

Figure 1. Graphics of the integrand $x \mapsto \cos(2\pi\omega x)\varphi(x)$ for $\omega = 80$ (left) and $\omega \mapsto \Re\{I(\omega)\}$ for $\omega \in [10, 130]$ (right).

analytic form,

$$I(\omega) = \int_0^1 \mathrm{e}^{2\pi \mathrm{i}\omega x}\varphi(x)\,\mathrm{d}x = \frac{2\mathrm{e} + \mathrm{e}^{2\mathrm{i}\omega\pi}(-5 + 8\mathrm{i}\omega\pi + 4\omega^2\pi^2)}{\mathrm{e}(1 - 2\mathrm{i}\omega\pi)^3},$$

and therefore, we can calculate the actual relative errors

$$\mathrm{err}_N(\omega) = \left|\frac{Q_N(\omega) - I(\omega)}{I(\omega)}\right|$$

in our optimal quadrature sums $Q_N(\omega) = \sum\limits_{\beta=0}^{N} C_\beta \varphi(h\beta)$.

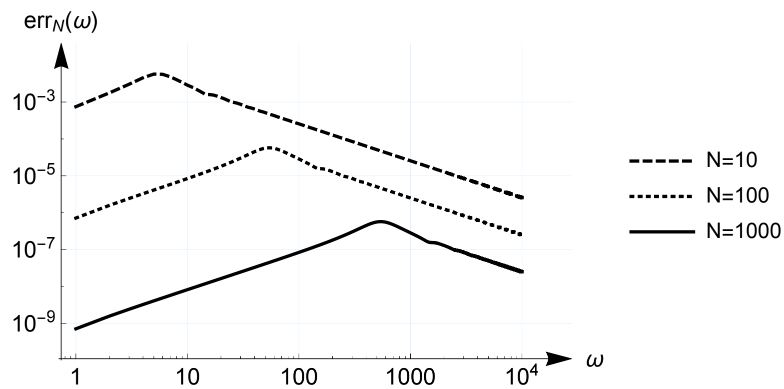

Figure 2. Relative errors $\omega \mapsto \mathrm{err}_N(\omega)$ for $N = 10, 100, 1000$.

The real part of the integral $I(\omega)$ is displayed in Figure 1 (right) for $\omega \in [10, 130]$.

Graphics of $\omega \mapsto \mathrm{err}_N(\omega)$ for $N = 10, 100, 1000$, when ω runs over $[1, 10^4]$, are presented in Figure 2 in log-log scale.

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