Journal of Applied Analysis and Computation
Volume 7, Number 3, August 2017, 884–898
Website:http://jaac-online.com/
DOI:10.11948/2017056

REGULARITY OF PULLBACK ATTRACTORS FOR NON-AUTONOMOUS STOCHASTIC COUPLED REACTION-DIFFUSION SYSTEMS*

Jinyan Yin[1], Yangrong Li[1] and Anhui Gu[1,†]

Abstract We consider the dynamical behavior of the typical non-autonomous autocatalytic stochastic coupled reaction-diffusion systems on the entire space $\mathbb{R}^n$. Some new uniform asymptotic estimates are implemented to investigate the existence of pullback attractors in the Sobolev space $H^1(\mathbb{R}^n)^3$ for the three-component reversible Gray-Scott system.

Keywords Non-autonomous cocycle, bi-spatial pullback attractor, regularity of attractor, three-component reversible Gray-Scott system.

MSC(2010) 37L55, 35R60, 60H15, 35B40, 35K57.

1. Introduction

This paper is to study the asymptotic behavior of solutions for the following non-autonomous stochastic three-component reversible Gray-Scott system with multiplicative noise defined on the entire space $\mathbb{R}^n$:

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t} = d_1\Delta\tilde{u} - (F+k)\tilde{u} + \tilde{u}^2\tilde{v} - G\tilde{u}^3 + N\tilde{w} + g_1(t,x) + \sigma\tilde{u}\circ\frac{d\omega}{dt}, \\ \frac{\partial \tilde{v}}{\partial t} = d_2\Delta\tilde{v} - F\tilde{v} - \tilde{u}^2\tilde{v} + G\tilde{u}^3 + g_2(t,x) + \sigma\tilde{v}\circ\frac{d\omega}{dt}, \\ \frac{\partial \tilde{w}}{\partial t} = d_3\Delta\tilde{w} - (F+N)\tilde{w} + k\tilde{u} + g_3(t,x) + \sigma\tilde{w}\circ\frac{d\omega}{dt}, \end{cases} \tag{1.1}$$

with initial data

$$(\tilde{u}(\tau,x), \tilde{v}(\tau,x), \tilde{w}(\tau,x)) = (\tilde{u}_\tau, \tilde{v}_\tau, \tilde{w}_\tau), \quad x\in\mathbb{R}^n, \tag{1.2}$$

where all given parameters are positive constants, g_i $(i=1,2,3)$ are external forces satisfying some certain integrable conditions, ω is a two-sided real-valued Wiener process and $\circ$ denotes the Stratonovich sense of the stochastic term.

Historically, the two-component Gray-Scott system was originated from describing some isothermal, cubic autocatalytic, continuously fed and diffusive reactions of two chemicals, which was signified one of the Brussels school led by the renowned physical chemist and Nobel Prize laureate (1977), Ilya Prigogine, see [9, 10, 17, 19].

† the corresponding author. Email address:gahui@swu.edu.cn(A. Gu)
[1] School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
* This work is supported by NSF of China (11571283), the Research Program of Frontier and Applied Basic Research of Chongqing (cstc2015jcyjA00018), the Fundamental Research Funds for the Central Universities (SWU115092, XDJK2016C168) and the Postgraduate Research and Innovation Project of Chongqing (CYB2015057).

The three-component reversible Gray-Scott model, first introduced by H. Mahara et.al., was based on the scheme of two reversible chemical or biochemical reactions, see [16]. Recently, You [23] took some non-dimensional transformations to make the three-component model reduce to the following deterministic system without the non-autonomous forces:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1\Delta u - (F+k)u + u^2 v - Gu^3 + Nw, \\ \frac{\partial v}{\partial t} = d_2\Delta v + F(1-v) - u^2 v + Gu^3, \\ \frac{\partial w}{\partial t} = d_3\Delta w - (F+N)w + ku. \end{cases} \tag{1.3}$$

By the re-scaling and grouping estimates, the existence or robustness (i.e. upper semi-continuity) of global attractors for system (1.3) was considered in [8, 23, 24] on a bounded domain with space dimension $n \leq 3$, so did the stochastic model in [25, 26].

However, as pointed out in [27], the study of the same or similar coupled reaction-diffusion system on a higher space dimensional domain $n > 3$ whether bounded or unbound, to the best of our knowledge, is still open. First of all, we cannot use the Sobolev embedding $H^1 \hookrightarrow L^6$ any more. Next, when dealing with such system consisting of two or more equations, the common difficulty is that the following sign-preserving property of the nonlinearity $f(s)$ in vector version is not satisfied:

$$\limsup_{|s|\to+\infty} f(s)s \leq c, \quad \text{for } c \geq 0.$$

This plays a key role in the theory of attractors, since it prevents the existing method of tackling the asymptotic behavior even for the classical reaction-diffusion equation. Once again, the approach of constructing positively invariant regions does not work since we do not assume solutions are nonnegative. Moreover, even if we wish, the positively invariant region method is hard to implement for such multiple reaction-diffusion equations.

Stochastic effects taken into consideration are of central importance for the development of mathematical models of complex phenomena under uncertainty arising in applications. Such random influences are not just compensation of defects in deterministic models but intrinsic phenomena. Then pullback random attractors were introduced for these random dynamical systems, see [4, 11, 18] with some applications in [2, 3, 20]. When system (1.3) perturbed by white noise in a bounded domain, the existence or upper semi-continuity of random attractors was investigated in [25, 26]. When dealing with the unbounded cases, the constant F presenting in the second equation of (1.3) will cause a new insuperable obstacle for estimating solutions. However, we can overcome this difficulty by affiliating F with the spatial variable x such that $F(x) \in L^2(\mathbb{R}^n) \cap L^6(\mathbb{R}^n)$ or just letting it in the external force $g_2(\cdot, x)$, then system (1.1) follows. In particular, the existence of random attractors in $L^2(\mathbb{R}^n)^3$ for autonomous system (1.1) was shown in [6, 7]. As far as we know, there are few results on the regularity of random attractors for the multi-component stochastic systems.

In this paper, we study the regularity of pullback attractors for non-autonomous system (1.1). Such regularity of random attractors seems to be first investigated in [12] and then extended to [13, 14, 28]. In fact, for a single deterministic reaction-diffusion equation, an H^1-attractor can be achieved by differentiating on both sides of the equation and making some additional assumptions for the derivation of the external force, see [5, 15, 21, 22, 29]. Unfortunately, this method cannot be generalized to the stochastic case because the Wiener process is almost everywhere non-differentiatial. We will obtain pullback attractors in $H^1(\mathbb{R}^n)^3$ by applying some new Gronwall inequalities and uniform absorptions both in $L^6(\mathbb{R}^n)^3$ and in $H^1(\mathbb{R}^n)^3$ respectively. These estimates are essential to present the asymptotic compactness inside a ball under the norm of $(H^1)^3$, while the tail estimate outside a ball will be achieved by using a cut-off technique. We remark here that our method is different from [14] of stochastic Fitzhugh-Nagumo system, in which the integral of L^6-norm can be proved through the nonlinearity and then the truncation estimates follow. This is why we claim initial data are in $L^2(\mathbb{R}^n)^3 \cap L^6(\mathbb{R}^n)^3$, even for the existence of attractors in $L^2(\mathbb{R}^n)^3$ when space dimension $n > 3$.

We begin this paper with a brief summary in Section 2 of the standard (non-autonomous) cocycle framework in which the theory of the regularity of pullback random attractors can be developed. In Section 3, we use the unique solution for system (1.1) to define a continuous cocycle. We then review in Section 4 some known results of solutions from [7] with some generalizations. In Sections 5-7, we derive uniform absorptions, tail estimates outside a ball and asymptotic estimates inside a ball respectively. We prove the existence of bi-spatial pullback attractors in the last section.

2. Preliminaries

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two separable Banach spaces and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ a metric dynamical system, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$ with the Borel σ-algebra $\mathcal{F}$ induced by the compact open topology of Ω and with the corresponding Wiener measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, and the group is defined by $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$ for all $(t, \omega) \in \mathbb{R} \times \Omega$, see [1].

