

# EXISTENCE AND GLOBAL EXPONENTIAL STABILITY OF ALMOST PERIODIC SOLUTIONS FOR BAM NEURAL NETWORKS WITH DISTRIBUTED LEAKAGE DELAYS ON TIME SCALES\*

Changjin Xu<sup>1,†</sup>, Peiluan Li<sup>2</sup> and Yicheng Pang<sup>3</sup>

**Abstract** In this paper, we deal with a class of BAM neural networks with distributed leakage delays on time scales. Some sufficient conditions which ensure the existence and exponential stability of almost periodic solutions for such class of BAM neural networks are obtained by applying the exponential dichotomy of linear differential equations, Lapunov functional method and contraction mapping principle. An example is given to illustrate the effectiveness of the theoretical predictions. The obtained results in this paper are completely new and complement the previously known publications.

**Keywords** BAM neural networks, almost periodic solution, exponential stability, exponential dichotomy, distributed leakage delay.

**MSC(2010)** 34K20, 34C25, 34K14.

## 1. Introduction

It is well known that bidirectional associative memory (BAM) neural networks have been paid much attention in the past decades due to their widely application prospect in many communities such as pattern recognition, speed detection of moving objects, image processing, automatic control engineering, optimization problems and so on [10, 35, 39, 43]. Considering that time delays are unavoidable due to the finite switching of amplifiers in practical implementation of neural networks, and the time delay may result in oscillation and instability, numerous scholars deal with the dynamics of BAM neural networks with time delays. For instance, Sakthivel et al. [32] focused on the design of state estimator for bidirectional as-

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<sup>†</sup>the corresponding author. Email address: [xcj403@126.com](mailto:xcj403@126.com) (C. Xu)

<sup>1</sup>Guizhou Key Laboratory of Economics System Simulation, Guizhou University of Finance and Economics, Guiyang 550004, China

<sup>2</sup>School of Mathematics and Statistics, Henan University of Science and Technology Luoyang 471023, China

<sup>3</sup>School of Mathematics and Statistics, Guizhou University of Finance and Economics Guiyang 550004, China

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sociative memory neural networks with leakage delays. By constructing a suitable Lyapunov-Krasovskii functional (LKF) together with free-weighting matrix technique, a new delay dependent sufficient condition is derived to estimate the neuron states through available output measurements, Li and Jian [21] investigated the exponential  $p$ -convergence for stochastic BAM neural networks with time-varying and infinite distributed delays. By constructing a new delay differential-integral inequality and a novel  $L$ -operator differential-integral inequality, and coupling with stochastic analysis techniques, some delay-dependent sufficient conditions are derived to guarantee exponential  $p$ -convergence of the stochastic BAM neural networks, Cao and Wan [9] presented several sufficient conditions for the global exponential stability of the equilibrium for inertial BAM neural network with time delays by using matrix measure and Halanay inequality, Berezansky et al. [5] established a new global exponential stability criteria for delayed BAM neural networks Via the M-matrix method. For more results on this aspect, we refer the readers to [1, 2, 4, 5, 8, 22, 24, 25, 41, 42].

Some authors point out that a typical time delay called Leakage (or “forgetting”) delay may exist in the negative feedback term of the neural networks (these terms are variously known as forgetting or leakage terms) and has an important impact on the dynamical behavior of neural networks [28, 38, 40]. For example, time delay in the stabilizing negative feedback term has a tendency to destabilize a system [29], Li et al. [4] pointed out that the existence and uniqueness of the equilibrium point are independent of time delays and initial conditions. For the effect of leakage delay on the stability of neural networks, we refer to the readers to [14, 17, 23, 27, 31, 33, 34, 36].

We would like to mention that most of the studies on the stability of neural networks are concerned with continuous-time and discrete-time. It is troublesome to investigate the dynamics of neural networks for continuous and discrete systems, respectively. Thus it is important for us to discuss that on time scales which can unify the continuous and discrete models. In addition, compared with periodicity, almost-periodicity occurs more frequently and it can reflect the nature law of neural networks more accurately [13]. To the best of our knowledge, there are few papers published on the existence and exponential stability of almost periodic solutions for BAM neural networks with distributed leakage delays on time scales. Thus the study on the almost periodic solutions for BAM neural networks with distributed leakage delays on time scales has important theoretical and practical significance. Inspired by the discussion above, in this paper, we are to consider the following BAM neural networks with distributed leakage delays on time scales

$$\left\{ \begin{array}{l} x_i^\Delta(t) = -a_i(t) \int_0^\infty k_i(s)x_i(t-s)\Delta s + \sum_{j=1}^m b_{ij}(t)g_j(y_j(t-\tau_j(t))) \\ \quad + \sum_{j=1}^m \sum_{l=1}^m e_{ijl}(t)p_j(y_j(t-\tau_j(t)))q_l(y_l(t-\tau_l(t))) + I_i(t), \\ y_j^\Delta(t) = -d_j(t) \int_0^\infty h_j(s)y_j(t-s)\Delta s + \sum_{i=1}^n c_{ji}(t)f_i(x_i(t-\omega_i(t))) \\ \quad + \sum_{i=1}^n \sum_{l=1}^n s_{jil}(t)v_i(x_i(t-\omega_i(t)))w_l(x_l(t-\omega_l(t))) + J_j(t), \end{array} \right. \quad (1.1)$$

where  $i = 1, 2, \dots, n, j = 1, 2, \dots, m, t \in \mathbb{R}$ ,  $x_i(t)$  and  $y_j(t)$  denote the potential (or voltage) of the cell  $i$  and  $j$  at time  $t$ ,  $a_i(t)$  and  $d_j(t)$  denote the rate with which

the cell  $i$  and  $j$  reset their potential to the resting state when isolate from the other cells and inputs,  $\tau_j(t)$  and  $\omega_i(t)$  are non-negative and satisfy  $t - \tau_j(t) \in \mathbb{T}$  and  $t - \omega_j(t) \in \mathbb{T}$ , they correspond to finite speed of axonal signal transmission,  $b_{ij}, c_{ji}, e_{ijl}$  and  $s_{jil}$  are the first- and second-order connection weights of the neural network, respectively,  $I_i$  and  $J_j$  denote the  $i$ th and the  $j$ th component of an external input source that introduce from outside the network to the cell  $i$  and  $j$ , respectively, continuous leakage delays kernel functions  $k_i > 0$  and  $h_j > 0$  satisfy that  $k_i(s)e^{r_1s}$  and  $h_j(s)e^{r_2s}$  are integrable on  $\mathbb{R}^+$  for certain positive constants  $r_1$  and  $r_2$ ,  $g_j, p_j, q_l, f_i, v_i$  and  $w_l$  are activation functions.

Our main object of this paper is to investigate the existence and global exponential stability of almost periodic solutions for system (1.1) by the theory of exponential dichotomy, fixed point theorems and Lyapunov functional method. We believe that this research on the existence and exponential stability of almost periodic solutions of system (1.1) has important theoretical value and tremendous potential for application in designing the BAM neural networks with distributed leakage delays. Our results are new and a good complement to the work of [26].

Let  $\mathbb{T}$  denote an almost periodic time scale. For the sake of simplification, denote  $= (-\infty, +\infty)$  and  $\mathbb{R}^+ = (0, +\infty)$ ,  $f^+ = \sup_{t \in \mathbb{T}} |f(t)|, f^- = \inf_{t \in \mathbb{T}} |f(t)|$ , where  $f : \mathbb{T} \rightarrow \mathbb{R}$  is an almost periodic function. Let  $X = \{\phi = (\varphi_1, \varphi_2, \dots, \varphi_n, \psi_1, \psi_2, \dots, \psi_m)^T | \varphi_i, \psi_j \in C^1(\mathbb{T}, \mathbb{R}), \varphi_i, \psi_j$  are almost periodic functions on  $\mathbb{T}, i = 1, 2, \dots, n, j = 1, 2, \dots, m\}$  with the norm  $\|\phi\| = \max\{|\varphi|_1, |\psi|_1\}$ , where  $|\varphi|_1 = \max\{|\varphi|_0, |\varphi^\Delta|_0\}, |\psi|_1 = \max\{|\psi|_0, |\psi^\Delta|_0\}, |\varphi|_0 = \max_{1 \leq i \leq n} \varphi_i^+, |\varphi^\Delta|_0 = \max_{1 \leq i \leq n} (\varphi_i^\Delta)^+, |\psi|_0 = \max_{1 \leq j \leq m} \psi_j^+, |\psi^\Delta|_0 = \max_{1 \leq j \leq m} (\psi_j^\Delta)^+, C^1(\mathbb{T}, \mathbb{R})$  is the set of continuous functions with nabla derivatives on  $\mathbb{T}$ . Then  $X$  is a Banach space.

The initial value of system (1.1) is given by

$$x_i(s) = \varphi_i(s), y_j(s) = \psi_j(s), s \in [-\infty, 0]_{\mathbb{T}} = \{t | t \in (-\infty, 0] \cap \mathbb{T}\}, \tag{1.2}$$

where  $\varphi_i, \psi_j \in C^1((-\infty, 0]_{\mathbb{T}}, \mathbb{R}), i = 1, 2, \dots, n, j = 1, 2, \dots, m$ .

Throughout this paper, we assume that the following conditions are satisfied.

(A1) For  $i = 1, 2, \dots, n, j = 1, 2, \dots, m, l = 1, 2, \dots, m, a_i(t), b_{ij}(t), e_{ijl}(t), I_i(t), \tau_j(t) \in C(\mathbb{R}, \mathbb{R}^+)$  are all almost periodic functions.

(A2) For  $i = 1, 2, \dots, n, j = 1, 2, \dots, m, l = 1, 2, \dots, n, d_j(t), c_{ji}(t), s_{jil}(t), J_j(t), \omega_i(t) \in C(\mathbb{R}, \mathbb{R}^+)$ . are all almost periodic functions.

(A3) For  $i = 1, 2, \dots, n, j = 1, 2, \dots, m, f_j, g_i \in C(\mathbb{R}, \mathbb{R})$  and there exist positive constants  $L_j^f, L_i^g, L_j^p, L_j^q, L_i^v, L_i^w, P_j, Q_j, V_i$  and  $W_i$  such that

$$\begin{aligned} |f_j(t_1) - f_j(t_2)| &\leq L_j^f |t_1 - t_2|, |g_i(t_1) - g_i(t_2)| \leq L_i^g |s_1 - s_2|, \\ |p_j(t_1) - p_j(t_2)| &\leq L_j^p |t_1 - t_2|, |q_j(t_1) - q_j(t_2)| \leq L_j^q |t_1 - t_2|, \\ |v_i(t_1) - v_i(t_2)| &\leq L_i^v |s_1 - s_2|, |w_i(t_1) - w_i(t_2)| \leq L_i^w |t_1 - t_2|, \\ |p_j(t)| &\leq P_j, |q_j(t)| \leq Q_j, |v_i(t)| \leq V_i, |w_i(t)| \leq W_i \end{aligned}$$

for  $t_1, t_2, t \in \mathbb{R}$ .

The remainder of the paper is organized as follows: in Section 2, we introduce several useful definitions and lemmas. In Section 3, some sufficient conditions which ensure the existence and a unique almost periodic solution of model (1.1) are established. The global exponential stability of model (1.1) is obtained in Section 4. In Section 5, an example which illustrate the theoretical findings is given. A brief conclusion is drawn in Section 6.

## 2. Preliminaries

In this section, we present some definitions and notations on time scales which can be found in the literatures [3, 6, 7, 11, 12, 15, 16, 18–20, 30, 37].

**Definition 2.1** ([7]). A time scale is an arbitrary nonempty closed subset  $\mathbb{T}$  of  $\mathbb{R}$ . The set  $\mathbb{T}$  inherits the standard topology of  $\mathbb{R}$ .

**Definition 2.2** ([7]). The forward jump operator  $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ , the backward jump operator  $\rho : \mathbb{T} \rightarrow \mathbb{T}$ , and the graininess  $\mu : \mathbb{T} \rightarrow \mathbb{R}^+ = [0, \infty)$  are defined, respectively, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t \text{ for } t \in \mathbb{T}.$$

If  $\sigma(t) = t$ , then  $t$  is called right-dense (otherwise: right-scattered), and if  $\rho(t) = t$ , then  $t$  is called left-dense (otherwise: left-scattered). If  $\mathbb{T}$  has a left-scattered maximum  $m$ , then we defined  $\mathbb{T}^k$  to be  $\mathbb{T} \setminus \{m\}$ ; otherwise  $\mathbb{T}^k = \mathbb{T}$ . If  $\mathbb{T}$  has a right-scattered minimum  $m$ , then we defined  $\mathbb{T}_k$  to be  $\mathbb{T} \setminus \{m\}$ ; otherwise,  $\mathbb{T}^k = \mathbb{T}$ .

**Definition 2.3.** A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is said to be rd-continuous if it is continuous at right-dense points in  $\mathbb{T}$  and its left-sides limits exists(finite) at left-dense points in  $\mathbb{T}$ . The set rd-continuous functions is shown by  $C_{rd}^1 = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$ .

**Definition 2.4.** For  $f : \mathbb{T} \rightarrow \mathbb{R}$  and  $t \in \mathbb{T}$ , we define  $f^\Delta(t)$ , the delta-derivative of  $f$  at  $t$ , to be the number(provided it exists) with the property that, given any  $\varepsilon > 0$ , there is a neighborhood  $U$  of  $t$  in  $\mathbb{T}$  such that

$$|[f(\rho(t)) - f(s)] - f^\Delta(t)[\rho(t) - s]| \leq \varepsilon|\rho(t) - s| \text{ for all } s \in U.$$

Thus  $f$  is said to be delta-differentiable if its nabla-derivative exists. The set of functions  $f : \mathbb{T} \rightarrow \mathbb{R}$  that are nabla-differentiable and whose delta-derivative are rd-continuous functions is denoted by  $C_{rd} = C_{rd}^1(\mathbb{T}) = C_{rd}^1(\mathbb{T}, \mathbb{R})$ .

**Definition 2.5.** A function  $F : \mathbb{T} \rightarrow \mathbb{R}$  is called a delta-antiderivative of  $f : \mathbb{T} \rightarrow \mathbb{R}$  provided  $F^\Delta(t) = f(t)$ , for all  $t \in \mathbb{T}$ . Then we write  $\int_r^s f(t)\Delta t := F(s) - F(r)$  for all  $s, t \in \mathbb{T}$ .

**Definition 2.6** ([18]). A time scale  $\mathbb{T}$  is called an almost periodic time scale if  $\Pi := \{p \in \mathbb{R} : t \pm p \in \mathbb{T}, \forall t \in \mathbb{T}\}$ .

A function  $r : \mathbb{T} \rightarrow \mathbb{R}$  is called regressive if  $1 + \mu(t)r(t) \neq 0$  for all  $t \in \mathbb{T}^k$ . If  $r$  is regressive function, then the generalized exponential function  $e_r$  is defined by

$$e_r(t, s) = \exp \left\{ \int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau \right\}, \text{ for } s, t \in \mathbb{T},$$

with the cylindrical transformation

$$\xi_h(z) = \begin{cases} \frac{\log(1+hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases}$$

Let  $p, q : \mathbb{T} \rightarrow \mathbb{R}$  be two regressive functions, we define

$$p \oplus q := p + q + \mu pq, \quad \ominus p := -\frac{p}{1 + \mu p}, \quad p \ominus q := p \oplus (\ominus q).$$

**Lemma 2.1** ([7]). Assume that  $p, q : \mathbb{T} \neq \mathbb{R}$  are two regressive functions, then

- (i)  $e_0(t, s) \equiv 1$  and  $e_p(t, t) \equiv 1$ ;
- (ii)  $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$ ;
- (iii)  $e_p(t, s) = \frac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$ ;
- (iv)  $e_p(t, s)e_p(s, r) = e_p(t, r)$ ;
- (v)  $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$ ;
- (vi)  $\frac{e_p(t, s)}{e_q(t, s)} = e_{p \ominus q}(t, s)$ .

