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BIFURCATION OF LIMIT CYCLES IN PIECEWISE SMOOTH SYSTEMS VIA MELNIKOV FUNCTION

Maoan Han[1,†] and Lijuan Sheng[1]

Abstract In this short paper, we present some remarks on the role of the first order Melnikov functions in studying the number of limit cycles of piecewise smooth near-Hamiltonian systems on the plane.

Keywords Non-smooth system, Melnikov function, limit cycle bifurcation.

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1. A fundamental result

There have been many articles concerning the problem of limit cycle bifurcations for piecewise smooth systems on the plane. One of the main subjects studied widely is the so-called near-Hamiltonian systems with the form

$$\dot{x} = H_y + \varepsilon f(x, y), \quad \dot{y} = -H_x + \varepsilon g(x, y), \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter,

$$H(x, y) = \begin{cases} H^+(x, y), & x > 0, \\ H^-(x, y), & x \le 0, \end{cases}$$

$$f(x, y) = \begin{cases} f^+(x, y), & x > 0, \\ f^-(x, y), & x \le 0, \end{cases}$$

and

$$g(x, y) = \begin{cases} g^+(x, y), & x > 0, \\ g^-(x, y), & x \le 0, \end{cases}$$

with the functions $H^\pm, f^\pm, g^\pm$ being C^∞ smooth.

To our knowledge, there are two methods which can be used to study the number of limit cycles of (1.1). One is to use the Melnikov function established in [6], the other averaging method developed in [1, 8, 12]. In the following we focus on the Melnikov function method.

To establish the first order Melnikov function, one must first make the following assumptions as in [6]:

(I) There exist an interval $J = (\alpha, \beta)$ and two points $A(h) = (0, a(h))$ and $A_1(h) = (0, a_1(h))$ such that for $h \in J$

$$H^+(A(h)) = H^+(A_1(h)) = h, \; H^-(A(h)) = H^-(A_1(h)), \; a(h) > a_1(h).$$

[†]the corresponding author. Email address: mahan@shnu.edu.cn(M. Han)
[1]Department of Mathematics, Shanghai Normal University, Shanghai, 200234, P.R.China

(II) The equation $H^+(x,y) = h,\ x \geq 0$, defines an orbital arc L_h^+ starting from $A(h)$ and ending at $A_1(h)$; the equation $H^-(x,y) = H^-(A_1(h)),\ x \leq 0$, defines an orbital arc L_h^- starting from $A_1(h)$ and ending at $A(h)$, such that $(1.1)|_{\varepsilon=0}$ has a family clockwise oriented periodic orbits $L_h = L_h^+ \bigcup L_h^-,\ h \in J$.

We also say that the unperturbed system $(1.1)|_{\varepsilon=0}$ has a period annulus $L_h,\ h \in J$. If (1.1) has a limit cycle Γ_ε satisfying

$$\lim_{\varepsilon \to 0} \Gamma_\varepsilon = L_h$$

for some $h \in J$, we say that the limit cycle is bifurcated from the period annulus. Then a widely interested and studied problem is the following: For a given system of the form (1.1), how many limit cycles can be bifurcated from a period annulus of its unperturbed system?

In many cases the system (1.1) contains a vector parameter $\delta \in \mathbf{D} \subset R^m$ with $\mathbf{D}$ compact. In this case a limit cycle $\Gamma_{\varepsilon,\delta}$ of (1.1) is said to be bifurcated from the period annulus $L_h,\ h \in J$ if

$$\lim_{\varepsilon \to 0,\ \delta \in \mathbf{D}} \Gamma_{\varepsilon,\delta} = L_h.$$

for some $h \in J$. Then we can ask the same question as above for all $\delta \in \mathbf{D}$ and ε small.

Let the inner boundary of the period annulus be a center which is the limit of L_h as h goes to, say, α. The center can be denoted by L_α. If

$$\lim_{\varepsilon \to 0,\ \delta \in \mathbf{D}} \Gamma_{\varepsilon,\delta} = L_\alpha,$$

we say the limit cycle $\Gamma_{\varepsilon,\delta}$ is bifurcated from the center L_α. Then another problem appears as follows:

How many limit cycles can be bifurcated from the center L_α for all $\delta \in \mathbf{D}$ and ε small? This is called the problem of Hopf bifurcation for piecewise smooth systems. Usually, we suppose that the center L_α is elementary (see [2] for the definition of elementary singular point for piecewise smooth systems on the plane).

