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THREE-DIMENSIONAL DYNAMICAL SYSTEMS WITH FOUR-DIMENSIONAL VESSIOT-GULDBERG-LIE ALGEBRAS

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Abstract Dynamical systems attract much attention due to their wide applications. Many significant results have been obtained in this field from various points of view. The present paper is devoted to an algebraic method of integration of three-dimensional nonlinear time dependent dynamical systems admitting nonlinear superposition with four-dimensional Vessiot-Guldberg-Lie algebras L_4. The invariance of the relation between a dynamical system admitting nonlinear superposition and its Vessiot-Guldberg-Lie algebra is the core of the integration method. It allows to simplify the dynamical systems in question by reducing them to *standard forms*. We reduce the three-dimensional dynamical systems with four-dimensional Vessiot-Guldberg-Lie algebras to 98 standard types and show that 86 of them are integrable by quadratures.

Keywords Time dependent dynamical system, nonlinear superposition of solutions, Vessiot-Guldberg-Lie algebra L_4, standard forms of L_4.

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1. Introduction

We consider time dependent nonlinear dynamical systems given by first-order ordinary differential equations

$$\frac{dx^i}{dt} = f^i(t,x), \quad i = 1,\dots,n, \tag{1.1}$$

with $n > 1$ dependent variables $x^1,\dots,x^n$. We denoted by x the n-dimensional vector

$$x = (x^1,\dots,x^n)$$

and refer to the system (1.1) as an *n-dimensional dynamical system.*

A major obstacle in investigating *nonlinear* dynamical systems is that they do not obey the usual superposition principle which provides powerful tools in dealing with linear systems. Furthermore, integration methods based on Lie group analysis of differential equations are not effective for dealing with the system (1.1) because

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determining equations for calculating Lie symmetries are not over-determined in the case of first-order ordinary differential equations. Therefore only narrow categories of particular nonlinear dynamical systems can be solved analytically. Consequently, nonlinear systems are mostly analyzed by numerical methods or by approximating them by linear systems using, e.g. perturbation theory.

In this paper we are concerned with constructing wide classes of integrable three-dimensional dynamical systems (1.1) using, instead of the linear superposition principle, the more general concept of *nonlinear superpositions* introduced in 1893 by Vessiot [1], Guldberg [2] and Lie [3].

S. Lie [4] noticed in 1885 that the key features of linear ordinary differential equations

$$\frac{dx^i}{dt} = a^i_k(t)x^k, \quad i = 1, \ldots, n,$$

are based on the fact that the differential operators

$$X_{ik} = x^i \frac{\partial}{\partial x^k}\,, \quad i, k = 1, \ldots, n,$$

generate a finite continuous group, namely the linear homogeneous group with n variables x^i. This observation led him (see [4], §8) to believe that the main properties of the linear equations can be extended to the nonlinear equations having the form of *generalized separation of variables*:

$$\frac{dx^i}{dt} = T_1(t)\xi^i_1(x) + \cdots + T_r(t)\xi^i_r(x), \quad i = 1, \ldots, n, \tag{1.2}$$

provided that the linear span L_r of the first-order differential operators

$$X_\alpha = \xi^i_\alpha(x)\frac{\partial}{\partial x^i}\,, \quad \alpha = 1, \ldots, r, \tag{1.3}$$

is closed under the commutator:

$$[X_\alpha, X_\beta] = c^\gamma_{\alpha\beta}X_\gamma. \tag{1.4}$$

It means that L_r is a finite-dimensional Lie algebra. The coefficients $T_\alpha(t)$ in Equations (1.2) are any smooth functions of the variable t. S. Lie showed (see [4, p.128]) that the general solution of his system (1.2) can be expressed via a certain finite number m of particular solutions

$$x_1 = (x^1_1, \ldots, x^n_1),\ \ldots,\ x_m = (x^1_m, \ldots, x^n_m) \tag{1.5}$$

of the system (1.2) and that the expression (nonlinear superposition)

$$x = \varphi(x_1, \ldots, x_m, C_1, \ldots, C_n) \tag{1.6}$$

for the general solution $x = (x^1, \ldots, x^n)$ as a function of the particular solutions (1.5) and arbitrary constants $C_1, \ldots, C_n$ is obtained by solving the equations

$$J_i(x,\ x_1, \ldots,\ x_m) = C_i, \quad i = 1, \ldots, n, \tag{1.7}$$

with respect to $x = (x^1, \ldots, x^n)$, where J_i are invariants of $m+1$ points $x,\ x_1, \ldots,\ x_m$ with respect to the group with the basic generators $X_1, \ldots, X_r$.

Later E. Vessiot [1] and A. Guldberg [2] came to a lucky idea to look for all systems of ordinary differential equations possessing *fundamental systems of integrals*, or in modern terminology, admitting *nonlinear superpositions*.

Definition 1.1. A dynamical system (1.1) admits a nonlinear superposition if the general solution of the system (1.1) can be written as a vector function (1.6) of a finite number of its particular solutions (1.5) and n arbitrary constants $C_1, \ldots, C_n$.

The solution to Vessiot-Guldberg's problem was given by S. Lie. He announced in [3] the following statement (the detailed proof is published in [5, Chapter 24, pp. 793-804]; see also [6, Sect. 6.7]).

Theorem 1.1. *The system (1.1) admits a nonlinear superposition if and only if it has the form of generalized separation of variables (1.2). The number m of necessary particular solutions (1.5) is estimated by*

$$nm \geq r. \tag{1.8}$$

The nonlinear superposition (1.6) for the system (1.2) is given by the equation (1.7).

The algebra L_r spanned by the operators (1.3) is called the *Vessiot-Guldberg-Lie algebra* for the dynamical system (1.2).