**Lemma 2.2** ([7]). Assume that  $f, g : \mathbb{T} \neq \mathbb{R}$  are delta differentiable at  $t \in \mathbb{T}^k$ , then

- (i)  $(\nu_1 f + \nu_2 g)^\Delta = \nu_1 f^\Delta + \nu_2 g^\Delta$ , for any constants  $\nu_1, \nu_2$ .
- (ii)  $(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t))$ ;
- (iii) if  $f$  and  $f^\Delta$  are continuous, then  $(\int_a^t f(t, s)f^\Delta s)^\Delta = f(\sigma(t), t) + \int_a^t f(t, s)\Delta s$ .

**Lemma 2.3** ([12, 15]). Let  $u : R \rightarrow R^n$  be continuous in  $t$ .  $u(t)$  is said to be almost periodic on  $R$  if for any  $\varepsilon$ , the set  $T(u, \varepsilon) = \{\sigma : |u(t + \sigma) - u(t)| < \varepsilon, \text{ for any } t \in R\}$  is relatively dense, i.e., for any  $\varepsilon > 0$ , it is possible to find a real number  $l = l(\varepsilon)$ , for any interval with length  $l(\varepsilon)$ , there exists a number  $\sigma = \sigma(\varepsilon)$  in this interval such that  $|u(t + \sigma) - u(t)| < \varepsilon$  for all  $t \in R$ .

**Definition 2.7** ([12, 15]). Let  $x \in \mathbb{R}^n$  and  $Q(t)$  be a  $n \times n$  continuous matrix defined on  $\mathbb{R}$ . The linear system

$$\dot{X}(t) = Q(t)X(t) \quad (2.1)$$

is said to admit an exponential dichotomy on  $\mathbb{R}$  if there exist positive constants  $k, \alpha_i$  projection  $P$  and the fundamental solution matrix  $X(t)$  of Eq. (2.1) satisfying  $|X(t)PX^{-1}(s)| \leq ke_{\ominus\alpha}(t, \sigma(s))$  for all  $s, t \in \mathbb{R}, t \geq \sigma(s)$  and  $|X(t)(I - P)X^{-1}(s)| \leq ke_{\ominus\alpha}(\sigma(s), t)$  for all  $s, t \in \mathbb{R}, t \leq \sigma(s)$ .

**Lemma 2.4** ([12, 15]). If the linear system (2.1) admits an exponential dichotomy, the almost periodic system

$$X^\Delta(t) = Q(t)X(t) + g(t) \quad (2.2)$$

has a unique almost periodic solution  $x(t)$ , and

$$x(t) = \int_{-\infty}^t X(t)PX^{-1}(\rho(s))g(s)\Delta s - \int_t^{+\infty} X(t)(I - P)X^{-1}(\rho(s))g(s)\Delta s,$$

where  $X(t)$  is the fundamental solution matrix of Eq. (2.1).

**Lemma 2.5.** Let  $c_i(t)$  be an almost periodic function on  $\mathbb{T}$ , where  $c_i(t) > 0, -c_i(t) \in \mathbb{T}^+$ , for all  $t \in \mathbb{T}$  and  $\min_{1 \leq i \leq n} \{\inf_{t \in \mathbb{T}} c_i(t)\} = \bar{m} > 0$ , then the linear system  $z^\Delta(t) = \text{diag}(-c_1(t), -c_2(t), \dots, -c_n(t))z(t)$  admits an exponential dichotomy on  $\mathbb{T}$ .

**Definition 2.8.** Let  $u^* = (x_1^*, x_2^*, \dots, x_n^*, y_1^*, y_2^*, \dots, y_m^*)^T$  be the almost periodic solution of system (1.1). If there exists a constant  $\lambda > 0$  such that for every solution  $u(t) = (x_1(t), x_2(t), \dots, x_n(t), y_1(t), y_2(t), \dots, y_m(t))^T$  of Eq. (1.1) with initial value  $\phi(s) = (\varphi_1(s), \varphi_2(s), \dots, \varphi_n(s), \psi_1(s), \psi_2(s), \dots, \psi_m(s))^T$  satisfying

$$x_i(t) - x_i^*(t) = O(e_{\ominus\lambda}(t, 0)), y_j(t) - y_j^*(t) = O(e_{\ominus\lambda}(t, 0)),$$

where  $i = 1, 2, \dots, n, j = 1, 2, \dots, m$ . Then the solution  $u^*$  is said to be global exponential stable.

### 3. Existence of almost periodic solution

In this section, we will consider the existence of the almost periodic solution of system (1.1).

Set  $\phi^0(t) = (\varphi_1^0(t), \varphi_2^0(t), \dots, \varphi_n^0(t), \psi_1^0(t), \psi_2^0(t), \dots, \psi_m^0(t))^T$ , where

$$\begin{aligned}\varphi_i^0(t) &= \int_{-\infty}^t e_{-a_i}(t, \rho(s)) I_i(s) \Delta s, i = 1, 2, \dots, n. \\ \psi_j^0(t) &= \int_{-\infty}^t e_{-d_j}(t, \rho(s)) J_j(s) \Delta s, j = 1, 2, \dots, m.\end{aligned}$$

Let  $\chi$  be a constant that satisfies

$$\begin{aligned}\chi \geq & \max\{|\phi^0|, \max_{1 \leq j \leq m} |g_j(0)|, \max_{1 \leq i \leq n} |f_i(0)|, \max_{1 \leq j \leq m} |p_j(0)|, \max_{1 \leq j \leq m} |q_j(0)|, \\ & \max_{1 \leq i \leq n} |v_i(0)|, \max_{1 \leq i \leq n} |w_i(0)|\}.\end{aligned}$$

**Theorem 3.1.** *In addition to (A1)-(A3), assume that (A4) for  $i = 1, 2, \dots, n, j = 1, 2, \dots, m$ ,*

$$\begin{aligned}\max \left\{ \max_{1 \leq i \leq n} \left\{ \frac{\Theta_i}{a_i^- \int_0^\infty k_i(u) \Delta u}, \left(1 + \frac{a_i^+}{a_i^-}\right) \Theta_i \right\}, \right. \\ \left. \max_{1 \leq j \leq m} \left\{ \frac{\Pi_j}{d_j^- \int_0^\infty h_j(u) \Delta u}, \left(1 + \frac{d_j^+}{d_j^-}\right) \Pi_j \right\} \right\} \leq 1,\end{aligned}$$

and  $\max\{\max_{1 \leq i \leq n} \{\epsilon_{i1}, \epsilon_{i2}\}, \max_{1 \leq j \leq m} \{\varrho_{j1}, \varrho_{j2}\}\} < 1$ , where

$$\begin{aligned}\Theta_i &= 2a_i^+ \int_0^\infty k_i(s) s \Delta s + 2\gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ \\ &+ \gamma_i^{-1} \chi \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ (2L_j^p \bar{\gamma}_j + 1) (2\bar{\gamma}_l L_l^q + 1), \\ \Pi_j &= 2d_j^+ \int_0^\infty h_j(s) s \Delta s + 2\bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ L_i^f \gamma_i + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ \\ &+ \bar{\gamma}_j^{-1} \chi \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ (2L_i^v \gamma_i + 1) (2\gamma_l L_l^w + 1), \\ \epsilon_{i1} &= \frac{1}{a_i^- \int_0^\infty k_i(u) \Delta u} \left[ a_i^+ \int_0^\infty k_i(u) u \Delta u + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j^{-1} \right. \\ &+ \left. \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ Q_l L_j^p \bar{\gamma}_j^{-1} + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ P_j L_l^q \bar{\gamma}_l^{-1} \right], \\ \epsilon_{i2} &= \left(1 + \frac{a_i^+}{a_i^-}\right) \left[ a_i^+ \int_0^\infty k_i(u) u \Delta u + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j^{-1} \right. \\ &+ \left. \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ Q_l L_j^p \bar{\gamma}_j^{-1} + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ P_j L_l^q \bar{\gamma}_l^{-1} \right],\end{aligned}$$

$$\begin{aligned} \varrho_{j1} &= \frac{1}{d_j^- \int_0^\infty h_j(u) \Delta u} \left[ d_j^+ \int_0^\infty h_j(u) u \Delta u + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ L_i^f \gamma_i^{-1} \right. \\ &\quad \left. + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ W_l L_i^v \gamma_i^{-1} + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ V_l L_i^w \gamma_l^{-1} \right], \\ \varrho_{j2} &= \left( 1 + \frac{d_j^+}{d_j^-} \right) \left[ d_j^+ \int_0^\infty h_j(u) u \Delta u + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ L_i^f \gamma_i^{-1} \right. \\ &\quad \left. + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ W_l L_i^v \gamma_i^{-1} + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ V_l L_i^w \gamma_l^{-1} \right]. \end{aligned}$$

Then system (1.1) has a unique almost periodic solution in  $\Omega = \{\phi \in X \mid \|\phi - \phi^0\| \leq \chi\}$ , where

$$\phi(t) = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t), \psi_1(t), \psi_2(t), \dots, \psi_m(t)).$$

**Proof.** Let

$$\bar{x}_i(t) = \gamma_i^{-1} x_i(t), \bar{y}_j(t) = \bar{\gamma}_j^{-1} y_j(t), i = 1, 2, \dots, n, j = 1, 2, \dots, m, \tag{3.1}$$

then system (1.1) can be transformed into

$$\left\{ \begin{aligned} \bar{x}_i^\Delta(t) &= -a_i(t) \int_0^\infty k_i(s) \Delta s \bar{x}_i(t) + a_i(t) \int_0^\infty k_i(s) \int_{t-s}^t \bar{x}_i^\Delta(u) \Delta u \Delta s \\ &\quad + \gamma_i^{-1} \sum_{j=1}^m b_{ij}(t) g_j(\bar{\gamma}_j \bar{y}_j(t - \tau_j(t))) \\ &\quad + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}(t) p_j(\bar{\gamma}_j \bar{y}_j(t - \tau_j(t))) q_l(\bar{\gamma}_l \bar{y}_l(t - \tau_l(t))) + \gamma_i^{-1} I_i(t), \\ \bar{y}_j^\Delta(t) &= -d_j(t) \int_0^\infty h_j(s) \Delta s \bar{y}_j(t) + d_j(t) \int_0^\infty h_j(s) \int_{t-s}^t \bar{y}_j^\Delta(u) \Delta u \Delta s \\ &\quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}(t) f_i(\gamma_i \bar{x}_i(t - \omega_i(t))) \\ &\quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}(t) v_i(\gamma_i \bar{x}_i(t - \omega_i(t))) w_l(\gamma_l \bar{x}_l(t - \omega_l(t))) + \bar{\gamma}_j^{-1} J_j(t). \end{aligned} \right. \tag{3.2}$$

For any given  $\phi \in X$ , we consider the following almost periodic system:

$$\left\{ \begin{aligned} \bar{x}_i^\Delta(t) &= -a_i(t) \int_0^\infty k_i(s) \Delta s \bar{x}_i(t) + M_i(t, \varphi, \psi) + \gamma_i^{-1} I_i(t), \\ \bar{y}_j^\Delta(t) &= -d_j(t) \int_0^\infty h_j(s) \Delta s \bar{y}_j(t) + N_j(t, \varphi, \psi) + \bar{\gamma}_j^{-1} J_j(t), \end{aligned} \right. \tag{3.3}$$

where  $i = 1, 2, \dots, n, j = 1, 2, \dots, m$  and

$$\left\{ \begin{array}{l} M_i(t, \varphi, \psi) = a_i(t) \int_0^\infty k_i(s) \int_{t-s}^t \bar{\varphi}_i^\Delta(u) \Delta u \Delta s + \gamma_i^{-1} \sum_{j=1}^m b_{ij}(t) g_j(\bar{\gamma}_j \bar{\psi}_j(t - \tau_j(t))) \\ \quad + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}(t) p_j(\bar{\gamma}_j \bar{\psi}_j(t - \tau_j(t))) q_l(\bar{\gamma}_l \bar{\psi}_l(t - \tau_l(t))), \\ N_j(t, \varphi, \psi) = d_j(t) \int_0^\infty h_j(s) \int_{t-s}^t \bar{\psi}_j^\Delta(u) \Delta u \Delta s + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}(t) f_i(\gamma_i \bar{\varphi}_i(t - \omega_i(t))) \\ \quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}(t) v_i(\gamma_i \bar{\varphi}_i(t - \omega_i(t))) w_l(\gamma_l \bar{\varphi}_l(t - \omega_l(t))). \end{array} \right. \quad (3.4)$$

Notice that  $a_i^- \int_0^\infty k_i(s) \Delta s > 0, d_j^- \int_0^\infty h_j(s) \Delta s > 0$ , it follows from Lemma 2.5 that

$$\left\{ \begin{array}{l} \bar{x}_i^\Delta(t) = -a_i(t) \int_0^\infty k_i(s) \Delta s \bar{x}_i(t), i = 1, 2, \dots, n, \\ \bar{y}_j^\Delta(t) = -d_j(t) \int_0^\infty h_j(s) \Delta s \bar{y}_j(t), j = 1, 2, \dots, m \end{array} \right. \quad (3.5)$$

admits an exponential dichotomy on  $\mathbb{T}$ . Thus it follows from Lemma 2.5 that (3.3) has a unique almost periodic solution, which takes the following form:

$$\left\{ \begin{array}{l} x_i^\varphi(t) = \int_{-\infty}^t e_{-a_i \int_0^\infty k_i(u) \Delta u}(t, \rho(s)) [(M_i(s, \varphi, \psi) + \gamma_i^{-1} I_i(s)) \Delta s, i = 1, 2, \dots, n, \\ y_j^\psi(t) = \int_{-\infty}^t e_{-d_j \int_0^\infty h_j(u) \Delta u}(t, \rho(s)) [(N_j(s, \varphi, \psi) + \bar{\gamma}_j^{-1} J_j(s)) \Delta s, j = 1, 2, \dots, m. \end{array} \right. \quad (3.6)$$

