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NONTRIVIAL PERIODIC SOLUTIONS FOR SECOND-ORDER DIFFERENTIAL DELAY EQUATIONS*

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Abstract In this paper, we consider the existence of periodic solutions for second-order differential delay equations. Some existence results are obtained using Malsov-type index and Morse theory, which extends and complements some existing results.

Keywords Differential delay equations, periodic solutions, variational calculus, Malsov-type index, Morse theory.

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1. Introduction

In recent decades, there are some results on the periodic solutions to delay differential equations that were established via variational calculus([1–15]). To the authors' knowledge, there are a few results on the periodic solutions to second-order delay differential equations that were established by Morse theory [16, 17], the Galerkin approximation scheme [18, 19] and Maslov-type index theory [20, 21]. Motivated by [9, 15], we consider existence of 2τ-periodic solutions for the following delay differential equations

$$x''(t) = -f(x(t-\tau)), \tag{1.1}$$

where $\tau > 0$ is a given constant.

Throughout this paper, we assume that $(f_1) - (f_3)$ as following:

(f_1) $f(x) \in C(\mathbf{R}^N, \mathbf{R}^N)$ is odd, i.e. for any $x \in \mathbf{R}^N$, $f(x) = -f(x)$.

(f_2) There exists a C^1-differentiable function $F(x)$, such that $\nabla F(x) = f(x)$, $\forall x \in \mathbf{R}^N$ and $F(0) = 0$.

(f_3) There are real symmetric $N \times N$ matrices A and B such that

(i) $f(x) = Ax + o(|x|)$ as $x \to \infty$,

(ii) $f(x) = Bx + o(|x|)$ as $x \to 0$,

that is, (1.1) is asymptotically linear both at infinity and origin.

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Throughout this paper, we also make some assumptions as follows:

For some positive numbers $R, c_1, c_2 > 0$ and $0 < s < 1$, let $F_\infty(x) = F'(x) - Ax$, which satisfies

(F_∞^+) $(F_\infty(x), x) \geq c_1|x|^{1+s},\ |F_\infty(x)| \leq c_2|x|^s, x \in \mathbf{R}^N$ with $|x| \geq R$;

(F_∞^-) $(F_\infty(x), x) \leq 0$ and $(F_\infty(x), x) \geq c_1|x|^{1+s},\ |F_\infty(x)| \leq c_2|x|^s, x \in \mathbf{R}^N$ with $|x| \geq R$.

For some positive numbers $\rho, c_3, c_4 > 0$ and $r > 1, F_0(x) := F'(x) - Bx$ satisfies

(F_0^+) $(F_0(x), x) \geq c_3|x|^{1+r},\ |F_\infty(x)| \leq c_4|x|^r, x \in \mathbf{R}^N$ with $|x| \leq \rho$;

(F_0^-) $(F_0(x), x) \leq 0$ and $|(F_0(x), x)| \geq c_3|x|^{1+r},\ |F_0(x)| \leq c_4|x|^r, x \in \mathbf{R}^N$ with $|x| \leq \rho$.

As that in [22], for given $N \times N$ real symmetric matrices S, T and positive integer k, we set

$z_k(S) =$ the number of negative eigenvalues of $(-1)^k(2k-1)^2 I + S$,

$\bar{z}_k(S) =$ the number of non-positive eigenvalues of $(-1)^k(2k-1)^2 I + S$,

and

$$\rho(S,T) = \sum_{k=1}^{\infty}[z_k(S) - z_k(T)], \quad \rho_1(S,T) = \sum_{k=1}^{\infty}[\bar{z}_k(S) - z_k(T)],$$
$$\rho_2(S,T) = \sum_{k=1}^{\infty}[\bar{z}_k(S) - \bar{z}_k(T)],$$

where I denotes the $N \times N$ identity matrix. It is known that $\rho(\cdot,\cdot), \rho_i(\cdot,\cdot), i = 1, 2$ are well defined, since for large k, $z_k(S) - z_k(T) = \bar{z}_k(S) - z_k(T) = \bar{z}_k(S) - \bar{z}_k(T) = 0$.

The rest of the paper is organized as follows. In Section 2, we shall state some lemmas. Criteria for the existence of τ-periodic solution for (1.1) is established in Section 3.

2. Preliminaries

In the following, we introduce some basic preliminary results on critical groups and Morse theory [17, 18].

Let E be a Hilbert space with its inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $\psi \in C^1(E, \mathbf{R})$. Let $K = \{x \in E | \psi(x) = \theta\}, \psi_m = \{x \in E | \psi(x) \leq m\}$.

Definition 2.1. Suppose that $\psi(K)$ is bounded from below by $a \in R$ and that ψ satisfies $(PS)_c$ for all $c \leq a$. The group $C_q(\psi, \infty) =: H_q(E, \psi_m), q \in Z$ is said to be the qth critical group of ψ at infinity. Here $H_*(\cdot,\cdot)$ denotes singular relative homology groups with Abelian coefficient groups.

We will work on the following framework as in [23, 24], where the group $C_*(\cdot,\cdot)$ can be described precisely.

