Journal of Applied Analysis and Computation
Volume 7, Number 2, May 2017, 439–454

Website:http://jaac-online.com/
DOI:10.11948/2017028

ON PERIOD-K SOLUTION FOR A POPULATION SYSTEM WITH STATE-DEPENDENT IMPULSIVE EFFECT*

Xiaowei Tang[1,2] and Xilin Fu[1,†]

Abstract The period-k solutions of population differential system with state-dependent impulsive effect are investigated by the theory of discontinuous dynamical system. Through G-function theory, the necessary and sufficient conditions are obtained for trajectory direction of a population differential system, and the results are better than the previous work. Also, the local stability of period-k solutions is studied by the mapping structure and the theory of eigenvalue analysis. Furthermore, the existence of period-1 solution is investigated for a special impulsive population differential system, and the analytical condition is established. Finally, the trajectory of period-1 solution and the relationship between different parameters and the module of eigenvalues are illustrated.

Keywords Period-k solution, local stability, mapping structure, trajectory direction.

MSC(2010) 34K13, 49N25.

1. Introduction

Pest control has been the focus of all governments in the world. In 2014, the occurrence area of corn borer in China was 3.53 million acres. In 2015, rice planthopper, cnaphalocrocismedinalis, corn borer, aphids in wheat spike caused a lot of damage. Pest control affects the yield of crops and farmers' income, which is not only in China, but also other countries .

Pest control has a close relation with population differential system. The traditional population differential system is continuous. In production, control measures are usually taken to make the pest population under the Economic Threshold (ET). For example, the traditional approach for pest control relies on the seasonal use of chemical pesticides, which have an inherent discontinuity. Usually, we describe this discontinuity by impulse, and it is necessary to establish the model of impulsive population differential system [21, 28, 33].

Since the 1990's, impulsive systems have significant development from theory to application [6–9, 12, 19, 20, 24, 26, 36]. Impulsive differential systems and impulsive functional differential systems are involved. Some of the systems have impulse at fixed time and others are with state-dependent impulse. Existence of solutions,

[†]the corresponding author. Email address:xilinfu@gmail.com(X. Fu)
[1]School of Mathematical Sciences, Shandong Normal University, Ji'nan, 250014, China
[2]Mathematical School, Qilu Normal University, Jinan, 250013, China
*The authors were supported by National Natural Science Foundation of China (11571208) and the Specialized Research Fund for the Doctoral Program of Higher Education of China(23704110001).

stability of trivial solutions, boundedness, existence and stability of periodic solutions are involved. And in many other fields, such as physics, engineering, control, finance, economics, impulsive differential systems have wide applications [9,12,24].

In the last 20 years, many scholars devoted themselves to the study of impulsive population differential systems [2,4,14,16,22,29,34]. By establishing impulsive condition according with reality, the existence, persistence and stability of periodic solution for impulsive population differential systems were studied. In paper [4], a predator-prey system with stocking of prey and harvesting of predator impulsively was studied. The prey population was stocked with a constant quantity and the predator population was harvested at a rate proportional to the species itself at fixed moments. The existence and global asymptotic stability of the periodic solution were proved under some conditions. Liu and Zhang [22] dealt with the effects of pulse toxicant input with constant rate on two-species Lotka-Volterra competition system in a polluted environment. The thresholds between persistence and extinction of each population were obtained. Jiang et al. [16] considered a stage-structured pest management system with impulsive effects and constant coefficients. The sufficient conditions of the existence, uniqueness and orbitally asymptotic stability of periodic solution were obtained.

As we can see, many scholars choose constant and fixed moment impulsive condition [4,16,22]. But in reality, state-dependent impulse are more practical. By using the properties of the Lambert W function and Poincare map, Tang and Cheke [29] proved that there was no periodic solution of a state-dependent impulsive models with order larger than or equal to three, except for one special case. Jiang [14] studied a Holling type II prey-predator impulsive system with state feedback control. The existence and stability of the semi-trivial periodic solution were presented by Poincare map. Zeng et al. [34] got an existence theorem of order one periodic solution for a general planar autonomous impulsive system by the qualitative theory of ordinary differential equations and geometry method. By using Floquet's theory and comparison techniques, Baek [2] analyzed the dynamics of the Holling-type IV two-competitive-prey one-predator system with impulsive perturbations and seasonal effects. Sufficient conditions for the local and global stabilities of the two-prey-free periodic solution were established. Huang and Song [13] used differential geometric theory and subsequent function studied the existence of an order-1 periodic solution for a population system with state impulsive feedback and gave the existence of the periodic solutions for a special system. Wang et al. [31] analyzed a model concerning biologically-based impulsive control strategy for pest control. It showed that there existed a globally stable susceptible pest eradication periodic solution when the impulsive period was less than some critical value. Liang et al. [23] developed a novel pest population growth model incorporating the evolution of pesticide resistance. Three pesticide switching methods, threshold condition-guided, density-guided and EIL-guided were modelled to determine the best choice under different condition. By using qualitative analysis method, Nie et al. [25] obtained that the control model exhibited two stable positive period-1 solution under some general conditions. For more works, one can refer to [3,10,11,15,27,32,35] and the references therein. In the study of impulsive population differential systems, the existence and stability of periodic solutions are particularly important. All of these works provide a theoretical support for the production of agriculture and animal husbandry.

When investigating the motion in Coulomb friction oscillator, Filippov [5] presented a differential dynamical system with discontinuous right-hand sides. Since then, Filippov's discontinuous theory has been applied to many other fields [1,30]. Albert Luo in paper [17] investigated the dynamical behavior of a discontinuous dynamical system. He gave the sufficient and necessary conditions for a flow to be a semi-passable flow or a non-passable flow of the first kind from Ω_i to Ω_j at point (X_m, t_m) on the boundary $\partial\Omega_{ij}$. On the basis of [13,17], we consider a general population differential system and a population differential system with state-dependent impulsive effect.

In part 2, we will establish necessary and sufficient conditions for trajectory direction of a population differential system. Using such conditions, we consider a special two species predator-prey differential system. And the sufficient condition for the uniqueness of period-1 solution of the system is given. In part 3, with mapping dynamical theory, we will present the local stability for period-k solutions of a population differential system with state-dependent impulsive effect. In part 4, the existence and local stability will be given for a special impulsive population differential system. The reliability of regression analysis is illustrated through the numerical simulation at last.