For  $\phi \in \Omega$ , we have  $\|\phi\| \leq \|\phi - \phi^0\| + \|\phi^0\| \leq 2\chi$ . Define a linear operator as follows:

$$\Phi : \Omega \rightarrow \Omega, (\varphi_1, \varphi_2, \dots, \varphi_n, \psi_1, \psi_2, \dots, \psi_m)^T \Rightarrow (x_1^\varphi, x_2^\varphi, \dots, x_n^\varphi, y_1^\psi, y_2^\psi, \dots, y_m^\psi)^T, \quad (3.7)$$

where  $x_i^\varphi, y_j^\psi (i = 1, 2, \dots, n; j = 1, 2, \dots, m)$  are defined by (3.6). In what follows, we will prove that  $\Phi$  is a contraction mapping. First we show that for any  $\phi \in \Omega$ , we have  $\Phi\phi \in \Omega$ . It follows from (3.4) that

$$\begin{aligned} |M_i(t, \varphi, \psi)| &= \left| a_i(t) \int_0^\infty k_i(s) \int_{t-s}^t \varphi_i^\Delta(u) \Delta u \Delta s + \gamma_i^{-1} \sum_{j=1}^m b_{ij}(t) g_j(\bar{\gamma}_j \bar{\psi}_j(t - \tau_j(t))) \right. \\ &\quad \left. + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}(t) p_j(\bar{\gamma}_j \bar{\psi}_j(t - \tau_j(t))) q_l(\bar{\gamma}_l \bar{\psi}_l(t - \tau_l(t))) \right| \\ &\leq a_i(t) \int_0^\infty k_i(s) \int_{t-s}^t |\varphi_i^\Delta(u)| \Delta u \Delta s + \gamma_i^{-1} \sum_{j=1}^m b_{ij}(t) |g_j(\bar{\gamma}_j \bar{\psi}_j(t - \tau_j(t)))| \\ &\quad + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}(t) |p_j(\bar{\gamma}_j \bar{\psi}_j(t - \tau_j(t)))| |q_l(\bar{\gamma}_l \bar{\psi}_l(t - \tau_l(t)))| \\ &\leq a_i^+ \int_0^\infty k_i(s) s |\Delta s| \bar{\varphi}^\Delta|_0 + \gamma_i^{-1} \sum_{j=1}^m b_{ij}(t) (|g_j(\bar{\gamma}_j \bar{\psi}_j(t - \tau_j(t))) - g_j(0)| + |g_j(0)|) \end{aligned}$$



$$\begin{aligned}
& + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}(t) (|p_j(\bar{\gamma}_j \psi_j(t - \tau_j(t))) - p_j(0)| + |p_j(0)|) \\
& \quad \times (|q_l(\bar{\gamma}_l \psi_l(t - \tau_l(t))) - q_l(0)| + |q_l(0)|) \\
& \leq a_i^+ \int_0^\infty k_i(s) s \Delta s |\varphi^\Delta|_0 + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j |\psi_j(t - \tau_j(t))| + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ |g_j(0)| \\
& \quad + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+(L_j^p \bar{\gamma}_j |\psi_j(t - \tau_j(t))| + |p_j(0)|) (L_l^q \bar{\gamma}_l |\psi_l(t - \tau_l(t))| + |q_l(0)|) \\
& \leq 2a_i^+ \int_0^\infty k_i(s) s \Delta s \chi + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j |\psi|_0 + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ |g_j(0)| \\
& \quad + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+(L_j^p \bar{\gamma}_j |\psi|_0 + |p_j(0)|) (\bar{\gamma}_l L_l^q |\psi|_0 + |q_l(0)|) \\
& \leq 2a_i^+ \int_0^\infty k_i(s) s \Delta s \chi + 2\gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j \chi + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ \chi \\
& \quad + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+(2L_j^p \bar{\gamma}_j \chi + \chi) (2\bar{\gamma}_l L_l^q \chi + \chi) \\
& = \chi \left[ 2a_i^+ \int_0^\infty k_i(s) s \Delta s + 2\gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ \right. \\
& \quad \left. + \gamma_i^{-1} \chi \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+(2L_j^p \bar{\gamma}_j + 1) (2\bar{\gamma}_l L_l^q + 1) \right] := \Theta_i \chi, \tag{3.8}
\end{aligned}$$

$$\begin{aligned}
|N_j(t, \varphi, \psi)| & = \left| d_j(t) \int_0^\infty h_j(s) \int_{t-s}^t \psi_j^\Delta(u) \Delta u \Delta s + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}(t) f_i(\gamma_i \bar{\varphi}_i(t - \omega_i(t))) \right. \\
& \quad \left. + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}(t) v_i(\gamma_i \bar{\varphi}_i(t - \omega_i(t))) w_l(\gamma_l \bar{\varphi}_l(t - \omega_l(t))) \right| \\
& \leq d_j(t) \int_0^\infty h_j(s) \int_{t-s}^t |\psi_j^\Delta(u)| \Delta u \Delta s + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}(t) |f_i(\gamma_i \bar{\varphi}_i(t - \omega_i(t)))| \\
& \quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}(t) |v_i(\gamma_i \bar{\varphi}_i(t - \omega_i(t)))| |w_l(\gamma_l \bar{\varphi}_l(t - \omega_l(t)))| \\
& \leq d_j^+ \int_0^\infty h_j(s) s \Delta s |\psi^\Delta|_0 + \bar{\gamma}_j^{-1} \sum_{i=1}^n v_{ji}(t) (|f_i(\gamma_i \bar{\varphi}_i(t - \omega_i(t))) - f_i(0)| + |f_i(0)|) \\
& \quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}(t) (|v_i(\gamma_i \bar{\varphi}_i(t - \omega_i(t))) - v_i(0)| + |v_i(0)|) \\
& \quad \times (|w_l(\gamma_l \bar{\varphi}_l(t - \omega_l(t))) - w_l(0)| + |w_l(0)|) \\
& \leq d_j^+ \int_0^\infty h_j(s) s \Delta s |\psi^\Delta|_0 + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ L_i^f \gamma_i |\varphi_i(t - \omega_i(t))| + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ |f_i(0)|
\end{aligned}$$

$$\begin{aligned}
& + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ (L_i^v \gamma_i |\varphi_i(t - \omega_i(t))| + |v_i(0)|) (L_l^w \gamma_l |\varphi_l(t - \omega_l(t))| + |w_l(0)|) \\
& \leq 2d_j^+ \int_0^\infty h_j(s) s \Delta s \chi + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ L_i^f \gamma_i |\varphi|_0 + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ |f_i(0)| \\
& \quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ (L_i^v \gamma_i |\varphi|_0 + |v_i(0)|) (\gamma_l L_l^w |\varphi|_0 + |w_l(0)|) \\
& \leq 2d_j^+ \int_0^\infty h_j(s) s \Delta s \chi + 2\bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ L_i^f \gamma_i \chi + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ \chi \\
& \quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ (2L_i^v \gamma_i \chi + \chi) (2\gamma_l L_l^w \chi + \chi) \\
& = \chi \left[ 2d_j^+ \int_0^\infty h_j(s) s \Delta s + 2\bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ L_i^f \gamma_i + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ \right. \\
& \quad \left. + \bar{\gamma}_j^{-1} \chi \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ (2L_i^v \gamma_i + 1) (2\gamma_l L_l^w + 1) \right] := \Pi_j \chi. \tag{3.9}
\end{aligned}$$

By (3.7)-(3.9), we get

$$\begin{aligned}
|(\Phi\phi - \phi^0)_i(t)| & = \left| \int_{-\infty}^t e_{-a_i \int_0^\infty k_i(u) \Delta u}(t, \rho(s)) M_i(s, \varphi, \psi) \Delta s \right| \\
& \leq \int_{-\infty}^t e_{-a_i \int_0^\infty k_i(u) \Delta u}(t, \rho(s)) |M_i(s, \varphi, \psi)| \Delta s \\
& \leq \int_{-\infty}^t e_{-a_i \int_0^\infty k_i(u) \Delta u}(t, \rho(s)) \Theta_i \chi \Delta s \\
& \leq \frac{\Theta_i \chi}{a_i^- \int_0^\infty k_i(u) \Delta u}, i = 1, 2, \dots, n, \tag{3.10}
\end{aligned}$$

$$\begin{aligned}
|(\Phi\phi - \phi^0)_{n+j}(t)| & = \left| \int_{-\infty}^t e_{-d_j \int_0^\infty h_j(u) \Delta u}(t, \rho(s)) N_j(s, \varphi, \psi) \Delta s \right| \\
& \leq \int_{-\infty}^t e_{-d_j \int_0^\infty h_j(u) \Delta u}(t, \rho(s)) |N_j(s, \varphi, \psi)| \Delta s \\
& \leq \int_{-\infty}^t e_{-d_j \int_0^\infty h_j(u) \Delta u}(t, \rho(s)) \Pi_j \chi \Delta s \\
& \leq \frac{\Pi_j \chi}{d_j^- \int_0^\infty h_j(u) \Delta u}, j = 1, 2, \dots, m. \tag{3.11}
\end{aligned}$$

In view of (3.7)-(3.9), we also get

$$\begin{aligned}
|(\Phi\phi - \phi^0)_i^\Delta(t)| & = \left| \left( \int_{-\infty}^t e_{-a_i \int_0^\infty k_i(u) \Delta u}(t, \rho(s)) M_i(s, \varphi, \psi) \Delta s \right)^\Delta \right| \\
& = \left| M_i(t, \varphi, \psi) - \int_0^\infty k_i(u) \Delta u a_i(t) \int_{-\infty}^t e_{-a_i \int_0^\infty k_i(u) \Delta u}(t, \rho(s)) \right.
\end{aligned}$$

$$\begin{aligned}
& \times |M_i(s, \varphi, \psi) \Delta s| \\
& \leq |M_i(t, \varphi, \psi)| + \int_0^\infty k_i(u) \Delta u a_i(t) \int_{-\infty}^t e_{-a_i, \int_0^\infty k_i(u) \Delta u}(t, \rho(s)) \\
& \quad \times |M_i(s, \varphi, \psi) \Delta s| \\
& \leq \left(1 + \frac{a_i^+}{a_i^-}\right) \Theta_i \chi, i = 1, 2, \dots, n, \tag{3.12}
\end{aligned}$$

$$\begin{aligned}
|(\Phi\phi - \phi^0)_{n+j}^\Delta(t)| &= \left| \left( \int_{-\infty}^t e_{-d_j, \int_0^\infty h_j(u) \Delta u}(t, \rho(s)) N_j(s, \varphi, \psi) \Delta s \right)^\Delta \right| \\
&= \left| N_j(t, \varphi, \psi) - \int_0^\infty h_j(u) \Delta u d_j(t) \int_{-\infty}^t e_{-d_j, \int_0^\infty h_j(u) \Delta u}(t, \rho(s)) \right. \\
& \quad \left. \times N_j(s, \varphi, \psi) \Delta s \right| \\
&\leq |N_j(t, \varphi, \psi)| + \int_0^\infty h_j(u) \Delta u d_j(t) \int_{-\infty}^t e_{-d_j, \int_0^\infty h_j(u) \Delta u}(t, \rho(s)) \\
& \quad \times |N_j(s, \varphi, \psi) \Delta s| \\
&\leq \left(1 + \frac{d_j^+}{d_j^-}\right) \Pi_j \chi, j = 1, 2, \dots, m. \tag{3.13}
\end{aligned}$$

According to (3.10)-(3.13), we have

$$\begin{aligned}
\|\Phi\phi - \phi^0\| &= \max \left\{ \max_{1 \leq i \leq n} \left\{ \frac{\Theta_i \chi}{a_i^- \int_0^\infty k_i(u) \Delta u}, \left(1 + \frac{a_i^+}{a_i^-}\right) \Theta_i \chi \right\}, \right. \\
& \quad \left. \max_{1 \leq j \leq m} \left\{ \frac{\Pi_j \chi}{d_j^- \int_0^\infty h_j(u) \Delta u}, \left(1 + \frac{d_j^+}{d_j^-}\right) \Pi_j \chi \right\} \right\} \leq \chi, \tag{3.14}
\end{aligned}$$

which implies that  $\Phi\phi \in \Omega$ . Next we prove that  $\Phi$  is a contraction. For  $\phi = (\varphi_1, \varphi_2, \dots, \varphi_n, \psi_1, \psi_2, \dots, \psi_m)^T$ ,  $\tilde{\phi} = (\tilde{\varphi}_1, \tilde{\varphi}_2, \dots, \tilde{\varphi}_n, \tilde{\psi}_1, \tilde{\psi}_2, \dots, \tilde{\psi}_m)^T \in \Omega$  and  $i = 1, 2, \dots, n, j = 1, 2, \dots, m$ , we have

$$\begin{aligned}
|(\Phi\phi - \Phi\tilde{\phi})_i(t)| &= \left| \int_{-\infty}^t e_{-a_i, \int_0^\infty k_i(u) \Delta u}(t, \rho(s)) \right. \\
& \quad \times \left\{ a_i(s) \int_0^\infty k_i(u) \int_{s-u}^s (\varphi_i^\Delta(\kappa) - \tilde{\varphi}_i^\Delta(\kappa)) \Delta \kappa \Delta u \right. \\
& \quad + \gamma_i^{-1} \sum_{j=1}^m b_{ij}(s) [g_j(\tilde{\gamma}_j \psi_j(s - \tau_j(s))) - g_j(\tilde{\gamma}_j \tilde{\psi}_j(s - \tau_j(s)))] \\
& \quad + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}(s) [p_j(\tilde{\gamma}_j \psi_j(s - \tau_j(s))) q_l(\tilde{\gamma}_l \psi_l(s - \tau_l(s))) \\
& \quad \left. \left. - p_j(\tilde{\gamma}_j \tilde{\psi}_j(s - \tau_j(s))) q_l(\tilde{\gamma}_l \tilde{\psi}_l(s - \tau_l(s))) \right] \right\} \Delta s \Big|
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{-\infty}^t e_{-a_i \int_0^\infty k_i(u) \Delta u}(t, \rho(s)) \\
&\quad \times \left\{ a_i(s) \int_0^\infty k_i(u) \int_{s-u}^s |\varphi_i^\Delta(\kappa) - \bar{\varphi}_i^\Delta(\kappa)| \Delta \kappa \Delta u \right. \\
&\quad + \gamma_i^{-1} \sum_{j=1}^m b_{ij}(s) |g_j(\bar{\gamma}_j \psi_j(s - \tau_j(s))) - g_j(\bar{\gamma}_j \bar{\psi}_j(s - \tau_j(s)))| \\
&\quad + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}(s) |p_j(\bar{\gamma}_j \psi_j(s - \tau_j(s))) q_l(\bar{\gamma}_l \psi_l(s - \tau_l(s))) \\
&\quad \left. - p_j(\bar{\gamma}_j \bar{\psi}_j(s - \tau_j(s))) q_l(\bar{\gamma}_l \bar{\psi}_l(s - \tau_l(s))) \right\} \Delta s \\
&\leq \int_{-\infty}^t e_{-a_i \int_0^\infty k_i(u) \Delta u}(t, \rho(s)) \left[ a_i^+ \int_0^\infty k_i(u) u \Delta u |\varphi - \bar{\varphi}|_1 \right. \\
&\quad + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j^{-1} |\psi_j(s - \tau_j(s)) - \tilde{\psi}_j(s - \tau_j(s))| \\
&\quad + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ Q_l L_j^p \bar{\gamma}_j^{-1} |\psi_j(s - \tau_j(s)) - \tilde{\psi}_j(s - \tau_j(s))| \\
&\quad \left. + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ P_j L_l^q \bar{\gamma}_l^{-1} |\psi_l(s - \tau_l(s)) - \tilde{\psi}_l(s - \tau_l(s))| \right] \Delta s \\
&\leq \int_{-\infty}^t e_{-a_i \int_0^\infty k_i(u) \Delta u}(t, \rho(s)) \left[ a_i^+ \int_0^\infty k_i(u) u \Delta u |\varphi - \bar{\varphi}|_1 \right. \\
&\quad + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j^{-1} |\psi_j(s - \tau_j(s)) - \tilde{\psi}_j(s - \tau_j(s))| \\
&\quad + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ Q_l L_j^p \bar{\gamma}_j^{-1} |\psi_j(s - \tau_j(s)) - \tilde{\psi}_j(s - \tau_j(s))| \\
&\quad \left. + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ P_j L_l^q \bar{\gamma}_l^{-1} |\psi_l(s - \tau_l(s)) - \tilde{\psi}_l(s - \tau_l(s))| \right] ds \\
&\leq \int_{-\infty}^t e_{-a_i \int_0^\infty k_i(u) \Delta u}(t, \rho(s)) \left[ a_i^+ \int_0^\infty k_i(u) u \Delta u |\varphi - \bar{\varphi}|_1 \right. \\
&\quad + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j^{-1} |\psi - \tilde{\psi}|_1 + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ Q_l L_j^p \bar{\gamma}_j^{-1} |\psi - \tilde{\psi}|_1 \\
&\quad \left. + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ P_j L_l^q \bar{\gamma}_l^{-1} |\psi - \tilde{\psi}|_1 \right] \Delta s \\
&\leq \frac{1}{a_i^- \int_0^\infty k_i(u) \Delta u} \left[ a_i^+ \int_0^\infty k_i(u) u \Delta u + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j^{-1} \right.
\end{aligned}$$