(A'_∞) $\psi(x) = \frac{1}{2}\langle Lx, x\rangle + \Lambda x$, where $L : E \to E$ is a self-adjoint operator such that 0 is isolated in the spectrum of L. The map $\Lambda \in C^1(E, R)$ satisfies $\Lambda'(x) = o(\|x\|)$ as $\|x\| \to \infty$. Λ, Λ' map bounded sets into bounded sets. $\psi(x)$ is bounded from below and ψ satisfies $(PS)_c$ for $c \ll 0$.

If (A'_∞) is satisfied. Let $V = Ker(L)$ and $W = V^\perp$. We split W as $W = W^+ \oplus W^-$ such that $L|_{W^+}$ is positive definite and $L|_{W^-}$ is negative definite. Denote $\mu := \dim W^-, \nu = \dim W$, the Morse index and the nullity of ψ at infinity respectively.

Proposition 2.1. *Let* (A'_∞) *hold. Then*

(i) $C_q(\psi, \infty) \cong \delta_{q,\mu} G$ *provided* ψ *satisfies the angle condition at infinity:*

(AC_∞^+) $\exists M>0, \alpha\in(0,\frac{\pi}{2})$ *such that* $\langle\psi'(x),v\rangle\geq 0$ *for any* $x=v+w\in X$ *with* $\|x\|\geq M, \|w\|\leq\|x\|\sin\alpha$.

(ii) $C_q(\psi,\infty)\cong\delta_{q,\mu+\nu}G$ *provided* ψ *satisfies the angle condition at infinity:*

(AC_∞^-) $\exists M>0, \alpha\in(0,\frac{\pi}{2})$ *such that* $\langle\psi'(x),v\rangle\leq 0$ *for any* $x=v+w\in X$ *with* $\|x\|\geq M, \|w\|\leq\|x\|\sin\alpha$.

Proposition 2.2. *Suppose* ψ *has an isolated critical point* x_0 *and is of class* C^2 *near* x_0. 0 *is isolated in the spectrum of* $L_0:=\psi''(x_0)$ *and* $\mu_0<\infty,\nu_0<\infty$, *where* μ_0,ν_0 *denote the Morse index and the nullity of* ψ *at* x_0. *Then*

(i) $C_q(\psi,x_0)\cong\delta_{q,\mu_0}G$ *provided* ψ *satisfies the angle condition at infinity:*

(AC_0^+) $\exists\rho>0,\alpha\in(0,\frac{\pi}{2})$ *such that* $\langle\psi'(x+x_0),v\rangle>0$ *for any* $x=v+w\in E=V_0\oplus W_0$ *with* $\|x\|\leq\rho, \|w\|\leq\|x\|\sin\alpha$.

(ii) $C_q(\psi,x_0)\cong\delta_{q,\mu_0+\nu_0}G$ *provided* ψ *satisfies the angle condition at infinity:*

(AC_0^-) $\exists\rho>0,\alpha\in(0,\frac{\pi}{2})$ *such that* $\langle\psi'(x+x_0),v\rangle<0$ *for any* $x=v+w\in E=V_0\oplus W_0$ *with* $\|x\|\leq\rho, \|w\|\leq\|x\|\sin\alpha$, *where* $V_0=Ker(L_0), W_0=W_0^+\oplus W_0^-:=V_0^\perp$.

By the change of variable $\lambda=\frac{\tau}{\pi}, t=\lambda s$, (1.1) is transformed to

$$x''(t)=-\lambda^2 f(x(t-\pi)). \tag{2.0}$$

Thus to seek a 2τ-periodic solution for (1.1) is equivalent to seek a 2π-periodic solution for (2.0).

Let $C^\infty(S^1,\mathbf{R}^N)$ denote the space of 2π periodic C^∞ functions on $\mathbf{R}$ with values in $\mathbf{R}^N$. For any, let $E:=W^{1/2,2}(S^1,\mathbf{R}^N)=\overline{C^\infty(S^1,\mathbf{R}^N)}$, where $|\cdot|$ is the induced norm in $\mathbf{R}^N$. Then E consists of those $z(t)\in L^2(S^1,\mathbf{R}^N)$ with Fourier series

$$z(t)=\frac{a_0}{\sqrt{2\pi}}+\frac{1}{\sqrt{\pi}}\sum_{m=1}^{\infty}\left(a_m\cos mt+b_m\sin mt\right),$$

where $a_0,a_m,b_m\in\mathbf{R}^N$. E is also a Hilbert space with norm $\|z\|=\int_0^{2\pi}[|z(t)|^2+|z'(t)|^2]dt<+\infty$ induced by the inner product $\langle\cdot\rangle$ defined as

$$\langle z,\bar{z}\rangle=(a_0,\bar{a}_0)+\sum_{m=1}^{\infty}(1+m^2)[(a_m,\bar{a}_m)+(b_m,\bar{b}_m)]$$

with $\bar{z}(t)=\frac{\bar{a}_0}{\sqrt{2\pi}}+\frac{1}{\sqrt{\pi}}\sum_{m=1}^\infty\left(\bar{a}_m\cos mt+\bar{b}_m\sin mt\right)$, where $(\cdot,\cdot)$ denotes the standard inner product in $\mathbf{R}^N$.