Definition 2.1 ([20])**.** A (non-autonomous) cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a measurable mapping

$$\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X, \quad (t, \tau, \omega, x) \to \Phi(t, \tau, \omega)x$$

such that $\Phi(0, \tau, \omega) = id$ on X and

$$\Phi(t + s, \tau, \omega) = \Phi(t, \tau + s, \theta_s\omega)\Phi(s, \tau, \omega), \quad \text{for all } t, s \in \mathbb{R}^+, \tau \in \mathbb{R} \text{ and } \omega \in \Omega.$$

Moreover, Φ is continuous if $\Phi : x \to \Phi(t, \tau, \omega)x$ is continuous in X for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

Definition 2.2. The pair (X, Y) is limit-identical if

$$x_n \in X \cap Y \text{ and } \|x_n - x_0\|_X + \|x_n - y_0\|_Y \to 0 \text{ as } n \to +\infty, \text{ then } x_0 = y_0 \in X \cap Y.$$

In the sequel, $\mathfrak{D}$ always denotes a universe of some set-valued mappings from $\mathbb{R} \times \Omega$ to $2^X \setminus \{\emptyset\}$, and Φ is a continuous cocycle on X over $\mathbb{R} \times \Omega$ such that

$$\Phi(t, \tau, \omega)X \subset Y, \quad \text{for all } t > 0, \tau \in \mathbb{R} \text{ and } \omega \in \Omega. \tag{2.1}$$

Definition 2.3. A set-valued mapping $\mathcal{A} : \mathbb{R} \times \Omega \to 2^{X\cap Y} \setminus \{\emptyset\}$ is a $\mathfrak{D}$-pullback (X, Y)-attractor for the cocycle Φ enjoying (2.1) if

(i) There is a $\mathcal{K} \in \mathfrak{D}$ such that $\mathcal{A}(\tau,\omega) \subset \mathcal{K}(\tau,\omega)$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$;
(ii) $\omega \to d_X(x, \mathcal{A}(\tau,\omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for every $x \in X$ and $\tau \in \mathbb{R}$;
(iii) $\mathcal{A}$ is invariant, i.e. $\Phi(t,\tau,\omega)\mathcal{A}(\tau,\omega) = \mathcal{A}(t+\tau, \theta_t\omega)$;
(iv) $\mathcal{A}(\tau,\omega)$ is compact and $\mathcal{A}$ attracts every element in $\mathfrak{D}$ under the topology of Y, i.e. for each $\mathcal{D} \in \mathfrak{D}$,

$$\lim_{t\to+\infty} \mathrm{dist}_Y(\Phi(t,\tau-t,\theta_{-t}\omega)\mathcal{D}(\tau-t,\theta_{-t}\omega), \mathcal{A}(\tau,\omega)) = 0,$$

where $\mathrm{dist}_Y(A,B) := \sup_{a\in A}\inf_{b\in B}\|a-b\|_Y$ is the Hausdorff semi-distance in Y.

Definition 2.4. A set-valued mapping $\mathcal{K} : \mathbb{R}\times\Omega \to 2^X \setminus \{\emptyset\}$ is a $\mathfrak{D}$-pullback absorbing set for the cocycle Φ if for each $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\mathcal{D} \in \mathfrak{D}$, there is $T := T(\tau,\omega,\mathcal{D}) > 0$ such that

$$\bigcup_{t\geq T} \Phi(t,\tau-t,\theta_{-t}\omega)\mathcal{D}(\tau-t,\theta_{-t}\omega) \subset \mathcal{K}(\tau,\omega).$$

Definition 2.5. A cocycle Φ is $\mathfrak{D}$-pullback limit-set compact in Y if for each $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\mathcal{D} \in \mathfrak{D}$,

$$\lim_{T\to+\infty} \kappa_Y(\bigcup_{t\geq T} \Phi(t,\tau-t,\theta_{-t}\omega)\mathcal{D}(\tau-t,\theta_{-t}\omega)) = 0, \tag{2.2}$$

where $\kappa_Y(A)$ $(A \subset Y)$ denotes the (non-compact) Kuratowski measure in Y defined by

$$\kappa_Y(A) = \inf\{d > 0 \mid A \text{ has a finite cover } \{A_i\} \text{ with } \mathrm{diam}_Y(A_i) \leq d\}.$$

Similarly, a cocycle Φ is $\mathfrak{D}$-pullback limit-set compact in X if (2.2) holds for $\kappa_X(\cdot)$.

Note that $A \subset Y$, then $\kappa_Y(A) < +\infty$ iff A is bounded, $\kappa_Y(A) = 0$ iff A is pre-compact, so is a set $B \subset X$, see [12, 29].

Also, the limit-identity ensures that $(X\cap Y, \|\cdot\|_{X\cap Y})$ is a Banach space, see [13].

Proposition 2.1 ([14])**.** *Let $\mathfrak{D}$ be a universe in X, (X,Y) a limit-identical pair of Banach spaces and Φ a continuous cocycle on X over $\mathbb{R}\times\Omega$ and take values in Y, then Φ has a $\mathfrak{D}$-pullback (X,Y)-attractor if*
(i) Φ has a closed, measurable absorbing set $\mathcal{K} \in \mathfrak{D}$, and
(ii) Φ is $\mathfrak{D}$-pullback limit-set compact both in X and in Y.

3. The cocycles and main results

We will establish the existence of bi-spatial pullback attractors for system (1.1). To do this, for given $\omega \in \Omega$, let $z(t,\omega) = e^{-\sigma\omega(t)}$, then z solves the following stochastic equation in the sense of Stratonovich integration:

$$\frac{dz}{dt} + \sigma z \circ \frac{d\omega}{dt} = 0.$$

By [1], we know that $t \to z(t,\theta_t\omega)$ is continuous for each $\omega \in \Omega$ and $|z(t,\omega)|$ is tempered. Therefore, we make the following variable transformations:

$$(u(t,\omega), v(t,\omega), w(t,\omega)) = z(t,\omega)(\tilde{u}(t,\omega), \tilde{v}(t,\omega), \tilde{w}(t,\omega)),$$

then system (1.1) can be written as

$$\begin{cases} \frac{\partial u}{\partial t} = d_1\Delta u - (F+k)u + z^{-2}(t,\omega)u^2v - Gz^{-2}(t,\omega)u^3 + Nw \\ \qquad +z(t,\omega)g_1(t,x), \\ \frac{\partial v}{\partial t} = d_2\Delta v - Fv - z^{-2}(t,\omega)u^2v + Gz^{-2}(t,\omega)u^3 + z(t,\omega)g_2(t,x), \\ \frac{\partial w}{\partial t} = d_3\Delta w - (F+N)w + ku + z(t,\omega)g_3(t,x), \end{cases} \tag{3.1}$$

with initial data $(u(\tau,\omega), v(\tau,\omega), w(\tau,\omega)) = (u_\tau, v_\tau, w_\tau) = z(\tau,\omega)(\tilde{u}_\tau, \tilde{v}_\tau, \tilde{w}_\tau)$, $\tau \in \mathbb{R}$.