A renowned example of a first-order ordinary differential equation admitting a nonlinear superposition is provided by the general Riccati equation

$$\frac{dx}{dt} = P(t) + Q(t)x + R(t)x^2.$$

In this example $n = 1,\ r = 3$, the nonlinear superposition (1.6) is given by the cross-ratio theorem stating that any four solutions x_1, x_2, x_3 and x of the Riccati equation are connected by the equation

$$\frac{(x - x_2)(x_3 - x_1)}{(x_1 - x)(x_2 - x_3)} = C, \quad C = \text{const}.$$

The operators (1.3) of the Vessiot-Guldberg-Lie algebra have the form

$$X_1 = \frac{\partial}{\partial x}\,, \quad X_2 = x\frac{\partial}{\partial x}\,, \quad X_3 = x^2\frac{\partial}{\partial x}\,,$$

and the number of necessary particular solutions is $m = 3$, hence the estimation (1.8) is satisfied with the equality sign.

We have demonstrated in [7] that there are 31 standard forms of time dependent three-dimensional dynamical systems (1.1) admitting nonlinear superpositions with *three-dimensional* Vessiot-Guldberg-Lie algebras L_3. The solvable L_3 provide 24 standard forms that are integrable by quadratures.

The purpose of the present paper is to enumerate standard forms of three-dimensional nonlinear dynamical systems admitting nonlinear superpositions with *four-dimensional* Vessiot-Guldberg-Lie algebras L_4 and to single out the integrable systems. We adopt here the notation and terminology used in [7].

2. Integration method

A method for integration of dynamical systems admitting nonlinear superposition has been suggested in [8, Section 11.2]. The method is based on the fact that dynamical systems admitting nonlinear superposition and their Vessiot-Guldberg-Lie algebras behave coherently under any change of the dependent variables

$$\tilde{x}^i = \tilde{x}^i(x), \quad i = 1, \ldots, n. \tag{2.1}$$

Table 1. Non-isomorphic structures of four-dimensional real Lie algebras

Type	$[X_1, X_2]$	$[X_2, X_3]$	$[X_1, X_3]$	$[X_1, X_4]$	$[X_2, X_4]$	$[X_3, X_4]$
I	0	0	0	0	0	0
II	X_1	0	0	0	0	0
III	X_1	0	0	0	0	X_3
IV	0	X_1	0	0	0	0
V	0	$X_1 + X_2$	X_1	0	0	0
VI	0	X_2	X_1	0	0	0
VII	0	hX_2	X_1	0	0	0
VIII	0	$X_1 + bX_2$	$bX_1 - X_2$	0	0	0
IX	X_1	X_3	$2X_2$	0	0	0
X	X_3	X_1	$-X_2$	0	0	0
XI	0	0	0	0	X_1	X_2
XII	0	0	0	aX_1	X_2	$X_2 + X_3$
XIII	0	0	0	X_1	0	X_2
XIV	0	0	0	X_1	$X_1 + X_2$	$X_2 + X_3$
XV	0	0	0	aX_1	bX_2	cX_3
XVI	0	0	0	aX_1	$bX_2 - X_1$	$X_2 + bX_3$
XVII	0	X_1	0	$2X_1$	X_2	$X_2 + X_3$
XVIII	0	X_1	0	$(1 + h)X_1$	X_2	hX_3
XIX	0	X_1	0	$2bX_1$	$bX_2 - X_3$	$X_2 + bX_3$
XX	0	X_2	X_1	$-X_2$	X_1	0

Namely, it can be shown ([9], see also [7]) that the system (1.2) is written in the new variables (2.1) in the form

$$\frac{d\tilde{x}^i}{dt} = T_1(t)\tilde{\xi}_1^i(\tilde{x}) + \cdots + T_r(t)\tilde{\xi}_r^i(\tilde{x}), \quad i = 1, \ldots, n, \tag{2.2}$$

where $\xi_\alpha = (\xi_\alpha^1, \ldots, \xi_\alpha^n)$ are transformed according to the vector law

$$\tilde{\xi}_\alpha^i = \frac{\partial \tilde{x}^i(x)}{\partial x^j}\, \xi_\alpha^j\,, \quad i = 1, \ldots, n, \tag{2.3}$$

and the coefficients $T_\alpha(t)$ in Equations (2.2) are the same as in Equations (1.2).

According to the above property, one can simplify dynamical systems admitting nonlinear superposition by reducing bases ξ_α of the Vessiot-Guldberg-Lie algebras to *simple* (standard) forms using appropriate changes of variables (2.1).

3. Four-dimensional Lie algebras in the real domain

Recall that the non-isomorphic structures of four-dimensional Lie algebras L_4 in the complex domain were described by S. Lie [10]. We will use the classification in the real domain given in [11]. It is presented in our paper in Table 1. Note that the constants in Table 1 satisfy the conditions $a \neq 0,\ |h| \leq 1,\ c \neq 0,\ b \geq 0.$

For our integration purposes we need realizations of Lie algebras L_4 by first-order linear partial differential operators in the three-dimensional space. The enumeration of non-similar realizations of four-dimensional real Lie algebras in the three-dimensional space can be found in recent publications [12,13]. The paper [12] deals with classification of systems of second-order ordinary differential equations. Paper [13] contains a useful short review of results and an extensive list of literature on classification of low-dimensional Lie algebras. In Tables 2 and 3 we use the realizations given in [13].

4. Standard forms of L_4 and associated dynamical systems

We write the three-dimensional dynamical systems (1.2) admitting nonlinear superposition with four-dimensional Vessiot-Guldberg-Lie algebras in the form

$$
\begin{aligned}
x' &= \xi_1^1(x,y,z)T_1(t) + \xi_2^1(x,y,z)T_2(t) + \xi_3^1(x,y,z)T_3(t) + \xi_4^1(x,y,z)T_4(t),\\
y' &= \xi_1^2(x,y,z)T_1(t) + \xi_2^2(x,y,z)T_2(t) + \xi_3^2(x,y,z)T_3(t) + \xi_4^2(x,y,z)T_4(t),\\
z' &= \xi_1^3(x,y,z)T_1(t) + \xi_2^3(x,y,z)T_2(t) + \xi_3^3(x,y,z)T_3(t) + \xi_4^3(x,y,z)T_4(t),
\end{aligned}
\tag{4.1}
$$

where x', y', z' denote the derivatives of the dependent variables x, y, z with respect to t. We reduce all systems (4.1) into standard forms by using the realizations of non-isomorphic real Lie algebras L_4. The result is given in Tables 2 and 3. Table 2 contains the standard forms of the system (4.1) associated with solvable Vessiot-Guldberg-Lie algebras L_4. In this table $\varphi(z), \psi(z), \theta(z)$ and $\theta(y,z)$ are arbitrary functions. Table 3 contains the standard forms of the system (4.1) associated with nonsolvable Vessiot-Guldberg-Lie algebras L_4.