$$\begin{aligned}
& + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ Q_l L_j^p \bar{\gamma}_j^{-1} + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ P_j L_l^q \bar{\gamma}_l^{-1} \Big] \|\phi - \tilde{\phi}\| \\
= & \epsilon_{i1} \|\phi - \tilde{\phi}\|, \tag{3.15} \\
|(\Phi\phi - \Phi\tilde{\phi})_i^\Delta(t)| = & \left| \left\{ \int_{-\infty}^t e_{-a_i \int_0^\infty k_i(u)\Delta u}(t, \rho(s)) \right. \right. \\
& \times \left[ a_i(s) \int_0^\infty k_i(u) \int_{s-u}^s (\varphi_i^\Delta(\kappa) - \bar{\varphi}_i^\Delta(\kappa)) \Delta\kappa \Delta u \right. \\
& + \gamma_i^{-1} \sum_{j=1}^m b_{ij}(s) (g_j(\bar{\gamma}_j \psi_j(s - \tau_j(s))) - g_j(\bar{\gamma}_j \bar{\psi}_j(s - \tau_j(s)))) \\
& + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}(s) (p_j(\bar{\gamma}_j \psi_j(s - \tau_j(s))) q_l(\bar{\gamma}_l \psi_l(s - \tau_l(s))) \\
& \left. \left. - p_j(\bar{\gamma}_j \bar{\psi}_j(s - \tau_j(s))) q_l(\bar{\gamma}_l \bar{\psi}_l(s - \tau_l(s)))) \right] \Delta s \right\}^\Delta \Big| \\
\leq & \left| a_i(t) \int_0^\infty k_i(u) \int_{t-u}^t (\varphi_i^\Delta(\kappa) - \bar{\varphi}_i^\Delta(\kappa)) \Delta\kappa \Delta u \right. \\
& + \gamma_i^{-1} \sum_{j=1}^m b_{ij}(t) (g_j(\bar{\gamma}_j \psi_j(t - \tau_j(t))) - g_j(\bar{\gamma}_j \bar{\psi}_j(t - \tau_j(t))) \\
& + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}(t) (p_j(\bar{\gamma}_j \psi_j(t - \tau_j(t))) q_l(\bar{\gamma}_l \psi_l(t - \tau_l(t))) \\
& \left. - p_j(\bar{\gamma}_j \bar{\psi}_j(t - \tau_j(t))) q_l(\bar{\gamma}_l \bar{\psi}_l(t - \tau_l(t)))) \right. \\
& - a_i(t) \int_0^\infty k_i(u) \Delta u \int_{-\infty}^t e_{-a_i \int_0^\infty k_i(u)\Delta u}(t, \rho(s)) \\
& \times \left[ a_i(s) \int_0^\infty k_i(u) \int_{s-u}^s (\varphi_i^\Delta(\kappa) - \bar{\varphi}_i^\Delta(\kappa)) \Delta\kappa \Delta u \right. \\
& + \gamma_i^{-1} \sum_{j=1}^m b_{ij}(s) (g_j(\bar{\gamma}_j \psi_j(s - \tau_j(s))) - g_j(\bar{\gamma}_j \bar{\psi}_j(s - \tau_j(s))) \\
& + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}(s) (p_j(\bar{\gamma}_j \psi_j(s - \tau_j(s))) q_l(\bar{\gamma}_l \psi_l(s - \tau_l(s))) \\
& \left. \left. - p_j(\bar{\gamma}_j \bar{\psi}_j(s - \tau_j(s))) q_l(\bar{\gamma}_l \bar{\psi}_l(s - \tau_l(s)))) \right] \Delta s \right| \\
\leq & a_i(t) \int_0^\infty k_i(u) \int_{t-u}^t |\varphi_i^\Delta(\kappa) - \bar{\varphi}_i^\Delta(\kappa)| \Delta\kappa \Delta u \\
& + \gamma_i^{-1} \sum_{j=1}^m b_{ij}(t) |g_j(\bar{\gamma}_j \psi_j(t - \tau_j(t))) - g_j(\bar{\gamma}_j \bar{\psi}_j(t - \tau_j(t)))|
\end{aligned}$$

$$\begin{aligned}
& + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}(t) |p_j(\bar{\gamma}_j \psi_j(t - \tau_j(t))) q_l(\bar{\gamma}_l \psi_l(t - \tau_l(t))) \\
& - p_j(\bar{\gamma}_j \bar{\psi}_j(t - \tau_j(t))) q_l(\bar{\gamma}_l \bar{\psi}_l(t - \tau_l(t)))| \\
& + a_i(t) \int_0^\infty k_i(u) \Delta u \int_{-\infty}^t e_{-a_i \int_0^\infty k_i(u) \Delta u}(t, \rho(s)) \\
& \times \left[ a_i(s) \int_0^\infty k_i(u) \int_{s-u}^s |\varphi_i^\Delta(\kappa) - \bar{\varphi}_i^\Delta(\kappa)| \Delta \kappa \Delta u \right. \\
& + \gamma_i^{-1} \sum_{j=1}^m b_{ij}(s) |g_j(\bar{\gamma}_j \psi_j(s - \tau_j(s))) - g_j(\bar{\gamma}_j \bar{\psi}_j(s - \tau_j(s)))| \\
& + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}(s) |p_j(\bar{\gamma}_j \psi_j(s - \tau_j(s))) q_l(\bar{\gamma}_l \psi_l(s - \tau_l(s))) \\
& - p_j(\bar{\gamma}_j \bar{\psi}_j(s - \tau_j(s))) q_l(\bar{\gamma}_l \bar{\psi}_l(s - \tau_l(s)))| \left. \right] \Delta s \\
\leq & a_i^+ \int_0^\infty k_i(u) u \Delta u |\varphi - \bar{\varphi}|_1 \\
& + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j^{-1} |\psi_j(s - \tau_j(s)) - \tilde{\psi}_j(s - \tau_j(s))| \\
& + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ Q_l L_j^p \bar{\gamma}_j^{-1} |\psi_j(s - \tau_j(s)) - \tilde{\psi}_j(s - \tau_j(s))| \\
& + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ P_j L_l^q \bar{\gamma}_l^{-1} |\psi_l(s - \tau_l(s)) - \tilde{\psi}_l(s - \tau_l(s))| \\
& + \frac{a_i^+}{a_i^-} \left[ a_i^+ \int_0^\infty k_i(u) u \Delta u |\varphi - \bar{\varphi}|_1 \right. \\
& + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j^{-1} |\psi_j(s - \tau_j(s)) - \tilde{\psi}_j(s - \tau_j(s))| \\
& + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ Q_l L_j^p \bar{\gamma}_j^{-1} |\psi_j(s - \tau_j(s)) - \tilde{\psi}_j(s - \tau_j(s))| \\
& \left. + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ P_j L_l^q \bar{\gamma}_l^{-1} |\psi_l(s - \tau_l(s)) - \tilde{\psi}_l(s - \tau_l(s))| \right] \\
\leq & a_i^+ \int_0^\infty k_i(u) u \Delta u |\varphi - \bar{\varphi}|_1 + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j^{-1} |\psi - \tilde{\psi}|_1 \\
& + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ Q_l L_j^p \bar{\gamma}_j^{-1} |\psi - \tilde{\psi}|_1 + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ P_j L_l^q \bar{\gamma}_l^{-1} |\psi - \tilde{\psi}|_1 \\
& + \frac{a_i^+}{a_i^-} \left[ a_i^+ \int_0^\infty k_i(u) u \Delta u |\varphi - \bar{\varphi}|_1 + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j^{-1} |\psi - \tilde{\psi}|_1 \right]
\end{aligned}$$

$$\begin{aligned}
 & + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ Q_l L_j^p \bar{\gamma}_j^{-1} |\psi - \tilde{\psi}|_1 + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ P_j L_l^q \bar{\gamma}_l^{-1} |\psi - \tilde{\psi}|_1 \Big] \\
 & \leq \left\{ a_i^+ \int_0^\infty k_i(u) u \Delta u + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j^{-1} \right. \\
 & \quad + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ Q_l L_j^p \bar{\gamma}_j^{-1} + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ P_j L_l^q \bar{\gamma}_l^{-1} \\
 & \quad + \frac{a_i^+}{a_i^-} \left[ a_i^+ \int_0^\infty k_i(u) u \Delta u + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j^{-1} \right. \\
 & \quad \left. \left. + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ Q_l L_j^p \bar{\gamma}_j^{-1} + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ P_j L_l^q \bar{\gamma}_l^{-1} \right] \right\} \|\phi - \tilde{\phi}\| \\
 & \leq \left( 1 + \frac{a_i^+}{a_i^-} \right) \left[ a_i^+ \int_0^\infty k_i(u) u \Delta u + \gamma_i^{-1} \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j^{-1} \right. \\
 & \quad \left. + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ Q_l L_j^p \bar{\gamma}_j^{-1} + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ P_j L_l^q \bar{\gamma}_l^{-1} \right] \|\phi - \tilde{\phi}\| \\
 & = \epsilon_{i2} \|\phi - \tilde{\phi}\|, \tag{3.16}
 \end{aligned}$$

$$\begin{aligned}
 |(\Phi\phi - \Phi\tilde{\phi})_{n+j}(t)| & = \left| \int_{-\infty}^t e_{-d_j} \int_0^\infty h_j(u) \Delta u(t, \rho(s)) \right. \\
 & \quad \times \left\{ d_j(s) \int_0^\infty h_j(u) \int_{s-u}^s (\psi_j^\Delta(\kappa) - \tilde{\psi}_j^\Delta(\kappa)) \Delta \kappa \Delta u \right. \\
 & \quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}(s) [f_i(\gamma_i \varphi_i(s - \omega_i(s))) - f_i(\gamma_i \bar{\varphi}_i(s - \omega_i(s)))] \\
 & \quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}(s) [v_i(\gamma_i \varphi_i(s - \omega_i(s))) w_l(\gamma_l \varphi_l(s - \omega_l(s))) \\
 & \quad \left. \left. - v_i(\gamma_i \bar{\varphi}_i(s - \omega_i(s))) w_l(\gamma_l \bar{\varphi}_l(s - \omega_l(s)))] \right\} \Delta s \Big| \\
 & \leq \int_{-\infty}^t e_{-d_j} \int_0^\infty h_j(u) \Delta u(t, \rho(s)) \\
 & \quad \times \left\{ d_j(s) \int_0^\infty h_j(u) \int_{s-u}^s |\psi_j^\Delta(\kappa) - \tilde{\psi}_j^\Delta(\kappa)| \Delta \kappa \Delta u \right. \\
 & \quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}(s) |f_i(\gamma_i \varphi_i(s - \omega_i(s))) - f_i(\gamma_i \bar{\varphi}_i(s - \omega_i(s))| \\
 & \quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}(s) |v_i(\gamma_i \varphi_i(s - \omega_i(s))) w_l(\gamma_l \varphi_l(s - \omega_l(s))) \\
 & \quad \left. \left. - v_i(\gamma_i \bar{\varphi}_i(s - \omega_i(s))) w_l(\gamma_l \bar{\varphi}_l(s - \omega_l(s))| \right\} \Delta s
 \end{aligned}$$

$$\begin{aligned}
&\leq \int_{-\infty}^t e_{-d_j} \int_0^\infty h_j(u) \Delta u(t, \rho(s)) \left[ d_j^+ \int_0^\infty h_j(u) u \Delta u |\psi - \tilde{\psi}|_1 \right. \\
&\quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ L_i^f \gamma_i^{-1} |\varphi_i(s - \omega_i(s)) - \tilde{\varphi}_i(s - \omega_i(s))| \\
&\quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ W_l L_i^p \gamma_i^{-1} |\varphi_i(s - \omega_i(s)) - \tilde{\varphi}_i(s - \omega_i(s))| \\
&\quad \left. + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ V_i L_j^w \gamma_l^{-1} |\varphi_l(s - \omega_l(s)) - \tilde{\varphi}_l(s - \omega_l(s))| \right] \Delta s \\
&\leq \int_{-\infty}^t e_{-d_j} \int_0^\infty h_j(u) \Delta u(t, \rho(s)) \left[ d_j^+ \int_0^\infty h_j(u) u \Delta u |\psi - \tilde{\psi}|_1 \right. \\
&\quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ L_i^f \gamma_i^{-1} |\varphi_i(s - \omega_i(s)) - \tilde{\varphi}_i(s - \omega_i(s))| \\
&\quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ W_l L_i^v \gamma_i^{-1} |\varphi_i(s - \omega_i(s)) - \tilde{\varphi}_i(s - \omega_i(s))| \\
&\quad \left. + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ V_i L_l^w \gamma_l^{-1} |\varphi_l(s - \omega_l(s)) - \tilde{\varphi}_l(s - \omega_l(s))| \right] \Delta s \\
&\leq \int_{-\infty}^t e_{-d_j(t)} \int_0^\infty h_j(u) \Delta u(t, \rho(s)) \left[ d_j^+ \int_0^\infty h_j(u) u \Delta u |\varphi - \tilde{\varphi}|_1 \right. \\
&\quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ L_i^f \gamma_i^{-1} |\varphi - \tilde{\varphi}|_1 + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ W_l L_i^v \gamma_i^{-1} |\varphi - \tilde{\varphi}|_1 \\
&\quad \left. + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ V_i L_l^w \gamma_l^{-1} |\varphi - \tilde{\varphi}|_1 \right] \Delta s \\
&\leq \frac{1}{d_j^- \int_0^\infty h_j(u) \Delta u} \left[ d_j^+ \int_0^\infty h_j(u) u \Delta u + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ L_i^f \gamma_i^{-1} \right. \\
&\quad \left. + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ W_l L_i^v \gamma_i^{-1} + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ V_i L_l^w \gamma_l^{-1} \right] \|\phi - \tilde{\phi}\| \\
&= \varrho_{j1} \|\phi - \tilde{\phi}\|, \tag{3.17}
\end{aligned}$$