From now on, let $\mathbb{H} = L^2(\mathbb{R}^n)^3$, $\mathbb{U} = L^6(\mathbb{R}^n)^3$, $X = \mathbb{H} \cap \mathbb{U}$ and $Y = H^1(\mathbb{R}^n)^3$. The external force $g(t,x) := (g_1(t,x), g_2(t,x), g_3(t,x))$ belongs to $L^2_{loc}(\mathbb{R},\mathbb{H}) \cap L^6_{loc}(\mathbb{R},\mathbb{U})$ and satisfies the following integrable conditions: for all $\tau \in \mathbb{R}$,

$$\int_{-\infty}^{0} e^{Fs}(\|g(s+\tau,\cdot)\|^2_{\mathbb{H}} + \|g(s+\tau,\cdot)\|^6_{\mathbb{U}})ds < +\infty. \tag{3.2}$$

By the Galerkin approximation, similar to the autonomous system (1.3) considered in [24], for each $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $(u_\tau, v_\tau, w_\tau) \in X$, system (3.1) has a unique solution such that $(u(\cdot,\tau,\omega,u_\tau), v(\cdot,\tau,\omega,v_\tau), w(\cdot,\tau,\omega,w_\tau)) \in C([\tau,\infty), X) \cap L^2((\tau,\infty), Y)$. The unique solution generates two continuous cocycles $\Phi, \tilde{\Phi} : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$, for all $(t,\tau,\omega) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega$ and $(u_\tau, v_\tau, w_\tau) = z(\tau,\omega)(\tilde{u}_\tau, \tilde{v}_\tau, \tilde{w}_\tau) \in X$,

$$\begin{aligned} \Phi(t,\tau,\omega)(u_\tau, v_\tau, w_\tau) :=& (u(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau), v(t+\tau,\tau,\theta_{-\tau}\omega,v_\tau), \\ & w(t+\tau,\tau,\theta_{-\tau}\omega,w_\tau)), \\ \tilde{\Phi}(t,\tau,\omega)(\tilde{u}_\tau, \tilde{v}_\tau, \tilde{w}_\tau) :=& z^{-1}(t,\omega)\Phi(t,\tau,\omega)(u_\tau, v_\tau, w_\tau). \end{aligned} \tag{3.3}$$

We remark that both cocycles Φ and $\tilde{\Phi}$ are (topologically) conjugated, in what follows, we only need to consider the cocycle Φ associated with system (3.1). Moreover, $\mathfrak{D}$ is always a universe whose elements $\mathcal{D}$ satisfy, for $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\mathcal{D}(\tau,\omega) \subset X \text{ and } \lim_{t\to+\infty} e^{-\delta t}(\|\mathcal{D}(\tau-t,\theta_{-t}\omega)\|^2_{\mathbb{H}} + \|\mathcal{D}(\tau-t,\theta_{-t}\omega)\|^6_{\mathbb{U}}) = 0, \tag{3.4}$$

where $\delta \in (0,F)$ and $\|D\|_X = \sup_{u\in D}\|u\|_X$.

The main result of this paper is stated as follows:

Theorem 3.1. *Assume* (3.2) *holds, the (non-autonomous) cocycle* Φ *generated by system* (3.1) *has a unique* $\mathfrak{D}$*-pullback bi-spatial attractor* $\mathcal{A}$ *with the initial space* $X = L^2(\mathbb{R}^n)^3 \cap L^6(\mathbb{R}^n)^3$ *and the non-initial space* $Y = H^1(\mathbb{R}^n)^3$*. In particular, the cocycle* $\tilde{\Phi}$ *induced by system* (1.1) *possesses a unique* $\mathfrak{D}$*-pullback* (X,Y)*-attractor* $\tilde{\mathcal{A}}$*.*

In the sequel, we prove Theorem 3.1 by using Proposition 2.1. Very similar to the autonomous system discussed in [7], we can prove that the condition (i) in Proposition 2.1 holds and that $\widetilde{\Phi}$ is $\mathfrak{D}$-pullback limit-set compact in $\mathbb{H} = L^2(\mathbb{R}^n)^3$. We remark here that the universe $\mathfrak{D}$ in [7], actually, is a collection of all tempered subsets in $X = L^2(\mathbb{R}^n)^3 \cap L^6(\mathbb{R}^n)^3$ rather than in $\mathbb{H}$, see [7, Proposition 4.2]. Then, our main work is to prove the $\mathfrak{D}$-pullback limit-set compactness of $\widetilde{\Phi}$ in $Y = H^1(\mathbb{R}^n)^3$. To do this, we prove $\widetilde{\Phi}$ is asymptotically small in Y by using a cut-off technique outside a large ball and $\widetilde{\Phi}$ is $\mathfrak{D}$-pullback limit-set compact in Y by applying some uniform absorption estimates to show the flattening property inside a ball.

4. Some auxiliary lemmas

We need the following Gronwall-type inequalities [14, Lemma 3.3], which will be used frequently in this paper.

Lemma 4.1. *Let y, y_1 and y_2 be three nonnegative and locally integrable functions on $\mathbb{R}$ such that $\frac{dy}{ds}$ is also locally integrable and*

$$\frac{dy}{ds} + ay(s) + y_1(s) \leq y_2(s), \;\; s \in \mathbb{R},$$

where the constant $a \geq 0$. If $\tau \in \mathbb{R}$ and $\mu > 0$, then

$$\sup_{s\in[\tau-\mu,\tau]} y(s) \leq \frac{e^{-a\mu}}{\mu}\int_{\tau-3\mu}^{\tau} y(r)dr + \int_{\tau-3\mu}^{\tau} y_2(r)dr, \tag{4.1}$$

$$\int_{\tau-\mu}^{\tau} y_1(r)dr \leq \frac{e^{-a\mu}}{\mu}\int_{\tau-3\mu}^{\tau} y(r)dr + \int_{\tau-3\mu}^{\tau} y_2(r)dr, \tag{4.2}$$

$$y(\tau) \leq \frac{1}{\mu}\int_{\tau-\mu}^{\tau} e^{a(r-\tau)}y(r)dr + \int_{\tau-\mu}^{\tau} e^{a(r-\tau)}y_2(r)dr. \tag{4.3}$$

In the next sections, we always denote

$$(u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}), v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}), w(s,\tau-t,\theta_{-\tau}\omega,w_{\tau-t})), \tag{4.4}$$

for $s \in [\tau-t,\tau]$, $t \geq 0, \tau \in \mathbb{R}$ and $(u_{\tau-t}, v_{\tau-t}, w_{\tau-t}) \in X$. Sometimes, write (4.4) as $u(\text{or } v,w)(s,\tau-t,\theta_{-\tau}\omega)$, $u(\text{or } v,w)(s,\tau-t)$ or $u(\text{or } v,w)(s)$ even $u(\text{or } v,w)$ if no confusions. Then we replace ω by $\theta_{-\tau}\omega$ in system (3.1), and find that (4.4) are solutions of the following system:

$$\begin{cases} u_s = d_1\Delta u - (F+k)u + z^{-2}(s,\theta_{-\tau}\omega)u^2v - Gz^{-2}(s,\theta_{-\tau}\omega)u^3 \\ \qquad +Nw + z(s,\theta_{-\tau}\omega)g_1(s,x), \\ v_s = d_2\Delta v - Fv - z^{-2}(s,\theta_{-\tau}\omega)u^2v + Gz^{-2}(s,\theta_{-\tau}\omega)u^3 \\ \qquad +z(s,\theta_{-\tau}\omega)g_2(s,x), \\ w_s = d_3\Delta w - (F+N)w + ku + z(s,\theta_{-\tau}\omega)g_3(s,x). \end{cases} \tag{4.5}$$

Similar to [7], the following useful estimates can be proved with some slight modifications.