5. Discussion of Tables 2 and 3

Tables 2 and 3 show that there exist 98 distinctly different classes of systems (4.1). Applying an arbitrary change of the dependent variables x, y, z to each class of the equations from Tables 2 and 3 one obtains an infinite set of three-dimensional dynamical systems admitting nonlinear superposition with four-dimensional Vessiot-Guldberg-Lie algebras. After a detailed inspection of Tables 2 and 3 we conclude that 86 classes of the systems (4.1) with solvable L_4, namely, the systems 1-84, 86 and 89 from Table 2 are integrable by quadratures. Note that the system 32 can be integrated reducing it to the system

$$
x' = \widetilde{T}_1(\tilde{t}) + bx + y, \quad y' = \widetilde{T}_2(\tilde{t}) + by - y, \quad z' = \widetilde{T}_3(\tilde{t})
$$

by introducing the new independent variable $\tilde{t} = \int T_3(t)dt$. The systems 33, 68 and 84 are integrated likewise. The systems 85, 87 and 88 from Table 2 provide linear systems with variable coefficients and, in general, are not integrable by quadratures. The systems with nonsolvable L_4 (Table 3) are nonlinear and not integrable by quadratures.

6. Conclusions

The three-dimensional dynamical systems (4.1) admitting nonlinear superposition with four-dimensional Vessiot-Guldberg-Lie algebras are mapped to standard forms. They are classified into 98 classes presented in Tables 2 and 3. Each class is a representative of an infinite set of equations involving three arbitrary functions of three variables. Systems associated with solvable four-dimensional Vessiot-Guldberg-Lie algebras are classified into 89 classes presented in Table 2, among them 86 are integrable by quadratures.

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Tables

Table 2. Dynamical systems with solvable L_4

Type	#	Operators	Systems
I	1	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ z\frac{\partial}{\partial x}+\varphi(z)\frac{\partial}{\partial y},\ \theta(z)\frac{\partial}{\partial x}+\psi(z)\frac{\partial}{\partial y}$	$x'=T_1(t)+zT_3(t)+\theta(z)T_4(t),$ $y'=T_2(t)+\varphi(z)T_3(t)+\psi(z)T_4(t),$ $z'=0$
	2	$\frac{\partial}{\partial x},\ y\frac{\partial}{\partial x},\ z\frac{\partial}{\partial x},\ \theta(y,z)\frac{\partial}{\partial x}$	$x'=T_1(t)+yT_2(t)+zT_3(t)+\theta(y,z)T_4(t),$ $y'=0,$ $z'=0$
	3	$\frac{\partial}{\partial x},\ y\frac{\partial}{\partial x},\ \varphi(y)\frac{\partial}{\partial x},\ \psi(y)\frac{\partial}{\partial x}$	$x'=T_1(t)+yT_2(t)+\varphi(y)T_3(t)+\psi(y)T_4(t),$ $y'=0,$ $z'=0$
II	4	$\frac{\partial}{\partial x},\ x\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}$	$x'=T_1(t)+xT_2(t),$ $y'=T_3(t),$ $z'=T_4(t)$
	5	$\frac{\partial}{\partial x},\ x\frac{\partial}{\partial x}+z\frac{\partial}{\partial z},\ \frac{\partial}{\partial y},\ z\frac{\partial}{\partial x}$	$x'=T_1(t)+xT_2(t)+zT_4(t),$ $y'=T_3(t),$ $z'=zT_2(t)$
	6	$\frac{\partial}{\partial x},\ x\frac{\partial}{\partial x}+\varphi(z)\frac{\partial}{\partial y},\ \frac{\partial}{\partial y},\ z\frac{\partial}{\partial y}$	$x'=T_1(t)+xT_2(t),$ $y'=\varphi(z)T_2(t)+T_3(t)+zT_4(t),$ $z'=0$
	7	$\frac{\partial}{\partial x},\ x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z},\ y\frac{\partial}{\partial x},\ z\frac{\partial}{\partial x}$	$x'=T_1(t)+xT_2(t)+yT_3(t)+zT_4(t),$ $y'=yT_2(t),$ $z'=zT_2(t)$
III	8	$\frac{\partial}{\partial x},\ x\frac{\partial}{\partial x}+\frac{\partial}{\partial z},\ \frac{\partial}{\partial y},\ y\frac{\partial}{\partial y}+C\frac{\partial}{\partial z}$	$x'=T_1(t)+xT_2(t),$ $y'=T_3(t)+yT_4(t),$ $z'=T_2(t)+CT_4(t)$
	9	$\frac{\partial}{\partial x},\ x\frac{\partial}{\partial x}+z\frac{\partial}{\partial y},\ \frac{\partial}{\partial y},\ y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}$	$x'=T_1(t)+xT_2(t),$ $y'=zT_2(t)+T_3(t)+yT_4(t),$ $z'=zT_4(t)$
	10	$\frac{\partial}{\partial x},\ x\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ y\frac{\partial}{\partial y}$	$x'=T_1(t)+xT_2(t),$ $y'=T_3(t)+yT_4(t),$ $z'=0$
	11	$\frac{\partial}{\partial x},\ x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},\ y\frac{\partial}{\partial x},\ -y\frac{\partial}{\partial y}+\frac{\partial}{\partial z}$	$x'=T_1(t)+xT_2(t)+yT_3(t),$ $y'=y[T_2(t)-T_4(t)],$ $z'=T_4(t)$
	12	$\frac{\partial}{\partial x},\ x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},\ y\frac{\partial}{\partial x},\ -y\frac{\partial}{\partial y}$	$x'=T_1(t)+xT_2(t)+yT_3(t),$ $y'=y[T_2(t)-T_4(t)],$ $z'=0$
IV	13	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial z},\ z\frac{\partial}{\partial x},\ \frac{\partial}{\partial y}$	$x'=T_1(t)+zT_3(t),$ $y'=T_4(t),$ $z'=T_2(t)$
	14	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial z},\ z\frac{\partial}{\partial x}+\varphi(y)\frac{\partial}{\partial z},\ y\frac{\partial}{\partial x}$	$x'=T_1(t)+zT_3(t)+yT_4(t),$ $y'=0,$ $z'=T_2(t)+\varphi(y)T_3(t)$
V	15	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ (x+y)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},\ \frac{\partial}{\partial z}$	$x'=T_1(t)+(x+y)T_3(t),$ $y'=T_2(t)+yT_3(t),$ $z'=T_4(t)$
	16	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ (x+y)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\frac{\partial}{\partial z},\ e^z\left(z\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)$	$x'=T_1(t)+(x+y)T_3(t)+ze^zT_4(t),$ $y'=T_2(t)+yT_3(t)+e^zT_4(t),$ $z'=T_3(t)$
	17	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ (x+y)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\frac{\partial}{\partial z},\ e^z\frac{\partial}{\partial x}$	$x'=T_1(t)+(x+y)T_3(t)+e^zT_4(t),$ $y'=T_2(t)+yT_3(t),$ $z'=T_3(t)$