For $\forall x\in E$, define a variational functional $I:E\to\mathbf{R}$ as

$$I(x)=\frac{1}{2}\int_0^{\frac{\pi}{2}}(x'(t+\pi),x'(t))dt+\lambda^2\int_0^{2\pi}F(x(t))dt=\langle Lx,x\rangle+\Phi(x). \tag{2.1}$$

Then by Riesz representation theorem, we can define a bounded linear self-adjoint operator on E as $\langle Lx,y\rangle=\frac{1}{2}\int_0^{2\pi}(x'(t+\pi),y'(t))dt=-\frac{1}{2}\int_0^{2\pi}(x''(t+\pi),y(t))dt$(which is obtained by integration by parts). We also define a compact operator $\Phi(x)=\lambda^2\int_0^{2\pi}F(x(t))dt, x\in E$ as [15]. Then $I(x)$ can be written as

$$I(x)=\langle Lx,x\rangle+\Phi(x). \tag{2.1$'$}$$

By a standard argument as in [16], we get that $I \in C^2(E, \mathbf{R})$ and critical points of I are solutions of (1.1). Thus, looking for nontrivial 2τ-periodic solution of (1.1) is equivalent to finding nonzero critical point of I.

For any $x, y \in E$, an easy computation of [15] gives the gradient of I as

$$\langle I'(x), y\rangle = \int_0^{2\pi} (x'(t+\pi), y'(t))dt + \lambda^2 \int_0^{2\pi} (f(x(t)), y(t))dt. \tag{2.2}$$

Let $\bar{E}$ be a subspace of E by $\bar{E} = \{x \in E \mid x(t+\pi) = -x(t)\}$, then

$$\bar{E} = \left\{ x \in E \mid x(t) = \frac{1}{\sqrt{\pi}} \sum_{m=1}^{\infty} [a_{2m-1}\cos(2m-1)t + b_{2m-1}\sin(2m-1)t] \right\}.$$

Under the norm $\int_0^{2\pi}[|z(t)|^2 + |z'(t)|^2 + |z''(t)|^2]dt < +\infty$, $\bar{E}$ is also a Hilbert space. Let the subspace $E_k (k = 1, 2, \cdots)$ of $\bar{E}$ be defined by

$$E_k = \left\{ x \in E \mid x(t) = \frac{1}{\sqrt{\pi}} \sum_{m=1}^{k} [a_{2m-1}\cos(2m-1)t + b_{2m-1}\sin(2m-1)t] \right\}.$$

Then $E_k \subset E_{k+1}$ and $dim E_k = 2kN, \forall k \in \mathbf{N}$.

Lemma 2.1. *Let D be an self-adjoint operator defined by a $N \times N$ symmetric matrix D and $m^+(L+D)_k, m^0(L+D)_k$ and $m^-(L+D)_k$ denote the dimension of the subspaces of E_k, where $(L+D)_k$ is positive definite, zero and negative definite respectively. Then*

(1) $m^-(L+D)_k = \sum_{j=1}^{k} z_j(D)$, $\quad m^0(L+D)_k = \sum_{j=1}^{k}(\bar{z}_j(D) - z_j(D)$, $\quad m^-(L+D)_k + m^0(L+D)_k + m^+(L+D)_k = 4Nk$.

(2) *there exists $\eta > 0$ independent of k such that $(I_\eta \setminus \{0\}) \cap (L+D)_k = \emptyset$ for all $k \in \mathbf{N}$, where $I_\eta = (-\eta, \eta)$ and $\sigma((L+D)_k)$ denote the spectrum of $(L+D)_k$.*

(3) *there exists a positive integer $\bar{k} > 0$ such that $Ker(L+D) \subset E_k$ for $k \geq \bar{k}$.*

Proof. Let $\mu_{ji}, i = 1, 2, \cdots, N$ be the eigenvalues of $(-1)^j(2j-1)^2 I + D$ and $u_{ji}, i = 1, 2, \cdots, N$ be the corresponding eigenvectors, which form an orthogonal basis of $\mathbf{R}^N$ for every $j \in \mathbf{N}$. By the argument in [16, Section 5], $\mu_{ji}/(1 + 2j - 1 + (2j-1)^2), j \in \mathbf{N}, i = 1, 2, \cdots, N$ are all the eigenvalues of $L + D$ and

$$e_{ji}^{(c)} = u_{ji}\cos(2j-1)t, \quad \tilde{e}_{ji}^{(c)} = u_{ji}\sin(2j-1)t, \quad j \in \mathbf{N}, i = 1, 2, \cdots, N,$$

form a complete orthogonal basis of E. We also get $\frac{\mu_{ji}}{(1+2j-1+(2j-1)^2)}, j = 1, 2, \cdots, k$, $i = 1, 2, \cdots, N$ are all the eigenvalues of $(L+D)_k$ and $e_{ji}^{(c)} = u_{ji}\cos jt, \tilde{e}_{ji}^{(c)} = u_{ji}\sin(2j-1)t, j = 1, 2, \cdots, k, i = 1, 2, \cdots, N$, form a complete orthogonal basis of E_k. It follows that the conclusion (1) holds.