$$\begin{aligned}
|(\Phi\phi - \Phi\tilde{\phi})_{n+j}^\Delta(t)| &= \left| \left\{ \int_{-\infty}^t e_{-d_j} \int_0^\infty h_j(u) \Delta u(t, \rho(s)) \right. \right. \\
&\quad \times \left[ d_j(s) \int_0^\infty h_j(u) \int_{s-u}^s (\psi_j^\Delta(\kappa) - \tilde{\psi}_j^\Delta(\kappa)) \Delta \kappa \Delta u \right. \\
&\quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}(s) (f_i(\gamma_i \varphi_i(s - \omega_i(s))) - f_i(\gamma_i \tilde{\varphi}_i(s - \omega_i(s)))) \\
&\quad \left. \left. + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}(s) (v_i(\gamma_i \varphi_i(s - \omega_i(s))) w_l(\gamma_l \varphi_l(s - \omega_l(s)))) \right. \right.
\end{aligned}$$



$$\begin{aligned}
& \left. - v_i(\gamma_i \bar{\varphi}_i(s - \omega_i(s))) w_l(\gamma_l \bar{\varphi}_l(s - \omega_l(s))) \right] \Delta s \Bigg\}^{\Delta} \Bigg| \\
\leq & \left| d_j(t) \int_0^\infty h_j(u) \int_{t-u}^t (\psi_j^\Delta(\kappa) - \bar{\psi}_j^\Delta(\kappa)) \Delta \kappa \Delta u \right. \\
& + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}(t) (f_i(\gamma_i \varphi_i(t - \omega_i(t))) - f_i(\gamma_i \bar{\varphi}_i(t - \omega_i(t)))) \\
& + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}(t) (v_i(\gamma_i \varphi_i(t - \omega_i(t))) w_l(\gamma_l \varphi_l(t - \omega_l(t))) \\
& - v_i(\gamma_i \bar{\varphi}_i(t - \omega_i(t))) w_l(\gamma_l \bar{\varphi}_l(t - \omega_l(t)))) \\
& - d_j(t) \int_0^\infty h_j(u) \Delta u \int_{-\infty}^t e_{-d_j} \int_0^\infty h_j(u) \Delta u (t, \rho(s)) \\
& \times \left[ d_j(s) \int_0^\infty h_j(u) \int_{s-u}^s (\psi_j^\Delta(\kappa) - \bar{\psi}_j^\Delta(\kappa)) \Delta \kappa \Delta u \right. \\
& + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}(s) (f_i(\gamma_i \varphi_i(s - \omega_i(s))) - f_i(\gamma_i \bar{\varphi}_i(s - \omega_i(s)))) \\
& + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}(s) (v_i(\gamma_i \varphi_i(s - \omega_i(s))) w_l(\gamma_l \varphi_l(s - \omega_l(s))) \\
& \left. - v_i(\gamma_i \bar{\varphi}_i(s - \omega_i(s))) w_l(\gamma_l \bar{\varphi}_l(s - \omega_l(s))) \right] \Delta s \Bigg| \\
\leq & d_j(t) \int_0^\infty h_j(u) \int_{t-u}^t |\psi_j^\Delta(\kappa) - \bar{\psi}_j^\Delta(\kappa)| \Delta \kappa \Delta u \\
& + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}(t) |f_i(\gamma_i \varphi_i(t - \omega_i(t))) - f_i(\gamma_i \bar{\varphi}_i(t - \omega_i(t)))| \\
& + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}(t) |v_i(\gamma_i \varphi_i(t - \omega_i(t))) w_l(\gamma_l \varphi_l(t - \omega_l(t))) \\
& - v_i(\gamma_i \bar{\varphi}_i(t - \omega_i(t))) w_l(\gamma_l \bar{\varphi}_l(t - \omega_l(t)))| \\
& + d_j(t) \int_0^\infty h_j(u) \Delta u \int_{-\infty}^t e_{-d_j} \int_0^\infty h_j(u) \Delta u (t, \rho(s)) \\
& \times \left[ d_j(s) \int_0^\infty h_j(u) \int_{s-u}^s |\psi_j^\Delta(\kappa) - \bar{\psi}_j^\Delta(\kappa)| \Delta \kappa \Delta u \right. \\
& + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}(s) |f_i(\gamma_i \varphi_i(s - \omega_i(s))) - f_i(\gamma_i \bar{\varphi}_i(s - \omega_i(s)))| \\
& + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}(s) |v_i(\gamma_i \varphi_i(s - \omega_i(s))) w_l(\gamma_l \varphi_l(s - \omega_l(s))) \\
& \left. - v_i(\gamma_i \bar{\varphi}_i(s - \omega_i(s))) w_l(\gamma_l \bar{\varphi}_l(s - \omega_l(s))) \right] \Delta s
\end{aligned}$$

$$\begin{aligned}
&\leq d_j^+ \int_0^\infty h_j(u) u \Delta u |\psi - \tilde{\psi}|_1 \\
&\quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ L_i^f \gamma_i^{-1} |\varphi_i(s - \omega_i(s)) - \tilde{\varphi}_i(s - \omega_i(s))| \\
&\quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ w_l L_i^v \gamma_i^{-1} |\varphi_i(s - \omega_i(s)) - \tilde{\varphi}_i(s - \omega_i(s))| \\
&\quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ V_i L_l^w \gamma_l^{-1} |\varphi_l(s - \omega_l(s)) - \tilde{\varphi}_l(s - \omega_l(s))| \\
&\quad + \frac{d_j^+}{d_j^-} \left[ d_j^+ \int_0^\infty h_j(u) u \Delta u |\psi - \tilde{\psi}|_1 \right. \\
&\quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ L_i^f \gamma_i^{-1} |\varphi_i(s - \omega_i(s)) - \tilde{\varphi}_i(s - \omega_i(s))| \\
&\quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ W_l L_i^v \gamma_i^{-1} |\varphi_i(s - \omega_i(s)) - \tilde{\varphi}_i(s - \omega_i(s))| \\
&\quad \left. + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ V_i L_l^w \gamma_l^{-1} |\varphi_l(s - \omega_l(s)) - \tilde{\varphi}_l(s - \omega_l(s))| \right] \\
&\leq d_j^+ \int_0^\infty h_j(u) u \Delta u |\psi - \tilde{\psi}|_1 + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ L_i^f \gamma_i^{-1} |\varphi - \tilde{\varphi}|_1 \\
&\quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ W_l L_i^v \gamma_i^{-1} |\psi - \tilde{\psi}|_1 + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ V_i L_l^w \gamma_l^{-1} |\varphi - \tilde{\varphi}|_1 \\
&\quad + \frac{d_j^+}{d_j^-} \left[ d_j^+ \int_0^\infty h_j(u) u \Delta u |\psi - \tilde{\psi}|_1 + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ L_i^f \gamma_i^{-1} |\varphi - \tilde{\varphi}|_1 \right. \\
&\quad \left. + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ W_l L_i^v \gamma_i^{-1} |\varphi - \tilde{\varphi}|_1 + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ V_i L_l^w \gamma_l^{-1} |\varphi - \tilde{\varphi}|_1 \right] \\
&\leq \left\{ d_j^+ \int_0^\infty h_j(u) u \Delta u + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ L_i^f \gamma_i^{-1} \right. \\
&\quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ W_l L_i^v \gamma_i^{-1} + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ V_i L_l^w \gamma_l^{-1} \\
&\quad + \frac{d_j^+}{d_j^-} \left[ d_j^+ \int_0^\infty h_j(u) u \Delta u + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ L_i^f \gamma_i^{-1} \right. \\
&\quad \left. + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ W_l L_i^v \gamma_i^{-1} + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ V_i L_l^w \gamma_l^{-1} \right] \left. \right\} \|\phi - \tilde{\phi}\| \\
&\leq \left( 1 + \frac{d_j^+}{d_j^-} \right) \left[ d_j^+ \int_0^\infty h_j(u) u \Delta u + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}^+ L_i^f \gamma_i^{-1} \right.
\end{aligned}$$

$$\begin{aligned}
 & + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ W_l L_i^v \gamma_i^{-1} + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}^+ V_l L_i^w \gamma_l^{-1} \Big] \|\phi - \tilde{\phi}\| \\
 & = \varrho_{j2} \|\phi - \tilde{\phi}\|. \tag{3.18}
 \end{aligned}$$

It follows from (A4) that

$$\max\left\{ \max_{1 \leq i \leq n} \{\epsilon_{i1}, \epsilon_{i2}\}, \max_{1 \leq j \leq m} \{\varrho_{j1}, \varrho_{j2}\} \right\} < 1. \tag{3.19}$$

By (3.15)-(3.19), we get  $\|\Phi\phi - \Phi\tilde{\phi}\| < \|\phi - \tilde{\phi}\|$ , which implies that  $\Phi$  is a contraction mapping. By Brouwer fixed point theorem,  $\Phi$  has a fixed point  $\phi^*$  such that  $\Phi\phi^* = \phi^*$ . Namely, (3.1) has a unique almost periodic solution in  $\Omega$ . The proof of Theorem 3.1 is complete.  $\square$

### 4. Exponential stability of almost periodic solution

In this section, we will discuss the exponential stability of almost periodic solution of system (1.1).

**Theorem 4.1.** *In addition to (A1)-(A4), assume that*

(A5) *For all  $t \in [0, \infty)_{\mathbb{T}}$  and  $i = 1, 2, \dots, n, j = 1, 2, \dots, m$ , there exist positive constants  $\lambda \in \mathbb{T}_\mu, \delta_i$  and  $\eta_j$  such that*

$$\begin{cases}
 - \left[ a_i^- \int_0^\infty k_i(s) \Delta s - a_i^+ \int_0^\infty k_i(s) \Delta s \right] \delta_i \\
 + \bar{\gamma}_i^{-1} \left[ \sum_{j=1}^m \bar{\gamma}_j \eta_j b_{ij}^+ L_j^g + \sum_{j=1}^m \sum_{l=1}^m \bar{\gamma}_j \eta_j e_{ijl}^+ (L_j^p Q_l + L_l^q P_j) \right] < 0, \\
 - \left[ d_j^- \int_0^\infty h_j(s) \Delta s - d_j^+ \int_0^\infty h_j(s) \Delta s \right] \eta_j \\
 + \bar{\gamma}_j^{-1} \left[ \sum_{i=1}^n \gamma_i \delta_i c_{ji}^+ L_i^f + \sum_{i=1}^n \sum_{l=1}^n \gamma_i \delta_i s_{jil}^+ (L_i^v W_l + L_l^w V_i) \right] < 0,
 \end{cases}$$

and

$$\begin{cases}
 \left[ \left[ (1 - \mu\lambda) a_i^- \int_0^\infty k_i(s) \Delta s + \left( \lambda + (1 - \mu\lambda) a_i^+ \int_0^\infty k_i(s) \int_{t-s}^t e_\lambda(t^*, \theta) \Delta \theta \right) \right] \delta_i \right. \\
 + e_\varsigma(\rho(t), 0) \bar{\gamma}_i^{-1} \left[ \sum_{j=1}^m \bar{\gamma}_j \eta_j b_{ij}^+ L_j^g e_\lambda(0, \tau_j(t^*)) + \sum_{j=1}^m \sum_{l=1}^m \bar{\gamma}_j \eta_j e_{ijl}^+ L_j^p Q_l e_\lambda(0, \tau_j(t^*)) \right. \\
 \left. \left. + \sum_{j=1}^m \sum_{l=1}^m \bar{\gamma}_j \eta_j e_{ijl}^+ L_l^q P_j e_\lambda(0, \tau_l(t^*)) \right] < 1, \right. \\
 \left[ \left[ (1 - \mu\lambda) d_j^- \int_0^\infty h_j(s) \Delta s + \left( \lambda + (1 - \mu\lambda) d_j^+ \int_0^\infty h_j(s) \int_{t-s}^t e_\lambda(t^*, \theta) \Delta \theta \right) \right] \eta_j \right. \\
 + e_\varsigma(\rho(t), 0) \bar{\gamma}_j^{-1} \left[ \sum_{i=1}^n \gamma_i \delta_i c_{ji}^+ L_i^f e_\lambda(0, \omega_i(t^*)) + \sum_{i=1}^n \sum_{l=1}^n \gamma_i \delta_i s_{jil}^+ L_i^v W_l e_\lambda(0, \omega_i(t^*)) \right. \\
 \left. \left. + \sum_{i=1}^n \sum_{l=1}^n \gamma_i \delta_i s_{jil}^+ L_l^w V_i e_\lambda(0, \omega_l(t^*)) \right] < 1. \right.
 \end{cases}$$

Then the almost periodic solution of system (1.1) is exponentially stable.

**Proof.** Assume that  $u(t) = (x_1(t), x_2(t), \dots, x_n(t), y_1(t), y_2(t), \dots, y_m(t))^T$  is an arbitrary solution of system (1.1) with the initial condition  $\phi(s) = (\varphi_1(s), \varphi_2(s), \dots, \varphi_n(s), \psi_1(s), \psi_2(s), \dots, \psi_m(s))^T$ . In view of Theorem 3.1, system (1.1) has an almost periodic solution  $u^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t), y_1^*(t), y_2^*(t), \dots, y_m^*(t))^T$  with the initial condition  $\phi^*(s) = (\varphi_1^*(s), \varphi_2^*(s), \dots, \varphi_n^*(s), \psi_1^*(s), \psi_2^*(s), \dots, \psi_m^*(s))^T$ . Let

$$\begin{cases} \zeta_i(t) = \gamma_i^{-1}(x_i(t) - x_i^*(t)), i = 1, 2, \dots, n, \\ \vartheta_j(t) = \bar{\gamma}_j^{-1}(y_j(t) - y_j^*(t)), j = 1, 2, \dots, m. \end{cases} \tag{4.1}$$

By (1.1) and (4.1), we have

$$\left\{ \begin{aligned} \zeta_i^\Delta(t) &= -a_i(t) \int_0^\infty k_i(s) \Delta s \zeta_i(t) + a_i(t) \int_0^\infty k_i(s) \int_{t-s}^t \zeta_i^\Delta(\theta) \Delta \theta \Delta s \\ &\quad + \gamma_i^{-1} \sum_{j=1}^m b_{ij}(t) [g_j(\vartheta_j(t - \tau_j(t))) - g_j(\vartheta_j^*(t - \tau_j(t)))] \\ &\quad + \gamma_i^{-1} \sum_{j=1}^m \sum_{l=1}^m e_{ijl}(t) [p_j(\vartheta_j(t - \tau_j(t))) q_l(\vartheta_l(t - \tau_l(t))) \\ &\quad \quad - p_j(\vartheta_j^*(t - \tau_j(t))) q_l(\vartheta_l^*(t - \tau_l(t)))] \\ \vartheta_j^\Delta(t) &= -d_j(t) \int_0^\infty h_j(s) \Delta s \vartheta_j(t) + d_j(t) \int_0^\infty h_j(s) \int_{t-s}^t \vartheta_j^\Delta(\theta) \Delta \theta \Delta s \\ &\quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}(t) [f_i(\zeta_i(t - \omega_i(t))) - f_i(\zeta_i^*(t - \omega_i(t)))] \\ &\quad + \bar{\gamma}_j^{-1} \sum_{i=1}^n \sum_{l=1}^n s_{jil}(t) [v_i(\zeta_i(t - \omega_i(t))) w_l(\zeta_l(t - \omega_l(t))) \\ &\quad \quad - w_i(\zeta_i^*(t - \omega_i(t))) w_l(\zeta_l^*(t - \omega_l(t)))] \end{aligned} \right. \tag{4.2}$$

where  $i = 1, 2, \dots, n, j = 1, 2, \dots, m$ . Let

$$\begin{cases} \mu_i(t) = e_\lambda(t, 0) \zeta_i(t), i = 1, 2, \dots, n, \\ \nu_j(t) = e_\lambda(t, 0) \vartheta_j(t), j = 1, 2, \dots, m. \end{cases} \tag{4.3}$$