Table 2 (Continued)

Type	#	Operators	Systems
	18	$\frac{\partial}{\partial x},\ y\frac{\partial}{\partial x},\ x\frac{\partial}{\partial x}-\frac{\partial}{\partial y},\ \frac{\partial}{\partial z}$	$x'=T_1(t)+yT_2(t)+xT_3(t),$ $y'=-T_2(t),$ $z'=T_4(t)$
	19	$\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}-\frac{\partial}{\partial y}, ze^{-y}\frac{\partial}{\partial x}$	$x'=T_1(t)+yT_2(t)+xT_3(t)+ze^{-y}T_4(t),$ $y'=-T_2(t),$ $z'=0$
	20	$\frac{\partial}{\partial x},\ y\frac{\partial}{\partial x},\ x\frac{\partial}{\partial x}-\frac{\partial}{\partial y},\ e^{-y}\frac{\partial}{\partial x}$	$x'=T_1(t)+yT_2(t)+xT_3(t)+e^{-y}T_4(t),$ $y'=-T_2(t),$ $z'=0$
VI	21	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},\ \frac{\partial}{\partial z}$	$x'=T_1(t)+xT_3(t),$ $y'=T_2(t)+yT_3(t),$ $z'=T_4(t)$
	22	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\frac{\partial}{\partial z},$ $e^z\frac{\partial}{\partial x}$	$x'=T_1(t)+xT_3(t)+e^zT_4(t),$ $y'=T_2(t)+yT_3(t),$ $z'=T_3(t)$
	23	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ x\frac{\partial}{\partial x}+\frac{\partial}{\partial z},\ \varphi(y)\frac{\partial}{\partial z}$	$x'=T_1(t)+xT_3(t),$ $y'=T_2(t),$ $z'=T_3(t)+\varphi(y)T_4(t)$
	24	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ x\frac{\partial}{\partial x}+\frac{\partial}{\partial z},\ e^z\frac{\partial}{\partial x}$	$x'=T_1(t)+xT_3(t)+e^zT_4(t),$ $y'=T_2(t),$ $z'=T_3(t)$
VII	25	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ x\frac{\partial}{\partial x}+ay\frac{\partial}{\partial y},\ \frac{\partial}{\partial z}$	$x'=T_1(t)+xT_3(t),$ $y'=T_2(t)+ayT_3(t),$ $z'=T_4(t)$
	26	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ x\frac{\partial}{\partial x}+ay\frac{\partial}{\partial y}+\frac{\partial}{\partial z},$ $e^z\frac{\partial}{\partial x}+e^{az}\frac{\partial}{\partial y}$	$x'=T_1(t)+xT_3(t)+e^zT_4(t),$ $y'=T_2(t)+ayT_3(t)+e^{az}T_4(t),$ $z'=T_3(t)$
	27	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ x\frac{\partial}{\partial x}+ay\frac{\partial}{\partial y}+\frac{\partial}{\partial z},$ $e^z\frac{\partial}{\partial x}$	$x'=T_1(t)+xT_3(t)+e^zT_4(t),$ $y'=T_2(t)+ayT_3(t),$ $z'=T_3(t)$
	28	$\frac{\partial}{\partial x}, y\frac{\partial}{\partial x},\ x\frac{\partial}{\partial x}+(1-a)y\frac{\partial}{\partial y},$ $\frac{\partial}{\partial z}$	$x'=T_1(t)+yT_2(t)+xT_3(t),$ $y'=(1-a)yT_3(t),$ $z'=T_4(t)$
	29	$\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}+(1-a)y\frac{\partial}{\partial y},$ $z\lvert y\rvert^{\frac{1}{1-a}}\frac{\partial}{\partial x}$	$x'=T_1(t)+yT_2(t)+xT_3(t)+z\lvert y\rvert^{\frac{1}{1-a}}T_4(t),$ $y'=(1-a)yT_3(t),$ $z'=0$
	30	$\frac{\partial}{\partial x},\ y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}+(1-a)y\frac{\partial}{\partial y},$ $\lvert y\rvert^{\frac{1}{1-a}}\frac{\partial}{\partial x}$	$x'=T_1(t)+yT_2(t)+xT_3(t)+z\lvert y\rvert^{\frac{1}{1-a}}T_4(t),$ $y'=(1-a)yT_3(t),$ $z'=0$
	31	$\frac{\partial}{\partial x},\frac{\partial}{\partial y},x\frac{\partial}{\partial x}+ay\frac{\partial}{\partial y}+\frac{\partial}{\partial z},$ $e^{az}\frac{\partial}{\partial y}$	$x'=T_1(t)+xT_3(t)+e^zT_4(t),$ $y'=T_2(t)+ayT_3(t)+e^{az}T_4(t),$ $z'=T_3(t)$
VIII	32	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ (bx+y)\frac{\partial}{\partial x}+(by-x)\frac{\partial}{\partial y},\ \frac{\partial}{\partial z}$	$x'=T_1(t)+(bx+y)T_3(t),$ $y'=T_2(t)+(by-x)T_3(t),$ $z'=T_4(t)$
	33	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ (bx+y)\frac{\partial}{\partial x}+(by-x)\frac{\partial}{\partial y}+\frac{\partial}{\partial z},\ e^{bz}(\cos(z)\frac{\partial}{\partial x}-\sin(z)\frac{\partial}{\partial y})$	$x'=T_1(t)+(bx+y)T_3(t)+e^{bz}\cos(z)T_4(t),$ $y'=T_2(t)+(by-x)T_3(t)-e^{bz}\sin(z)T_4(t),$ $z'=T_3(t)$
	34	$\frac{\partial}{\partial x},\ y\frac{\partial}{\partial x},\ (b-y)x\frac{\partial}{\partial x}-(1+y^2)\frac{\partial}{\partial y},\ \frac{\partial}{\partial z}$	$x'=T_1(t)+yT_2(t)+(b-y)xT_3(t),$ $y'=-(1+y^2)T_3(t),$ $z'=T_4(t)$