Denote the eigenvalues of the matrix D are $d_i, i = 1, 2, \cdots, N$, where d_i are finite. Then by the argument in [16, Section 5], $\mu_{ji} = (-1)^j(2j-1)^2 + d_i, j \in \mathbf{N}, i = 1, 2, \cdots, N$ and

$$\frac{\mu_{ji}}{(1+2j-1+(2j-1)^2)} = \frac{(-1)^j(2j-1)^2 + d_i}{(1+2j-1+(2j-1)^2)} \to \pm 1, \quad \left\{ \left| \frac{\mu_{ji}}{j+1} \right| \right\} \setminus \{0\}$$

has positive minimum $\eta > 0$. Thus we have $I_\eta \setminus \{0\} \cap \sigma((L+D)_k) = \emptyset$ for all $k \in \mathbf{N}$, which shows that (2) holds.

For sufficiently large $j, (-1)^j(2j-1)^2 I + D$ is non-degenerate, then $\{u_{ji}|\mu_{ji} = 0, j \in \mathbf{N}, i = 1, 2, \cdots, N\}$ is a finite set. Since $Ker(L+D) = span\{e_{ji}^{(c)}, \tilde{e}_{ji}^{(c)}|\mu_{ji} = 0, j \in \mathbf{N}, i = 1, 2, \cdots, N\}$, it follows that (3) holds and the proof is completed. □

The following Lemma 2.2 can be showed in the same way as that of [23, Lemmas 3.1 and 3.2], so we omit the proof of it.

Lemma 2.2. *Suppose thatf satisfies* $(f_1)-(f_3)$ *and let the functional* I *be defined by* (2.1). *We have the following propositions:*

(1) *If* $(F_\infty^+)(or(F_\infty^-))$ *holds, then* I *satisfies the angle condition* $(AC_\infty^+)(or(AC_\infty^-))$ *at infinity, i.e., there are* $M > 0, \beta > 0, \alpha \in (0, \frac{\pi}{2})$ *such that*

$$\langle I'(x), \frac{v}{\|v\|}\rangle \geq \beta > 0, \quad (or\ \langle I'(x), \frac{v}{\|v\|}\rangle \leq -\beta < 0,)$$

for any $x = v + w \in \bar{E}$ *with* $\|x\| \geq M, \|w\| \leq \|x\| sin\alpha$, *where* $V = Ker(L + \lambda^2 A)$, *the null space of the self-adjoint operator* $L + \lambda^2 A$ *and* $W = V^\perp$.

(2) *If* $(F_0^+)(or(F_0^-))$ *holds, then there are* $\rho > 0, 0 < \epsilon < 1$ *such that*

$$\int_0^{2\pi} (F_0(x), v,)dt > 0, \quad (or \int_0^{2\pi} (F_0(x), v)dt < 0,)$$

for any $x = v + w \in \bar{E} = V_0 \oplus W_0$ *with* $x \neq 0, \|x\| \leq \rho, \|w\| \leq \epsilon\|x\|$, *where* $V_0 = Ker(L + \lambda^2 B)$, *the null space of the self-adjoint operator* $L + \lambda^2 B$ *and* $W_0 = V_0^\perp$. *Hence the function* I *defined by* (2.1) *satisfies the angle condition* $(AC_0^+)(or(AC_0^-))$ *at the origin.*

Lemma 2.3. *Suppose thatf satisfies* $(f_1)-(f_3)$ *and the functional* $I_k = I|_{E_k}$. *We have the following propositions:*

(1) *If* $(F_\infty^\pm)$ *holds, then* $\exists k_\infty \in \mathbf{N}$ *such that* $I_k, k \geq k_\infty$ *satisfies the angle condition* $(AC_\infty^\pm)$ *at infinity.*

(2) *If* $(F_0^\pm)$ *holds, then* $\exists k_0 \in \mathbf{N}$ *such that* $I_k, k \geq k_0$ *satisfies the angle condition* $(AC_0^\pm)$ *at origin.*

Proof. Proof of (1). By Lemma 2.1(3), there exists $k_\infty > 0$ such that E_k can be divided as $E_k = V \oplus W_k$ for $k \geq k_\infty$, where W_k is the orthogonal projection of W on to E_k. Since, by Lemma 2.2(1), I satisfies $(AC_\infty^\pm)$ for $k \geq k_\infty$ at infinity for any $x = v + w \in V \oplus W_k = E_k$ with $\|x\| \geq M, \|w\| \leq \|x\| \sin\alpha$ with $x \in E_k, v \in V, w \in W_k$, then we have

$$\pm\langle I'(x), \frac{v}{\|v\|}\rangle = \pm\langle I_k'(x), \frac{v}{\|v\|}\rangle = \pm\langle I'(x), \frac{v}{\|v\|}\rangle \geq \beta > 0,$$

i.e. I_k satisfies $(AC_\infty^\pm)$ for $k \geq k_\infty$ at infinity.

