Journal of Applied Analysis and Computation
Volume 5, Number 4, November 2015, 781–792
Website:http://jaac-online.com/
doi:10.11948/2015059

AFFINE-PERIODIC SOLUTIONS FOR DISCRETE DYNAMICAL SYSTEMS*

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Abstract The paper concerns the existence of affine-periodic solutions for discrete dynamical systems. This kind of solutions might be periodic, harmonic, quasi-periodic, even non-periodic. We prove the existence of affine-periodic solutions for discrete dynamical systems by using the theory of Brouwer degree. As applications, another existence theorem is given via Lyapnov function.

Keywords Discrete dynamical systems, affine-periodic solutions, Brouwer degree.

MSC(2010) 37J45, 39A11, 47H11.

1. Introduction and main results

Garrett Birkhoff introduced the concept of dynamical system [1], it vividly describes the physical background of differential equations. A dynamical system might be defined as a deterministic mathematical description of a system forward in time. Time here either may be a continuous variable, or else it may be a discrete integer-valued variable. An example of a dynamical system in which time is a continuous variable is a system of m-dimensional, first-order, autonomous, ordinary differential equations

$$\frac{dx}{dt} = M(x), x \in \mathbb{R}^m, \tag{1.1}$$

where $M : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous. In the case of discrete and integer-valued time, an example of a dynamical system is

$$x_{n+1} = F(n, x_n), \tag{1.2}$$

where x_n, x_{n+1} are m-dimensional vectors, $F : \mathbb{N}_+ \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous with respect to x_n.

The problem of periodic solution of continuous dynamical system has been a main subject of investigation. By using various methods and techniques, such as fixed point theory, the Kaplan-Yorke method, coincidence degree theory and topological degree theory [3, 5–8, 10]. In general, it is much more difficult to investigate

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*The authors were supported by National Basic Research Program of China(grant No. 2013CB834100), and National Natural Science Foundation of China(grant No. 11171132).

the periodic solutions of discrete dynamical system than continuous dynamical system, because there are probably more complicated behaviors. In 1964, Alexander Nicoli Sharkovsky introduced his fundamental theorem on the periods of continuous maps on the real line [9]. Part of Sharkovsky Theorem was later discovered in 1975, independently, by Tianyan Li and James Yorke [14]. In addition to introducing "chaos" in mathematics, the Li-Yorke paper was instrumental in introducing Shakovsky Theorem in English which made it accessible to more scientists. In 1978, Mitchell Feigenbaum discovered a universal constant, the "Feigenbaum number" [2], that is shared by unimodal continuous maps on the real line. However, the study in higher dimensional systems is generally more difficult. Since 2000, Jianshe Yu et al. investigated the periodic solution of discrete dynamical system by developing Kaplan-Yorke method and using critical point theory [4, 13, 15].

In recent years, the conception of affine-periodic solutions was proposed, and the existence of solutions was studied for continuous dynamical system [11, 12, 16]. Affine-periodic solution is a kind of periodic or quasi-periodic solutions with symmetry, more precisely, is some quasi-periodic solutions with symmetry. In this paper, we are concerned with the existence of affine-periodic solutions for discrete dynamical systems.

For simplicity, we consider the following system.

$$x_{n+1} - x_n = f(n, x_n), \tag{1.3}$$

where $n \in \mathbb{N}_+$, $f : \mathbb{N}_+ \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous with respect to x_n, and for some $N \in \mathbb{N}_+$, and $Q \in GL(m)$,

$$f(n+N, x_n) = Qf(n, Q^{-1}x_n). \tag{1.4}$$

We call (1.3) a (Q, N)-affine-periodic system.