Lemma 4.2. *For every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\mathcal{D} \in \mathfrak{D}$, there exists $T := T(\tau,\omega,\mathcal{D}) \geq 3$ such that for all $t \geq T$ and $(u_{\tau-t}, v_{\tau-t}, w_{\tau-t}) \in \mathcal{D}(\tau-t,\theta_{-t}\omega)$,*

$$\|u(\tau)\|^2 + \|v(\tau)\|^2 + \|w(\tau)\|^2 \leq R_1(\tau,\omega), \tag{4.6}$$

$$\|u(\tau)\|_6^6 + \|v(\tau)\|_6^6 + \|w(\tau)\|_6^6 \leq R_2(\tau,\omega), \tag{4.7}$$

and

$$\int_{\tau-t}^{\tau} e^{F(s-\tau)}(\|u(s)\|_{H^1}^2 + \|v(s)\|_{H^1}^2 + \|w(s)\|_{H^1}^2)ds \leq R_1(\tau,\omega),$$

$$\int_{\tau-t}^{\tau} e^{F(s-\tau)}(\|u(s)\|_6^6 + \|v(s)\|_6^6 + \|w(s)\|_6^6)ds \leq R_2(\tau,\omega),$$

where $R_1(\tau,\omega) := M + Mz^{-2}(-\tau,\omega)\int_{-\infty}^0 e^{Fs}z^2(s,\omega)\|g(s+\tau,\cdot)\|^2ds$, $R_2(\tau,\omega) := M+Mz^{-6}(-\tau,\omega)\int_{-\infty}^0 e^{Fs}z^6(s,\omega)\|g(s+\tau,\cdot)\|_{\mathbb{U}}^6ds$ *and* $M := M(F,G,N,k,d_1,d_2,d_3)$. *In particular, we have*

$$\int_{\tau-3}^{\tau}(\|u(s)\|_{H^1}^2 + \|v(s)\|_{H^1}^2 + \|w(s)\|_{H^1}^2)ds \le R_1(\tau,\omega), \tag{4.8}$$

$$\int_{\tau-3}^{\tau}(\|u(s)\|_6^6 + \|v(s)\|_6^6 + \|w(s)\|_6^6)ds \le R_2(\tau,\omega). \tag{4.9}$$

Lemma 4.3. *For every* $\tau\in\mathbb{R}$, $\omega\in\Omega$ *and* $\mathcal{D}\in\mathfrak{D}$, *there exists* $T := T(\tau,\omega,\mathcal{D}) \ge 3$ *such that for all* $t \ge T$ *and* $(u_{\tau-t}, v_{\tau-t}, w_{\tau-t}) \in \mathcal{D}(\tau-t,\theta_{-t}\omega)$,

$$\|\nabla u(\tau)\|^2 + \|\nabla v(\tau)\|^2 + \|\nabla w(\tau)\|^2 \le R_3(\tau,\omega),$$

where $R_3(\tau,\omega) := cR_1(\tau,\omega)+cz^4(-\tau,\omega)R_2(\tau,\omega)$ *with* $R_1(\tau,\omega)$ *and* $R_2(\tau,\omega)$ *defined in Lemma 4.2.*

Lemma 4.4. *For* $\tau\in\mathbb{R}$, $\omega\in\Omega$ *and* $\mathcal{D}\in\mathfrak{D}$ *are fixed, then for each* $\varepsilon > 0$, *there exist* $T := T(\varepsilon) \ge 3$ *and* $K := K(\varepsilon) \ge 1$ *such that for all* $t \ge T$ *and* $(u_{\tau-t}, v_{\tau-t}, w_{\tau-t}) \in \mathcal{D}(\tau-t,\theta_{-t}\omega)$,

$$\int_{|x|\ge K}(u^2(\tau) + v^2(\tau) + w^2(\tau))dx < \varepsilon, \tag{4.10}$$

$$\int_{|x|\ge K}(u^6(\tau) + v^6(\tau) + w^6(\tau))dx < \varepsilon, \tag{4.11}$$

and

$$\begin{aligned}&\int_{\tau-1}^{\tau}\int_{|x|\ge K}(u^2(s) + v^2(s) + w^2(s) + |\nabla u(s)|^2 + |\nabla v(s)|^2 + |\nabla w(s)|^2)dxds\\ &+\int_{\tau-1}^{\tau}\int_{|x|\ge K}(u^6(s) + v^6(s))dxds < \varepsilon.\end{aligned} \tag{4.12}$$

5. Uniform absorptions both in $\mathbb{U}$ and Y

This section establishes some uniform absorptions when the time belongs to a compact interval, which are vital to show the $\mathfrak{D}$-pullback limit-set compactness in Y. We always denote c and $C := C(\omega)$ by a generic constant and a generic random variable respectively, which may alter values everywhere.

Lemma 5.1. *For every* $\tau\in\mathbb{R}$, $\omega\in\Omega$ *and* $\mathcal{D}\in\mathfrak{D}$, *there exist* $T := T(\tau,\omega,\mathcal{D}) \ge 3$ *and a finite function* $R_4 : \mathbb{R}\times\Omega \to (0,+\infty)$ *such that for all* $t \ge T$ *and* $(u_{\tau-t}, v_{\tau-t}, w_{\tau-t}) \in \mathcal{D}(\tau-t,\theta_{-t}\omega)$,

$$\sup_{s\in[\tau-1,\tau]}(\|u(s)\|_6^6 + \|v(s)\|_6^6 + \|w(s)\|_6^6) \le R_4(\tau,\omega), \tag{5.1}$$

$$\begin{aligned}&\sup_{s\in[\tau-1,\tau]}(\|u(s)\|_{H^1}^2 + \|v(s)\|_{H^1}^2 + \|w(s)\|_{H^1}^2)\\ &+\int_{\tau-1}^{\tau}(\|u_r(r)\|^2 + \|v_r(r)\|^2 + \|w_r(r)\|^2)dr \le R_4(\tau,\omega).\end{aligned} \tag{5.2}$$

Proof. Let

$$V(s,\tau-t)=\frac{1}{G}v(s,\tau-t),\quad W(s,\tau-t)=\frac{N}{k}w(s,\tau-t),\quad \mu=\frac{k}{N},$$

then (4.5) can be rewritten as

$$\begin{cases} u_s=d_1\Delta u-(F+k)u+Gz^{-2}(s,\theta_{-\tau}\omega)u^2V\\ \qquad -Gz^{-2}(s,\theta_{-\tau}\omega)u^3+kW+z(s,\theta_{-\tau}\omega)g_1(s,x),\\ V_s=d_2\Delta V-FV-z^{-2}(s,\theta_{-\tau}\omega)u^2V+z^{-2}(s,\theta_{-\tau}\omega)u^3\\ \qquad +\frac{1}{G}z(s,\theta_{-\tau}\omega)g_2(s,x),\\ \mu W_s=\mu d_3\Delta W-(\mu F+k)W+ku+z(s,\theta_{-\tau}\omega)g_3(s,x). \end{cases} \tag{5.3}$$

Take the inner products of (5.3) with (u^5, GV^5, W^5), and add them together to derive

$$\begin{aligned}&\frac{1}{6}\frac{d}{ds}(\|u\|_6^6+G\|V\|_6^6+\mu\|W\|_6^6)+(F+k)\|u\|_6^6+GF\|V\|_6^6+(\mu F+k)\|W\|_6^6\\ \le&-Gz^{-2}(s,\theta_{-\tau}\omega)\int_{\mathbb{R}^n}(u^8-u^7V-u^3V^5+u^2V^6)dx+k(W,u^5)\\ &+k(u,W^5)+z(s,\theta_{-\tau}\omega)\left((g_1(s,x),u^5)+(g_2(s,x),V^5)+(g_3(s,x),W^5)\right),\end{aligned}$$

which, along with $u^8-u^7V-u^3V^5+u^2V^6\ge 0$, the continuity of $r\to z(\cdot,\theta_r\omega)$ and the Young inequality, shows that for all $s\in[\tau-3,\tau]$,

$$\frac{d}{ds}(\|u\|_6^6+G\|V\|_6^6+\mu\|W\|_6^6)+F(\|u\|_6^6+G\|V\|_6^6+\mu\|W\|_6^6)\le C\|g(s,\cdot)\|_6^6. \tag{5.4}$$

Using the Gronwall inequality (4.1) (with $\mu=1$) to (5.4), we obtain from (4.9) that there exists $T:=T(\tau,\omega,\mathcal{D})\ge 3$ such that for all $t\ge T$,

$$\begin{aligned}&\sup_{s\in[\tau-1,\tau]}(\|u(s)\|_6^6+G\|V(s)\|_6^6+\mu\|W(s)\|_6^6)\\ \le&C\int_{\tau-3}^{\tau}(\|u(r)\|_6^6+G\|V(r)\|_6^6+\mu\|W(r)\|_6^6)dr+C\int_{\tau-3}^{\tau}\|g(r,\cdot)\|_6^6dr\\ \le&CR_2(\tau,\omega)+C\int_{-3}^{0}\|g(r+\tau,\cdot)\|_6^6dr,\end{aligned}$$

which implies (5.1) immediately.