2. Trajectory direction of population differential system

Consider a general population differential system

$$\begin{cases} \dfrac{dx}{dt} = f_1(x, y), \\ \dfrac{dy}{dt} = f_2(x, y), \end{cases} \tag{2.1}$$

and a population differential system with state-dependent impulsive effect

$$\begin{cases} \left.\begin{aligned} \dfrac{dx}{dt} &= f_1(x, y), \\ \dfrac{dy}{dt} &= f_2(x, y), \end{aligned}\right\} X \notin M(x, y), \\ \left.\begin{aligned} \Delta x &= g_1(x, y), \\ \Delta y &= g_2(x, y), \end{aligned}\right\} X \in M(x, y), \end{cases} \tag{2.2}$$

where $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $X \in \Omega$, Ω is an open set in R_+^2. $f_1(x, y)$, $f_2(x, y)$, $g_1(x, y)$ and $g_2(x, y)$ are continuous in Ω. $M = M(x, y)$ denotes impulsive set and $N = \{(x, y)|x + g_1(x, y), y + g_2(x, y)\} \in R_2^+$ denotes phase set.

Suppose that domain Ω is divided into several small domains $\Omega_i, i \in I$ by the isoclinic lines of the vector fields of system (2.1). Let $\Omega = \bigcup\limits_{i \in I} \Omega_i$, and the separation boundary of adjacent domain Ω_i and Ω_j is defined as $\partial\Omega_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j = \{(x, y)|\phi_{ij}(x, y) = 0\}$.

Let $F(X) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$, then system (2.1) can be written as

$$\frac{dX}{dt} = F(X) = F^{(i)}(X),\ i \in I. \tag{2.3}$$

Lemma 2.1. *Suppose that there is a point $X_m \in \partial\Omega_{ij}$ at time t_m between adjacent domain Ω_i and Ω_j, then the trajectory of system* (2.1) *from Ω_i will enter into Ω_j if and only if*

$$\left.\begin{array}{ll} \text{either} & \left.\begin{array}{l} n^T_{\partial\Omega_{ij}} \cdot F^{(i)}(X_m) > 0, \\ n^T_{\partial\Omega_{ij}} \cdot F^{(j)}(X_m) > 0, \end{array}\right\} n_{\partial\Omega_{ij}} \to \Omega_j, \\ \text{or} & \left.\begin{array}{l} n^T_{\partial\Omega_{ij}} \cdot F^{(i)}(X_m) < 0, \\ n^T_{\partial\Omega_{ij}} \cdot F^{(j)}(X_m) < 0, \end{array}\right\} n_{\partial\Omega_{ij}} \to \Omega_i. \end{array}\right\} \tag{2.4}$$

Proof. For a point $X_m \in \partial\Omega_{ij}$ with $n_{\partial\Omega_{ij}} \to \Omega_j$. Suppose the two flows $X^{(i)}(t)$ and $X^{(j)}(t)$ are in domain Ω_i and Ω_j. For an arbitrarily small $0 < \varepsilon < 1$, let $a \in [t_{m-\varepsilon}, t_{m-}]$, $b \in [t_{m+}, t_{m+\varepsilon}]$, where t_{m-} and t_{m+} reflect the responses in domain Ω_i and domain Ω_j. Using Taylor series, we have

$$\begin{aligned} X^{(i)}(t_{m-\varepsilon}) &\equiv X^{(i)}(t_{m-} - \varepsilon) = X^{(i)}(a) + \dot{X}^{(i)}(a)(t_{m_-} - \varepsilon - a) + o(t_{m_-} - \varepsilon - a), \\ X^{(j)}(t_{m+\varepsilon}) &\equiv X^{(j)}(t_{m+} + \varepsilon) = X^{(j)}(b) + \dot{X}^{(j)}(b)(t_{m_+} + \varepsilon - b) + o(t_{m_+} + \varepsilon - b). \end{aligned}$$

Let $a \to t_{m_-}$ and $b \to t_{m_+}$, the above equations lead to

$$\begin{aligned} X^{(i)}(t_{m-\varepsilon}) &= X^{(i)}(t_{m_-}) - \dot{X}^{(i)}(t_{m_-})\varepsilon + o(\varepsilon), \\ X^{(j)}(t_{m+\varepsilon}) &= X^{(j)}(t_{m_+}) + \dot{X}^{(j)}(t_{m_+})\varepsilon + o(\varepsilon). \end{aligned}$$

Since $0 < \varepsilon < 1$, the following relation exist,

$$\begin{aligned} n^T{}_{\partial\Omega_{ij}} \cdot [X^{(i)}(t_{m_-}) - X^{(i)}(t_{m-\varepsilon})] &= n^T{}_{\partial\Omega_{ij}} \cdot \dot{X}^{(i)}(t_{m_-})\varepsilon = n^T{}_{\partial\Omega_{ij}} \cdot F^{(i)}(X(t_{m_-}))\varepsilon, \\ n^T{}_{\partial\Omega_{ij}} \cdot [X^{(j)}(t_{m+\varepsilon}) - X^{(j)}(t_{m_+})] &= n^T{}_{\partial\Omega_{ij}} \cdot \dot{X}^{(j)}(t_{m+})\varepsilon = n^T{}_{\partial\Omega_{ij}} \cdot F^{(j)}(X(t_{m_+}))\varepsilon. \end{aligned}$$

If the trajectory from domain Ω_i enters into Ω_j, we will have

$$\begin{aligned} n^T{}_{\partial\Omega_{ij}} \cdot [X^{(i)}(t_{m_-}) - X^{(i)}(t_{m-\varepsilon})] &> 0, \\ n^T{}_{\partial\Omega_{ij}} \cdot [X^{(j)}(t_{m+\varepsilon}) - X^{(j)}(t_{m_+})] &> 0, \end{aligned}$$

which implies

$$\begin{aligned} n^T{}_{\partial\Omega_{ij}} \cdot F^{(i)}(X(t_{m_-})) &= n^T{}_{\partial\Omega_{ij}} \cdot F^{(i)}(X_m) > 0, \\ n^T{}_{\partial\Omega_{ij}} \cdot F^{(j)}(X(t_{m_+})) &= n^T{}_{\partial\Omega_{ij}} \cdot F^{(j)}(X_m) > 0. \end{aligned}$$

In a similar manner, the flow at point (X_m, t_m) to the boundary $\partial\Omega_{ij}$ with $n_{\partial\Omega_{ij}} \to \Omega_i$ will enter into Ω_j with the second inequality equation in (2.4), and vice versa. $\square$

Remark 2.1. The function $n^T_{\partial\Omega_{ij}} \cdot [X^{(i)}(t_{m_-}) - X^{(i)}(t_{m-\varepsilon})]$ is defined as a G-function in [17]. So we call this method being the G-function method.