Proof of (2). By Lemma 2.1(3), there exists $k_\infty > 0$ such that E_k can be divided as $E_k = V_0 \oplus W_0$ for $k \geq k_\infty$, where W_0 is the orthogonal projection of W on to E_k. Since, by Lemma 2.2(2), I satisfies $(AC_0^\pm)$ for $k \geq k_\infty$ at origin for any $x = v+w \in V_0 \oplus W_0 = E_k$, with $x \neq 0, \|x\| \leq \rho, \|w\| \leq \epsilon\|x\|, x \in E_0, v \in V_0, w \in W_0$, where $V_0 = Ker(L + \lambda^2 B)$, the null space of the self-adjoint operator $L + \lambda^2 B$ and $W_0 = V_0^\perp$, then we have

$$\mp\langle I'(x), v\rangle = \mp\langle I_k'(x), v\rangle = \pm \int_0^{2\pi} (F_0(x), v)dt \geq \beta > 0,$$

i.e. I_k satisfies $(AC_0^\pm)$ for $k \geq k_\infty$ at origin. Then the proof is completed. □

Definition 2.2. We say that I satisfies the $(PS)^*$ condition, if any sequence $x_n \in E_m$ such that $I_m(x_n)$ being bounded and $I'_m(x_n)$ possesses a subsequence convergent in E_m, where $I_m = I|_{E_m}$ is the restriction of I on E_m.

Lemma 2.4. *Assume that* $(f_1)-(f_3)$ *and* $(F^{\pm}_{\infty})$ *hold. Then*

(1) *Each* I_k *satisfies the* (PS) *condition for* $k \geq k_\infty (\in \mathbf{N})$.

(2) I *satisfies the* $(PS)^*$ *condition, i.e. any sequence* $\{x_k\} \in E_k$ *such that* $I'_k(x_k) \to 0$ *as* $k \to \infty$ *and that* $I_k(x_k)$ *is bounded from above, possesses a subsequence convergent in* $\bar{E}$.

Proof. (1) Let k_∞ be as Lemma 2.2 and $E_k = V \oplus W_k$, where V and E_k be defined as above. For any fixed $k \geq k_\infty$, let $\{x_m\} \in E_k$ be such that

$$I'_k(x_n) \to 0 \text{ as } n \to \infty. \tag{2.4}$$

Now, we will show that $\{x_m\} \in E_k$ is bounded. If not, then

$$\|x_n\| \to \infty \text{ as } n \to \infty. \tag{2.5}$$

Let $x_n = v_n + w_n = v_n + w_n^- + w_n^+ \in E_k = V \oplus W_k = V \oplus W_k^- \oplus W_k^+$. Then for any $y \in E_k$, we get

$$\langle I'(x_n), y\rangle = \langle (L+\lambda^2 A)_k x_n, y\rangle + \langle I'_k(x_n) - (L+\lambda^2 A)_k x_n, y\rangle. \tag{2.6}$$

By $(f_1), (f_3)$, for any $r > 0$, there exists $c(r) > 0$ such that

$$|f(t,y) - Ay| \leq r|y| + c(r), \forall y \in \mathbf{R}^N. \tag{2.7}$$

Then by (2.7) and compactly imbedding theorem($E, \bar{E}$ are compactly embedded in $L^2(S^1, \mathbf{R}^N)$), we have

$$\|I'(x) - (L+\lambda^2 A)x\| \leq c_1 \|f(t,x) - \lambda^2 Ax\|_{L^2} \leq c_1(r|x| + c(r)), \forall x \in \bar{E}, \tag{2.8}$$

where c_1 is a constant independent of r. Since that r can be arbitrarily small, it follows that

$$\frac{\|I'(x) - (L+\lambda^2 A)x\|}{\|x\|} \to 0, \text{ as } x \to \infty, \tag{2.9}$$

then we get

$$\frac{\|I'(x_n) - (L+\lambda^2 A)_k x_n\|}{\|x_n\|} \to 0, \ \forall x_n \in E_k, \text{as } x_n \to \infty. \tag{2.10}$$

Let $\eta > 0$ be as in Lemma 2.1(2) and take $y = w_n^- + w_n^+$ as above. Then from (2.4) and (2.10), for any given $\delta > 0$, we have

$$\eta \|w_n\|^2 \leq \delta \|w_n\| \|x_n\| + \delta \|w_n\|, \tag{2.11}$$

for n sufficiently large. It follows from (2.11) and δ is arbitrary small that

$$\frac{\|w_n\|}{\|x_n\|} \to 0, \text{as } n \to \infty. \tag{2.12}$$

From (2.5) and (2.12), we get that

$$\|x_n\| \geq M, \quad \|w_n\| \leq \|x_n\| \sin\alpha, \text{ for sufficiently large } n.$$

Since I_k satisfies the angle condition $(AC^{\pm}_{\infty})$, which implies that

$$\left|\left\langle I'(x), \frac{v}{\|v\|}\right\rangle\right| \geq \beta > 0. \tag{2.13}$$

By (2.4),

$$\left\langle I'(x), \frac{v}{\|v\|}\right\rangle \to 0, \quad \text{as} \quad n \to \infty, \tag{2.14}$$

which is contract with (2.13). Thus $\{x_m\} \in E_k$ is bounded.