Consider system (1.3). If there is a linear transformation of coordinates B, which makes $y_n = Bx_n$, then

$$y_{n+1} - y_n = B(x_{n+1} - x_n) = Bf(n, B^{-1}y_n).$$

Let $g(n, y_n) = Bf(n, B^{-1}y_n)$, we have

$$\begin{aligned} g(n+N, y_n) &= Bf(n+N, B^{-1}y_n) = BQf(n, Q^{-1}B^{-1}y_n), \\ \hat{Q}g(n, \hat{Q}^{-1}y_n) &= \hat{Q}Bf(n, B^{-1}\hat{Q}^{-1}y_n) = BQf(n, Q^{-1}B^{-1}y_n), \end{aligned}$$

where $\hat{Q} = BQB^{-1}$. Hence

$$g(n+N, y_n) = \hat{Q}g(n, \hat{Q}^{-1}y_n).$$

It means that linear transformation of coordinates keep the affine-periodicity of system (1.3). Obviously, for general nonlinear transformation of coordinates, the affine-periodicity will not keep anymore. It is easy to see that this affine-periodic invariance exhibits two characters: periodicity in time and symmetry in space.

Now a basic topic is to investigate the existence of (Q, N)-affine-periodic solutions x_n of system (1.3), i.e.

$$x_{n+N} = Qx_n. \tag{1.5}$$

Let I be identity matrix. If $Q = I$, $Q = -I$, $Q^N = I$, $Q \in SO(m)$, then the (Q, N)-affine-periodic solutions are periodic solutions, antiperiodic solutions, harmonic solutions, quasi-periodic solutions, respectively. If Q is not orthogonal matrix, then the (Q, N)-affine-periodic solutions might be even non-periodic.

In fact, this problem is equivalent to proving the existence of solutions of the BVP in the following.

Proposition 1.1. *The existence of* (Q,N)*-affine-periodic solutions of* (1.3) *is equivalent to the existence of solutions of* (1.3) *with* $x_N = Qx_0$.

Indeed, for any solution x_n of (1.3), let $u_n = Q^{-1}x_{n+N}$. Then

$$\begin{aligned} u_{n+1} - u_n &= Q^{-1}(x_{n+N+1} - x_{n+N}) = Q^{-1}f(n+N, x_{n+N}) \\ &= Q^{-1}(Qf(n, Q^{-1}x_{n+N})) = f(n, u_n). \end{aligned}$$

This shows that u_n is a solution of (1.3), and $u_0 = Q^{-1}x_N$, we know that $u_n = Q^{-1}x_{n+N} \equiv x_n$ if and only if $x_0 = Q^{-1}x_N$.

The purpose of this paper is to investigate the existence of affine-periodic solutions for discrete dynamical system (1.3), where $Q \in O(m)$.

Let us introduce our main result as follows.

Theorem 1.1. *Consider the following auxiliary system*

$$x_{n+1} - x_n = \lambda f(n, x_n), \tag{1.6}$$

where $\lambda \in [0,1]$.

Let $D \subset \mathbb{R}^m$ *be a bounded open set. Assume the following hold for system* (1.6).

(H_1) *For each* $\lambda \in [0,1]$, *every possible affine-periodic solution* x_n *of* (1.6) *satisfies*

$$x_n \notin \partial D, \ \forall n \in \mathbb{N}_+;$$

(H_2) *The Brouwer degree*

$$deg(g, D \cap \ker(I-Q), 0) \neq 0, \ if \ \ker(I-Q) \neq \{0\},$$

where $g(a) = \frac{1}{N}\sum_{k=0}^{N} Pf(a)$, *and* $P : \mathbb{R}^m \to \ker(I-Q)$ *is an orthogonal projection.*

Then system (1.3) *has at least one* (Q,N)*-affine-periodic solution.*

The rest of the paper is organized as follows. We first give a proof of Theorem 1.1 in section 2. In section 3, we give another result, which shows that Lyapunov's method is applicable to study the existence of affine-periodic solutions. There we also give an example.

2. Proof of Theorem 1.1

Proof. Consider the auxiliary system

$$x_{n+1} - x_n = \lambda f(n, x_n),$$

with the boundary value condition $x_N = Qx_0$, where $\lambda \in [0,1]$. Let x_n be any solution of (1.6) with $x_N = Qx_0$. We have

$$x_N = x_0 + \lambda \sum_{k=0}^{N} f(k, x_k) = Qx_0.$$

Then

$$(I-Q)x_0 = -\lambda \sum_{k=0}^{N} f(k, x_k), \tag{2.1}$$

where I is identity matrix.

Case 1 : $\mathrm{Ker}(I-Q)\neq\{0\}$.