Now, consider the following two species predator-prey differential system

$$\begin{cases} \dfrac{dx}{dt} = x(a - rx - by), \\ \dfrac{dy}{dt} = y(cx - d). \end{cases} \tag{2.5}$$

System (2.5) can be used to describe the interaction between predator and prey. $x(t)$ is the population density of prey, $y(t)$ is the population density of predator. a, b, c, d are positive constants, which respectively denote the growth rate of prey, the predatory rate of predator, the conversion rate of prey into a predator and the death rate of predator. $r \geq 0$ is the population competition coefficient.

Suppose (H$_1$): $ac - dr > 0$, then system (2.5) has an unique positive equilibrium point $Z = (\dfrac{d}{c}, \dfrac{ac - dr}{bc})$, which is global asymptotically stable.

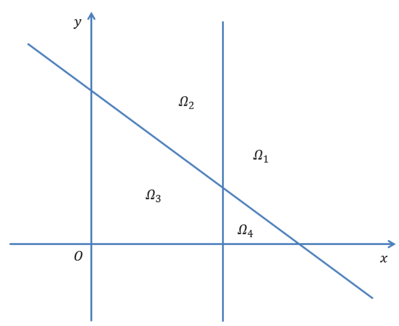


Figure 1. four subdomains

There are two isoclinic lines of system (2.5): $a - rx - by = 0$ and $cx - d = 0$, which divide the first quadrant R^2_+ into four domains

$$\Omega_1 = \{(x, y)|cx - d > 0, a - rx - by < 0\} \cap R^2_+,$$
$$\Omega_2 = \{(x, y)|cx - d < 0, a - rx - by < 0\} \cap R^2_+,$$
$$\Omega_3 = \{(x, y)|cx - d < 0, a - rx - by > 0\} \cap R^2_+,$$
$$\Omega_4 = \{(x, y)|cx - d > 0, a - rx - by > 0\} \cap R^2_+,$$

which are sketched in Fig. 1. The line perpendicular to the X axis is $cx - d = 0$, and the other line is $a - rx - by = 0$.

The separation boundary of adjacent domain Ω_i and Ω_j is defined as

$$\partial\Omega_{12} = \partial\Omega_{21} = \left\{(x, y)|x = \frac{d}{c}, y \geq \frac{ac - dr}{bc}\right\},$$
$$\partial\Omega_{23} = \partial\Omega_{32} = \left\{(x, y)|0 \leq x \leq \frac{d}{c}, a - rx - by = 0\right\},$$
$$\partial\Omega_{34} = \partial\Omega_{43} = \left\{(x, y)|x = \frac{d}{c}, 0 \leq y \leq \frac{ac - dr}{bc}\right\},$$
$$\partial\Omega_{41} = \partial\Omega_{14} = \left\{(x, y)|\frac{d}{c} \leq x \leq \frac{a}{r}, a - rx - by = 0\right\}.$$

Let $F(X) = \begin{pmatrix} x(a - rx - by) \\ y(cx - d) \end{pmatrix}$, then system (2.5) can be described as

$$\frac{dX}{dt} = F(X) = F^{(i)}(X),\ i = 1, 2, 3, 4. \tag{2.6}$$

Theorem 2.1. *Suppose that system* (2.5) *satisfies* (H_1), *then the trajectory of system* (2.5) *starting from domain* Ω_1 *will enter into* Ω_2, *from domain* Ω_2 *will enter into* Ω_3, *from domain* Ω_3 *will enter into* Ω_4, *and the trajectory of system* (2.5) *starting from domain* Ω_4 *will enter into* Ω_1.

Proof. The normal vectors of corresponding separation boundary are

$$n^T_{\partial\Omega_{12}} = (1, 0),\ n^T_{\partial\Omega_{23}} = (-r, -b),\ n^T_{\partial\Omega_{34}} = (1, 0),\ n^T_{\partial\Omega_{41}} = (-r, -b).$$

Suppose that the trajectory of system (2.5) from domain Ω_1 reaches the separation boundary $\partial\Omega_{12}$ at t_m, then

$$\begin{aligned} n^T_{\partial\Omega_{12}} \cdot F^{(1)}(X(t_{m-})) &= (1, 0) \cdot \begin{pmatrix} x(t_{m-}) \cdot (a - rx(t_{m-}) - by(t_{m-})) \\ y(t_{m-}) \cdot (cx(t_{m-}) - d) \end{pmatrix} \\ &= x(t_{m-}) \cdot (a - rx(t_{m-}) - by(t_{m-})) < 0. \end{aligned}$$

Similarly, $n^T_{\partial\Omega_{12}} \cdot F^{(2)}(X(t_{m+})) < 0$, where t_{m-} and t_{m+} reflect the responses in domain Ω_1 and domain Ω_2 rather than the separation boundary $\partial\Omega_{12}$. So if the trajectory of system (2.5) from domain Ω_1 reached the separation boundary $\partial\Omega_{12}$, it would enter into domain Ω_2.

On the other hand,

$$\begin{aligned} &n^T_{\partial\Omega_{14}} \cdot F^{(1)}(X(t_{n-})) = (-r, -b) \cdot \begin{pmatrix} x(t_{n-}) \cdot (a - rx(t_{n-}) - by(t_{n-})) \\ y(t_{n-}) \cdot (cx(t_{n-}) - d) \end{pmatrix} \\ &= (-r) \cdot [x(t_{n-}) \cdot (a - rx(t_{n-}) - by(t_{n-}))] - by(t_{n-}) \cdot [cx(t_{n-}) - d] < 0, \end{aligned}$$

where t_{n-} reflects the response in domain Ω_1. So if the trajectory of system (2.5) from domain Ω_1 reached the separation boundary $\partial\Omega_{14}$, it would not enter into domain Ω_4. Since Z is global asymptotically stable, so the trajectory of system (2.5) from domain Ω_1 will enter into domain Ω_2. Similarly, the left result is true. □

Definition 2.1. Suppose $P(x, y)$ is a point belonging to M. If the trajectory of system (2.1) from $Q : (x + g_1(x, y), y + g_2(x, y)) \in N$ would enter to point P, then system (2.2) has a period-1 solution $Z(t)$. At this time, we can say $Z(t)$ is after one time impulsive effect.

If the trajectory of solution $Z(t)$ from $P(x, y) \in M$ would enter to point P after k times impulsive effect, then $Z(t)$ is a period-k solution of system (2.2).

When the density of one species reaches the population capacity ET, people can take measures to maintain the ecological balance, such as capturing another species or increasing this species. Let $M = \{(x, y) | x = ET, y > 0\}$, consider the following two species predator-prey impulsive differential system

$$\begin{cases} \left.\begin{aligned} \frac{dx}{dt} &= x(a - rx - by), \\ \frac{dy}{dt} &= y(cx - d), \end{aligned}\right\} x < ET, \\ \left.\begin{aligned} \Delta x &= -px, \\ \Delta y &= q, \end{aligned}\right\} x = ET, \end{cases} \tag{2.7}$$

where $0 < p < 1, q > 0$.