Under the assumptions (I) and (II), the authors [6] established a bifurcation function $F(h, \varepsilon)$ for (1.1). From [6] we see that the function F has the following properties:

(A) For any given interval $[a,b] \subset J$, there exists an $\varepsilon_0 = \varepsilon_0(a,b) > 0$ such that $F \in C^\infty$ for $h \in [a,b]$ and $|\varepsilon| \leq \varepsilon_0$.

(B) The period annulus $\{L_h, h \in J\}$ bifurcates a limit cycle if and only if $F(h,\varepsilon)$ has a zero in h near some $h_0 \in J$.

(C) Let $F(h,0) = M(h)$. Then

$$M(h) = M^+(h) + \frac{H_y^+(A)}{H_y^-(A)} M^-(h), \tag{1.2}$$

where by Theorem 1.1 in [6] and Lemma 2.2 in [7]

$$M^\pm(h) = \int_{L_h^\pm} g^\pm dx - f^\pm dy.$$

Based on the above conclusions (A)-(C), it is direct to obtain the following fundamental theorem as in the smooth case.

Theorem 1.1. *Under the assumptions (I) and (II), we have*

(1) if $M(h)$ has k zeros in h on the interval J with each having an odd multiplicity, then (1.1) *has at least k limit cycles bifurcating from the period annulus for ε small;*

(2) if the function $M(h)$ has at most k zeros in h on the interval J, taking into multiplicities account, then there exist at most k limit cycles of (1.1) *bifurcating from the period annulus.*

The theorem can be proved by contradiction. It tells us that when $M(h)$ is not zero identically it can be used to find the maximum number of limit cycles bifurcated from the period annulus.

This theorem has also many applications to Hopf bifurcation and homoclinic and heteroclinic bifurcations, see [4, 10, 11, 14] for instance. In the following section we proceed a further study on Hopf bifurcation.

2. Hopf bifurcation

Now let

$$\begin{aligned} &H_x^{\pm}(0,0) = H_y^{\pm}(0,0) = 0, \\ &\det \frac{\partial(H_y^{\pm}, -H_x^{\pm})}{\partial(x,y)}(0,0) > 0. \end{aligned} \tag{2.1}$$

This ensures that the origin is an elementary singular point of $(1.1)|_{\varepsilon=0}$. Consider

$$\begin{aligned} \dot{x} &= H_y + \varepsilon f(x,y,\delta), \\ \dot{y} &= -H_x + \varepsilon g(x,y,\delta), \end{aligned} \tag{2.2}$$

where H is as before, ε is small, $\delta \in R^m$,

$$f(x,y,\delta) = \begin{cases} f^+(x,y,\delta), & x > 0, \\ f^-(x,y,\delta), & x \le 0, \end{cases}$$

and

$$g(x,y,\delta) = \begin{cases} g^+(x,y,\delta), & x > 0, \\ g^-(x,y,\delta), & x \le 0. \end{cases}$$

For (2.2), the first order Melnikov M depends on δ, denoted by $M(h,\delta)$. The following theorem was obtained in [6].

Theorem 2.1. *Let the assumptions (I) and (II) and* (2.1) *hold with $J = (0,\beta)$, $\beta > 0$. If further,*

$$f^{\pm}(0,0,\delta) = g^{\pm}(0,0,\delta) = 0, \tag{2.3}$$

then

(1) the function $M(h,\delta)$ *has an expansion of the form*

$$M(h,\delta) = \sum_{j\geq 2} b_{j-1}(\delta) h^{\frac{j}{2}}; \tag{2.4}$$

(2) system (2.2) *has at most* k *limit cycles in a neighborhood of the origin for all* (ε,δ) *near* $(0,\delta_0)$ *if*

$$b_j(\delta_0) = 0,\ j = 1,\cdots,k,\quad b_{k+1}(\delta_0) \neq 0; \tag{2.5}$$

(3) system (2.2) *has* k *limit cycles in an arbitrary neighborhood of the origin for some* (ε,δ) *near* $(0,\delta_0)$ *if* (2.5) *holds and*

$$\operatorname{rank} \frac{\partial(b_1,\cdots,b_k)}{\partial(\delta_1,\cdots,\delta_m)}(\delta_0) = k.$$