It from (4.2) and (4.3) that

$$\begin{aligned} \mu_i^\Delta(t) &= \lambda e_\lambda(t, 0) \zeta_i(t) + e_\lambda(\rho(t), 0) \zeta_i^\Delta(t) \\ &= \lambda \zeta_i(t) + e_\lambda(\rho(t), 0) \left\{ -a_i(t) \int_0^\infty k_i(s) \Delta s \zeta_i(t) \right. \\ &\quad + a_i(t) \int_0^\infty k_i(s) \int_{t-s}^t \zeta_i^\Delta(\theta) \Delta \theta \Delta s \\ &\quad + \gamma_i^{-1} \sum_{j=1}^m b_{ij}(t) [g_j(\nu_j(t - \tau_j(t))) - g_j(\nu_j^*(t - \tau_j(t)))] \\ &\quad \left. + \sum_{j=1}^m \sum_{l=1}^m e_{ijl}(t) [p_j(\nu_j(t - \tau_j(t))) q_l(\nu_l(t - \tau_l(t))) \right. \\ &\quad \left. - w_i(\zeta_i^*(t - \omega_i(t))) w_l(\zeta_l^*(t - \omega_l(t)))] \right\} \end{aligned}$$

$$\begin{aligned}
& - p_j(\nu_j^*(t - \tau_j(t)))q_l(\nu_l^*(t - \tau_l(t))) \Big\} \\
& = \lambda \zeta_i(t) - (1 - \mu\lambda)a_i(t) \int_0^\infty k_i(s) \Delta s \mu_i(t) \\
& + (1 - \mu\lambda)a_i(t) \int_0^\infty k_i(s) \int_{t-s}^t e_\lambda(t, \kappa) \mu_i^\Delta(\kappa) \Delta \kappa \Delta s \\
& + e_\lambda(\rho(t), 0) \gamma_i^{-1} \left\{ \sum_{j=1}^m b_{ij}(t) [g_j(\nu_j(t - \tau_j(t))) - g_j(\nu_j^*(t - \tau_j(t)))] \right. \\
& + \sum_{j=1}^m \sum_{l=1}^m e_{ijl}(t) [p_j(\nu_j(t - \tau_j(t)))q_l(\nu_l(t - \tau_l(t))) \\
& \left. - p_j(\nu_j^*(t - \tau_j(t)))q_l(\nu_l^*(t - \tau_l(t)))] \right\} \tag{4.4}
\end{aligned}$$

and

$$\begin{aligned}
\nu_j^\Delta(t) & = \lambda e_\lambda(t, 0) \nu_j(t) + e_\lambda(\rho(t), 0) \nu_j^\Delta(t) \\
& = \lambda \nu_j(t) + e_\lambda(\rho(t), 0) \left\{ - d_j(t) \int_0^\infty h_j(s) \Delta s \nu_j(t) \right. \\
& + d_j(t) \int_0^\infty h_j(s) \int_{t-s}^t \nu_j^\Delta(\theta) \Delta \theta \Delta s \\
& + \bar{\gamma}_j^{-1} \sum_{i=1}^n c_{ji}(t) [f_i(\mu_i(t - \omega_i(t))) - f_i(\mu_i^*(t - \omega_i(t)))] \\
& + \sum_{i=1}^n \sum_{l=1}^n s_{jil}(t) [v_i(\mu_i(t - \omega_i(t))) w_l(\mu_l(t - \omega_l(t))) \\
& \left. - v_i(\mu_i^*(t - \omega_i(t))) w_l(\mu_l^*(t - \omega_l(t)))] \right\} \\
& = \lambda \nu_j(t) - (1 - \mu\lambda) d_j(t) \int_0^\infty k_i(s) \Delta s \nu_j(t) \\
& + (1 - \mu\lambda) d_j(t) \int_0^\infty h_j(s) \int_{t-s}^t e_\lambda(t, \kappa) \nu_j^\Delta(\kappa) \Delta \kappa \Delta s \\
& + e_\lambda(\rho(t), 0) \bar{\gamma}_j^{-1} \left\{ \sum_{i=1}^n c_{ji}(t) [f_i(\mu_i(t - \omega_i(t))) - f_i(\mu_i^*(t - \omega_i(t)))] \right. \\
& + \sum_{i=1}^n \sum_{l=1}^n s_{jil}(t) [v_i(\mu_i(t - \omega_i(t))) w_l(\mu_l(t - \omega_l(t))) \\
& \left. - w_i(\mu_i^*(t - \omega_i(t))) w_l(\mu_l^*(t - \omega_l(t)))] \right\}. \tag{4.5}
\end{aligned}$$

We define continuous functions  $\Psi_i(\varsigma)$  ( $i = 1, 2, \dots, n$ ) and  $\Lambda_j(\varsigma)$  ( $j = 1, 2, \dots, m$ ) as

follows:

$$\left\{ \begin{aligned} \Psi_i(\varsigma) &= - \left[ (1-\mu\varsigma)a_i^- \int_0^\infty k_i(s)\Delta s - \left( \varsigma + (1-\mu\varsigma)a_i^+ \int_0^\infty k_i(s) \int_{t-s}^t e_\varsigma(t^*, \theta)\Delta\theta \right) \right] \delta_i \\ &\quad + e_\varsigma(\rho(t), 0)\gamma_i^{-1} \left[ \sum_{j=1}^m \tilde{\gamma}_j \eta_j b_{ij}^+ L_j^g e_\varsigma(0, \tau_j(t^*)) + \sum_{j=1}^m \sum_{l=1}^m \tilde{\gamma}_j \eta_j e_{ijl}^+ L_j^p Q_l e_\varsigma(0, \tau_j(t^*)) \right. \\ &\quad \left. + \sum_{j=1}^m \sum_{l=1}^m \tilde{\gamma}_j \eta_j e_{ijl}^+ L_l^q P_j e_\varsigma(0, \tau_l(t^*)) \right], \\ \Lambda_j(\varsigma) &= - \left[ (1-\mu\varsigma)d_j^- \int_0^\infty h_j(s)\Delta s - \left( \varsigma + (1-\mu\varsigma)d_j^+ \int_0^\infty h_j(s) \int_{t-s}^t e_\varsigma(t^*, \theta)\Delta\theta \right) \right] \eta_j \\ &\quad + e_\varsigma(\rho(t), 0)\tilde{\gamma}_j^{-1} \left[ \sum_{i=1}^n \gamma_i \delta_i c_{ji}^+ L_i^f e_\varsigma(0, \omega_i(t^*)) + \sum_{i=1}^n \sum_{l=1}^n \gamma_i \delta_i s_{jil}^+ L_i^v W_l e_\varsigma(0, \omega_i(t^*)) \right. \\ &\quad \left. + \sum_{i=1}^n \sum_{l=1}^n \gamma_i \delta_i s_{jil}^+ L_l^w V_i e_\varsigma(0, \omega_l(t^*)) \right]. \end{aligned} \right. \tag{4.6}$$

Then we have

$$\left\{ \begin{aligned} \Psi_i(0) &= - \left[ a_i^- \int_0^\infty k_i(s)\Delta s - a_i^+ \int_0^\infty k_i(s)\Delta s \right] \delta_i \\ &\quad + \gamma_i^{-1} \left[ \sum_{j=1}^m \tilde{\gamma}_j \eta_j b_{ij}^+ L_j^g + \sum_{j=1}^m \sum_{l=1}^m \tilde{\gamma}_j \eta_j e_{ijl}^+ (L_j^p Q_l + L_l^q P_j) \right] < 0, \\ \Lambda_j(0) &= - \left[ d_j^- \int_0^\infty h_j(s)\Delta s - d_j^+ \int_0^\infty h_j(s)\Delta s \right] \eta_j \\ &\quad + \tilde{\gamma}_j^{-1} \left[ \sum_{i=1}^n \gamma_i \delta_i c_{ji}^+ L_i^f + \sum_{i=1}^n \sum_{l=1}^n \gamma_i \delta_i s_{jil}^+ (L_i^v W_l + L_l^w V_i) \right] < 0. \end{aligned} \right. \tag{4.7}$$

In view of the continuity of  $\Psi_i(\varsigma)(i = 1, 2, \dots, n)$  and  $\Lambda_j(\varsigma)(j = 1, 2, \dots, m)$ , then exists positive constant  $\lambda$  such that

$$\left\{ \begin{aligned} \Psi_i(\lambda) &= - \left[ (1-\mu\lambda)a_i^- \int_0^\infty k_i(s)\Delta s - \left( \lambda + (1-\mu\lambda)a_i^+ \int_0^\infty k_i(s) \int_{t-s}^t e_\lambda(t^*, \theta)\Delta\theta \right) \right] \delta_i \\ &\quad + e_\lambda(\rho(t), 0)\gamma_i^{-1} \left[ \sum_{j=1}^m \tilde{\gamma}_j \eta_j b_{ij}^+ L_j^g e_\lambda(0, \tau_j(t^*)) + \sum_{j=1}^m \sum_{l=1}^m \tilde{\gamma}_j \eta_j e_{ijl}^+ L_j^p Q_l e_\lambda(0, \tau_j(t^*)) \right. \\ &\quad \left. + \sum_{j=1}^m \sum_{l=1}^m \tilde{\gamma}_j \eta_j e_{ijl}^+ L_l^q P_j e_\lambda(0, \tau_l(t^*)) \right] < 0, \\ \Lambda_j(\lambda) &= - \left[ (1-\mu\lambda)d_j^- \int_0^\infty h_j(s)\Delta s - \left( \lambda + (1-\mu\lambda)d_j^+ \int_0^\infty h_j(s) \int_{t-s}^t e_\lambda(t^*, \theta)\Delta\theta \right) \right] \eta_j \\ &\quad + e_\lambda(\rho(t), 0)\tilde{\gamma}_j^{-1} \left[ \sum_{i=1}^n \gamma_i \delta_i c_{ji}^+ L_i^f e_\lambda(0, \omega_i(t^*)) + \sum_{i=1}^n \sum_{l=1}^n \gamma_i \delta_i s_{jil}^+ L_i^v W_l e_\lambda(0, \omega_i(t^*)) \right. \\ &\quad \left. + \sum_{i=1}^n \sum_{l=1}^n \gamma_i \delta_i s_{jil}^+ L_l^w V_i e_\lambda(0, \omega_l(t^*)) \right] < 0, \end{aligned} \right. \tag{4.8}$$

where  $i = 1, 2, \dots, n, j = 1, 2, \dots, m$ . Let

$$\Theta = \max \left\{ \max_{1 \leq i \leq n} \{ |\mu_i(s)|, |\mu_i^\Delta(s)| \}, \max_{1 \leq j \leq m} \{ |\nu_j(s)|, |\nu_j^\Delta(s)| \}, s \in [-\infty, 0]_{\mathbb{T}} \right\}.$$

and  $\varrho$  be a positive number such that for  $t \in (-\infty, 0]_{\mathbb{T}}$  and  $i = 1, 2, \dots, n, j = 1, 2, \dots, m$ ,

$$\begin{cases} |\mu_i(t)| \leq \Theta < \varrho\delta_i, |\mu_i^\Delta(t)| \leq \Theta < \varrho\delta_i, \\ |\nu_j(t)| \leq \Theta < \varrho\eta_j, |\nu_j^\Delta(t)| \leq \Theta < \varrho\eta_j. \end{cases} \tag{4.9}$$

Next we prove that for  $t \in \mathbb{T}$  and  $i = 1, 2, \dots, n, j = 1, 2, \dots, m$ ,

$$\begin{cases} |\mu_i(t)| < \varrho\delta_i, |\mu_i^\Delta(t)| < \varrho\delta_i, \\ |\nu_j(t)| < \varrho\eta_j, |\nu_j^\Delta(t)| < \varrho\eta_j. \end{cases} \tag{4.10}$$

If (4.10) does not hold true, then there exist  $i \in \{1, 2, \dots, n\}, j \in \{1, 2, \dots, m\}$  and a first time  $t^* > 0$  such that one of the following cases is satisfied.

$$\begin{aligned} &|\mu_i(t^*)| = \varrho\delta_i \text{ and } |\mu_i(t)| < \varrho\delta_i \text{ for all } t \in (-\infty, t^*)_{\mathbb{T}}, \text{ and} \\ &|\mu_i^\Delta(t^*)| < \varrho\delta_i, |\nu_j(t^*)| < \varrho\eta_j, |\nu_j^\Delta(t^*)| < \varrho\eta_j \text{ for all } t \in (-\infty, t^*)_{\mathbb{T}}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} &|\mu_i^\Delta(t^*)| = \varrho\delta_i \text{ and } |\mu_i^\Delta(t)| < \varrho\delta_i \text{ for all } t \in (-\infty, t^*)_{\mathbb{T}}, \text{ and} \\ &|\mu_i(t^*)| < \varrho\delta_i, |\nu_j(t^*)| < \varrho\eta_j, |\nu_j^\Delta(t^*)| < \varrho\eta_j \text{ for all } t \in (-\infty, t^*)_{\mathbb{T}}, \end{aligned} \tag{4.12}$$

$$\begin{aligned} &|\nu_j(t^*)| = \varrho\eta_j \text{ and } |\nu_j(t)| < \varrho\eta_j \text{ for all } t \in (-\infty, t^*)_{\mathbb{T}}, \text{ and} \\ &|\mu_i^\Delta(t^*)| < \varrho\delta_i \text{ and } |\nu_j(t^*)| < \varrho\eta_j \text{ for all } t \in (-\infty, t^*)_{\mathbb{T}}, \end{aligned} \tag{4.13}$$

$$\begin{aligned} &|\nu_j(t^*)| = \varrho\eta_j \text{ and } |\nu_j(t)| < \varrho\eta_j \text{ for all } t \in (-\infty, t^*)_{\mathbb{T}}, \text{ and} \\ &|\mu_i(t^*)| < \varrho\delta_i, |\mu_i^\Delta(t^*)| < \varrho\delta_i, |\nu_j(t^*)| < \varrho\eta_j \text{ for all } t \in (-\infty, t^*)_{\mathbb{T}}. \end{aligned} \tag{4.14}$$

If (4.11) holds, then either

$$\begin{aligned} (a) \quad &\mu_i(t^*) = \varrho\delta_i, \mu_i^\Delta(t^*) \geq 0 \text{ and } |\mu_i(t)| < \varrho\delta_i \text{ for all } t \in (-\infty, t^*)_{\mathbb{T}}, \text{ and} \\ &|\mu_i^\Delta(t)| < \varrho\delta_i, |\nu_j(t)| < \varrho\eta_j, |\nu_j^\Delta(t)| < \varrho\eta_j \text{ for all } t \in (-\infty, t^*)_{\mathbb{T}}, \end{aligned}$$

or

$$\begin{aligned} (b) \quad &|\mu_i^\Delta(t^*)| = \varrho\delta_i \text{ and } |\mu_i^\Delta(t)| < \varrho\delta_i \text{ for all } t \in (-\infty, t^*)_{\mathbb{T}}, \text{ and} \\ &|\mu_i(t)| < \varrho\delta_i, |\nu_j(t)| < \varrho\eta_j, |\nu_j^\Delta(t)| < \varrho\eta_j \text{ for all } t \in (-\infty, t^*)_{\mathbb{T}}. \end{aligned}$$