Theorem 2.2. *Suppose that system* (2.7) *has a period-1 solution and* (H_1), (H_2), (H_3) *are satisfied, where*

(H_2) $\frac{a}{r} > ET > \frac{d}{c}$,

(H_3) $q > \frac{a}{b} - \frac{a}{r}(1-p)ET$,

then the period-1 solution of system (2.7) *is unique.*

Proof. Let

$$\Omega_5 = \left\{(x,y)|\frac{d}{c} < x < ET, a - rx - by < 0\right\} \cap R_+^2,$$
$$\Omega_6 = \left\{(x,y)|\frac{d}{c} < x < ET, a - rx - by > 0\right\} \cap R_+^2.$$

Suppose $Z_1(t)$ and $Z_2(t)$ are two period-1 solutions of system (2.7). Point $A = (ET, y_1)$ and point $B = (ET, y_2)$ are on the respective trajectories. Let $y_1 < y_2$. After one time impulsive effect, we get point $\overline{A} = ((1-p)ET, y_1 + q)$ and point $\overline{B} = ((1-p)ET, y_2 + q)$.

Since $y_1 + q < y_2 + q$, by Theorem 2.1, $Z_1(t)$ and $Z_2(t)$ must intersect in domain Ω_2, Ω_3 or Ω_6, which is impossible. Then the period-1 solution of system (2.7) is unique. □

3. Mapping structure and local stability analysis of state-dependent impulsive population system

Definition 3.1. Assume that $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is a period-k solution of system (2.2). If there exists a neighborhood U sufficiently small such that W limit set of trajectory starting from any point $Q \in U$ is always $X(t)$, then the period-k solution $X(t)$ of system (2.2) is stable, otherwise $X(t)$ is unstable.

Let $X(t)$ is a period-k solution of system (2.2) from point $P = (x_0, y_0) \in M$, whose period is T. Obviously, $X(t)$ reaches the impulsive set M finite times in a period T. Let impulsive time be $t_1, t_2, \ldots, t_k$, and impulsive point set $\{(x_i, y_i), i = 1, 2, \cdots, k\}$, then $t_k = t_0 + T$.

Let $\Omega = \{(x,y)|(x,y) \notin M\} \cap R_+^2$. Making mapping P_{i+1}^+ and P_{i+1} as followed:

$P^+_{i+1} : M \to \Omega$, satisfying

$$P^+_{i+1}(x_i, y_i) \equiv (x^+_i, y^+_i) = (x_i + g_1(x_i, y_i), y_i + g_2(x_i, y_i)), i = 0, 1, 2, \ldots, k-1.$$

Then the governing equations of mapping P^+_{i+1} can be expressed by

$$\begin{cases} x^+_i = x_i + g_1(x_i, y_i), \\ y^+_i = y_i + g_2(x_i, y_i), \\ t^+_i = t_i. \end{cases} \tag{3.1}$$

$P_{i+1} : \Omega \to M$, satisfying

$$P_{i+1}(x^+_i, y^+_i) = (x_{i+1}, y_{i+1}), i = 0, 1, 2, \ldots, k-1,$$

where (x_{i+1}, y_{i+1}) is the point on the trajectory of system (2.1) starting from (x^+_i, y^+_i, t_i). Then the governing equations of mapping P_{i+1} can be expressed by

$$\begin{cases} H(t^+_i, x^+_i, y^+_i, t_{i+1}, x_{i+1}, y_{i+1}) = 0, \\ G(t^+_i, x^+_i, y^+_i, t_{i+1}, x_{i+1}, y_{i+1}) = 0. \end{cases} \tag{3.2}$$

If system (2.1) is a linear differential system, Eq. (3.2) can be obtained quickly through elementary integral method. If system (2.1) is a nonlinear differential system, using the method in paper [18], we also can get Eq. (3.2) which is an implicit function.

Theorem 3.1. *The local stability for the period-k solution $X(t)$ of system* (2.2) *can be obtained by generalized eigenvalue analysis.*

Proof. For the above period-k solution $X(t)$, let

$$P = \underbrace{(P_1 \circ P^+_1) \circ (P_2 \circ P^+_2) \circ \ldots \circ (P_k \circ P^+_k)}_{k-\text{terms}},$$

then the mapping structure P satisfying

$$P(x_0, ET) = (x_0, ET). \tag{3.3}$$

For a little perturbation $\Delta X_i = \Delta(x_i, y_i)^T$, the variational equation of $X(t)$ for the local stability analysis is

$$\Delta X_{i+T} = DP \cdot \Delta X_i,$$

where

$$DP^+_{i+1} = \left.\begin{pmatrix} \frac{\partial x^+_i}{\partial x_i} & \frac{\partial x^+_i}{\partial y_i} \\ \frac{\partial y^+_i}{\partial x_i} & \frac{\partial y^+_i}{\partial y_i} \end{pmatrix}\right|_{(x_i, y_i, x^+_i, y^+_i)},$$

$$DP_{i+1} = \left.\begin{pmatrix} \frac{\partial x_{i+1}}{\partial x^+_i} & \frac{\partial x_{i+1}}{\partial y^+_i} \\ \frac{\partial y_{i+1}}{\partial x^+_i} & \frac{\partial y_{i+1}}{\partial y^+_i} \end{pmatrix}\right|_{(x^+_i, y^+_i, x_{i+1}, y_{i+1})}, \ i = 0, 1, 2, \ldots, k-1,$$

$$DP = \underbrace{(DP^+_1 \cdot DP_1) \cdot (DP^+_2 \cdot DP_2) \cdot \ldots \cdot (DP^+_k \cdot DP_k)}_{k-\text{terms}}.$$

From Eq. (3.1) and Eq. (3.2), the generalized characteristic equation of mapping structure P is

$$\det(DP - \lambda I) = 0, \tag{3.4}$$

where I is an unit matrix. Suppose the eigenvalues for the mapping structure of the period-k solution $X(t)$ are λ_1 and λ_2. Without the local singularity involving with discontinuity causing by impulse, the eigenvalue analysis can give an accurate prediction:

(i) If $|\lambda_i| < 1, i = 1, 2$, then the period-$k$ solution $X(t)$ of system (2.2) is stable.

(ii) If $|\lambda_i| > 1, i = 1$ or 2, then the period-k solution $X(t)$ of system (2.2) is unstable.

(iii) If $|\lambda_i| = 1, i = 1$ or 2, then the bifurcation phenomenon occurs.