The proof of the above theorem depends on the formula of M and the fact below under the conditions $f^{\pm}(0,0,\delta) = g^{\pm}(0,0,\delta) = 0$:

$$F(h,\varepsilon) = \sum_{j\geq 2} \widetilde{b}_{j-1}(\varepsilon,\delta) h^{\frac{j}{2}},\quad \widetilde{b}_{j-1}(0,\delta) = b_{j-1}(\delta). \tag{2.6}$$

By using the above formula we can obtain more results on the maximum number of limit cycles near the origin. For example, similar to the proof of Theorems 2.4.2 and 2.4.3 in [5], we have

Theorem 2.2. *Let the assumptions (I), (II) and* (2.1) *hold with* $J = (0,\beta)$, $\beta > 0$, *also let* (2.3) *hold. If*

(i) $H^{\pm}, f^{\pm}, g^{\pm}$ *are analytic in* (x,y) *at the origin, and* $f^{\pm}, g^{\pm}$ *depend on* δ *linearly;*

(ii) there exists an integer k *such that* (2.2) *has a center at the origin as* $b_1 = \cdots = b_k = 0$, *where* $\{b_j\}$ *are the coefficients appeared in* (2.4)*;*

(iii) $b_1(\delta_0) = \cdots = b_k(\delta_0) = 0$ *for some* $\delta_0 \in R^m$ *and*

$$\operatorname{rank} \frac{\partial(b_1,\cdots,b_k)}{\partial(\delta_1,\cdots,\delta_m)} = k,$$

then for any given $N > 0$, *there exists* $\varepsilon_0 > 0$ *such that* (2.2) *has at most* $k-1$ *limit cycles near a neighborhood of the origin for all* $0 < |\varepsilon| \leq \varepsilon_0$, $\|\delta\| \leq N$. *Moreover,* $k-1$ *limit cycles can appear in an arbitrary neighborhood of the origin.*

In fact, under our assumptions, we have

$$\begin{aligned} &\widetilde{b}_j = b_j(1+O(\varepsilon)),\quad j = 1,\cdots,k,\\ &\widetilde{b}_j = b_1\varphi_{1j}(\varepsilon,\delta) + \cdots + b_k\varphi_{kj}(\varepsilon,\delta),\ j \geq k+1. \end{aligned}$$

Hence, by (2.6), the function F can be rewritten as

$$F(h,\varepsilon) = h\sum_{j=1}^{k} b_j(1+P_j(h^{\frac{1}{2}},\varepsilon,\delta))h^{\frac{j-1}{2}},$$

where $\{P_j\}$ are analytic functions with $P_j(0,0,\delta)=0, j=1,\cdots,k$. It implies from the above form of F that F has at most $k-1$ positive zeros in h. Also, $k-1$ positive zeros can appear.

The condition (2.3) means that the origin is always a singular point under perturbation. If we do not require it, we have

Theorem 2.3. *Consider* (2.2). *Let the assumptions (I) and (II) and* (2.1) *hold with* $J=(0,\beta),\ \beta>0$. *Then*

(1) We have formally

$$M(h,\delta)=\sum_{j\ge 1} b_{j-1}(\delta)h^{\frac{j}{2}}.$$

(2) If there exist k *and* δ_0 *such that*

$$b_0(\delta_0)=\cdots=b_k(\delta_0)=0,\ \ b_{k+1}(\delta_0)\neq 0,$$

and

$$\operatorname{rank}\frac{\partial(b_0,\cdots,b_k)}{\partial(\delta_1,\cdots,\delta_m)}(\delta_0)=k+1,$$

then (2.2) *can have at least* $k+1$ *limit cycles in an arbitrary neighborhood of the origin.*