Table 2 (Continued)

Type	#	Operators	Systems
	35	$\frac{\partial}{\partial x}, y\frac{\partial}{\partial x},$ $(b-y)x\frac{\partial}{\partial x} - (1+y^2)\frac{\partial}{\partial y},$ $z\sqrt{1+y^2}\exp(-b\arctan(y))\frac{\partial}{\partial x}$	$x' = T_1(t) + yT_2(t) + (b-y)xT_3(t) + z\sqrt{1-y^2}\exp(-b\arctan(y))T_4(t),$ $y' = -(1+y^2)T_3(t),$ $z' = 0$
	36	$\frac{\partial}{\partial x}, y\frac{\partial}{\partial x},$ $(b-y)x\frac{\partial}{\partial x} - (1+y^2)\frac{\partial}{\partial y},$ $\sqrt{1+y^2}\exp(-b\arctan(y))\frac{\partial}{\partial x}$	$x' = T_1(t) + yT_2(t) + (b-y)xT_3(t) + \sqrt{1-y^2}\exp(-b\arctan(y))T_4(t),$ $y' = -(1+y^2)T_3(t),$ $z' = 0$
XI	37	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z},\ y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}$	$x' = T_1(t) + yT_4(t),$ $y' = T_2(t) + zT_4(t),$ $z' = T_3(t)$
	38	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, -\frac{1}{2}z^2\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, y\frac{\partial}{\partial x} - \frac{\partial}{\partial z}$	$x' = T_1(t) - \frac{1}{2}z^2T_3(t) + yT_4(t),$ $y' = T_2(t) + zT_3(t),$ $z' = -T_4(t)$
	39	$\frac{\partial}{\partial x},\ y\frac{\partial}{\partial x},\ \frac{\partial}{\partial z},\ yz\frac{\partial}{\partial x} - \frac{\partial}{\partial y}$	$x' = T_1(t) + yT_2(t) + yzT_4(t),$ $y' = -T_4(t),$ $z' = T_3(t)$
	40	$\frac{\partial}{\partial x},\ y\frac{\partial}{\partial x},\ z\frac{\partial}{\partial x},\ -\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$	$x' = T_1(t) + yT_2(t) + zT_4(t),$ $y' = -T_4(t),$ $z' = -yT_4(t)$
	41	$\frac{\partial}{\partial x},\ y\frac{\partial}{\partial x},\ \frac{1}{2}y^2\frac{\partial}{\partial x},\ -\frac{\partial}{\partial y}$	$x' = T_1(t) + yT_2(t) + \frac{1}{2}y^2T_3(t),$ $y' = -T_4(t),$ $z' = 0$
XII	42	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z},$ $bx\frac{\partial}{\partial x} + (y+z)\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$	$x' = T_1(t) + bxT_4(t),$ $y' = T_2(t) + (y+z)T_4(t),$ $z' = T_3(t) + zT_4(t)$
	43	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ z\frac{\partial}{\partial y}, bx\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{\partial}{\partial z}$	$x' = T_1(t) + bxT_4(t),$ $y' = T_2(t) + zT_3(t) + yT_4(t),$ $z' = -T_4(t)$
	44	$\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, \frac{\partial}{\partial z}, (bx+yz)\frac{\partial}{\partial x} + (b-1)y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$	$x' = T_1(t) + yT_2(t) + (bx+yz)T_4(t),$ $y' = (b-1)yT_4(t),$ $z' = T_3(t) + zT_4(t)$
	45	$\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, z\frac{\partial}{\partial x}, bx\frac{\partial}{\partial x} + (b-1)y\frac{\partial}{\partial y} + ((b-1)z - y)\frac{\partial}{\partial z}$	$x' = T_1(t) + yT_2(t) + zT_3(t) + bxT_4(t),$ $y' = (b-1)yT_4(t),$ $z' = ((b-1)z - y)T_4(t)$
	46	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ e^{(1-b)z}\frac{\partial}{\partial x} + z\frac{\partial}{\partial y},$ $bx\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{\partial}{\partial z}$	$x' = T_1(t) + e^{(1-b)z}T_3(t) + bxT_4(t),$ $y' = T_2(t) + zT_3(t) + yT_4(t),$ $z' = -T_4(t)$
	47	$\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, \frac{y}{1-b}\ln\|y\|\frac{\partial}{\partial x}, bx\frac{\partial}{\partial x} + (b-1)y\frac{\partial}{\partial y}$	$x' = T_1(t) + yT_2(t) + \frac{y}{1-b}\ln\|y\|T_3(t) + bxT_4(t),$ $y' = (b-1)yT_4(t),$ $z' = 0$
XIII	48	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z},\ x\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}$	$x' = T_1(t) + xT_4(t),$ $y' = T_2(t) + zT_4(t),$ $z' = T_3(t)$
	49	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \varepsilon e^{-z}\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - \frac{\partial}{\partial z}$	$x' = T_1(t) + \varepsilon e^{-z}T_3(t) + xT_4(t),$ $y' = T_2(t) + zT_3(t),$ $z' = -T_4(t)$
	50	$\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, \frac{\partial}{\partial z}, (x+yz)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$	$x' = T_1(t) + yT_2(y) + (x+yz)T_4(t),$ $y' = yT_4(t),$ $z' = T_3(t)$
	51	$\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, z\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + (z-y)\frac{\partial}{\partial z}$	$x' = T_1(t) + yT_2(t) + zT_3(t) + xT_4(t),$ $y' = yT_4(t),$ $z' = (z-y)T_4(t)$