If (a) holds, then it follows from (A6) that

$$\begin{aligned} 0 \leq \mu_i^\Delta(t^*) = &\lambda\zeta_i(t^*) - (1 - \mu\lambda)a_i(t^*) \int_0^\infty k_i(s)\Delta s \mu_i(t^*) \\ &+ (1 - \mu\lambda)a_i(t^*) \int_0^\infty k_i(s) \int_{t^*-s}^{t^*} e_\lambda(t^*, \kappa)\mu_i^\Delta(\kappa)\Delta\kappa\Delta s \\ &+ e_\lambda(\rho(t^*), 0)\gamma_i^{-1} \left\{ \sum_{j=1}^m b_{ij}(t^*) [g_j(\nu_j(t^* - \tau_j(t^*))) - g_j(\nu_j^*(t^* - \tau_j(t^*)))] \right. \\ &+ \sum_{j=1}^m \sum_{l=1}^m e_{ijl}(t^*) [p_j(\nu_j(t^* - \tau_j(t^*)))q_l(\nu_l(t^* - \tau_l(t^*))) \\ &\left. - p_j(\nu_j^*(t^* - \tau_j(t^*)))q_l(\nu_l^*(t^* - \tau_l(t^*)))] \right\} \end{aligned}$$

$$\begin{aligned}
&\leq - \left[ (1 - \mu\lambda)a_i^- \int_0^\infty k_i(s)\Delta s - \lambda \right] \mu_i(t^*) \\
&\quad + a_i^+(1 - \mu\lambda) \int_0^\infty k_i(s) \int_{t^*-s}^{t^*} e_\lambda(t^*, \kappa) |\mu_i^\Delta(\kappa)| \Delta\kappa \Delta s \\
&\quad + e_\lambda(\rho(t^*), 0) \gamma_i^{-1} \left\{ \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j e_\lambda(0, t^* - \tau_j(t^*)) |\nu_j(t^* - \tau_j(t^*))| \right. \\
&\quad \left. + \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ [L_j^p Q_l \bar{\gamma}_j |\nu_j(t^* - \tau_j(t^*))| + L_l^q P_l \bar{\gamma}_l |\nu_l(t^* - \tau_l(t^*))|] \right\} \\
&\leq - \left[ (1 - \mu\lambda)a_i^- \int_0^\infty k_i(s)\Delta s - \lambda \right] \varrho \epsilon_i \\
&\quad + a_i^+(1 - \mu\lambda) \int_0^\infty k_i(s) \int_{t^*-s}^{t^*} e_\lambda(t^*, \kappa) \varrho \epsilon_i \Delta\kappa \Delta s \\
&\quad + e_\lambda(\rho(t^*), 0) \gamma_i^{-1} \left\{ \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j e_\lambda(0, t^* - \tau_j(t^*)) \varrho \eta_j \right. \\
&\quad \left. + \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ L_j^p Q_l \bar{\gamma}_j \varrho \eta_j + \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ L_l^q P_l \bar{\gamma}_l \varrho \eta_l \right\} \\
&= - \left[ (1 - \mu\lambda)a_i^- \int_0^\infty k_i(s)\Delta s \right. \\
&\quad \left. - \left( \lambda + (1 - \mu\lambda)a_i^+ \int_0^\infty k_i(s) \int_{t-s}^t e_\lambda(t^*, \theta) \Delta\theta \right) \right] \delta_i \\
&\quad + e_\varsigma(\rho(t), 0) \gamma_i^{-1} \left[ \sum_{j=1}^m \bar{\gamma}_j \eta_j b_{ij}^+ L_j^g e_\lambda(0, \tau_j(t^*)) \right. \\
&\quad + \sum_{j=1}^m \sum_{l=1}^m \bar{\gamma}_j \eta_j e_{ijl}^+ L_j^p Q_l e_\lambda(0, \tau_j(t^*)) \\
&\quad \left. + \sum_{j=1}^m \sum_{l=1}^m \bar{\gamma}_j \eta_j e_{ijl}^+ L_l^q P_l e_\lambda(0, \tau_l(t^*)) \right] \varrho < 0, \tag{4.15}
\end{aligned}$$

which is a contradiction. Thus (4.11) is not hold true. If (b) holds, then according to (A5), we have

$$\begin{aligned}
&0 \geq \mu_i^\Delta(t^*) \\
&= \lambda \zeta_i(t^*) - (1 - \mu\lambda)a_i(t^*) \int_0^\infty k_i(s)\Delta s \mu_i(t^*) \\
&\quad + (1 - \mu\lambda)a_i(t^*) \int_0^\infty k_i(s) \int_{t^*-s}^{t^*} e_\lambda(t^*, \kappa) \mu_i^\Delta(\kappa) \Delta\kappa \Delta s \\
&\quad + e_\lambda(\rho(t^*), 0) \gamma_i^{-1} \left\{ \sum_{j=1}^m b_{ij}(t^*) [g_j(\nu_j(t^* - \tau_j(t^*))) - g_j(\nu_j^*(t^* - \tau_j(t^*)))] \right\}
\end{aligned}$$



$$\begin{aligned}
 & + \left. \sum_{j=1}^m \sum_{l=1}^m e_{ijl}(t^*) [p_j(\nu_j(t^* - \tau_j(t^*)))q_l(\nu_l(t^* - \tau_l(t^*))) \right. \\
 & \quad \left. - p_j(\nu_j^*(t^* - \tau_j(t^*)))q_l(\nu_l^*(t - \tau_l(t^*))) \right] \Big\} \\
 & > \left[ (1 - \mu\lambda)a_i^- \int_0^\infty k_i(s)\Delta s - \lambda \right] \mu_i(t^*) \\
 & \quad - a_i^+(1 - \mu\lambda) \int_0^\infty k_i(s) \int_{t^*-s}^{t^*} e_\lambda(t^*, \kappa) |\mu_i^\Delta(\kappa)| \Delta\kappa \Delta s \\
 & \quad - e_\lambda(\rho(t^*), 0) \gamma_i^{-1} \left\{ \sum_{j=1}^m b_{ij}^+ L_j^g \tilde{\gamma}_j e_\lambda(0, t^* - \tau_j(t^*)) |\nu_j(t^* - \tau_j(t^*))| \right. \\
 & \quad \left. + \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ [L_j^p Q_l \tilde{\gamma}_j |\nu_j(t^* - \tau_j(t^*))| + L_l^q P_l \tilde{\gamma}_l |\nu_l(t^* - \tau_l(t^*))|] \right\} \\
 & \geq - \left[ (1 - \mu\lambda)a_i^- \int_0^\infty k_i(s)\Delta s \lambda \right] \varrho \epsilon_i \\
 & \quad - a_i^+(1 - \mu\lambda) \int_0^\infty k_i(s) \int_{t^*-s}^{t^*} e_\lambda(t^*, \kappa) \varrho \epsilon_i \Delta\kappa \Delta s \\
 & \quad - e_\lambda(\rho(t^*), 0) \gamma_i^{-1} \left\{ \sum_{j=1}^m b_{ij}^+ L_j^g \tilde{\gamma}_j e_\lambda(0, t^* - \tau_j(t^*)) \varrho \eta_j \right. \\
 & \quad \left. + \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ L_j^p Q_l \tilde{\gamma}_j \varrho \eta_j + \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ L_l^q P_l \tilde{\gamma}_l \varrho \eta_l \right\} \\
 & = \left[ (1 - \mu\lambda)a_i^- \int_0^\infty k_i(s)\Delta s - \left( \lambda + (1 - \mu\lambda)a_i^+ \int_0^\infty k_i(s) \int_{t-s}^t e_\lambda(t^*, \theta) \Delta\theta \right) \right] \delta_i \\
 & \quad - e_\lambda(\rho(t), 0) \gamma_i^{-1} \left[ \sum_{j=1}^m \tilde{\gamma}_j \eta_j b_{ij}^+ L_j^g e_\lambda(0, \tau_j(t^*)) + \sum_{j=1}^m \sum_{l=1}^m \tilde{\gamma}_j \eta_j e_{ijl}^+ L_j^p Q_l e_\lambda(0, \tau_j(t^*)) \right. \\
 & \quad \left. + \sum_{j=1}^m \sum_{l=1}^m \tilde{\gamma}_j \eta_j e_{ijl}^+ L_l^q P_j e_\lambda(0, \tau_l(t^*)) \right] \varrho > 0, \tag{4.16}
 \end{aligned}$$

which is also a contradiction. thus (4.11) does not hold true. If (4.12) holds, then it follows from (A5) that

$$\begin{aligned}
 \varrho \delta_i = & \mu_i^\Delta(t^*) \leq (1 - \mu\lambda)a_i(t^*) \int_0^\infty k_i(s)\Delta s \mu_i(t^*) + \lambda |\zeta_i(t^*)| \\
 & + (1 - \mu\lambda)a_i(t^*) \int_0^\infty k_i(s) \int_{t^*-s}^{t^*} e_\lambda(t^*, \kappa) |\mu_i^\Delta(\kappa)| \Delta\kappa \Delta s \\
 & + e_\lambda(\rho(t^*), 0) \gamma_i^{-1} \left\{ \sum_{j=1}^m b_{ij}(t^*) |g_j(\nu_j(t^* - \tau_j(t^*))) - g_j(\nu_j^*(t^* - \tau_j(t^*)))| \right. \\
 & \left. + \sum_{j=1}^m \sum_{l=1}^m e_{ijl}(t^*) |p_j(\nu_j(t^* - \tau_j(t^*)))q_l(\nu_l(t^* - \tau_l(t^*))) \right.
 \end{aligned}$$

$$\begin{aligned}
& \left. - p_j(\nu_j^*(t^* - \tau_j(t^*)))q_l(\nu_l^*(t - \tau_l(t^*))) \right\} \\
\leq & \left[ (1 - \mu\lambda)a_i^+ \int_0^\infty k_i(s)\Delta s + \lambda \right] \mu_i(t^*) \\
& + a_i^+(1 - \mu\lambda) \int_0^\infty k_i(s) \int_{t^*-s}^{t^*} e_\lambda(t^*, \kappa) |\mu_i^\Delta(\kappa)| \Delta\kappa \Delta s \\
& + e_\lambda(\rho(t^*), 0) \gamma_i^{-1} \left\{ \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j e_\lambda(0, t^* - \tau_j(t^*)) |\nu_j(t^* - \tau_j(t^*))| \right. \\
& \left. + \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ [L_j^p Q_l \bar{\gamma}_j |\nu_j(t^* - \tau_j(t^*))| + L_l^q P_l \bar{\gamma}_l |\nu_l(t^* - \tau_l(t^*))|] \right\} \\
\leq & \left[ (1 - \mu\lambda)a_i^+ \int_0^\infty k_i(s)\Delta s + \lambda \right] \varrho\epsilon_i \\
& + a_i^+(1 - \mu\lambda) \int_0^\infty k_i(s) \int_{t^*-s}^{t^*} e_\lambda(t^*, \kappa) \varrho\epsilon_i \Delta\kappa \Delta s \\
& + e_\lambda(\rho(t^*), 0) \gamma_i^{-1} \left\{ \sum_{j=1}^m b_{ij}^+ L_j^g \bar{\gamma}_j e_\lambda(0, t^* - \tau_j(t^*)) \varrho\eta_j \right. \\
& \left. + \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ L_j^p Q_l \bar{\gamma}_j \varrho\eta_j + \sum_{j=1}^m \sum_{l=1}^m e_{ijl}^+ L_l^q P_l \bar{\gamma}_l \varrho\eta_l \right\} \\
= & \left[ (1 - \mu\lambda)a_i^+ \int_0^\infty k_i(s)\Delta s + \left( \lambda + (1 - \mu\lambda)a_i^+ \int_0^\infty k_i(s) \int_{t-s}^t e_\lambda(t^*, \theta) \Delta\theta \right) \right] \\
& + e_\lambda(\rho(t), 0) \gamma_i^{-1} \left[ \sum_{j=1}^m \bar{\gamma}_j \eta_j b_{ij}^+ L_j^g e_\lambda(0, \tau_j(t^*)) + \sum_{j=1}^m \sum_{l=1}^m \bar{\gamma}_j \eta_j e_{ijl}^+ L_j^p Q_l e_\lambda(0, \tau_j(t^*)) \right. \\
& \left. + \sum_{j=1}^m \sum_{l=1}^m \bar{\gamma}_j \eta_j e_{ijl}^+ L_l^q P_j e_\lambda(0, \tau_l(t^*)) \right] \varrho\delta_i < \varrho\delta_i, \tag{4.17}
\end{aligned}$$

which is a contradiction. Thus (4.12) does not hold true. In a similar way, we can also prove that (4.13) and (4.14) hold true. Then we have

$$\begin{cases} |x_i(t) - x_i^*(t)| \leq e_\lambda(t, 0) \varrho\delta_i, t \in \mathbb{T}, i = 1, 2, \dots, n, \\ |y_j(t) - y_j^*(t)| \leq e_\lambda(t, 0) \varrho\eta_j, t \in \mathbb{T}, j = 1, 2, \dots, m. \end{cases} \tag{4.18}$$

Therefore the almost periodic solution of system (1.1) is exponentially stable. This completes the proof of Theorem 4.1.  $\square$

**Remark 4.1.** Jiang et al. [17] studied the almost periodic solutions for a memristor-based neural networks with leakage, time-varying and distributed delays, Li and Fan [24] investigated the existence and globally exponential stability of almost periodic solution for Cohen-Grossberg BAM neural networks with variable coefficients, Li et al. [26] studied the existence and global exponential stability of almost periodic solution for high-order BAM neural networks with delays on time scales, Wang [35] analyzed the almost periodic solutions of impulsive BAM neural networks with

variable delays on time scales, all the authors of [17, 24, 26, 35] do not deal with the neural networks with distributed leakage delays. From this viewpoint, our results are completely new and complement some previous works.