(2). Suppose that $\{x_k\} \in E_k$ such that

$$I_k'(x_k) \to 0 \text{ as } k \to \infty. \tag{2.15}$$

Similar to above discussion, we obtain that $\{x_k\}$ is bounded.

Let $\Phi_k = \Phi\mid_{E_k}$ and P_k be the project mapping from $\bar{E}$ to E_k, then

$$L_k(x_k) = I_k'(x_k) - \Phi_k'(x_k) \quad \text{and} \quad \Phi_k'(x_k) = P_k\Phi'(x_k). \tag{2.16}$$

Since E_k is invariant with respect to L, (2.16) shows that

$$L(x_k) = I_k'(x_k) - P_k\Phi'(x_k). \tag{2.17}$$

Since I' is compact, so we can choose a subsequence of $\{x_k\}$, (we still named as $\{x_k\}$,) such that $\{-P_k\Phi'(x_k)\} \to y$ in $\bar{E}$. It follows from (2.17) that $\lim\limits_{k\to\infty} Lx_k = y$. It is easy to see that $L : \bar{E} \mapsto \bar{E}$ is invertible, so that $\lim\limits_{k\to\infty} x_k = L^{-}(y)$. It's obvious that x is a critical point of I. The proof is completed.

□

3. Main results.

Theorem 3.1. *Assume that f is C^1-differentiable in x near the origin $\mathbf{0} \in E$ and satisfies $(f_1) - (f_3)$. If $\rho_1(\lambda^2 B, \lambda^2 B) = 0$ and one of the following conditions holds:*

(1) (A^+_{∞}) *and* $\rho(\lambda^2 A, \lambda^2 B) \neq 0$,

(2) (A^-_{∞}) *and* $\rho_1(\lambda^2 A, \lambda^2 B) \neq 0$,

then (1.1) *has a nontrivial* 2τ*-periodic solution.*

Proof. Proof of (1). Step 1. For each $k \in \mathbf{N}$, define the map $\Lambda_k : E_k \mapsto \mathbf{R}$ as

$$\Lambda_k(x) = \Phi_k(x) - \frac{1}{2}\langle\lambda^2 Ax, x\rangle, \ x \in E_k. \tag{3.1}$$

Then I_k can be rewritten as

$$I_k(x) = \frac{1}{2}\langle(L + \lambda^2 Ax)_k x, x\rangle + \Phi_k(x), \ x \in E_k. \tag{3.2}$$

By (A^+_{∞}) and the standard arguments, we get that Λ_k is C^1 under the topology $C = C([0, 2\pi], \mathbf{R}^N)$ and satisfies

$$\|\Lambda_k'(x)\|_C = o(\|x\|_C), \quad \text{as } \|x\|_C \to \infty, x \in E_k, \tag{3.3}$$

which, combining with Lemma 2.1(2), implies that I_k satisfies (A'_∞). It follows by Lemma 2.3(1) and Proposition 2.1(1) that

$$C_q(I_k,\infty)\cong\delta_{q,\mu_k}G,\ k\geq k_\infty, \tag{3.4}$$

where $\mu_k=m^-(L+\lambda^2A)_k$.

Since the injection of E into $C=C([0,2\pi],\mathbf{R}^N)$, with its maximum norm $\|x\|_C$, is continuous and f is C^1-differentiable near $\mathbf{0}\in E_k$, we know that I_k is C^2-differentiable near the origin $\mathbf{0}\in E_k$. Further, we have

$$I'_k(\mathbf{0})=(L+\lambda^2B)_k. \tag{3.5}$$

Since $\rho_1(\lambda^2B,\lambda^2B)=0$, which implies that for every $k\in\mathbf{N}$, $\bar{z}_k(\lambda^2B)-z_k(\lambda^2B)=0$, we see that the origin $\mathbf{0}\in E_k$ is a non-degenerate critical point of I_k. Thus we have

$$C_q(I_k,\mathbf{0})\cong\delta_{q,\mu_k^0}G,\ k\in\mathbf{N}, \tag{3.6}$$

where $\mu_k^0=m^-(L+\lambda^2B)_k$. By $\rho(\lambda^2A,\lambda^2B)\neq 0$, there exists $\bar{k}\in\mathbf{N}$ such that for $k\geq\bar{k}$, $\mu_k\neq\mu_k^0$ for $k\geq\max\{k_\infty,\bar{k}\}$. Then from (3.4), (3.6), it follows that I_k has different critical groups at origin and at infinity respectively, which implies that I_k has at least one nontrivial critical point $x_k\neq\mathbf{0}$ for $k\geq\max\{k_\infty,\bar{k}\}$.