Table 2 (Continued)

Type	#	Operators	Systems
	52	$\frac{\partial}{\partial x}$, $y\frac{\partial}{\partial x}$, $-y\ln\lvert y\rvert\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$	$x'=T_1(t)+yT_2(t)-y\ln\lvert y\rvert T_3(t)+xT_4(t)$, $y'=yT_4(t)$, $z'=0$
XIV	53	$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$, $(x+y)\frac{\partial}{\partial x}+(y+z)\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}$	$x'=T_1(t)+(x+y)T_4(t)$, $y'=T_2(t)+(y+z)T_4(t)$, $z'=T_3(t)+zT_4(t)$
	54	$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $-\frac{1}{2}z^2\frac{\partial}{\partial x}+z\frac{\partial}{\partial y}$, $(x+y)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-\frac{\partial}{\partial z}$	$x'=T_1(t)-\frac{1}{2}z^2T_3(t)+(x+y)T_4(t)$, $y'=T_2(t)+zT_3(t)+yT_4(t)$, $z'=-T_4(t)$
	55	$\frac{\partial}{\partial x}$, $y\frac{\partial}{\partial x}$, $\frac{\partial}{\partial z}$, $(x+yz)\frac{\partial}{\partial x}-\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}$	$x'=T_1(t)+yT_2(t)+(x+yz)T_4(t)$, $y'=-T_4(t)$, $z'=T_3(t)+zT_4(t)$
	56	$\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, z\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}-\frac{\partial}{\partial y}-y\frac{\partial}{\partial z}$	$x'=T_1(t)+yT_2(t)+zT_3(t)+xT_4(t)$, $y'=-T_4(t)$, $z'=-yT_4(t)$
	57	$\frac{\partial}{\partial x}$, $y\frac{\partial}{\partial x}$, $\frac{1}{2}y^2\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial x}-\frac{\partial}{\partial y}$	$x'=T_1(t)+yT_2(t)+\frac{1}{2}y^2T_3(t)+xT_4(t)$, $y'=-T_4(t)$, $z'=0$
XV	58	$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$, $ax\frac{\partial}{\partial x}+by\frac{\partial}{\partial y}+cz\frac{\partial}{\partial z}$	$x'=T_1(t)+axT_4(t)$, $y'=T_2(t)+byT_4(t)$, $z'=T_3(t)+czT_4(t)$
	59	$\frac{\partial}{\partial x}$, $y\frac{\partial}{\partial x}$, $z\frac{\partial}{\partial x}$, $ax\frac{\partial}{\partial x}+(a-b)\,y\frac{\partial}{\partial y}+(a-c)\,z\frac{\partial}{\partial z}$	$x'=T_1(t)+yT_2(t)+zT_3(t)+axT_4(t)$, $y'=(a-b)yT_4(t)$, $z'=(a-c)zT_4(t)$
	60	$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $z\frac{\partial}{\partial x}+\varphi(z)\frac{\partial}{\partial y}$, $x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$	$x'=T_1(t)+zT_3(t)+axT_4(t)$, $y'=T_2(t)+\varphi(z)T_3(t)+yT_4(t)$, $z'=0$
	61	$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\varphi(y)\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial x}+\frac{\partial}{\partial z}$	$x'=T_1(t)+\varphi(y)T_3(t)+xT_4(t)$, $y'=T_2(t)$, $z'=T_4(t)$
	62	$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\varphi(y)\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial x}$	$x'=T_1(t)+\varphi(y)T_3(t)+xT_4(t)$, $y'=T_2(t)$, $z'=0$
	63	$\frac{\partial}{\partial x}$, $y\frac{\partial}{\partial x}$, $\frac{\partial}{\partial z}$, $x\frac{\partial}{\partial x}+cz\frac{\partial}{\partial z}$	$x'=T_1(t)+yT_2(t)+xT_4(t)$, $y'=0$, $z'=T_3(t)+czT_4(t)$
	64	$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\exp((1-c)z)\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\frac{\partial}{\partial z}$	$x'=T_1(t)+e^{(1-c)z}T_3(t)+xT_4(t)$, $y'=T_2(t)+yT_4(t)$, $z'=T_4(t)$
	65	$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\varepsilon_1\exp((a-1)z)\frac{\partial}{\partial x}+\varepsilon_2\exp((b-1)z\frac{\partial}{\partial y}$, $ax\frac{\partial}{\partial x}+by\frac{\partial}{\partial y}+\frac{\partial}{\partial z}$	$x'=T_1(t)+\varepsilon_1e^{(a-1)z}T_3(t)+axT_4(t)$, $y'=T_2(t)+\varepsilon_2e^{(b-1)z}T_3(t)+byT_4(t)$, $z'=T_4(t)$
	66	$\frac{\partial}{\partial x}$, $y\frac{\partial}{\partial x}$, $\frac{\partial}{\partial z}$, $ax\frac{\partial}{\partial x}+(a-b)\,y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}$	$x'=T_1(t)+yT_2(t)+axT_4(t)$, $y'=(a-b)yT_4(t)$, $z'=T_3(t)+zT_4(t)$
	67	$\frac{\partial}{\partial x}$, $\exp((a-b)y)\frac{\partial}{\partial x}$, $\exp((a-1)y)\frac{\partial}{\partial x}$, $ax\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$	$x'=T_1(t)+e^{(a-b)y}T_2(t)$ $+e^{(a-1)y}T_3(t)+axT_4(t)$, $y'=T_4(t)$, $z'=0$
XVI	68	$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$, $ax\frac{\partial}{\partial x}+(by+z)\,\frac{\partial}{\partial y}+(bz-y)\,\frac{\partial}{\partial z}$	$x'=T_1(t)+axT_4(t)$, $y'=T_2(t)+(by+z)T_4(t)$, $z'=T_3(t)+(bz-y)T_4(t)$