□

4. Existence of period-k solution for a special impulsive population system

Let $M = \{(x, y)|x > 0, y = ET\}$, consider a population differential system with state-dependent impulsive effect

$$\begin{cases} \left.\begin{aligned} \frac{dx}{dt} &= -ax + by, \\ \frac{dy}{dt} &= cx - dy, \end{aligned}\right\} y < ET, \\ \left.\begin{aligned} \Delta x &= px, \\ \Delta y &= qy, \end{aligned}\right\} y = ET, \end{cases} \tag{4.1}$$

where $-1 < p < 0, -1 < q < 0$ are constant. Take insect pest population as an example. $x = x(t)$ is the population density of larvae and $y = y(t)$ is the population density of adult. $b > 0$ is the birth rate of adult , $d > 0$ is the death rate of adult. Let $b > d, a = c_1 + c$, where c is the conversion rate from larvae to adult, and c_1 is the death rate of larvae, $a > c > 0$.

From Eq. (3.1)and Eq. (3.2), the governing equations of mapping P^+_{i+1} and P_{i+1} are

$$\begin{cases} x_i^+ = (1 + p)x_i, \\ y_i^+ = (1 + q)y_i, \\ t_i^+ = t_i, \end{cases}$$

and

$$\begin{cases} \begin{aligned} x_{i+1} = &\frac{d + \alpha_1}{\alpha_1 - \alpha_2}(x_i^+ - \frac{d + \alpha_2}{c} y_i^+)e^{\alpha_1(t_{i+1} - t_i^+)} \\ &+ \frac{d + \alpha_2}{\alpha_2 - \alpha_1}(x_i^+ - \frac{d + \alpha_1}{c} y_i^+)e^{\alpha_2(t_{i+1} - t_i^+)}, \\ y_{i+1} = &\frac{c}{\alpha_1 - \alpha_2}(x_i^+ - \frac{d + \alpha_2}{c} y_i^+)e^{\alpha_1(t_{i+1} - t_i^+)} \\ &+ \frac{c}{\alpha_2 - \alpha_1}(x_i^+ - \frac{d + \alpha_1}{c} y_i^+)e^{\alpha_2(t_{i+1} - t_i^+)}, \end{aligned} \end{cases}$$

where $\alpha_1 = \frac{-a-d+\sqrt{(a-d)^2+4bc}}{2}, \alpha_2 = \frac{-a-d-\sqrt{(a-d)^2+4bc}}{2}$.

Suppose that $X(t)$ is a period-k solution of system (4.1) starting from (x_0, ET), then

$$\begin{cases} x_0^+ = (1+p)x_0,\ y_0^+ = (1+q)ET,\ t_0^+ = t_0, \\ x_1 = \dfrac{d+\alpha_1}{\alpha_1-\alpha_2}(x_0^+ - \dfrac{d+\alpha_2}{c}y_0^+)e^{\alpha_1(t_1-t_0^+)} + \dfrac{d+\alpha_2}{\alpha_2-\alpha_1}(x_0^+ - \dfrac{d+\alpha_1}{c}y_0^+)e^{\alpha_2(t_1-t_0^+)}, \\ ET = \dfrac{c}{\alpha_1-\alpha_2}(x_0^+ - \dfrac{d+\alpha_2}{c}y_0^+)e^{\alpha_1(t_1-t_0^+)} + \dfrac{c}{\alpha_2-\alpha_1}(x_0^+ - \dfrac{d+\alpha_1}{c}y_0^+)e^{\alpha_2(t_1-t_0^+)}, \\ x_1^+ = (1+p)x_1,\ y_1^+ = (1+q)ET,\ t_1^+ = t_1, \\ x_2 = \dfrac{d+\alpha_1}{\alpha_1-\alpha_2}(x_1^+ - \dfrac{d+\alpha_2}{c}y_1^+)e^{\alpha_1(t_2-t_1^+)} + \dfrac{d+\alpha_2}{\alpha_2-\alpha_1}(x_1^+ - \dfrac{d+\alpha_1}{c}y_0^+)e^{\alpha_2(t_2-t_1^+)}, \\ ET = \dfrac{c}{\alpha_1-\alpha_2}(x_1^+ - \dfrac{d+\alpha_2}{c}y_1^+)e^{\alpha_1(t_2-t_1^+)} + \dfrac{c}{\alpha_2-\alpha_1}(x_1^+ - \dfrac{d+\alpha_1}{c}y_1^+)e^{\alpha_2(t_2-t_1^+)}, \\ \qquad\qquad \vdots \\ x_{k-1}^+ = (1+p)x_{k-1},\ y_1^+ \ = (1+q)ET,\ t_{k-1}^+ = t_{k-1}, \\ x_0 = \dfrac{d+\alpha_1}{\alpha_1-\alpha_2}(x_{k-1}^+ - \dfrac{d+\alpha_2}{c}y_{k-1}^+)e^{\alpha_1(t_k-t_{k-1}^+)} \\ \qquad + \dfrac{d+\alpha_2}{\alpha_2-\alpha_1}(x_{k-1}^+ - \dfrac{d+\alpha_1}{c}y_{k-1}^+)e^{\alpha_2(t_k-t_{k-1}^+)}, \\ ET = \dfrac{c}{\alpha_1-\alpha_2}(x_{k-1}^+ - \dfrac{d+\alpha_2}{c}y_{k-1}^+)e^{\alpha_1(t_k-t_{k-1}^+)} \\ \qquad + \dfrac{c}{\alpha_2-\alpha_1}(x_{k-1}^+ - \dfrac{d+\alpha_1}{c}y_{k-1}^+)e^{\alpha_2(t_k-t_{k-1}^+)}. \end{cases} \tag{4.2}$$

If Eq. (4.2) has a solution, then there must be a period-k solution of system (4.1). Especially, if $X(t)$ is a periodic solution of system (4.1), whose period is T, we will have the following theorem.