In this case, $(I-Q)^{-1}$ does not exist. By coordinate transformation, we can just let

$$Q=\left(\begin{array}{cc} I & 0\\ 0 & Q_1\end{array}\right),$$

without loss of generality, suppose $(I-Q_1)^{-1}$ exists.

Let $P:\mathbb{R}^m\to\mathrm{Ker}(I-Q)$ be the orthogonal projection. Then

$$\begin{aligned}(I-Q)x_0&=(I-Q)(x_0^{\ker}-x_0^{\perp})\\ &=-\lambda\sum_{k=0}^{N}f(k,x_k)\\ &=-\lambda\sum_{k=0}^{N}Pf(k,x_k)-\lambda\sum_{k=0}^{N}(I-P)f(k,x_k).\end{aligned}$$

We have

$$(I-Q)x_0=-\lambda\sum_{k=0}^{N}Pf(k,x_k)-\lambda\sum_{k=0}^{N}(I-P)f(k,x_k), \tag{2.2}$$

where $x_0^{\ker}\in\ker(I-Q)$, $x_0^{\perp}\in\mathrm{Im}(I-Q)$, and $x_0=x_0^{\ker}+x_0^{\perp}$.

Let $L_P=(I-Q)|_{\mathrm{Im}(I-Q)}$. It is easy to see that L_P^{-1} exists. Thus (2.2) is equivalent to

$$(I-Q)x_0^{\ker}=-\lambda\sum_{k=0}^{N}Pf(k,x_k)=0,$$

$$(I-Q)x_0^{\perp}=-\lambda\sum_{k=0}^{N}(I-P)f(k,x_k).$$

Thus we have

$$x_0^{\perp}=-\lambda L_P^{-1}(I-P)\sum_{k=0}^{N}f(k,x_k).$$

Let $X=\{x:\{0,1,2,\cdots,N\}\to\mathbb{R}^m\}$, and define the norm as $\|x\|=\max_{n\in\{0,1,\cdots,N\}}|x_n|$. It is easy to see that X is a Banach space with the norm $\|\cdot\|$.

For $x\in X$, which satisfies $x_n\in\overline{D}$ for all $n\in\{0,1,\cdots,N\}$, we define an operator $T(x_0^{\ker},x,\lambda)$ by

$$T(x_0^{\ker},x,\lambda)=\left(\begin{array}{c} x_0^{\ker}+\frac{1}{N}\sum_{k=0}^{N}Pf(k,x_k)\\ x_0^{\ker}-\lambda L_P^{-1}(I-P)\sum_{k=0}^{N}f(k,x_k)+\lambda\sum_{k=0}^{n}f(k,x_k)\end{array}\right), \tag{2.3}$$

where $\lambda\in[0,1]$.

We claim that each fixed point x of T in X is a solution of (1.6) with $x_N=Qx_0$.

In fact, if x is a fixed point of T, then

$$\left(\begin{array}{c} x_0^{\ker}\\ x_n\end{array}\right)=\left(\begin{array}{c} x_0^{\ker}+\frac{1}{N}\sum_{k=0}^{N}Pf(k,x_k)\\ x_0^{\ker}-\lambda L_P^{-1}(I-P)\sum_{k=0}^{N}f(k,x_k)+\lambda\sum_{k=0}^{n}f(k,x_k)\end{array}\right).$$

Thus

$$\frac{1}{N}\sum_{k=0}^{N} Pf(k, x_k) = 0, \tag{2.4}$$

$$x_n = x_0^{\text{ker}} - \lambda L_P^{-1}(I-P)\sum_{k=0}^{N} f(k, x_k) + \lambda\sum_{k=0}^{n} f(k, x_k). \tag{2.5}$$

By (2.5) we have

$$x_0 = x_0^{\text{ker}} - \lambda L_P^{-1}(I-P)\sum_{k=0}^{N} f(k, x_k).$$

Hence

$$\begin{aligned} Qx_0 &= Qx_0^{\text{ker}} - \lambda Q L_P^{-1}(I-P)\sum_{k=0}^{N} f(k, x_k) \\ &= x_0^{\text{ker}} - \lambda Q L_P^{-1}(I-P)\sum_{k=0}^{N} f(k, x_k). \end{aligned}$$