To prove the uniform estimate (5.2), we take the inner products of (4.5) with (u_s,v_s,w_s) and sum them up to have

$$\begin{aligned}&\frac{1}{2}\frac{d}{ds}((F+N)\|u\|^2+d_1\|\nabla u\|^2+F\|v\|^2+d_2\|\nabla v\|^2+(F+N)\|w\|^2\\ &+d_3\|\nabla w\|^2)+\|u_s\|^2+\|v_s\|^2+\|w_s\|^2\\ =&z^{-2}(s,\theta_{-\tau}\omega)\left((u^2v-Gu^3,u_s)-(u^2v-Gu^3,v_s)\right)+(Nw,u_s)+(ku,w_s)\\ &+z(s,\theta_{-\tau}\omega)\left((g_1(s,x),u_s)+(g_2(s,x),v_s)+(g_3(s,x),w_s)\right),\end{aligned}$$

which means for all $s\in[\tau-3,\tau]$,

$$\begin{aligned}&\frac{d}{ds}((F+N)\|u\|^2+d_1\|\nabla u\|^2+F\|v\|^2+d_2\|\nabla v\|^2+(F+N)\|w\|^2\\ &+d_3\|\nabla w\|^2)+\|u_s\|^2+\|v_s\|^2+\|w_s\|^2\\ \le&C(\|u\|_6^6+\|v\|_6^6+\|u\|^2+\|w\|^2+\|g(s,\cdot)\|^2).\end{aligned} \tag{5.5}$$

Applying the Gronwall inequalities (4.1) and (4.2) (with $\mu = 1$) to (5.5), it yields from (4.8) and (4.9) that

$$\begin{aligned}
&\sup_{s\in[\tau-1,\tau]}(\|u(s)\|_{H^1}^2+\|v(s)\|_{H^1}^2+\|w(s)\|_{H^1}^2)\\
&+\int_{\tau-1}^{\tau}(\|u_r(r)\|^2+\|v_r(r)\|^2+\|w_r(r)\|^2)dr\\
\le& C\int_{\tau-3}^{\tau}(\|u(r)\|_{H^1}^2+\|v(r)\|_{H^1}^2+\|w(r)\|_{H^1}^2)dr\\
&+C\int_{\tau-3}^{\tau}(\|u(r)\|_6^6+\|v(r)\|_6^6)dr+C\int_{-3}^{0}\|g(r+\tau,\cdot)\|^2dr\\
\le& C(R_1(\tau,\omega)+R_2(\tau,\omega)+\int_{-3}^{0}\|g(r+\tau,\cdot)\|^2dr),
\end{aligned}$$

which completes the proof. □

6. Uniform estimates outside a ball

We prove system (4.5) is asymptotically arbitrary small outside a ball under the norm of Y.

Lemma 6.1. *For $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\mathcal{D} \in \mathfrak{D}$ are fixed, then for each $\varepsilon > 0$, there exist $T := T(\varepsilon) \ge 3$ and $K := K(\varepsilon) \ge 1$ such that for all $t \ge T$, $\xi \ge K$ and $(u_{\tau-t}, v_{\tau-t}, w_{\tau-t}) \in \mathcal{D}(\tau - t, \theta_{-t}\omega)$,*

$$\int_{|x|\ge\xi}(|\nabla u(\tau)|^2+|\nabla v(\tau)|^2+|\nabla w(\tau)|^2)dx<\varepsilon.$$

Proof. Let $\rho_\xi(x) = \rho(\frac{|x|^2}{\xi^2})$, $x \in \mathbb{R}^n$ and $\xi > 0$, where $\rho : \mathbb{R}^+ \to [0, 1]$ is a smooth function satisfying

$$\rho(s)=\begin{cases}0, & \text{if } 0\le s\le 1,\\ 1, & \text{if } s\ge 2.\end{cases} \tag{6.1}$$

Taking the inner products of (4.5) with $(-\rho_\xi\Delta u, -\rho_\xi\Delta v, -\rho_\xi\Delta w)$ and adding them together, we deduce for all $s \in [\tau - 1, \tau]$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^n}\rho_\xi(|\nabla u|^2+|\nabla v|^2+|\nabla w|^2)dx+\int_{\mathbb{R}^n}\nabla\rho_\xi(u_s\nabla u+v_s\nabla v+w_s\nabla w)dx\\
&+\int_{\mathbb{R}^n}\rho_\xi(d_1|\Delta u|^2+d_2|\Delta v|^2+d_3|\Delta w|^2)dx\\
=&((F+k)u-Nw,\rho_\xi\Delta u)+(Fv,\rho_\xi\Delta v)+((F+N)w-ku,\rho_\xi\Delta w)\\
&+z^{-2}(s,\theta_{-\tau}\omega)\left((u^2v-Gu^3,-\rho_\xi\Delta u)-(u^2v-Gu^3,-\rho_\xi\Delta v)\right)\\
&-z(s,\theta_{-\tau}\omega)\left((g_1(s,x),\rho_\xi\Delta u)+(g_2(s,x),\rho_\xi\Delta v)+(g_3(s,x),\rho_\xi\Delta w)\right).
\end{aligned}\tag{6.2}$$

It is obvious to check that $\|\nabla\rho_\xi\|_\infty \le \frac{c}{\xi}$ for each $\xi > 0$, then there exists $K_1 :=$

$K_1(\varepsilon) \geq 1$ such that for all $\xi \geq K_1$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \nabla\rho_\xi(u_s\nabla u + v_s\nabla v + w_s\nabla w)dx \\
\geq& -\frac{c}{\xi}(\|u_s\|^2 + \|\nabla u\|^2 + \|v_s\|^2 + \|\nabla v\|^2 + \|w_s\|^2 + \|\nabla w\|^2) \\
\geq& -\varepsilon(\|u_s\|^2 + \|\nabla u\|^2 + \|v_s\|^2 + \|\nabla v\|^2 + \|w_s\|^2 + \|\nabla w\|^2).
\end{aligned} \tag{6.3}$$

We also choose $K_2 \geq K_1$ such that for all $\xi \geq K_2$, the right hand side of (6.2) is bounded by

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^n} \rho_\xi(d_1|\Delta u|^2 + d_2|\Delta v|^2 + d_3|\Delta w|^2)dx \\
&+ C\int_{|x|\geq\xi}(u^6 + v^6 + u^2 + v^2 + w^2 + g^2(s,x))dx,
\end{aligned} \tag{6.4}$$

whenever $s \in [\tau-1, \tau]$. Then it follows from (6.2)-(6.4) that for all $\xi \geq K_2$ and $s \in [\tau-1, \tau]$,

$$\begin{aligned}
&\frac{d}{ds}\int_{\mathbb{R}^n} \rho_\xi(|\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2)dx \\
\leq&\varepsilon(\|u_s\|^2 + \|\nabla u\|^2 + \|v_s\|^2 + \|\nabla v\|^2 + \|w_s\|^2 + \|\nabla w\|^2) \\
&+ C\int_{|x|\geq\xi}(u^6 + v^6 + u^2 + v^2 + w^2 + g^2(s,x))dx.
\end{aligned} \tag{6.5}$$