Step 2. From above discussion of Step 1, we know that $\mathbf{0}\notin\sigma((L+\lambda^2B)_k)$. By Lemma 2.1(2), which implies that there exists a constant $\eta>0$ independent of k, such that $(-\eta,\eta)\cap\sigma((L+\lambda^2B)_k)=\emptyset$. It follows that for $k\geq\max\{k_\infty,\bar{k}\}$,

$$\|(L+\lambda^2B)_kx\|\geq\eta\|x\|,\ x\in E_k,x\neq\mathbf{0}. \tag{3.7}$$

From (2.9),(2.10), for $k\geq k'\in\mathbf{N}$, it follows that there exists a constant $r_0>0$ such that

$$\|I'(x)-(L+\lambda^2B)x\|\leq\frac{\eta}{2}\|x\|,\ x\in E,\|x\|<r_0, \tag{3.8}$$

and

$$\|I'_k(x)-(L+\lambda^2B)_kx\|\leq\frac{\eta}{2}\|x\|,\ x\in E_k,\|x\|<r_0, \tag{3.9}$$

which, combining with (2.6) and (3.7), implies that for $x\in E_k,\|x\|<r_0$,

$$\begin{aligned}\|I'_k(x)\|&=\|I'_k(x)-(L+\lambda^2B)_kx+(L+\lambda^2B)_kx\|\\&\geq\|(L+\lambda^2B)_kx\|-\|I'_k(x)-(L+\lambda^2B)_kx\|\geq\tfrac{\eta}{2}\|x\|.\end{aligned} \tag{3.10}$$

Since x_k is a nontrivial critical point of I_k in E_k, we know from (3.10) that $\|x\|\geq r_0$ for $k\geq\hat{k}=\max\{k_\infty,\bar{k},k'\}$.

Step 3. By Lemma 2.4(2), I satisfies the $(PS)^*$ condition. Hence there is a limit point x of $\{x_k\}$, which is a critical point of I with $\|x\|\geq r_0$ for $k\geq\max\hat{k}$. Then the proof is completed.

Case (2) can be proved in the same way as the proof of (1). □

Theorem 3.2. *Assume that f is C^1-differentiable in x near the origin $\mathbf{0}\in E$ and satisfies $(f_1)-(f_3)$. If $\rho_1(\lambda^2B,\lambda^2B)>0$ and one of the following conditions holds:*

(1) $(A^+_\infty),(A^+_0)$ *and* $\rho(\lambda^2A,\lambda^2B)\neq 0$,
(2) $(A^+_\infty),(A^+_0)$ *and* $\rho_1(\lambda^2B,\lambda^2A)\neq 0$,
(3) $(A^-_\infty),(A^+_0)$ *and* $\rho(\lambda^2A,\lambda^2B)\neq 0$,
(4) $(A^-_\infty),(A^-_0)$ *and* $\rho_2(\lambda^2B,\lambda^2A)\neq 0$,

then (1.1) *has at least a nontrivial 2τ-periodic solution.*

Proof. Proof of (1). Step 1. Similar to the proof of Theorem 3.1, we can get (3.4). By (A_0^+), Lemma 2.3(2) and Proposition 2.2(i), we have

$$C_q(I_k, \mathbf{0}) \cong \delta_{q,\mu_k^0} G, \ k \geq k_0. \tag{3.11}$$

Note that

$$\rho(\lambda^2 A, \lambda^2 B) \neq 0$$

implies that $\mu_k \neq \mu_k^0$ for $k \geq k'' \in \mathbf{N}$, we have

$$C_q(I_k, \mathbf{0}) \ncong C_q(I_k, \infty), \ k \geq \tilde{k} = \max\{k_\infty, k_0, k''\}, \tag{3.12}$$

which implies that $I_k, k \geq \tilde{k}$ has at least one nontrivial critical point $x_k \neq \mathbf{0}$.

Step 2. By Lemma 2.1(2), we have

$$\|(L + \lambda^2 B)_k w\| \geq \eta \|w\|, \ w \in W_k, w \neq \mathbf{0}. \tag{3.13}$$

Consider that $x = v + w \in E_k = V \oplus W_k$ with $x \neq \mathbf{0}$ and $\|x\| \leq \rho$. If $\|w\| \leq \|x\| \sin\alpha$, then by (A_∞^+), we get

$$\langle I_k'(x), v\rangle > 0 \quad \text{for} \quad k \geq \max \tilde{k}. \tag{3.14}$$

If $\|w\| \geq \|x\| \sin\alpha$, then

$$\begin{aligned} \langle I_k'(x), w\rangle &= \langle (L + \lambda^2 B)_k(x), w\rangle + \langle I_k'(x) - (L + \lambda^2 B)_k(x), w\rangle \\ &\geq \eta\|w\| - o(\|x\|)\|w\| \geq \eta\|w\| - o(\|w\|^2) > 0. \end{aligned} \tag{3.15}$$

Thus there exists a constant $r_1 > 0$, independent of k, such that for any $k \geq \tilde{k}$,

$$\langle I_k'(x), w\rangle \neq 0, \ x \in E_k, \|x\| < r_1.$$

So we have $\|x_k\| \geq r_1$ for any $k \geq \tilde{k}$.

