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CHATTER DYNAMIC ANALYSIS FOR A PLANING MODEL WITH THE EFFECT OF PULSE*

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Abstract In this paper, the phenomena of chatter vibration in a typical planing process is discussed from impulsive point of view. Considering the instantaneous vibration as an impulse, we present the planing model as a second-order impulsive dynamical system, which is a specific discontinuous one. Then we investigate its chatter conditions via the method of flow theory in discontinuous systems, analyze flow's dynamical behaviors on separation surface and get general results on chatter criterion. Finally, such dynamical analysis and criterion are applied to a specific planing model of coal plough.

Keywords Chatter, planing, discontinuous, pulse.

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1. Introduction

In traditional process of mechanical machining, there are three basic factors, tool, workpiece and cutting motion. It is normal to encounter an instable dynamic phenomena which is called self-excited vibration due to the relative motion between tool and workpiece. Once the machining tool vibrates, the machine would cut into the wavy material surface in further cutting process and regenerate corresponding undulations. Such undulation of the system left by previous cut would affect the next cutting process as a kind of instantaneous impact does, which might cause more complicated phenomena named chatter as the most obstacle of all problems facing machinists. For a century, various techniques and methods have developed in mechanical manufacturing, such as turning, planing, milling, drilling and so on, but almost in all machining processes, such undesired chatter leading to vibrational instability is present. Therefore, a lot of work have been done to look for the cause of such phenomena. In [3], Arnold firstly found it was not induced by external periodic forces but the forces generated in dynamic machining process itself, which was the most important characteristic property of chatter vibration. Since then, several theories have been presented to interpret such phenomena, including regeneration theory, vibration coupling theory, negative friction theory, lag of cutting force theory and so on. Nowadays, Wiercigroch and Budak [20] concluded that there are two kinds of chatter vibration, primary chatter and secondary chatter.

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And primary chatter is caused by friction between tool and workpiece, mechanical effects or mode coupling; while the latter case is caused by the regeneration of wavy surface on the workpiece, called regenerative vibration. Actually, from the very beginning, Tobias [18] had studied the modeling of the dynamic response, structural aspects and stability limit aspects of regenerative vibration. And in [13], Merrit had considered regenerative chatter as a closed-loop interaction between the structural dynamics and the machining process.

For nearly fifty years, it has been a popular topic for academic and industrial research to investigate chatter phenomena, but chatter is still one of the main obstacles in achieving automation as shown in [2], because it is much more detrimental to finished surfaces and cutting tools due to its unstable behaviors which result in large relative displacements between tool and workpiece, especially in more complicated sticking and sliding cases in [1]. As we know, excessive vibration between tool and workpiece would bring several adverse effects to machining processes. Its catastrophic nature creates numerous problems to the total capacity usage of a machine tool in production, including unsteady process, poor surface quality, inefficient productivity rate, limited reliability and safety of the tool and so on. Since it causes so many inconveniences, nowadays, many scholars have proposed various techniques to avoid the occurrence of regenerative chatter in the machining process by either predicting or detecting as soon as it occurs. Many engineers have also tried active or passive control strategies to find conditions for chatter vibrations in order to obtain higher productivity and better surface finish of the product. Such chatter conditions include the alternation cutting force caused by the disturbed system, enough energy supply to keep chattering and so on. In [16], Rubenstein investigated the tool oscillation during a planing operation with the influence of nonlinear magnitude of the cutting stresses. In 1980, Sexton [17] indicated the proper selection of amplitude and frequency of the speed signal was dependent on dynamics of the cutting process and constrained by the drive system response. In 1993, without systematic variation of the speed, the above technique was utilized to generate chatter in an otherwise stable process in [19]. And some analytical treatments have been used to deal with the effects of nonlinearity, such as perturbation method in [14]. Recently in [4], Davies etc proposed a new stability theory for interrupted machining process from the angle of impact. And in [11], Li etc considered the occurrence of vibration as the instantaneous phenomena which could even regenerate chatter due to the instantaneously react cutting force components and the dynamic regenerative effects in milling process. Furthermore, Nayfeh etc [15] presented a state-of-the-art review of chatter in machining processes and classified current methods to ensure chatter-free cutting conditions, during which the dynamic characteristics could be obtained through model testing by using the impulse method. Based on the fact that, chatter phenomena arises due to the successive cutting motion between tool and workpiece over a previously machined undulated or wavy surface at the frequency of the most dominant mode of the machine tool structure, in [6], Fu and Zheng constructed a set for chatter conditions and studied the regeneration process for a single degree of freedom(SDOF) system in a turning process. If certain conditions were met, regenerative chatter would occur when small waves in the material left by cut at some moment during subsequent passes. By the theory of flow switchability, we systematically investigated the occurrence condition and the energy change of chatter from discontinuous point of view and extended the typical turning model into a second-order impulsive differential system. As is well known, second-order system as a typical mathematical equation, has provided many important mechanical models in the field of nonlinear vibration. In [7], Fu and Zheng presented a second-order vibration switched system and obtained several general results about chatter, also regarding it as the effect of impulse.