By using the Gronwall inequality (4.1) (with $\mu = \frac{1}{3}$ and $a = 0$) to (6.5) and combining with the Lebesgue theorem on (3.2), we have from (4.12) and (5.2) that

$$\begin{aligned}
&\int_{|x|\geq\sqrt{2}\xi}(|\nabla u(\tau)|^2 + |\nabla v(\tau)|^2 + |\nabla w(\tau)|^2)dx \\
\leq& C\int_{\tau-1}^{\tau}\int_{|x|\geq\xi}(|\nabla u(s)|^2 + |\nabla v(s)|^2 + |\nabla w(s)|^2)dxds + C\int_{-1}^{0}\int_{|x|\geq\xi}|g(s+\tau,x)|^2dxds \\
&+ C\int_{\tau-1}^{\tau}\int_{|x|\geq\xi}(u^6(s) + v^6(s) + u^2(s) + v^2(s) + w^2(s))dxds \\
&+ \varepsilon\int_{\tau-1}^{\tau}(\|u_s\|^2 + \|\nabla u\|^2 + \|v_s\|^2 + \|\nabla v\|^2 + \|w_s\|^2 + \|\nabla w\|^2)ds \\
\leq& C\varepsilon,
\end{aligned}$$

which completes the proof. □

7. Uniform estimates inside a ball

This section shows the $\mathfrak{D}$-pullback limit-set compactness of system (4.5) in $H_0^1(Q_{\sqrt{2}\xi})^3$, where $Q_{\sqrt{2}\xi} = \{x \in \mathbb{R}^n : |x| < \sqrt{2}\xi\}$ with $\xi \geq 1$. To do this, define $\varrho = 1 - \rho$, where ρ is given in (6.1). Given $\xi \geq 1$, let $\varrho_\xi(x) = \varrho(\frac{|x|^2}{\xi^2})$ for $x \in \mathbb{R}^n$, we can easily see that

$$(\hat{u}, \hat{v}, \hat{w}) := (\hat{u}(s), \hat{v}(s), \hat{w}(s)) = \varrho_\xi(u(s), v(s), w(s))$$

are solutions of (by (4.5))

$$\begin{cases}\hat{u}_s = d_1\Delta\hat{u} - (F+k)\hat{u} + \varrho_\xi z^{-2}(s,\theta_{-\tau}\omega)u^2v - G\varrho_\xi z^{-2}(s,\theta_{-\tau}\omega)u^3 \\ \qquad +N\varrho_\xi w + \varrho_\xi z(s,\theta_{-\tau}\omega)g_1(s,x) - d_1 u\Delta\varrho_\xi - 2d_1\nabla\varrho_\xi\nabla u, \\ \hat{v}_s = d_2\Delta\hat{v} - F\hat{v} - \varrho_\xi z^{-2}(s,\theta_{-\tau}\omega)u^2v + G\varrho_\xi z^{-2}(s,\theta_{-\tau}\omega)u^3 \\ \qquad +\varrho_\xi z(s,\theta_{-\tau}\omega)g_2(s,x) - d_2 v\Delta\varrho_\xi - 2d_2\nabla\varrho_\xi\nabla v, \\ \hat{w}_s = d_3\Delta\hat{w} - (F+N)\hat{w} + k\varrho_\xi u + \varrho_\xi z(s,\theta_{-\tau}\omega)g_3(s,x) \\ \qquad -d_3 w\Delta\varrho_\xi - 2d_3\nabla\varrho_\xi\nabla w. \end{cases} \tag{7.1}$$

with the boundary condition

$$(\hat{u},\hat{v},\hat{w}) = \mathbf{0}, \quad \text{for } \ |x| = \sqrt{2}\xi,$$

where $s\in[\tau-1,\tau]$ with $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$ and $x\in Q_{\sqrt{2}\xi}$.

Let $H_i = \operatorname{span}\{e_1,e_2,\cdots,e_i\}\subset H_0^1(Q_{\sqrt{2}\xi})$, $P_i : H_0^1(Q_{\sqrt{2}\xi})\to H_i$ the canonical projector, I the identity and $\{\lambda_i\}_{i\ge1}$ eigenvalues related to eigenfunctions $\{e_i\}_{i\ge1}$ of the operator $-\Delta$ with zero boundary condition in $Q_{\sqrt{2}\xi}$. For any $\hat{u}\in H_0^1(Q_{\sqrt{2}\xi})$, $\hat{u}$ has a unique orthogonal decomposition: $\hat{u} = P_i\hat{u}\oplus(I-P_i)\hat{u} = \hat{u}_1\oplus\hat{u}_2$.

Lemma 7.1. *For $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $\mathcal{D}\in\mathfrak{D}$ are fixed, then for each $\xi\ge1$ and $\varepsilon>0$, there exist $T:=T(\varepsilon)\ge3$ and $N:=N(\varepsilon)\ge1$ such that for all $t\ge T$, $i\ge N$ and $(u_{\tau-t},v_{\tau-t},w_{\tau-t})\in\mathcal{D}(\tau-t,\theta_{-t}\omega)$,*

$$\|(I-P_i)\left(\hat{u}(\tau)+\hat{v}(\tau)+\hat{w}(\tau)\right)\|^2_{H_0^1(Q_{\sqrt{2}\xi})} < \varepsilon.$$

Proof. Since $\|u\|+\|\nabla u\|$ is equivalent to $\|\nabla u\|$ in $H_0^1(Q_{\sqrt{2}\xi})$, it suffices to prove for all $t\ge T$ and $i\ge N$,

$$\|\nabla\hat{u}_2(\tau)\|^2_{L^2(Q_{\sqrt{2}\xi})} + \|\nabla\hat{v}_2(\tau)\|^2_{L^2(Q_{\sqrt{2}\xi})} + \|\nabla\hat{w}_2(\tau)\|^2_{L^2(Q_{\sqrt{2}\xi})} < \varepsilon.$$

To this end, taking the inner products of (7.1) with $(-\Delta\hat{u}_2,-\Delta\hat{v}_2,-\Delta\hat{w}_2)$ on $Q_{\sqrt{2}\xi}$ and summing them up, we find for all $s\in[\tau-1,\tau]$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{ds}\left(\|\nabla\hat{u}_2\|^2+\|\nabla\hat{v}_2\|^2+\|\nabla\hat{w}_2\|^2\right) + d_1\|\Delta\hat{u}_2\|^2 + d_2\|\Delta\hat{v}_2\|^2 \\ &+ d_3\|\Delta\hat{w}_2\|^2 + (F+k)\|\nabla\hat{u}_2\|^2 + F\|\nabla\hat{v}_2\|^2 + (F+N)\|\nabla\hat{w}_2\|^2 \\ =&(N\varrho_\xi w,-\Delta\hat{u}_2) + (k\varrho_\xi u,-\Delta\hat{w}_2) + (d_1 u\Delta\varrho_\xi + 2d_1\nabla\varrho_\xi\nabla u,\Delta\hat{u}_2) \\ &+ (d_2 v\Delta\varrho_\xi + 2d_2\nabla\varrho_\xi\nabla v,\Delta\hat{v}_2) + (d_3 w\Delta\varrho_\xi + 2d_3\nabla\varrho_\xi\nabla w,\Delta\hat{w}_2) \\ &+ z^{-2}(s,\theta_{-\tau}\omega)\left((\varrho_\xi u^2v - G\varrho_\xi u^3,-\Delta\hat{u}_2) - (\varrho_\xi u^2v - G\varrho_\xi u^3,-\Delta\hat{v}_2)\right) \\ &+ z(s,\theta_{-\tau}\omega)\left((\varrho_\xi g_1(s,x),-\Delta\hat{u}_2) + (\varrho_\xi g_2(s,x),-\Delta\hat{v}_2)\right) \\ &+ \left((\varrho_\xi g_3(s,x),-\Delta\hat{w}_2)\right).\end{aligned} \tag{7.2}$$

Since $s-\tau\in[-1,0]$ and $\|\varrho_\xi\|_\infty\le1$, the last three lines in (7.2) are bounded by

$$\frac{d_1}{6}\|\Delta\hat{u}_2\|^2 + \frac{d_2}{6}\|\Delta\hat{v}_2\|^2 + \frac{d_3}{6}\|\Delta\hat{w}_2\|^2 + C(\|u\|_6^6+\|v\|_6^6+\|g(s,\cdot)\|^2). \tag{7.3}$$

By the Young inequality, we have

$$\begin{aligned}&(N\varrho_\xi w,-\Delta\hat{u}_2) + (k\varrho_\xi u,-\Delta\hat{w}_2) \\ \le&\frac{d_1}{6}\|\Delta\hat{u}_2\|^2 + \frac{d_3}{6}\|\Delta\hat{w}_2\|^2 + c(\|u\|^2+\|w\|^2).\end{aligned} \tag{7.4}$$