It follows from (2.4) that

$$\begin{aligned} (I-Q)L_P^{-1}(I-P)\sum_{k=0}^{N} f(k, x_k) &= (I-P)\sum_{k=0}^{N} f(k, x_k) \\ &= (I-P)\sum_{k=0}^{N} f(k, x_k) + P\sum_{k=0}^{N} f(k, x_k) = \sum_{k=0}^{N} f(k, x_k). \end{aligned}$$

Thus

$$\lambda Q L_P^{-1}(I-P)\sum_{k=0}^{N} f(k, x_k) = \lambda L_P^{-1}(I-P)\sum_{k=0}^{N} f(k, x_k) - \lambda\sum_{k=0}^{N} f(k, x_k).$$

Then

$$\begin{aligned} Qx_0 &= Qx_0^{\text{ker}} - \lambda Q L_P^{-1}(I-P)\sum_{k=0}^{N} f(k, x_k) \\ &= x_0^{\text{ker}} - \lambda Q L_P^{-1}(I-P)\sum_{k=0}^{N} f(k, x_k) + \lambda\sum_{k=0}^{N} f(k, x_k) \\ &= x_N. \end{aligned}$$

Thereby,

$$Qx_0 = x_N. \tag{2.6}$$

By (2.5) and (2.6), we know that (2.1) holds. Thus,

$$x_0^{\perp} = -\lambda L_P^{-1}(I-P)\sum_{k=0}^{N} f(k, x_k).$$

Consequently,

$$\begin{aligned} x_n &= x_0^{\ker} - \lambda L_P^{-1}(I-P)\sum_{k=0}^{N} f(k,x_k) + \lambda \sum_{k=0}^{n} f(k,x_k) \\ &= x_0^{\ker} + x_0^{\perp} + \lambda \sum_{k=0}^{n} f(k,x_k) \\ &= x_0 + \lambda \sum_{k=0}^{n} f(k,x_k). \end{aligned}$$

This shows that the fixed point x is a solution of (1.6) with $x_N = Qx_0$.

Now, we are to prove the existence of fixed point of T.

Define

$$X_\lambda = \{x \in X : |\frac{x_n - x_s}{n-s}| \le \lambda M, \forall n \ne s\},$$

where constant M satisfies $M > \sup\{f(n,x_n) : n \in \{0,1,\cdots,N\}, x_n \in \overline{D}\}$, we make a retraction $\alpha_\lambda : X \to X_\lambda$.

Define an operator $\widehat{T}(x_0^{\ker}, x, \lambda)$ by

$$\widehat{T}(x_0^{\ker}, x, \lambda) = \begin{pmatrix} x_0^{\ker} + \frac{1}{N}\sum_{k=0}^{N} Pf(k, \alpha_\lambda \circ x_k) \\ \alpha_\lambda \circ x_0^{\ker} - \lambda L_P^{-1}(I-P)\sum_{k=0}^{N} f(k,\alpha_\lambda \circ x_k) + \lambda \sum_{k=0}^{n} f(k, \alpha_\lambda \circ x_k) \end{pmatrix}. \tag{2.7}$$

Note that $P : \mathbb{R}^m \to \ker(I-Q)$, we have

$$\frac{1}{N}\sum_{k=0}^{N} Pf(k,x_k) \in \ker(I-Q),$$
$$\frac{1}{N}\sum_{k=0}^{N} Pf(k,\alpha_\lambda \circ x_k) \in \ker(I-Q).$$

Consider the homotopy

$$H(x_0^{\ker}, x, \lambda) = \widehat{T}(x_0^{\ker}, x, \lambda), (x_0^{\ker}, x, \lambda) \in (D \cap \ker(I-Q)) \times \widetilde{D} \times [0,1], \tag{2.8}$$

where $\widetilde{D} = \{x \in X : x_n \in D, n \in \{0,1,\cdots,N\}\}$.