Theorem 4.1. *Suppose that $X(t)$ is a periodic solution of system* (4.1) *starting from (x_i, ET), then $X(t)$ is a period-1 solution unless* (4.3) *is satisfied, where*

$$\frac{\dfrac{(d+\alpha_1)(d+\alpha_2)}{c(\alpha_1-\alpha_2)}(1+q)(e^{\alpha_1 T} - e^{\alpha_2 T})}{\dfrac{d+\alpha_1}{\alpha_1-\alpha_2}(1+p)e^{\alpha_1 T} + \dfrac{d+\alpha_2}{\alpha_2-\alpha_1}(1+p)e^{\alpha_2 T} - 1} = \frac{1 + \dfrac{1+q}{\alpha_1-\alpha_2}[(d+\alpha_2)e^{\alpha_1 T} - (d+\alpha_1)e^{\alpha_2 T}]}{\dfrac{c(1+p)}{\alpha_1-\alpha_2}(e^{\alpha_1 T} - e^{\alpha_2 T})}. \tag{4.3}$$

At the same time,

$$DP = \begin{pmatrix} (1+p)(\frac{d+\alpha_1}{\alpha_1-\alpha_2}e^{\alpha_1 T} + \frac{d+\alpha_2}{\alpha_2-\alpha_1}e^{\alpha_2 T}) & \begin{array}{l} -(1+p)[\frac{(d+\alpha_1)(d+\alpha_2)}{c(\alpha_1-\alpha_2)}e^{\alpha_1 T} \\ +\frac{(d+\alpha_1)(d+\alpha_2)}{c(\alpha_2-\alpha_1)}e^{\alpha_2 T}] \end{array} \\ (1+q)(\frac{c}{\alpha_1-\alpha_2}e^{\alpha_1 T} + \frac{c}{\alpha_2-\alpha_1}e^{\alpha_2 T}) & -(1+q)(\frac{d+\alpha_2}{\alpha_1-\alpha_2}e^{\alpha_1 T} + \frac{d+\lambda_1}{\alpha_2-\alpha_1}e^{\alpha_2 T}) \end{pmatrix}_{2\times 2}.$$

Proof. If $X(t)$ is a period-k solution of system (4.1) starting from (x_i, ET), then from Eq. (4.2), we have

$$\begin{cases} x_{i+1} = \dfrac{d+\alpha_1}{\alpha_1-\alpha_2}[(1+p)x_i - \dfrac{d+\alpha_2}{c}(1+q)ET]e^{\alpha_1(t_{i+1}-t_i^+)} \\ \qquad + \dfrac{d+\alpha_2}{\alpha_2-\alpha_1}[(1+p)x_i - \dfrac{d+\alpha_1}{c}(1+q)ET]e^{\alpha_2(t_{i+1}-t_i^+)}, \\ ET = \dfrac{c}{\alpha_1-\alpha_2}[(1+p)x_i - \dfrac{d+\alpha_2}{c}(1+q)ET]e^{\alpha_1(t_{i+1}-t_i^+)} \\ \qquad + \dfrac{c}{\alpha_2-\alpha_1}[(1+p)x_i - \dfrac{d+\alpha_1}{c}(1+q)ET]e^{\alpha_2(t_{i+1}-t_i^+)}. \end{cases} \tag{4.4}$$

Combining with Eq. (4.3), we have $x_{i+1} = x_i$ and $t_{i+1} = t_i + T$, so $X(t)$ is a period-1 solution of system (4.1). And we can compute DP easily by the governing equations. □

Remark 4.1. Eq. (4.3) is also

$$\begin{aligned} \alpha_1 - \alpha_2 = {} & [(d+\alpha_1)(1+p) - (d+\alpha_2)(1+q)]e^{\alpha_1 T} \\ & +[(1+q)(d+\alpha_1) - (1+p)(d+\alpha_2)]e^{\alpha_2 T} \\ & -[(1+p)(1+q)(\alpha_1-\alpha_2)]e^{(\alpha_1+\alpha_2)T}. \end{aligned} \tag{4.5}$$

Remark 4.2. Eq. (4.3) and Eq. (4.5) describe the relationship between p, q and T. If Eq. (4.3) or Eq. (4.5) is satisfied, there must be a period-1 periodic solution of system (4.1).

Remark 4.3. The initial value (x_i, ET) of the order-1 periodic solution $X(t)$ satisfies

$$x_i = \frac{\dfrac{(d+\alpha_1)(d+\alpha_2)}{c(\alpha_1-\alpha_2)}(1+q)(e^{\alpha_1 T} - e^{\alpha_2 T})}{\dfrac{d+\alpha_1}{\alpha_1-\alpha_2}(1+p)e^{\alpha_1 T} + \dfrac{d+\alpha_2}{\alpha_2-\alpha_1}(1+p)e^{\alpha_2 T} - 1} \tag{4.6}$$

or

$$x_i = \frac{1 + \dfrac{1+q}{\alpha_1-\alpha_2}[(d+\alpha_2)e^{\alpha_1 T} - (d+\alpha_1)e^{\alpha_2 T}]}{\dfrac{c(1+p)}{\alpha_1-\alpha_2}(e^{\alpha_1 T} - e^{\alpha_2 T})}. \tag{4.7}$$

Now let $a = 1,\ b = 2,\ c = 0.8,\ d = 0.5,\ ET = 5$. We give the relationship of $p(q)$ and T in Figs. 2(a)-2(f) first. The blue (red) curve represents that there is (not) a period-1 solution of system (4.1) with corresponding parameters. And horizontal coordinate is the period T of the period-1 solution for system (4.1), vertical coordinate is the parameter p or q, respectively. (a)-(c) describe the relation between T and p. (d)-(f) depict the relationship between T and p. With the parameter $q = -0.2,\ q = -0.5,\ q = -0.8$ and $p = -0.2,\ p = -0.5,\ p = -0.8$, the range of period T for system (II-2) are (0.2016,1.33), (0.5968,2.2238), (1.148,3.94), (0.1008,2.086), (0.2982,2.986) and (0.5705,4.668) in (a)-(f).

Let $p = -0.5,\ q = -0.8,\ x_0 = 6.518,\ ET = y_0 = 5$, the period-1 solution of system (4.1) is sketched in Fig. 3. x_0 is decided by Eq. (4.6) or Eq. (4.7). We also can choose other p, q and x_0. Here, we won't describe more. Let $p = -0.5$,