Since

$$\|\nabla \varrho_\xi\|_\infty = \|\rho' \frac{2x}{\xi^2}\|_\infty \le \frac{c}{\xi} \le c,$$
$$\|\Delta \varrho_\xi\|_\infty = \|\rho'' \frac{4x^2}{\xi^4} + \rho' \frac{2}{\xi^2}\|_\infty \le \frac{c}{\xi^2} \le c,$$

then the remaining terms on the right side of (7.2) are bounded by

$$\begin{aligned}&\frac{d_1}{6}\|\Delta \hat{u}_2\|^2 + \frac{d_2}{6}\|\Delta \hat{v}_2\|^2 + \frac{d_3}{6}\|\Delta \hat{w}_2\|^2 \\ &+ c(\|u\|^2 + \|\nabla u\|^2 + \|v\|^2 + \|\nabla v\|^2 + \|w\|^2 + \|\nabla w\|^2).\end{aligned} \tag{7.5}$$

Note that $\|\Delta \hat{u}_2\|^2 + \|\Delta \hat{v}_2\|^2 + \|\Delta \hat{w}_2\|^2 \ge \lambda_{i+1}(\|\nabla \hat{u}_2\|^2 + \|\nabla \hat{v}_2\|^2 + \|\nabla \hat{w}_2\|^2)$ on $Q_{\sqrt{2}\xi}$, we obtain from (7.2)-(7.5) that for all $s \in [\tau - 1, \tau]$,

$$\begin{aligned}&\frac{d}{ds}\left(\|\nabla \hat{u}_2\|^2 + \|\nabla \hat{v}_2\|^2 + \|\nabla \hat{w}_2\|^2\right) + d_0\lambda_{i+1}\left(\|\nabla \hat{u}_2\|^2 + \|\nabla \hat{v}_2\|^2 + \|\nabla \hat{w}_2\|^2\right) \\ \le& C(\|u\|^2 + \|\nabla u\|^2 + \|v\|^2 + \|\nabla v\|^2 + \|w\|^2 + \|\nabla w\|^2 \\ &+ \|u\|_6^6 + \|v\|_6^6 + \|g(s,\cdot)\|^2),\end{aligned} \tag{7.6}$$

where $d_0 = \min\{d_1, d_2, d_3\}$. Applying the Gronwall inequality (4.3) (with $\mu = \frac{1}{3}$) to (7.6), we derive

$$\begin{aligned}&\|\nabla \hat{u}_2(\tau)\|^2 + \|\nabla \hat{v}_2(\tau)\|^2 + \|\nabla \hat{w}_2(\tau)\|^2 \\ \le& 3\int_{\tau-1}^{\tau} e^{-d_0\lambda_{i+1}(\tau-s)}(\|\nabla \hat{u}_2(s)\|^2 + \|\nabla \hat{v}_2(s)\|^2 + \|\nabla \hat{w}(s)\|^2)ds \\ &+ C\int_{\tau-1}^{\tau} e^{-d_0\lambda_{i+1}(\tau-s)}(\|u\|^2 + \|\nabla u\|^2 + \|v\|^2 \\ &+ \|\nabla v\|^2 + \|w\|^2 + \|\nabla w\|^2)ds \\ &+ C\int_{\tau-1}^{\tau} e^{-d_0\lambda_{i+1}(\tau-s)}(\|u(s)\|_6^6 + \|v(s)\|_6^6)ds \\ &+ C\int_{-1}^{0} e^{d_0\lambda_{i+1}s}\|g(s+\tau,\cdot)\|^2 ds.\end{aligned} \tag{7.7}$$

By using the Lebesgue theorem on (3.2), we have that there exists $N_1 := N_1(\varepsilon) \ge 1$ such that the last term in (7.7) is less than ε whenever $i \ge N_1$. By (5.1) and (5.2), there exist $T := T(\varepsilon) \ge 3$ and $N_2 \ge N_1$ such that for all $t \ge T$ and $i \ge N_2$, the second and third terms on the right side of (7.7) are bounded by

$$CR_4(\tau,\omega)\int_{\tau-1}^{\tau} e^{-d_0\lambda_{i+1}(\tau-s)}ds \le \frac{CR_4}{d_0\lambda_{i+1}} < \varepsilon. \tag{7.8}$$

By $\|\nabla \varrho_\xi\|_\infty \le c$ again, we obtain

$$\|\nabla \hat{u}_2\|^2 = \|u_2\nabla \varrho_\xi + \varrho_\xi \nabla u_2\|^2 \le c(\|u\|^2 + \|\nabla u\|^2).$$

Similarly, we can prove the boundedness of $\|\nabla \hat{v}_2\|^2$ and $\|\nabla \hat{w}_2\|^2$ for all $t \ge T$, in view of (5.2). Then there exists $N \ge N_2$ such that for all $i \ge N$, the first term on the right side of (7.7) is bounded by

$$
\begin{aligned}
& c\int_{\tau-1}^{\tau} e^{-d_0\lambda_{i+1}(\tau-s)}(\|u\|^2+\|\nabla u\|^2+\|v\|^2+\|\nabla v\|^2+\|w\|^2+\|\nabla w\|^2)ds \\
\leq & \frac{cR_4}{d_0\lambda_{i+1}} < \varepsilon.
\end{aligned} \tag{7.9}
$$

It follows from (7.7)-(7.9) that for all $i \geq N$ and $t \geq T$,

$$
\|\nabla\hat{u}_2(\tau)\|^2_{L^2(Q_{\sqrt{2}\xi})}+\|\nabla\hat{v}_2(\tau)\|^2_{L^2(Q_{\sqrt{2}\xi})}+\|\nabla\hat{w}_2(\tau)\|^2_{L^2(Q_{\sqrt{2}\xi})}<3\varepsilon,
$$

which shows the desired inequality. □

8. Proof of the main result

In this section, based on the above necessary estimates, we are now in a position to prove Theorem 3.1 by using Proposition 2.1.

Proof of Theorem 3.1. Recall $X = L^2(\mathbb{R}^n)^3 \cap L^6(\mathbb{R}^n)^3$ and $Y = H^1(\mathbb{R}^n)^3$, then it is obvious that (X, Y) is a limit-identical pair. The cocycle Φ defined in (3.3) is continuous on X and takes values in Y. By (4.6), Φ has a closed, measurable absorbing set $\mathcal{K} \in \mathfrak{D}$. By (4.10) and Lemma 4.3, it is simple to prove Φ is $\mathfrak{D}$-pullback limit-set compact in $L^2(\mathbb{R}^n)^3$. Therefore, by Proposition 2.1, it remains to show that Φ is $\mathfrak{D}$-pullback limit-set compact in Y.

For $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\mathcal{D} \in \mathfrak{D}$ are fixed, define

$$
B(T) = \bigcup_{t\geq T}\Phi(t,\tau-t,\theta_{-t}\omega)\mathcal{D}(\tau-t,\theta_{-t}\omega).
$$

Given $\varepsilon > 0$, by (4.10) and Lemma 6.1, there exist $T_1 := T_1(\varepsilon) \geq 3$ and $K_1 := K_1(\varepsilon) \geq 1$ such that for each $\xi \geq K_1$,

$$
\|u\|^2_{H^1(Q^c_\xi)}+\|v\|^2_{H^1(Q^c_\xi)}+\|w\|^2_{H^1(Q^c_\xi)}<\varepsilon, \quad \text{for all } (u,v,w)\in B(T_1). \tag{8.1}
$$