We claim that

$$0 \notin (id - H)(\partial((D \cap \ker(I-Q)) \times \widetilde{D}) \times [0,1]). \tag{2.9}$$

Suppose on the contrary that there exists $(\bar{x}_0^{\ker}, \bar{x}, \bar{\lambda}) \in \partial((D \cap \ker(I-Q)) \times \widetilde{D} \times [0,1])$, such that $(id - H)(\bar{x}_0^{\ker}, \bar{x}, \bar{\lambda}) = 0$. Since $\bar{x}_0^{\ker} \in \partial D$ is contrary to (H_1), and $\partial(D \cap \ker(I-Q)) \subset \partial D$, we have that $\bar{x}_0^{\ker} \notin \partial(D \cap \ker(I-Q))$. Hence $\bar{x} \in \partial\widetilde{D}$. We discuss it by two cases as follows.

(a) When $\bar{\lambda} = 0$, we have

$$X_0 = \{x \in X : |\frac{x_n - x_s}{n-s}| \le 0, \forall n \ne s\},$$

which implies that $\alpha_0 \circ x_n \equiv \alpha_0 \circ x_0, n \in \{0, 1, \cdots, N\}$. It follows from $(id - H)(\bar{x}_0^{\ker}, \bar{x}, \bar{\lambda}) = 0$ that

$$\begin{pmatrix} \bar{x}_0^{\ker} \\ \bar{x}_n \end{pmatrix} = \begin{pmatrix} \bar{x}_0^{\ker} + \frac{1}{N}\sum_{k=0}^{N} Pf(k, \alpha_0 \circ x_k) \\ \alpha_0 \circ \bar{x}_0^{\ker} \end{pmatrix}.$$

It means that $\bar{x}_n \equiv \bar{x}_0$, for all $n \in \{0, 1, \cdots, N\}$. Putting $\bar{x}_0 = p$, we have

$$\alpha_0 \circ \bar{x}_0^{\ker} = \bar{x}_n = p,$$
$$g(p) = \frac{1}{N}\sum_{k=0}^{N} Pf(p) = 0.$$

Notice that $\bar{x} \in \partial\widetilde{D}$, and $\widetilde{D} = \{x \in X : x_n \in D, n \in \{0, 1, \cdots, N\}\}$. Then there exists $k_0 \in \{0, 1, \cdots, N\}$, such that $\bar{x}_{k_0} \in \partial D$. As $\bar{x}_n \equiv p$, $n \in \{0, 1, \cdots, N\}$, we obtain that $p \in \partial D$, and $g(p) = 0$, which contradicts (H_2).

(b) When $\bar{\lambda} \in (0, 1]$, noticing that $(id - H)(\bar{x}_0^{\ker}, \bar{x}, \bar{\lambda}) = 0$, it follows that

$$\begin{pmatrix} \bar{x}_0^{\ker} \\ \bar{x}_n \end{pmatrix} = \begin{pmatrix} \bar{x}_0^{\ker} + \frac{1}{N}\sum_{k=0}^{N} Pf(k, \alpha_0 \circ x_k) \\ \alpha_0 \circ \bar{x}_0^{\ker} - \bar{\lambda} L_p^{-1}(I - P)\sum_{k=0}^{N} f(k, \alpha_0 \circ x_k) + \bar{\lambda}\sum_{k=0}^{n} f(k, \alpha_0 \circ x_k) \end{pmatrix}.$$

Thus

$$\frac{1}{N}\sum_{k=0}^{N} Pf(k, \alpha_0 \circ x_k) = 0,$$

$$\bar{x}_n = \alpha_0 \circ \bar{x}_0^{\ker} - \bar{\lambda} L_p^{-1}(I - P)\sum_{k=0}^{N} f(k, \alpha_0 \circ x_k) + \bar{\lambda}\sum_{k=0}^{n} f(k, \alpha_0 \circ x_k). \tag{2.10}$$

Note that

$$|\frac{\bar{x}_n - \bar{x}_s}{n - s}| = \frac{1}{|n - s|}|\bar{\lambda}\sum_{k=s}^{n} f(k, \alpha_0 \circ x_k)| \le \bar{\lambda} M.$$

Hence $\bar{x} \in X_{\bar{\lambda}}$, which implies that $\alpha_{\bar{\lambda}} \circ \bar{x} = \bar{x}$. Therefore we can rewrite (2.10) as

$$\bar{x}_n = \bar{x}_0^{\ker} - \bar{\lambda} L_p^{-1}(I - P)\sum_{k=0}^{N} f(k, x_k) + \bar{\lambda}\sum_{k=0}^{n} f(k, x_k).$$

Similarly to(2.5), we can prove that $\bar{x}_n$ is a solution of (1.6). Now the solution $\bar{x} \in \partial\widetilde{D}$, which contradicts (H_1).