Step 3. By Lemma 2.4, I satisfies $(PS)^*$, thus $\{x_k\}$ has a subsequence converging to some point $x \in \bar{E}$ with $\|x\| \geq r_1$, which is a nontrivial critical point of I, i.e. problem (1.1) has at least one nontrivial 2π-periodic solution.

Case (2)-(4) can be proved in the same way as the proof of (1). So we omit it. □

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References

[1] J. B. Li and X. Z. He, *Periodic solutions of some differential delay equations created by Hamiltonian systems*, Bull. Aust. Math. Soc., 1999, 60(3), 377–390.

[2] J. B. Li and X. Z. He, *Multiple periodic solutions of differential delay equations created by asymptotically linear Hamiltonian systems*, Nonlinear Anal. TMA., 1998, 31(1–2), 45–54.

[3] J. B. Li and X. Z. He, *Proof and generalization of Kaplan-Yorke's conjecture under the condition $f'(0) > 0$ on periodic solution of differential delay equations*, Sci. China Series A, 1999, 42(9), 957–964.

[4] J. B. Li, X. Z. He and Z. Liu, *Hamiltonian symmetric groups and multiple periodic solutions of differential delay equations*, Nonlinear Anal. TMA., 1999, 35(4), 457–474.

[5] Y. T. Xu and Z. M. Guo, *Applications of a Z_p index theory to periodic solutions for a class of functional differential equations*, J. Math. Anal. Appl., 2001, 257(1), 189–205.

[6] Z. M. Guo and J. S. Yu, *Multiplity results for periodic solutions to delay differential equations via critical point theory*, J. Differential equations, 2005, 218(1), 15–35.

[7] G. H. Fei, *Multiple periodic solutions of differential delay equations via Hamiltonian systems (I)*, Nonlinear Analysis, 2006, 65(1), 25–39.

[8] G. Fei, *Multiple periodic solutions of differential delay equations via Hamiltonian systems (II)*, Nonlinear Anal., 2006, 65(1), 40–58.

[9] X. S. Zhang and Q. Meng, *Nonlinear periodic solutions for delay differential systems via Morse theory*, Nonlinear Anal., 2011, 74(5), 1960–1968.

[10] C. G. Liu, *Periodic solutions of asymptotically linear delay differential systems via Hamiltonian systems*, J. Differential equations, 2012, 252(10), 5712-5734.

[11] J. S. Yu and H. F. Xiao, *Multiple periodic solutions with minimal period 4 of the delay differential equation $\dot{x}(t) = -f(t, x(t-1))$*, J. Differential Equations, 2013, 254(5), 2158–2172.

[12] Q. Wang and C. G. Liu, *Periodic solutions of delay differential systems via hamiltonian systems*, Nonlinear Anal. TMA., 2014, 102, 159–174.

[13] S. P. Lu, L. Zheng and L. J. Chen, *Homoclinic solutions for a class of second order neutral functional differential systems*, Acta Mathematica Scientia, 2013, 33(5), 1361–1374.

[14] D. Z. Chen and B. X. Dai, *Periodic solution of second order impulsive delay differential systems via variational method*, Applied Mathematics Letters, 2014, 38, 61–66.

[15] C. J. Guo and Z. M. Guo, *Existence of multiple periodic solutions for a class of second-order delay differential equations*, Nonlinear Analysis: RWA., 2009, 10(5), 3285–3297.

[16] K. C. Chang, *Infinite Dimensional Morse Theory and Multiple Solution Problem*, Birkhauser, Boston, 1993.

[17] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer-Verlag, Berlin, 1989.

[18] S. Li and J. Liu, *Morse theory and asymptotically linear Hamiltonian systems*, J. Differential Equations, 1989, 78(1), 53–73.

[19] P. H. Rabinowitz, *Periodic solutions of Hamiltonian systems*, Comm. Pure Appl. Math., 1978, 31(2), 157–184.

[20] C. Conley and E. Zehnder, *Morse type index theory for flows and periodic solutions for Hamiltonian equations*, Comm. Pure Appl. Math., 1984, 37(2), 207–253.

[21] Y. Long and E. Zehnder, *Morse theory for forced oscillations of asymptotically linear Hamiltonian systems, in: Stochastic Processes in Physics and Geometry*, World Scientific, Singapore, 1990, 528–563.

[22] V. Benci, *On critical point theory for indefinite functionals in the presence of symmetries*, Trans. Amer. Soc. 1982, 274(2), 533–572.

[23] J. B. Su, *Nontrivial periodic solutions for the asymptotically linear Hamiltonian systems with resonance at infinity*, J. Differential Equations, 1998, 145(2), 252–273.

[24] T. Bartsch and S. J. Li, *Critical point theory for asymptotically quadratic functionals and applications to problems with resonance*, Nonlinear Anal., 1997, 28(3), 419–441.