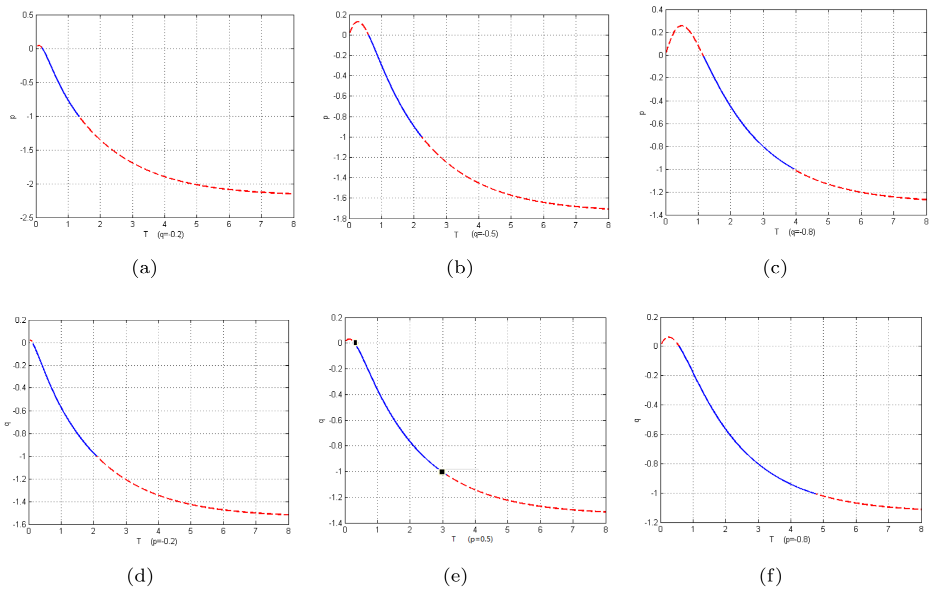


Figure 2. The relationship between T and p with parameter $q = -0.2,\ q = -0.5,\ q = -0.8$ in (a)-(c). The relationship between T and q with parameter $p = -0.2,\ p = -0.5$ and $p = -0.8$ in (d)-(f). $(a = 1,\ b = 2,\ c = 0.8,\ d = 0.5.)$

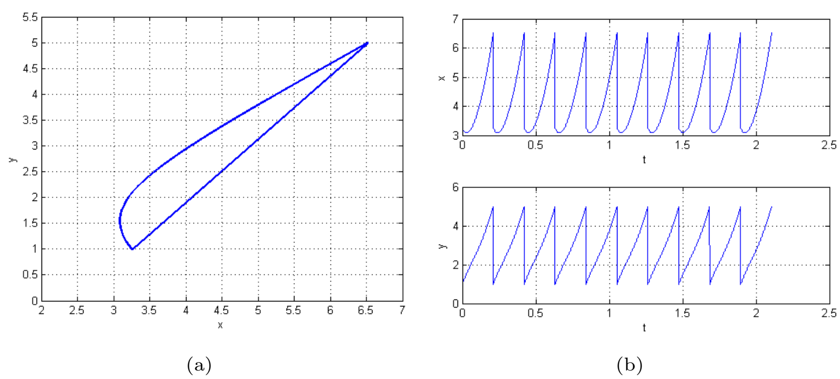
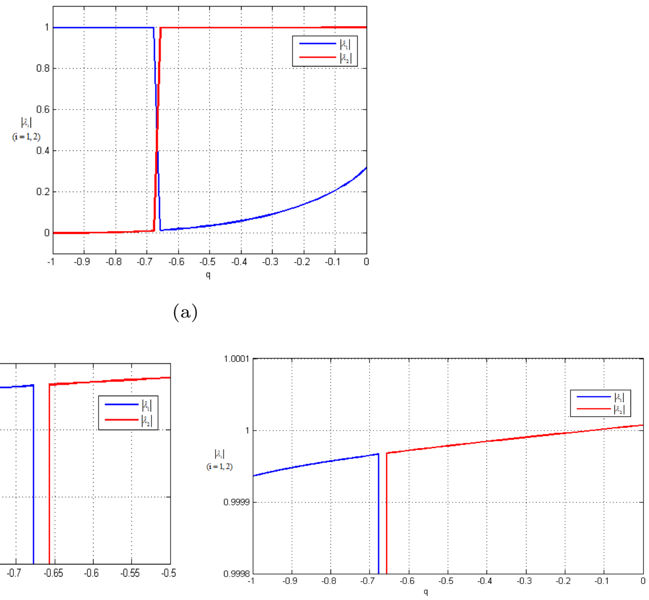


Figure 3. The trajectory of the period-1 solution of system (II-2) with parameter $a = 1,\ b = 2,\ c = 0.8,\ d = 0.5,\ p = -0.5,\ q = -0.8,\ x_0 = 6.518,\ ET = 5$.

the relationship between q and $|\lambda_1|, |\lambda_2|$ is sketched in Fig. 4. We can also choose other p, such $p = -0.2,\ p = -0.8$. Similarly, we can describe the relation between p and $|\lambda_1|, |\lambda_2|$ for given q.

When $q \in (-1, -0.159)$, both $|\lambda_1|, |\lambda_2|$ are smaller than 1, so the period-1 solution of system (II-2) is stable. When $q \in (-0.159, 0)$, one of $|\lambda_1|, |\lambda_2|$ is greater than 1, so the period-1 solution of system (II-2) is unstable,which are sketched in Fig. 4. When $q = -0.159$, one of $|\lambda_1|, |\lambda_2|$ is 1, bifurcation phenomenon will occur. Let $p = -0.5,\ q = -0.8,\ x_0 = 6.518,\ x_0' = 6.5,\ ET = y_0 = y_0' = 5$, the corresponding trajectories are sketched in Fig. 5, in which we can see that the period-1 solution starting from $(6.518, 5)$ is stable.

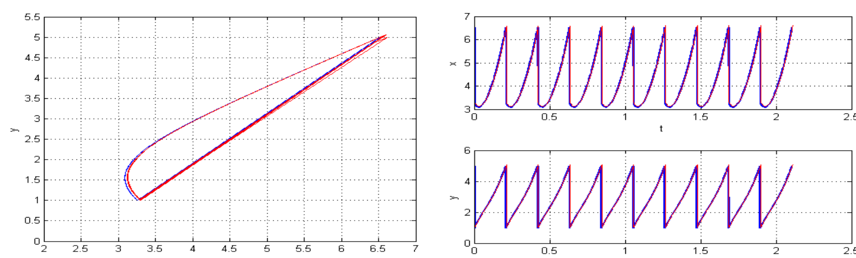


(a)

(b)

(c)

Figure 4. The relationship between q and $|\lambda_1|$, $|\lambda_2|$ with parameter $a = 1,\ b = 2,\ c = 0.8,\ d = 0.5,\ p = -0.5,\ ET = 5$ in (a), (b) and (c) are partial enlarged details.



(a)

(b)

Figure 5. Trajectories of the period-1 solution of system (II-2) starting from different points. The red curve is the trajectory from (6.5,5) and the blue one is (6.518,5) with parameter $a = 1,\ b = 2,\ c = 0.8,\ d = 0.5,\ p = -0.5,\ q = -0.8,\ ET = 5$.

5. Conclusion

The conditions we have obtained in part 2 are sufficient and necessary, which are better than the sufficient conditions in [4,32]. With the G-function method, we also can give conditions for a flow tangential to the boundary. The dynamical behavior of the population differential can be described more profound. With mapping structure, we analyze the period-k solution of a population differential system with state-dependent impulsive effect. This mapping structure can be used more widely. And the local stability of periodic solution can be obtained by eigenvalue analysis. Comparing with Lyapunov function method, this method is more intuitive and simpler.