By (a) and (b), we have

$$0 \notin (id - H)(\partial((D \cap \ker(I - Q)) \times \widetilde{D}) \times [0, 1]).$$

Thus by the homotopy invariance and the theory of Brouwer degree, we have

$$\begin{aligned} &\deg(id - H(x_0^{\ker}, x, 1), (D \cap \ker(I - Q)) \times \widetilde{D}, 0) \\ =&\deg(id - H(x_0^{\ker}, x, 0), (D \cap \ker(I - Q)) \times \widetilde{D}, 0) \\ =&\deg(g, D \cap \ker(I - Q), 0) \\ \neq&0. \end{aligned}$$

By the regularity of Brouwer degree, there exists $\bar{x}^* \in \widetilde{D}$, such that

$$\begin{pmatrix} \bar{x}_0^{*\,\ker} \\ \bar{x}_n^* \end{pmatrix} = \widehat{T}(\bar{x}_0^{*\,\ker}, \bar{x}_n^*, 1). \tag{2.11}$$

A similar proof in (b) yields that $\bar{x}^* \in X_1$, that is

$$\widehat{T}(\bar{x}_0^{*\,\ker}, \bar{x}_n^*, 1) = T(\bar{x}_0^{*\,\ker}, \bar{x}_n^*, 1). \tag{2.12}$$

By (2.11)and (2.12), we obtain that $\bar{x}^*$ is a fixed point of T in X. Then $\bar{x}^*$ is a solution of system (1.3) with $x_N = Qx_0$.

Case 2 : $\ker(I-Q) = \{0\}$.

In this case, $(I-Q)^{-1}$ exists, and

$$x_0 = -\lambda Q L_P^{-1}(I-P)\sum_{k=0}^{N} f(k, x_k).$$

Consider the homotopy

$$H(x,\lambda) = -\lambda L_P^{-1}(I-P)\sum_{k=0}^{N} f(k, \alpha_\lambda \circ x_k) + \lambda \sum_{k=0}^{N} f(k, \alpha_\lambda \circ x_k).$$

Similar to the proof when $\mathrm{Ker}(I-Q) \neq \{0\}$, we have $0 \notin (id - H)(\partial\widetilde{D} \times [0,1])$. Thereby

$$\begin{aligned} \deg(id - H(\cdot,1), \widetilde{D}, 0) &= \deg(id - H(\cdot,0), \widetilde{D}, 0) \\ &= \deg(id, \widetilde{D}, 0) \\ &= 1. \end{aligned}$$

Thus there exists $\bar{x}_n^*$ with $\bar{x}_n^* \in D, \forall n \in \{0,1,\cdots,N\}$ such that

$$\bar{x}_n^* = \bar{x}_0^* + \sum_{k=0}^{n} f(k, \bar{x}_k^*).$$

This shows that $\bar{x}_n^*$ is a solution of system (1.3) with boundary condition $x_N = Qx_0$.

By Proposition 1.1 system (1.3) has a (Q,N)-affine-periodic solution. This finishes the proof of Theorem 1.1.

□

3. Applications

Theorem 1.1 offers a topological method to study the existence of affine-periodic solutions in theory. When dealing with some specific problems, we hope to have a more directly method. The Lyapunov function method is a useful instrument in the study of solutions. In this section, we give a result on basis of Lyapnov functions.

Theorem 3.1. *Consider system* (1.3) *, assume that there exist functions* $V_i : \mathbb{R}^m \to \mathbb{R}$*,* $i = 1,2,\cdots,l$ *and* $\sigma > 0$*, such that*

(H_3) *For* M_i *large enough,*

$$|\langle \nabla V_i(x_n), f(n,x_n)\rangle| \geq \sigma > 0, \forall |x_n| \geq M_i, i = 1,2,\cdots,l, n \in \mathbb{N}_+.$$