Proof. By (1.2), the function $M(h,\delta)$ has the following formula

$$M(h,\delta)=M^+(h,\delta)+N(h)M^-(h,\delta), \tag{2.7}$$

where

$$M^{\pm}(h,\delta)=\int_{L_h^{\pm}} g^{\pm}dx-f^{\pm}dy,\quad N(h)=\frac{H_y^+(A)}{H_y^-(A)},\quad A=(0,a(h)).$$

By [6], there is a C^∞ function $\varphi(v)=\sum\limits_{i\ge 1} e_i v^i, e_1>0$, such that

$$a(h)=\varphi(h^{\frac12}),\quad a_1(h)=\varphi(-h^{\frac12}).$$

It follows from (2.1) that

$$N(h)=\sum_{j\ge 0} n_j h^{\frac{j}{2}},\quad n_0>0. \tag{2.8}$$

Introduce four functions $\widetilde{f}^{\pm}$ and $\widetilde{g}^{\pm}$ as follows

$$\widetilde{f}^{\pm}(x,y,\delta)=f^{\pm}(x,y,\delta)-f^{\pm}(0,0,\delta),$$
$$\widetilde{g}^{\pm}(x,y,\delta)=g^{\pm}(x,y,\delta)-g^{\pm}(0,0,\delta).$$

Then

$$M^{\pm}(h,\delta)=\widetilde{M}^{\pm}(h,\delta)+g^{\pm}(0,0,\delta)\int_{L_h^{\pm}}dx-f^{\pm}(0,0,\delta)\int_{L_h^{\pm}}dy, \tag{2.9}$$

where

$$\widetilde{M}^{\pm}(h,\delta) = \int_{L_h^{\pm}} \widetilde{g}^{\pm} dx - \widetilde{f}^{\pm} dy.$$

By the proof of Theorem 1.2 in [6], the functions $\widetilde{M}^{\pm}$ have the expansions

$$\widetilde{M}^{\pm}(h,\delta) = \sum_{j\geq 2} \widetilde{m}_{j-1}^{\pm}(\delta) h^{\frac{j}{2}}.$$

Further, we have $\int_{L_h^{\pm}} dx = 0$, and

$$\int_{L_h^{\pm}} dy = \pm(a_1(h) - a(h)) = \pm[\varphi(-h^{\frac{j}{2}}) - \varphi(h^{\frac{j}{2}})] = \pm \sum_{j\geq 1}((-1)^j - 1)e_j h^{\frac{j}{2}}.$$

Therefore, by (2.9)

$$M^{\pm}(h,\delta) = \pm 2e_1 f^{\pm}(0,0,\delta)h^{\frac{1}{2}} + \sum_{j\geq 2} m_{j-1}^{\pm}(\delta) h^{\frac{j}{2}},$$

where

$$m_{j-1}^{\pm}(\delta) = \widetilde{m}_{j-1}^{\pm}(\delta) \pm f^{\pm}(0,0,\delta)[1-(-1)^j]e_j.$$

Then the first conclusion follows by substituting (2.8) and (2.9) into (2.7). The proof for conclusion (2) is direct. The proof is completed. □

We remark that since the origin may no longer be a singular point of (2.2), the system may have more than $k+1$ limit cycles near the origin under the condition of conclusion (2). See [2].

If the functions H, f and g in (2.2) depend on another small parameter λ, then the function M also depends on λ. In this case, we have

$$M(h,\delta,\lambda) = M_0(h,\delta) + \lambda M_1(h,\delta) + \lambda^2 M_2(h,\delta) + O(\lambda^3).$$

The formulas of $M_1(h,\delta)$ and $M_2(h,\delta)$ were obtained in [3] for smooth case and in [13] for nonsmooth case. When $M_0 = 0, M_1 \neq 0$, then for $0 < |\varepsilon| \ll \lambda \ll 1$, we can study the number of limit cycles by using $M_1(h,\delta)$. Frequently, we can find more limit cycle using M_1 than using M_0 only. Similarly, when $M_0 = M_1 = 0$, we can find limit cycles by using M_2. The functions can be used to study not only Poincaré bifurcation (that is, bifurcation of limit cycles from period annulus) but also Hopf bifurcation and homoclinic and heteroclinic bifurcations. For more details, see [3, 13].

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