Actually, it is universal for a dynamical system to experience transient phenomenon in practical problems, which means the pulse occurs. Since early 1980s, many specialists have systematically investigated the stability of impulsive differential systems ([5]), especially from 1989 when Lakshmikantham etc [12] introduced a 'beating' phenomenon for the variable-impulse case, more attentions have been paid on this topic as it is more in line with the actual situation in virtue of the appearance of pulse phenomena([21]), namely the motion hits the same surface finite or infinite number of times, which causes rhythmical 'chatter' in differential equations. As for the conditions for pulse phenomena, traditional method was concentrated in arguing about the relationship between the motion curve and the impulse surfaces as well as the impulse functions, which needed detailed analysis on the global property of impulse. To avoid these inconveniences, in 2013, Zheng and Fu [22] first viewed the impulsive system as a global discontinuous one and investigated its chatter dynamics, which was just what we wanted to indicate in this paper. By noting the solution's dynamical behaviors near the impulse surface, some results on pulse phenomena of simple form were proposed, without discussing the rigorous property of impulse. In this paper, we will utilize the discontinuous theory and investigate chatter dynamics for a planing model.

Planing is a kind of machining method for the tool with relatively reciprocating movement on the workpiece surface. The height distribution of the surface machined by planing is homogeneous with a relatively low plough head, therefore, the process of planing is mainly used for machining plane and groove metal shape, as well as in mining process by breaking the coal seam in the way of planing. However, in virtue of its close tool about the material surface, it is difficult to mine hard coal seam, and with the frictional resistance for the plough head and conveyor, vibration even chatter phenomena flare at times. In this paper, we will focus on the planing process, introduce a planing model to investigate its chatter dynamics, and go along with the research of searching for chatter conditions, utilizing our method of impulsive system on the property of pulse phenomena. Besides, we will generalize the local theory of discontinuous systems([10]) and utilize the means into planing process further. By analysing the coherence between a solution of differential equations and the flow of dynamical systems, the model would be deemed as a discontinuous one consisting of two sub-systems with time-varying domains and a reference dynamical system based on the time-varying separation boundary.

The remaining contents of this paper consist of four parts. In Section 2, we will present a planing model as a certain kind of second-order impulsive equation and analyze its features from discontinuous point of view. In Section 3, several conceptions of flow theory as well as a flow's dynamical behaviors at a boundary in the normal direction to the chatter condition set will be presented. In Section 4, by using mapping structures, the flow's transversal property through a domain to the adjacent one will analyzed, and some sufficient conditions for the absence of chatter will be obtained. In Section 5, we will apply our analysis and criterion to a specific planing model of coal plough.

2. Model Analysis

In this section, we will discuss a typical planing model of coal plough. Take a mining machine as the example, and the following is a diagram for a comprehensive mechanical coal mining equipment for thin seam mining, which could not only mine but also load and transport the coal by the conveyer belt. In Figure 1, the planing model for the planing process with double-wheel driving has been formulated. The coal plough system includes a double-wheel driving system, a coal planing blade attached to the conveyer belt and a plough chain connecting the conveyer belt and wheels.

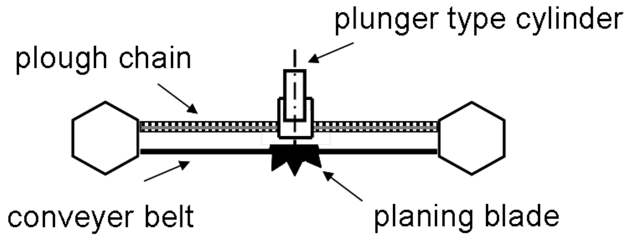

Figure 1. Diagram for a typical planing model of coal plough.