And if $Ker(I-Q) \neq \{0\}$*,*

$$|\langle \nabla V_i(x_n), Pf(n,x_n)\rangle| \geq \sigma > 0, \forall x_n \in \ker(I-Q) \text{ and } |x_n| \geq M_i, i = 1,2,\cdots,l, n \in \mathbb{N}_+,$$

where $P : \mathbb{R}^m \to \ker(I-Q)$ *is an orthogonal projection;*

(H_4) *If* $\langle \nabla V_i(x_n), f(n,x_n)\rangle > 0$*, then Hessian matrix* $(\frac{\partial^2 V}{\partial x_i \partial x_j})$ *is positive semidefinite, and if* $\langle \nabla V_i(x_n), f(n,x_n)\rangle < 0$*, then Hessian matrix* $(\frac{\partial^2 V}{\partial x_i \partial x_j})$ *is negative semidefinite;*

(H_5)

$$\sum_{i=0}^{l} |V_i(x_n)| \to \infty, as |x_n| \to \infty;$$

(H_6) *The Brouwer degree*

$$deg(\nabla V_0, B_{M_0} \cap \ker(I-Q), 0) \neq 0, \ if \ \ker(I-Q) \neq \{0\},$$

where $B_\rho = \{p \in \mathbb{R}^m : |p| < \rho\}$*.*

Then system (1.3) *has at least one* (Q,N)*-affine-periodic solution.*

Proof. Consider the auxiliary system

$$x_{n+1} - x_n = \lambda f(n,x_n),$$

where $\lambda \in [0,1]$. Set

$$L_i = \sup\{|V_i(x_n)| : |x_n| \leq M_i\},$$

$$L = \sum_{i=0}^{l} L_i,$$

$$D = \{p \in \mathbb{R}^m : \sum_{i=0}^{l} |V_i(p)| < L+1\},$$

$$V(x_n) = \sum_{i=0}^{l} |V_i(x_n)|.$$

By (H_5), we claim that D is bounded for $\lambda \in (0,1]$, and every possible (Q,N)-affine-periodic solution x_n of (1.6) satisfies

$$x_n \in D, \forall n \in \mathbb{N}_+.$$

In fact, since x_n is a (Q,N)-affine-periodic solution of (1.6) and $Q \in O(m)$, there exists a sequence $\{n_j\} \subset \mathbb{N}_+$, such that

$$V(x_{n_j}) \to \sup_{\mathbb{N}_+} V(x_n) < \infty, \ as \ j \to \infty. \tag{3.1}$$

Hence for some i,

$$|V_i(x_{n_j})| \to \sup_{\mathbb{N}_+} |V_i(x_n)|, \ as \ j \to \infty.$$

By (H_4) and Taylor's theorem we know that

$$\begin{aligned}&V_i(x_{n_j+1}) - V_i(x_{n_j})\\=&\langle \nabla V_i(x_{n_j}), f(n_j, x_{n_j})\rangle +\\&\frac{1}{2!}\langle (\frac{\partial^2 V_i(x_{n_j} + \theta(x_{n_j+1} - x_{n_j}))}{\partial x_k \partial x_s}) \cdot (x_{n_j+1} - x_{n_j}), x_{n_j+1} - x_{n_j}\rangle \to 0,\ as\ j \to \infty,\end{aligned}$$

where $\theta \in (0, 1)$. By (H_3), this result yields

$$|x_{n_j}| < M_i,\ as\ j \to \infty.$$

Consequently, by the definition of D and (3.1), we have

$$x_n \in D,\ \forall n \in \mathbb{N}_+.$$

Thus hypothesis (H_1) holds.

If $\ker(I - Q) = \{0\}$, by the proof of Theorem 1.1, we know that (1.3) admits a (Q, N)-affine-periodic solution.

Now we prove that if $\ker(I - Q) \neq \{0\}$,

$$\deg(g, D \cap \ker(I - Q), 0) \neq 0.$$

Indeed, consider the homotopy

$$H(p, \lambda) = \lambda \mathrm{sgn}(\langle \nabla V_0(x_n), Pf(n, x_n)\rangle|_{\partial(B_{M_0} \cap \ker(I-Q))}) \nabla V_0(p) + (1 - \lambda) g(p),$$

where $(p, \lambda) \in (B_{M_0} \cap \ker(I - Q)) \times [0, 1]$.