## 5. Examples

In this section, we present an example to verify the analytical predictions obtained in the previous section. Consider the following BAM neural networks with distributed leakage delays on time scales

$$\left\{ \begin{array}{l} x_1^\Delta(t) = -a_1(t) \int_0^\infty k_1(s)x_i(t-s)\Delta s + \sum_{j=1}^2 b_{1j}(t)g_j(y_j(t-\tau_j(t))) \\ \quad + \sum_{j=1}^2 \sum_{l=1}^2 e_{1jl}(t)p_j(y_j(t-\tau_j(t)))q_l(y_l(t-\tau_l(t))) + I_1(t), \\ x_2^\Delta(t) = -a_2(t) \int_0^\infty k_2(s)x_i(t-s)\Delta s + \sum_{j=1}^2 b_{2j}(t)g_j(y_j(t-\tau_j(t))) \\ \quad + \sum_{j=1}^2 \sum_{l=1}^2 e_{2jl}(t)p_j(y_j(t-\tau_j(t)))q_l(y_l(t-\tau_l(t))) + I_2(t), \\ y_1^\Delta(t) = -d_1(t) \int_0^\infty h_1(s)y_j(t-s)\Delta s + \sum_{i=1}^2 c_{1i}(t)f_i(x_i(t-\omega_i(t))) \\ \quad + \sum_{i=1}^2 \sum_{l=1}^2 s_{1il}(t)v_i(x_i(t-\omega_i(t)))w_l(x_l(t-\omega_l(t))) + J_1(t), \\ y_2^\Delta(t) = -d_2(t) \int_0^\infty h_2(s)y_j(t-s)\Delta s + \sum_{i=1}^2 c_{2i}(t)f_i(x_i(t-\omega_i(t))) \\ \quad + \sum_{i=1}^2 \sum_{l=1}^2 s_{2il}(t)v_i(x_i(t-\omega_i(t)))w_l(x_l(t-\omega_l(t))) + J_2(t), \end{array} \right. \quad (5.1)$$

where

$$\begin{aligned} \begin{bmatrix} a_1(t) & a_2(t) \\ d_1(t) & d_2(t) \end{bmatrix} &= \begin{bmatrix} 0.5 + 0.2|\sin t| & 0.4 + 0.1|\cos t| \\ 0.7 + 0.3|\sin t| & 0.5 + 0.3|\cos t| \end{bmatrix}, \\ \begin{bmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{bmatrix} &= \begin{bmatrix} 0.09 + 0.04 \sin t & 0.03 + 0.01 \cos t \\ 0.07 + 0.03|\cos t| & 0.07 + 0.04 \sin t \end{bmatrix}, \\ \begin{bmatrix} e_{111}(t) & e_{112}(t) \\ e_{121}(t) & e_{122}(t) \end{bmatrix} &= \begin{bmatrix} 0.06 + 0.03 \cos t & 0.09 + 0.04 \cos t \\ 0.08 + 0.05 \sin t & 0.09 + 0.06 \sin t \end{bmatrix}, \\ \begin{bmatrix} e_{211}(t) & e_{212}(t) \\ e_{221}(t) & e_{222}(t) \end{bmatrix} &= \begin{bmatrix} 0.08 + 0.05 \sin t & 0.08 + 0.04 \cos t \\ 0.06 + 0.01 \cos t & 0.09 + 0.05 \sin t \end{bmatrix}, \\ \begin{bmatrix} s_{111}(t) & s_{112}(t) \\ s_{121}(t) & s_{122}(t) \end{bmatrix} &= \begin{bmatrix} 0.06 + 0.03 \cos t & 0.09 + 0.04 \sin t \\ 0.08 + 0.04 \sin t & 0.09 + 0.05 \cos t \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} s_{211}(t) & s_{212}(t) \\ s_{221}(t) & s_{222}(t) \end{bmatrix} &= \begin{bmatrix} 0.08 + 0.05 \cos t & 0.09 + 0.05 \sin t \\ 0.08 + 0.02 \sin t & 0.08 + 0.05 \cos t \end{bmatrix}, \\ \begin{bmatrix} k_1(t) & k_2(t) \\ h_1(t) & h_2(t) \end{bmatrix} &= \begin{bmatrix} e^{-5t^2} & e^{-5t^2} \\ e^{-5t^2} & e^{-5t^2} \end{bmatrix}, \quad \begin{bmatrix} g_1(y_1) & g_2(y_2) \\ f_1(x_1) & f_2(x_2) \end{bmatrix} = \begin{bmatrix} |y_1| & |y_2| \\ |x_1| & |x_2| \end{bmatrix}, \\ \begin{bmatrix} \tau_1(t) & \tau_2(t) \\ \omega_1(t) & \omega_2(t) \end{bmatrix} &= \begin{bmatrix} 0.03 + 0.01 \sin t & 0.06 + 0.02 \sin t \\ 0.05 + 0.02 \sin t & 0.07 + 0.03 \sin t \end{bmatrix}, \\ \begin{bmatrix} p_1(y_1) & p_2(y_2) \\ q_1(y_1) & q_2(y_2) \end{bmatrix} &= \begin{bmatrix} |y_1| & |y_2| \\ |y_1| & |y_2| \end{bmatrix}, \quad \begin{bmatrix} v_1(x_1) & v_2(x_2) \\ w_1(x_1) & w_2(x_2) \end{bmatrix} = \begin{bmatrix} |x_1| & |x_2| \\ |x_1| & |x_2| \end{bmatrix}, \\ \begin{bmatrix} I_1(t) & I_2(t) \\ J_1(t) & J_2(t) \end{bmatrix} &= \begin{bmatrix} 0.0038 \cos t & 0.0065 \cos t \\ 0.0067 \cos t & 0.0089 \cos t \end{bmatrix}. \end{aligned}$$

Then

$$\begin{aligned} \begin{bmatrix} a_1^+ & a_2^+ \\ d_1^+ & d_2^+ \end{bmatrix} &= \begin{bmatrix} 0.7 & 0.5 \\ 1.0 & 0.8 \end{bmatrix}, \quad \begin{bmatrix} a_1^- & a_2^- \\ d_1^- & d_2^- \end{bmatrix} = \begin{bmatrix} 0.5 & 0.4 \\ 0.7 & 0.5 \end{bmatrix}, \\ \begin{bmatrix} b_{11}^+ & b_{12}^+ \\ b_{21}^+ & b_{22}^+ \end{bmatrix} &= \begin{bmatrix} 0.13 & 0.04 \\ 0.10 & 0.11 \end{bmatrix}, \quad \begin{bmatrix} e_{111}^+ & e_{112}^+ \\ e_{121}^+ & e_{122}^+ \end{bmatrix} = \begin{bmatrix} 0.09 & 0.13 \\ 0.13 & 0.15 \end{bmatrix}, \\ \begin{bmatrix} e_{211}^+ & e_{212}^+ \\ e_{221}^+ & e_{222}^+ \end{bmatrix} &= \begin{bmatrix} 0.13 & 0.12 \\ 0.07 & 0.14 \end{bmatrix}, \quad \begin{bmatrix} s_{111}^+ & s_{112}^+ \\ s_{121}^+ & s_{122}^+ \end{bmatrix} = \begin{bmatrix} 0.09 & 0.13 \\ 0.12 & 0.14 \end{bmatrix}, \\ \begin{bmatrix} s_{211}^+ & s_{212}^+ \\ s_{221}^+ & s_{222}^+ \end{bmatrix} &= \begin{bmatrix} 0.13 & 0.14 \\ 0.10 & 0.13 \end{bmatrix}, \quad \begin{bmatrix} L_1^g & L_2^g \\ L_1^f & L_2^f \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}, \\ \begin{bmatrix} L_1^p & l_2^p \\ L_1^q & l_2^q \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \\ \begin{bmatrix} L_1^v & L_1^v \\ L_1^w & L_2^w \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} V_1 & V_2 \\ W_1 & W_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

Here we consider the case  $\mathbb{T} = \mathbb{R}$ . Let  $\gamma_1 = \bar{\gamma} = \bar{\gamma}_1 = \bar{\gamma}_2 = \delta_1 = \delta_2 = \varsigma_1 = \eta_2 = 1$ ,  $\lambda = 0.0000005$  and  $\rho(t) = t, \mu(t) = 0$ . Then we have

$$\begin{aligned} &\max \left\{ \max_{1 \leq i \leq 2} \left\{ \frac{\Theta_i}{a_i^- \int_0^\infty k_i(u) \Delta u}, \left( 1 + \frac{a_i^+}{a_i^-} \right) \Theta_i \right\}, \right. \\ &\max_{1 \leq j \leq 2} \left\{ \frac{\Pi_j}{d_j^- \int_0^\infty h_j(u) \Delta u}, \left( 1 + \frac{d_j^+}{d_j^-} \right) \Pi_j \right\} \left. \right\} \approx 0.6099 \leq 1, \\ &\max \{ \max_{1 \leq i \leq 2} \{ \epsilon_{i1}, \epsilon_{i2} \}, \max_{1 \leq i \leq 2} \{ \varrho_{j1}, \varrho_{j2} \} \} \approx 0.3675 < 1, \\ &- \left[ a_1^- \int_0^\infty k_1(s) \Delta s - a_1^+ \int_0^\infty k_1(s) \Delta s \right] \delta_1 \end{aligned}$$

$$\begin{aligned}
& +\gamma_1^{-1} \left[ \sum_{j=1}^2 \bar{\gamma}_j \eta_j b_{1j}^+ L_j^g + \sum_{j=1}^2 \sum_{l=1}^2 \bar{\gamma}_j \eta_j e_{1jl}^+ (L_j^p Q_l + L_l^q P_j) \right] \approx -0.7038 < 0, \\
& - \left[ a_2^- \int_0^\infty k_2(s) \Delta s - a_2^+ \int_0^\infty k_2(s) \Delta s \right] \delta_2 \\
& +\gamma_2^{-1} \left[ \sum_{j=1}^2 \bar{\gamma}_j \eta_j b_{2j}^+ L_j^g + \sum_{j=1}^2 \sum_{l=1}^2 \bar{\gamma}_j \eta_j e_{2jl}^+ (L_j^p Q_l + L_l^q P_j) \right] \approx -0.5946 < 0, \\
& - \left[ d_1^- \int_0^\infty h_1(s) \Delta s - d_1^+ \int_0^\infty h_1(s) \Delta s \right] \eta_1 \\
& +\bar{\gamma}_1^{-1} \left[ \sum_{i=1}^2 \gamma_i \delta_i c_{1i}^+ L_i^f + \sum_{i=1}^2 \sum_{l=1}^2 \gamma_i \delta_i s_{1il}^+ (L_i^v W_l + L_l^w V_i) \right] \approx -0.8301 < 0, \\
& - \left[ d_2^- \int_0^\infty h_2(s) \Delta s - d_2^+ \int_0^\infty h_2(s) \Delta s \right] \eta_2 \\
& +\bar{\gamma}_2^{-1} \left[ \sum_{i=1}^2 \gamma_i \delta_i c_{2i}^+ L_i^f + \sum_{i=1}^2 \sum_{l=1}^2 \gamma_i \delta_i s_{2il}^+ (L_i^v W_l + L_l^w V_i) \right] \approx -0.9126 < 0, \\
& \left[ (1 - \mu\lambda) a_1^- \int_0^\infty k_1(s) \Delta s + \left( \lambda + (1 - \mu\lambda) a_1^+ \int_0^\infty k_1(s) \int_{t-s}^t e_\lambda(t^*, \theta) \Delta \theta \right) \right] \delta_1 \\
& + e_\varsigma(\rho(t), 0) \gamma_1^{-1} \left[ \sum_{j=1}^2 \bar{\gamma}_j \eta_j b_{1j}^+ L_j^g e_\lambda(0, \tau_j(t^*)) + \sum_{j=1}^2 \sum_{l=1}^2 \bar{\gamma}_j \eta_j e_{1jl}^+ L_j^p Q_l e_\lambda(0, \tau_j(t^*)) \right. \\
& \left. + \sum_{j=1}^2 \sum_{l=1}^2 \bar{\gamma}_j \eta_j e_{1jl}^+ L_l^q P_j e_\lambda(0, \tau_l(t^*)) \right] \approx 0.7993 < 1, \\
& \left[ (1 - \mu\lambda) a_2^- \int_0^\infty k_2(s) \Delta s + \left( \lambda + (1 - \mu\lambda) a_2^+ \int_0^\infty k_2(s) \int_{t-s}^t e_\lambda(t^*, \theta) \Delta \theta \right) \right] \delta_2 \\
& + e_\varsigma(\rho(t), 0) \gamma_2^{-1} \left[ \sum_{j=1}^2 \bar{\gamma}_j \eta_j b_{2j}^+ L_j^g e_\lambda(0, \tau_j(t^*)) + \sum_{j=1}^2 \sum_{l=1}^2 \bar{\gamma}_j \eta_j e_{2jl}^+ L_j^p Q_l e_\lambda(0, \tau_j(t^*)) \right. \\
& \left. + \sum_{j=1}^2 \sum_{l=1}^2 \bar{\gamma}_j \eta_j e_{2jl}^+ L_l^q P_j e_\lambda(0, \tau_l(t^*)) \right] \approx 0.8854 < 1, \\
& \left[ (1 - \mu\lambda) d_1^- \int_0^\infty h_1(s) \Delta s + \left( \lambda + (1 - \mu\lambda) d_1^+ \int_0^\infty h_1(s) \int_{t-s}^t e_\lambda(t^*, \theta) \Delta \theta \right) \right] \eta_1 \\
& + e_\varsigma(\rho(t), 0) \bar{\gamma}_1^{-1} \left[ \sum_{i=1}^2 \gamma_i \delta_i c_{1i}^+ L_i^f e_\lambda(0, \omega_i(t^*)) + \sum_{i=1}^2 \sum_{l=1}^2 \gamma_i \delta_i s_{1il}^+ L_i^v W_l e_\lambda(0, \omega_i(t^*)) \right. \\
& \left. + \sum_{i=1}^2 \sum_{l=1}^2 \gamma_i \delta_i s_{1il}^+ L_l^w V_i e_\lambda(0, \omega_l(t^*)) \right] \approx 0.5093 < 1, \\
& \left[ (1 - \mu\lambda) d_2^- \int_0^\infty h_2(s) \Delta s + \left( \lambda + (1 - \mu\lambda) d_2^+ \int_0^\infty h_2(s) \int_{t-s}^t e_\lambda(t^*, \theta) \Delta \theta \right) \right] \eta_2 \\
& + e_\varsigma(\rho(t), 0) \bar{\gamma}_2^{-1} \left[ \sum_{i=1}^2 \gamma_i \delta_i c_{2i}^+ L_i^f e_\lambda(0, \omega_i(t^*)) + \sum_{i=1}^2 \sum_{l=1}^2 \gamma_i \delta_i s_{2il}^+ L_i^v W_l e_\lambda(0, \omega_i(t^*)) \right. \\
& \left. + \sum_{i=1}^2 \sum_{l=1}^2 \gamma_i \delta_i s_{2il}^+ L_l^w V_i e_\lambda(0, \omega_l(t^*)) \right]
\end{aligned}$$

$$\left. + \sum_{i=1}^2 \sum_{l=1}^2 \gamma_i \delta_i s_{2il}^+ L_l^w V_i e_\lambda(0, \omega_l(t^*)) \right] \approx 0.6887 < 1.$$

It is easy to check that all the conditions in Theorem 4.1 are satisfied. Then we can conclude that system (5.1) has exactly one almost periodic solution which is exponentially stable.

## 6. Conclusions

In the present paper, we consider a class of BAM neural networks with distributed leakage delays on time scales. With the aid of the exponential dichotomy of linear differential equations, Lapunov functional method and contraction mapping principle, some sufficient conditions are derived to ensure the existence and exponential stability of almost periodic solutions for such class of BAM neural networks. An example is given to illustrate the effectiveness of our theoretical findings. The results obtained in this paper are completely new and complement the previously known ones and the method used in this paper can be applied to handle numerous cellular neural networks, Hopfield neural networks and so on. Recently, pseudo almost periodic and anti-periodic solutions of neural networks have also been paid more attention by many authors. However, there are rare results on pseudo almost periodic and anti-periodic solution of BAM neural networks with distributed leakage delays, which might be our future research topic.

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