In [8], the mathematical dynamical system is proceeded from dynamic analysis for characteristics of the machine tool structure for such model, which comes from an SDOF orthogonal cutting process with a flexible tool and relatively rigid workpiece. Except traction of the plough chain, the motivation for planing blade also comes from several coal breaking force, such as high pressure hydraulic function equipped with automatic control system, like plunger type cylinder in Figure 1, to realize the full automation of coal mining. The motion equation for the planing blade in the feed direction x is

$$mx^{''} = F_t(x, x') - f(t), \tag{2.1}$$

where x represents the displacement of the planing blade, m is the mass of the tool, $F_t(x, x')$ represents the traction of the plough chain, and $f(t)$ is the resistance originated from the cutting force and load.

Actually, in corresponding planing model, the cutting tool is directed perpendicular to the coal seam, shaving a thin piece of material as the spindle turns, especially when the planing blade is once surrounded by hard materials, it might vibrate more dramatically and lead to unstable chatter process in the planing operation. However, coal seam is naturally formed with nonuniform densities, making the model experience time-varying and space-varying forces acting on the physical system and incorporate inertia force and cutting force. As shown is Figure 2, the planing blade is affected by three dimensional load and quantitative friction resistances, including ploughing resistance F_x, the feed resistance F_z, lateral resistance F_y and the weight of its own. And the ploughing resistance is related to others with larger fluctuation. Besides, in the orthogonal planing, the cutting force is generally considered to be proportional to the chip area which depends on cutting time and frequency, so we denote $F_\mu(t)$ as the time-varying cutting force in feed direction. And since the coal planer rotates along the conveyer belt, it would generate friction as

$$f(t) = F_x + F_\mu(t), \tag{2.2}$$

where μ is the friction coefficient.

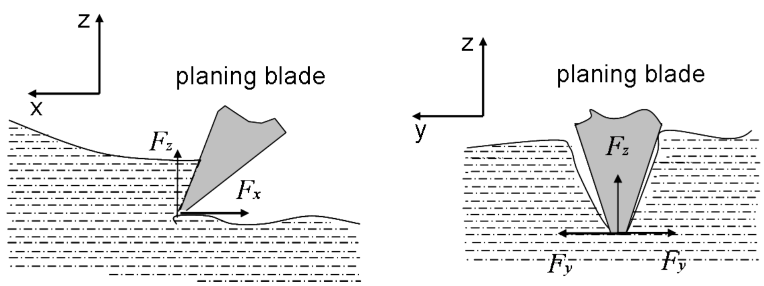

Figure 2. The main view and the lateral view for the planing blade in planing process.

Substituting equation (2.2) into equation (2.1) and dividing by m, it gives a form of second-order differential equation as

$$x^{''} = \frac{F_t(x,x')}{m} - \frac{F_x}{m} - \frac{F_\mu(t)}{m}. \tag{2.3}$$

Instead of searching for possible chatter frequencies by scanning from the transfer function during cutting, like [6], we denote $M(t)$ and $N(t)$ as two regions for the occurrence and nonoccurrence of vibration dependent with time t. It means that when at those moments satisfying the condition set $M(t)$, regenerative chatter occurs with a small fluctuation on displacement of the material. Subsequently, the machine would work on according to the rule of equation (2.2) till the state next encounters with the condition set $M(t)$. Besides, denote function $\Gamma(t,x,x') = 0$ portraying any element in set $M(t)$ and boundary Γ between $M(t)$ and $N(t)$ depicting the surface of the set, where $\Gamma(t,x,x') \neq 0$ indicates element (x,x') in $N(t)$. When the state stays in the region $N(t)$, the vibratory condition would not be switched on, while once the dynamic characteristics satisfy the condition, it means the state would pass through boundary Γ and switch into region $M(t)$.

Viewing the planing process including the occurrence of chatter vibration as a whole dynamic system, unlike the traditional research, in this paper, we would take a different approach of the chatter process in planing operation. Consider equation (2.3) including vibration as the following impulsive system

$$\begin{cases} x^{''} = \frac{F_t(x,x')}{m} - \frac{F_x}{m} - \frac{F_\mu(t)}{m}, & \Gamma(t,x,x') \neq 0, \\ \varphi : x(t) \longrightarrow \varphi(x(t)), & \Gamma(t,x,x') = 0, \end{cases} \tag{2.4}$$

where $M(t) \subset \Omega,\ \Omega \subset R$ is an open set, $x \in \Omega$, and the operator $\varphi \in C(M(t),\Omega)$ transfers the states in region $M(t)$ into another with small fluctuation. It means when the vibration happens, the small wave in the material would accumulate during subsequent passes of the cut due to the fluctuation depending on the original stateAs for the following planing movement, it can somewhere keep the normal proc., or be certain state satisfying the vibratory condition again, in which the latter case represents the occurrence of chatter vibration.