Table 2 (Continued)

Type	#	Operators	Systems
	69	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y}$, $\varepsilon\exp((b-a)\arctan(z))$ $\times\sqrt{1+z^2}\,\frac{\partial}{\partial x}+z\frac{\partial}{\partial y}$, $(ax-\varepsilon y\exp((b-a)\arctan(z))$ $\times\sqrt{1+z^2})\frac{\partial}{\partial x}+(b-z)y\frac{\partial}{\partial y}$ $-(1+z^2)\frac{\partial}{\partial z}$	$x'=T_1(t)$ $+\varepsilon\exp((b-a)\arctan(z))\sqrt{1+z^2}T_3(t)$ $+(ax-\varepsilon y\exp((b-a)\arctan(z))\sqrt{1+z^2})$ $\times T_4(t)$, $y'=T_2(t)+zT_3(t)+(b-z)yT_4(t)$, $z'=-(1+z^2)T_4(t)$
	70	$\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, z\frac{\partial}{\partial x}$, $ax\frac{\partial}{\partial x}+[(a-b)y+z]\,\frac{\partial}{\partial y}$ $+[(a-b)z-y]\,\frac{\partial}{\partial z}$	$x'=T_1(t)+yT_2(t)+zT_3(t)+axT_4(t)$, $y'=[(a-b)y+z]\,T_4(t)$, $z'=[(a-b)z-y]\,T_4(t)$
	71	$\frac{\partial}{\partial x}, \exp((a-b)y)\cos(y)\frac{\partial}{\partial x}$, $-\exp((a-b)y)\sin(y)\frac{\partial}{\partial x}$, $ax\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$	$x'=T_1(t)+e^{(a-b)y}\cos(y)T_2(t)$ $-e^{(a-b)y}\sin(y)T_3(t)+axT_4(t)$, $y'=T_4(t)$, $z'=0$
XVII	72	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ y\frac{\partial}{\partial x}+\frac{\partial}{\partial z}$, $\left(2x+\frac{1}{2}z^2\right)\frac{\partial}{\partial x}+(y+z)\,\frac{\partial}{\partial y}+$ $z\frac{\partial}{\partial z}$	$x'=T_1(t)+yT_3(t)+\left(2x+\frac{1}{2}z^2\right)T_4(t)$, $y'=T_2(t)+(y+z)T_4(t)$, $z'=T_3(t)+zT_4(t)$
	73	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ y\frac{\partial}{\partial x}+z\frac{\partial}{\partial y},\ 2x\frac{\partial}{\partial x}+$ $y\frac{\partial}{\partial y}-\frac{\partial}{\partial z}$	$x'=T_1(t)+yT_3(t)+2xT_4(t)$, $y'=T_2(t)+zT_3(t)+yT_4(t)$, $z'=-T_4(t)$
	74	$\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}$, $\left(2x+\frac{1}{2}y^2\right)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\frac{\partial}{\partial z}$	$x'=T_1(t)+yT_3(t)+\left(2x+\frac{1}{2}y^2\right)T_4(t)$, $y'=-T_3(t)+yT_4(t)$, $z'=T_4(t)$
	75	$\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}$, $\left(2x+\frac{1}{2}y^2\right)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$	$x'=T_1(t)+yT_3(t)+\left(2x+\frac{1}{2}y^2\right)T_4(t)$, $y'=-T_3(t)+yT_4(t)$, $z'=0$
XVIII	76	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial x}+\frac{\partial}{\partial z}$, $(1+b)\,x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+bz\frac{\partial}{\partial z}$	$x'=T_1(t)+yT_3(t)+(1+b)xT_4(t)$, $y'=T_2(t)+yT_4(t)$, $z'=T_3(t)+bzT_4(t)$
	77	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ y\frac{\partial}{\partial x}+z\frac{\partial}{\partial y}$, $(1+b)\,x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+$ $(1-b)\,z\frac{\partial}{\partial z}$	$x'=T_1(t)+yT_3(t)+(1+b)xT_4(t)$, $y'=T_2(t)+zT_3(t)+yT_4(t)$, $z'=(1-b)zT_4(t)$
	78	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ y\frac{\partial}{\partial x},\ (1+b)\,x\frac{\partial}{\partial x}+$ $y\frac{\partial}{\partial y}+\frac{\partial}{\partial z}$	$x'=T_1(t)+yT_3(t)+(1+b)xT_4(t)$, $y'=T_2(t)+yT_4(t)$, $z'=T_4(t)$
	79	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ y\frac{\partial}{\partial x},\ (1+b)\,x\frac{\partial}{\partial x}+$ $y\frac{\partial}{\partial y}$	$x'=T_1(t)+yT_3(t)+(1+b)xT_4(t)$, $y'=T_2(t)+yT_4(t)$, $z'=0$
	80	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ y\frac{\partial}{\partial x},\ z\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$	$x'=T_1(t)+yT_3(t)+zT_4(t)$, $y'=T_2(t)+yT_4(t)$, $z'=0$
	81	$\frac{\partial}{\partial x},\ y\frac{\partial}{\partial x},\ -\frac{\partial}{\partial y},\ (1+b)\,x\frac{\partial}{\partial x}+$ $by\frac{\partial}{\partial y}+\frac{\partial}{\partial z}$	$x'=T_1(t)+yT_2(t)+(1+b)xT_4(t)$, $y'=-T_3(t)+byT_4(t)$, $z'=T_4(t)$
	82	$\frac{\partial}{\partial x},\ y\frac{\partial}{\partial x},\ -\frac{\partial}{\partial y},\ (1+b)\,x\frac{\partial}{\partial x}+$ $by\frac{\partial}{\partial y}$	$x'=T_1(t)+yT_2(t)+(1+b)xT_4(t)$, $y'=-T_3(t)+byT_4(t)$, $z'=0$
	83	$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ y\frac{\partial}{\partial x}+\frac{\partial}{\partial z},\ x\frac{\partial}{\partial x}+$ $y\frac{\partial}{\partial y}+C\frac{\partial}{\partial z}$	$x'=T_1(t)+yT_3(t)+xT_4(t)$, $y'=T_2(t)+yT_4(t)$, $z'=T_3(t)+CT_4(t)$