References

[1] E. Angelis, *Optimal semi-active control and non-linear dynamics response of variable stiffness structures*, Journal of Vibration and Control, 2015, 11(10), 1253–1289.

[2] H. Baek, *An impulsive two-prey one-predator system with seasonal effects*, Discrete Dynamics in Nature and Society, 2009, 57(2), 332–337.

[3] H. Baek, *Qualitative analysis of Beddington-DeAngelis type impulsive predator-prey models*, Nonlinear Analysis, 2010, 11(1), 1312–1322.

[4] L. Dong, L. Chen and L. Sun, *Extinction and permanence of the predator–prey system with stocking of prey and harvesting of predator impulsively*, Mathematical Methods in the Applied Sciences, 2006, 29(4), 415–425.

[5] A. Filippov, *Differential equations with discontinuous right-hand side*, Amer. Math. Soc. Trans., 1964, 42(2), 199–231.

[6] X. Fu, J. Qi and Y. Liu, *The existence of periodic orbits for nonlinear impulsive differential systems*, Communications in Nonlinear Science and Numerical Simulation, 1999, 4(1), 50-53.

[7] X. Fu, B. Yan and Y. Liu, *Introduction of Impulsive Differential Systems*, Science Press, Beijing, 2005.

[8] X. Fu, B. Yan and Y. Liu, *Nonlinear Impulsive Differential Systems*, Science Press, Beijing, 2008.

[9] X. Fu and S. Zheng, *New approach in dynamics of regenerative chatter research of turning*, Communications in Nonlinear Science and Numerical Simulation, 2014, 19(11), 4013–4023.

[10] G. Grzegorz, *Viabe periodic solutions in state-dependent impulsive problems*, Collect. Math., 2015, 66(3), 1–15.

[11] P. Georgescua and G. Moroanub, *Impulsive perturbations of a three-trophic prey-dependent food chain system*, Mathematical and Computer Modelling, 2008, 48(7), 975-997.

[12] M. He, F. Chen and Z. Li, *Almost periodic solution of an impulsive differential equation model of plankton allelopathy*, Nonlinear Analysis, 2010, 11(4), 2296–2301.

[13] M. Huang and X. Song, *Study on species cooperative systems with impulsive state feedback control*, Journal of System Science and Mathematical Science, 2012, 32(3), 265–276.

[14] G. Jiang, *Complex dynamics of a Holling type II prey-predator system with state feedback control*, Chaos, Solutions and Fractals, 2007, 31(2), 448–461.

[15] G. Jiang and Q. Lu, *Impulsive state feedback control of predator-prey model*, Journal of Computational and Applied Mathematics, 2007, 200(1), 193–207.

[16] G. Jiang, Q. Lu and G. Luo, *Impulsive control of a stage-structured pest management system*, Journal of Mathematical Study, 2003, 36(4), 331–344.

[17] A. Luo, *Discontinuous Dynamical Systems*, Higher Education Press, Beijing, 2012.

[18] A. Luo and G. Yu, *A semi-analytical periodic motion in Duffing oscillator through mapping structures*, Discontinuity, Nonlinear and Complexity, 2015, 4(2), 121–150.

[19] Z. Liu, H. Chen and T. Zhou, *Variational methods to the second-order impulsive differential equation with Dirichlet boundary value problem*, Computers and Mathematics with Applications, 2011, 61(6), 1687–1699.

[20] Z. Luo, B. Dai and Q. Zhang, *Existence of positive periodic solutions for an impulsive semi-ratio-dependent predator-prey model with dispersion and time delays*, Applied Mathematics and Computation, 2010, 215(9), 3390–3398.

[21] X. Liu and K. Rohlf, *Impulsive control of a Lotka-Volterra system*, Ima Journal of Mathematical Control and Information, 1998, 15(3), 269–284.

[22] B. Liu and L. Zhang, *Dynamics of a two-species Lotka-Volterra competition system in a polluted environment with pulse toxicant input*, Applied Mathematics and Computation, 2009, 214(1), 155–162.

[23] J. Liang, S. Tang, J. Nieto and R. Cheke, *Analytical methods for detecting pesticide switches with evolution of pesticide resistance*, Mathemaical Biosciences, 2013, 245(2), 249–257.

[24] J. Nieto, R. Rodrguez-Lpez and M. Pesqueira, *Greens function for the periodic boundary value problem related to a first-order impulsive differential equation and applications to functional Problems*, Differential Equations and Dynamical Systems, 2011, 19(3), 199–210.

[25] L. Nie, Z. Teng, J. Nieto and I. Jung, *State impulsive control strategies for a two-languages competitive model with bilingualism and interlinguistic similarity*, Physica A, 2015, 430, 136–147.

[26] J. Qi and X. Fu, *Existence of limit cycles of impulsive differential equations with impulses at variable times*, Nonlinear Analysis, 2001, 44(3), 345–353.

[27] B. Shen and X. Wang, *Ninlinear state-dependent impulsive system in fed-batch culture and its optimal control*, Numerical Algebra, Control and Optimization, 2015, 5(4), 369–380.

[28] A. Terry, *Impulsive culling of a structured population on two patches*, Journal of Mathematical Biology, 2010, 61(6), 843–875.

[29] S. Tang and R. Cheke, *State-dependent impulsive models of integrated pest management strategies and their dynamic consequences*, Journal of Mathematical Biology, 2005, 50(3), 257–292.

[30] Utkin, *Variable structure systems with sliding modes*, IEEE Trans Automat control, 1977, 22(2), 212–222.

[31] L. Wang, L. Chen and J. Nieto, *The dynamics of an epidemic model for pest control with impulsive effect*, Nonlinear Analysis, 2010, 11(3), 1374–1386.

[32] Q. Xiao and B. Dai, *Dynamics of an impulsive predator –prey logistic population model with state-dependent*, Applied Mathematics and Computation, 2015, 259(5), 220–230.

[33] L. Zhao, *Impulsive control of a Lotka-Volterra predator-prey system*, Journal of Biomathematics, 2002, 17(1), 38–47.

[34] G. Zeng, L. Chen and L. Sun, *Existence of periodic solution of order one of planar impulsive autonomous*, Journal and Computational and Applied Mathematics, 2006, 186(2), 466–481.

[35] J. Zhao, X.Guo, Z. Han and Z. Chen, *Average conditions for competitive system in a nonautonomous two dimensional Lotka-Volterra system*, Mathematical and computer modeling, 2013, 57(5), 1131–1138.

[36] X. Zhang, J. Yan and A. Zhao, *Existence of positive periodic solutions for an impulsive differential equation*, Nonlinear Analysis, 2008, 68(10), 3209–3216.