Actually, from the mathematical point of view, equation (2.4) originally gives the evolution process subject to instantaneous impulsive effects, while in our model by introducing the former vibratory threshold function $\Gamma(t,x,x') = 0$, it means that, when at moments that do not satisfy the vibratory condition, the solution for equation (2.4) would move in accordance with the second-order differential equation (2.3), while when at those moments satisfying the threshold function, the solution would experience a transfer in obedience to the impulse operator representing the instantaneous changing on displacement. Those vibratory moments when the vibration occurs are also called the pulse moments, and corresponding implicit function generated by the threshold function $\Gamma(t, x, x') = 0$ can be regarded as a surface in the (t, x, x')-space and called pulse surface denoted by Γ. For the original continuous solution for equation (2.3), it may be (i) a continuous function, if the integral curve does not intersect $M(t)$ or hits it at the fixed points of the operator φ; (ii) a piecewise continuous function having countable number of discontinuities of the first kind, if the integral curve encounters the pulse surface Γ at countable number of non-fixed points of φ. The former case stands for the normal planing process or the situation that just some little vibrations happen without any extra waves left in the subsequent planing, and the latter represents the occurrence of chatter which may be combined with different type of flow curves.

For a transformation, $x_1 = x,\ x_2 = x^{'}$ was used and an equivalent system of equation (2.4) is given as

$$\begin{cases} x_1^{'} = x_2, & \Gamma(t, x_1, x_2) \neq 0, \\ x_2^{'} = \frac{1}{m}[F_t(x_1, x_2) - F_x - F_\mu(t)], & \Gamma(t, x_1, x_2) \neq 0, \\ \varphi : x_i(t) \longrightarrow \varphi(x_i(t)), & \Gamma(t, x_1, x_2)) = 0,\ i = 1, 2. \end{cases} \tag{2.5}$$

To obtain a standard format, we introduce vector $\mathbf{x} = (x_1, x_2)^{\mathrm{T}}$, and equation (2.5) becomes

$$\begin{cases} \mathbf{x}^{'} = \mathbf{F}(t, \mathbf{x}), & \Gamma(t, \mathbf{x}) \neq 0, \\ \Phi : \mathbf{x}(t) \longrightarrow \Phi(\mathbf{x}(t)), & \Gamma(t, \mathbf{x}) = 0, \end{cases} \tag{2.6}$$

where the vector field function $\mathbf{F} = (x_2, \frac{1}{m}[F_t(x_1, x_2) - F_x - F_\mu(t)])^{\mathrm{T}}$ and the impulsive operator $\Phi = (\varphi(x_1(t)), \varphi(x_2(t)))^{\mathrm{T}} \in C(R \times R, R \times R)$.

As for the vibratory threshold, implicit function $\Gamma(t, \mathbf{x}) = 0$ can generate a corresponding dynamical time-varying boundary. If $\Gamma(t, \mathbf{x}) = 0$ has a countable number of roots as explicit functions $t = \tau_k(\mathbf{x})$ for each $\mathbf{x},\ k = 1, 2...$, satisfying necessary assumptions, those points $(\tau_k(\mathbf{x}), \mathbf{x})$ in the whole space compose the pulse surfaces Σ_ks as the impulsive theory in [12] mentioned. On the other hand, the implicit function also divides the state space into two domains $M(t)$ and $N(t)$ varying as time passes. The relationship between the separation boundary and $M(t)$ can also be deemed that, $M(t)$ is the kernel of the function $\Gamma(t, \mathbf{x})$ about state $\mathbf{x}$ as

$$M(t) = \ker(\Gamma) = \{\mathbf{x} \in \Omega |\ \Gamma(t, \mathbf{x}(t)) = 0\},$$

while the surface Γ for $M(t)$ stands for the hyperplane-pulse surface in the whole space.

Suppose that for any $(t, \mathbf{x}) \in \Gamma$, corresponding normal vector on the time-varying separation boundary is denoted as

$${}^t\mathbf{n}_\Gamma\big|_{\mathbf{x}} = (\nabla\Gamma(t, \mathbf{x}))\big|_{\mathbf{x}},$$

where $\nabla = (\partial/\partial x_1, \partial/\partial x_2)^{\mathrm{T}}$ is the Hamiltonian operator.

By analyzing features of the impulsive equation we presented in planing process from discontinuous point of view, we could discuss the occurrence and nonoccurrence of chatter vibration for its equivalent dynamical system (2.6). Next, we would give the conception of vibration point first of all.