It follows that

$$\begin{aligned}&\langle \nabla V_0(p), H(p, \lambda)\rangle\\=&\lambda \mathrm{sgn}(\langle \nabla V_0(x_n), Pf(n, x_n)\rangle|_{\partial(B_{M_0} \cap \ker(I-Q))}) |\nabla V_0(p)|^2 + (1 - \lambda)\langle \nabla V_0(p), g(p)\rangle.\end{aligned} \tag{3.2}$$

For any $(p, n) \in (B_{M_0} \cap \ker(I - Q)) \times \mathbb{N}_+$, by (H_3), we know that the sign of $\langle \nabla V_0(p), Pf(n, p)\rangle$ does not change. And by the definition of $g(a)$, we have

$$\begin{aligned}\langle \nabla V_0(p), g(p)\rangle =&\langle \nabla V_0(p), \frac{1}{N}\sum_{k=1}^{N} Pf(k, p)\rangle\\=&\frac{1}{N}\langle \nabla V_0(p), \sum_{k=1}^{N} Pf(k, p)\rangle.\end{aligned}$$

It means that $\langle \nabla V_0(p), g(p)\rangle$ always has the same sign with $\langle \nabla V_0(p), \sum_{k=1}^{N} Pf(k, p)\rangle$. Also, by (H_3), we know that $|\nabla V_0(p)| \neq 0$, when $p \in \partial(B_{M_0} \cap \ker(I - Q))$. Consequently, the right hand of (3.2) is nonzero.

Thus

$$\langle \nabla V_0(p), H(p, \lambda)\rangle \neq 0, \forall (p, \lambda) \in \partial(B_{M_0} \cap \ker(I - Q)) \times [0, 1],$$

which implies that $0 \notin H(\partial(B_{M_0} \cap \ker(I - Q)) \times [0, 1])$.

The homotopy invariance of the Brouwer degree implies

$$\begin{aligned}0 \neq& \deg(g, D \cap \ker(I - Q), 0)\\=& \deg(\mathrm{sgn}(\langle \nabla V_0(x_n), Pf(n, x_n)\rangle|_{\partial(B_{M_0} \cap \ker(I-Q))}) \nabla V_0(p), B_{M_0} \cap \ker(I - Q), 0).\end{aligned}$$

Hence hypothesis (H_2) holds. Thus Theorem 3.1 follows from Theorem 1.1. □

Example 3.1. Consider the following system

$$x_{n+1} - x_n = \nabla V(x_n) - y_n, \tag{3.3}$$

where $V : \mathbb{R}^m \to \mathbb{R}$ is an even function, and $y_{n+N} = Qy_n,\ Q \in O(m)$.

Also if $\langle \nabla V_i(x_n), f(n, x_n)\rangle > 0$, then Hessian matrix $(\frac{\partial^2 V}{\partial x_i \partial x_j})$ is positive semidefinite, and if $\langle \nabla V_i(x_n), f(n, x_n)\rangle < 0$, then Hessian matrix $(\frac{\partial^2 V}{\partial x_i \partial x_j})$ is negative semidefinite.

In addition,

$$|V(x_n)| \to \infty, |\nabla V(x_n)| \to \infty,\ as\ |x_n| \to \infty.$$

Then the system $x_{n+1} - x_n = \nabla V(x_n) - y_n$ has an affine-periodic solution.

Proof. Let $V_0(x_n) = V(x_n)$. Then if $x_n \gg 1$, we have

$$\begin{aligned}\langle \nabla V(x_n), \nabla V(x_n) - y_n\rangle &= |\nabla V(x_n)|^2 - \langle \nabla V(x_n), y_n\rangle \\ &\geq |\nabla V(x_n)|^2 - \frac{1}{2}|\nabla V(x_n)|^2 - \frac{1}{2}|y_n|^2 \\ &= \frac{1}{2}|\nabla V(x_n)|^2 - \frac{1}{2}|y_n|^2 \\ &> 0.\end{aligned}$$

As $V(x_n)$ is even, it is clear that $\nabla V(x_n)$ is odd. According to Borsuk Theorem, for M large enough, we have

$$\deg(\nabla V(x_n), B_M, 0) \neq 0.$$

The conclusion follows from Theorem 3.1. □

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