By Lemma 7.1, there exist $i := i(\varepsilon) \geq 1$ and $T_2 \geq T_1$ such that

$$
\begin{aligned}
&\|(I-P_i)\hat{u}\|^2_{H^1_0(Q_{\sqrt{2}\xi})}+\|(I-P_i)\hat{v}\|^2_{H^1_0(Q_{\sqrt{2}\xi})}+\|(I-P_i)\hat{w}\|^2_{H^1_0(Q_{\sqrt{2}\xi})}<\varepsilon,\\
&\text{for all } (u,v,w)\in B(T_2),
\end{aligned} \tag{8.2}
$$

where $(\hat{u}, \hat{v}, \hat{w}) := \varrho_\xi(u, v, w)$ and ϱ_ξ is the bounded function given in Lemma 7.1. By (4.6) and Lemma 4.3, there is $T_3 \geq T_2$ such that $B(T_3)$ is bounded in Y and thus $\hat{B}(T_3) := \{(\hat{u}, \hat{v}, \hat{w}); (u, v, w) \in B(T_3)\}$ is bounded in $H^1_0(Q_{\sqrt{2}\xi})^3$, which shows $P_i\hat{B}(T_3)$ is bounded in the finite dimensional subspace $\Big(P_i(H^1_0(Q_{\sqrt{2}\xi}))\Big)^3$. Therefore, $P_i\hat{B}(T_3)$ is pre-compact in $H^1_0(Q_{\sqrt{2}\xi})^3$ and thus $\kappa_{H^1_0(Q_{\sqrt{2}\xi})^3}(P_i\hat{B}(T_3)) = 0$, where $\kappa(\cdot)$ is the Kuratowski measure. This, combining with (8.2), implies

$$
\kappa_{H^1_0(Q_{\sqrt{2}\xi})^3}(\hat{B}(T_3))\leq\kappa_{H^1_0(Q_{\sqrt{2}\xi})^3}(P_i\hat{B}(T_3))+\kappa_{H^1_0(Q_{\sqrt{2}\xi})^3}((I-P_i)\hat{B}(T_3))<2\varepsilon,
$$

which shows $\hat{B}(T_3)$ is pre-compact in $H^1_0(Q_{\sqrt{2}\xi})^3$. Since $(\hat{u}, \hat{v}, \hat{w}) = \varrho_\xi(u, v, w) = (u, v, w)$ on Q_ξ, we obtain

$$
\kappa_{H^1_0(Q_\xi)^3}(B(T_3))=\kappa_{H^1_0(Q_\xi)^3}(\hat{B}(T_3))\leq\kappa_{H^1_0(Q_{\sqrt{2}\xi})^3}(\hat{B}(T_3))<2\varepsilon,
$$

which, along with (8.1), implies

$$\kappa_Y(B(T_3)) \leq \kappa_{H_0^1(Q_\xi)^3}(B(T_3)) + \varepsilon < 3\varepsilon.$$

Thus Φ is $\mathfrak{D}$-pullback limit-set compact in Y, and Φ has indeed an (X, Y)-pullback attractor, so is $\hat{\Phi}$ in view of the equivalence of Φ and $\hat{\Phi}$. □

References

[1] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, Berlin, 1998.

[2] P. Bates, H. Lisei and K. Lu, *Attractors for stochastic lattice dynamical systems*, Stoch. Dyn., 2006, 6(1), 1–21.

[3] P. Bates, K. Lu and B. Wang, *Random attractors for stochastic reaction-diffusion equations on unbounded domains*, J. Differ. Equations, 2009, 246(2), 845–869.

[4] H. Crauel and F. Flandoli, *Attractors for random dynamical systems*, Probab. Theory Relat. Fields, 1994, 100(3), 365–393.

[5] A. Gu, *Pullback $\mathcal{D}$-attractors of non-autonomous three-component reversible Gray-Scott system on unbounded domains*, Abstr. Appl. Anal., 2013, 2013(2), 1–13.

[6] A. Gu, *Random attractors of stochastic three-component reversible Gray-Scott system on unbounded domains,* Abstr. Appl. Anal., 2012, 2012(7), 1–22.

[7] A. Gu and H. Xiang, *Upper semicontinuity of random attractors for stochastic three-component reversible Gray-Scott system,* Appl. Math. Comput., 2013, 225(12), 387–400.

[8] A. Gu, S. Zhou and Z. Wang, *Uniform attractor of non-autonomous three-component reversible Gray-Scott system*, Appl. Math. Comput., 2013, 219(16), 8718–8729.

[9] P. Gray and S. Scott, *Autocatalytic reactions in the isothermal continuous stirred tank reactor: Isolas and other forms of multistability*, Chem. Eng. Sci., 1983, 38(1), 29–43.

[10] P. Gray and S. Scott, *Autocatalytic reactions in the isothermal, continuous stirred tank reactor: Oscillations and instabilities in the system* $a+2b \to 3b, b \to c$, Chem. Eng. Sci., 1984, 39(6), 1087–1097.

[11] P. Kloeden and M. Rasmussen, *Nonautonomous Dynamical Systems*, American Mathematical Society, Providence, 2011.

[12] Y. Li and B. Guo, *Random attractors for quasi-continuous random dynamical systems and applications to stochastic reaction-diffusion equations*, J. Differ. Equations, 2008, 245(7), 1775–1800.

[13] Y. Li, A. Gu and J. Li, *Existence and continuity of bi-spatial random attractors and application to stochastic semilinear Laplacian equations*, J. Differ. Equations, 2015, 258(2), 5040-534.

[14] Y. Li and J. Yin, *A modified proof of pullback attractors in a sobolev space for stochastic Fitzhugh-Nagumo equations*, Disc. Cont. Dyn. Syst. Serie B., 2016, 21(4), 1203–1223.

[15] G. Lukaszewicz, *On pullback attractors in H_0^1 for nonautonomous reaction-diffusion equations*, Int. J. Bifurcat. Chaos, 2010, 20(9), 2637–2644.

[16] H. Mahara, N. Suematsu, T. Yamaguchi, K. Ohgane, Y. Nishiura and M. Shimomura, *Three-variable reversible Gray-Scott model,* J. Chem. Physics, 2004, 121(18), 8968–8972.

[17] I. Prigogine and R. Lefever, *Symmetry-breaking instabilities in dissipative systems*, J. Chem. Physics, 1968, 48(4), 1665–1700.

[18] B. Schmalfuss, *Backward cocycle and attractors of stochastic differential equations (in: V. Reitmann, T. Riedrich, N. Koksch, eds.),* International Seminar on Applied Mathematics-Nonlinear Dynamics: Attractor Approximation and Global Behavior, Technische Universität, Dresden, 1992, 185–192.

[19] S. Scott and K. Showalter, *Simple and complex reaction-diffusion fronts, Chemical Waves and Patterns (R. Kapral and K. Showalter, eds.),* Understanding Chemical Reactivity, Springer, 1995, 10, 485–516.

[20] B. Wang, *Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems*, J. Differ. Equations, 2012, 253(5), 1544–1583.

[21] B. Wang, *Pullback attractors for non-autonomous reaction-diffusion equations on $\mathbb{R}^n$,* Front. Math. China, 2009, 4(3), 563–583.

[22] F. Yin and L. Liu, *$\mathcal{D}$-pullback attractor for a non-autonomous wave equation with additive noise on unbounded domains*, Comput. Math. Appl., 2014, 68(3), 424–438.

[23] Y. You, *Dynamics of three-component reversible Gray-Stott model*, Disc. Cont. Dyn. Syst. Serie B., 2010, 14(4), 1671–1688.

[24] Y. You, *Robustness of global attractors for reversible Gray-Scott systems*, J. Dyn. Differ. Equat., 2012, 24(3), 495–520.

[25] Y. You, *Random attractors and robustness for stochastic reversible reaction-diffusion systems*, Disc. Cont. Dyn. Syst. Serie A., 2014, 34(1), 301–333.

[26] Y. You, *Random dynamics of stochastic reaction-diffusion systems with additive noise*, J. Dyn. Differ. Equat., 2015, 29(1), 82–112. DOI: 10.1007/s10884-015-9431-4.

[27] Y. You, *Dynamics of two-compartment Gray-Scott equations*, Nonlinear Anal., 2011, 74(5), 1969–1986.

[28] W. Zhao, *Regularity of random attractors for a degenerate parabolic equations driven by additive noise*, Appl. Math. Comput., 2014, 239(15), 358–374.

[29] C. Zhong, M. Yang and C. Sun, *The existence of global attractors for the norm-to-weak continuous semigroup and application to the nonlinear reaction-diffusion equations*, J. Differ. Equations, 2006, 223(2), 367–399.