Table 2 (Continued)

Type	#	Operators	Systems
XIX	84	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial x} + \frac{\partial}{\partial z}$, $\frac{1}{2}\left(4ax + z^2 - y^2\right)\frac{\partial}{\partial x} + (ay+z)\frac{\partial}{\partial y} + (az-y)\frac{\partial}{\partial z}$	$x' = T_1(t) + yT_3(t) + \frac{1}{2}(4ax - y^2 + z^2)T_4(t)$, $y' = T_2(t) + (ay+z)T_4(t)$, $z' = T_3(t) + (az-y)T_4(t)$
XX	85	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$, $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + C\frac{\partial}{\partial z}$	$x' = T_1(t) + xT_3(t) + yT_4(t)$, $y' = T_2(t) + yT_3(t) - xT_4(t)$, $z' = T_3(t) + CT_4(t)$
	86	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + \frac{\partial}{\partial z}, -xy\frac{\partial}{\partial x} - \left(1+y^2\right)\frac{\partial}{\partial y}$	$x' = T_1(t) + xT_3(t) - xyT_4(t)$, $y' = T_2(t) - (1+y^2)T_4(t)$, $z' = T_3(t)$
	87	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$	$x' = T_1(t) + xT_3(t) + yT_4(t)$, $y' = T_2(t) + yT_3(t) - xT_4(t)$, $z' = T_4(t)$
	88	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$	$x' = T_1(t) + xT_3(t) + yT_4(t)$, $y' = T_2(t) + yT_3(t) - xT_4(t)$, $z' = 0$
	89	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}$, $-xy\frac{\partial}{\partial x} - \left(1+y^2\right)\frac{\partial}{\partial y}$	$x' = T_1(t) + xT_3(t) - xyT_4(t)$, $y' = T_2(t) - (1+y^2)T_4(t)$, $z' = 0$

Table 3. Dynamical systems with nonsolvable L_4

Type	#	Operators	Systems
IX	90	$\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}, y\frac{\partial}{\partial x} + 2yz\frac{\partial}{\partial y} + \left(z^2+c\right)\frac{\partial}{\partial z}$, $c \in \{-1; 0; 1\}$	$x' = T_1(t) + xT_2(t) + x^2T_3(t) + yT_4(t)$, $y' = yT_2(t) + 2xyT_3(t) + 2yzT_4(t)$, $z' = yT_3(t) + (z^2+c)T_4(t)$
	91	$\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, $\left(x^2+y^2\right)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}, \frac{\partial}{\partial z}$	$x' = T_1(t) + xT_2(t) + (x^2+y^2)T_3(t)$, $y' = yT_2(t) + 2xyT_3(t)$, $z' = T_4(t)$
	92	$\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, $\left(x^2-y^2\right)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}, \frac{\partial}{\partial z}$	$x' = T_1(t) + xT_2(t) + (x^2-y^2)T_3(t)$, $y' = yT_2(t) + 2xyT_3(t)$, $z' = T_4(t)$
	93	$\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, $x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}, \frac{\partial}{\partial z}$	$x' = T_1(t) + xT_2(t) + x^2T_3(t)$, $y' = yT_2(t) + 2xyT_3(t)$, $z' = T_4(t)$
	94	$\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, $x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}, yz\frac{\partial}{\partial y}$	$x' = T_1(t) + xT_2(t) + x^2T_3(t)$, $y' = yT_2(t) + 2xyT_3(t) + yzT_4(t)$, $z' = 0$
	95	$\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, $x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}$	$x' = T_1(t) + xT_2(t) + x^2T_3(t)$, $y' = yT_2(t) + 2xyT_3(t) + yT_4(t)$, $z' = 0$
	96	$\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$	$x' = T_1(t) + xT_2(t) + x^2T_3(t)$, $y' = T_4(t)$, $z' = 0$
X	97	$-\sin(x)\tan(y)\frac{\partial}{\partial x} - \cos(x)\frac{\partial}{\partial y}$, $\frac{\partial}{\partial x}$, $\sin(x)\frac{\partial}{\partial y} - \cos(x)\tan(y)\frac{\partial}{\partial x}$, $\frac{\partial}{\partial z}$	$x' = T_2(t) - \sin(x)\tan(y)T_1(t) - \cos(x)\tan(y)T_3(t)$, $y' = \sin(x)T_3(t) - \cos(x)T_1(t)$, $z' = T_4(t)$
	98	$\sin(x)\sec(y)\frac{\partial}{\partial z} - \sin(x)\tan(y)\frac{\partial}{\partial x} - \cos(x)\frac{\partial}{\partial y}$, $\sin(x)\frac{\partial}{\partial y} - \cos(x)\tan(y)\frac{\partial}{\partial x} + \cos(x)\sec(y)\frac{\partial}{\partial z}$, $\frac{\partial}{\partial x}, \frac{\partial}{\partial z}$	$x' = T_3(t) - \sin(x)\tan(y)T_1(t) - \cos(x)\tan(y)T_2(t)$, $y' = \sin(x)T_2(t) - \cos(x)T_1(t)$, $z' = T_4(t) + \sin(x)\sec(y)T_1(t) + \cos(x)\sec(y)T_2(t)$