Definition 2.1. Consider a solution $\mathbf{x}(t) = \mathbf{x}(t_0, \mathbf{x}_0, t)$ of dynamical system (2.6) with initial condition $(t_0, \mathbf{x}_0)$. We call the point $(t^*, \mathbf{x}(t^*))$ is a vibration point of the solution $\mathbf{x}(t)$ for the impulse surface Γ in the whole space, if there is a t^*, such that $\Gamma(t^*, \mathbf{x}(t^*)) = 0$. And we call $\mathbf{x}^*$ is an intersection point of the trajectory $\mathbf{x}$ for $M(t)$, where $\mathbf{x}^* \doteq \mathbf{x}(t^*)$. We denote that $(t^*, \mathbf{x}^*) \in \Gamma$ and $\mathbf{x}^* \in M(t)$.

Then, we give definitions of regenerative chatter phenomena based on the occurrence of vibration.

Definition 2.2. For dynamical system (2.6), suppose $(t^*, \mathbf{x}(t^*))$ is a vibration point of the solution $\mathbf{x}(t)$, if there exists some certain small $\varepsilon > 0$, such that for any $t \in (t^*, t^* + \varepsilon)$, it gives $\Gamma(t, \mathbf{x}(t)) = 0$, we say that the phenomena of chatter would be regenerated; otherwise, if such ε can not be found, we say that the phenomena of chatter could not be regenerated.

Remark 2.1. The occurrence of regenerative chatter phenomena is that, once there is a vibration point of any solution starting from $N(t)$, the state in $M(t)$ is not the fixed points of the operator Φ, causing some fluctuations on the material and resulting in extra waves left in subsequent planing, which accumulates vibration. The nonoccurrence of regenerative chatter phenomena is that, when the vibration happens, the intersection point of the trajectory in $M(t)$ is the fixed point of Φ, making the tool go along the original curve in subsequent planing, which remains a continuous process.

3. Dynamical Analysis

In this section, we will introduce the flow theory of discontinuous systems. At first, the basic flow structures including interior flow and reference flow for discontinuous dynamical systems as well as their several geometric relations to the reference surface Γ will be introduced. Consider equation (2.6) as a dynamical system, and the solution can be treated as flow

$$\mathbf{x}_t = \mathbf{x}(\mathbf{x}_0, t - t_0),$$

where $\mathbf{x}(\mathbf{x}_0, t - t_0)$ is C^1-smooth on open subset $M(t)$ and $T \subset R$, satisfying initial condition

$$\mathbf{x}(t_0) = \mathbf{x}(\mathbf{x}_0, 0).$$

Therefore, any continuous solution of the dynamical system (2.6) starting from $N(t)$ can be termed as *interior flow* $\mathbf{x}_t$ in $N(t)$ regarding to the equation without encountering $M(t)$. To analyze and predict the precise motion for the flow once encountering separation boundary of $M(t)$, suppose that $(t^*, \mathbf{x}(t^*))$ is a vibration point of the solution $\mathbf{x}(t)$ for the impulse surface Γ. To distinguish the intersection points on the boundary out of those inside the domains, we denote that $\mathbf{x}(t^*) \doteq \overline{\mathbf{x}} \in M(t)$. Meanwhile, the time-varying boundary Γ can be employed as a well-behaved dynamical system as a *reference surface*. The *reference flow* $\overline{\mathbf{x}}_t$ which always stays in $M(t)$ is determined by the *interior* $\mathbf{x}_t$ at the same instant, satisfying

$$\begin{cases} \Gamma(t, \overline{\mathbf{x}}_t) = 0, \\ \Gamma(t, \Phi(\overline{\mathbf{x}}_t)) = 0, \end{cases} \tag{3.1}$$

where the later equation means $\overline{\mathbf{x}}_t$ is the fixed element of the operator Φ in the kernel of Γ.

In this paper, we focus on the nonoccurrence of chatter phenomena of the impulsive differential system (2.6). Next, we would subsequently introduce some conceptions to utilize the flow theory in discontinuous systems([9]).

Definition 3.1. Consider dynamical systems (2.6) and (3.1) with corresponding flows $\mathbf{x}_t$ and $\overline{\mathbf{x}}_t$. Suppose that flow $\mathbf{x}_t$ in $N(t)$ does not contact with the reference flow $\overline{\mathbf{x}}_t$ in $M(t)$. For an instant moment $\tilde{t}$ with $(\tilde{t}, \mathbf{x}_{\tilde{t}}) \notin \Gamma$ and an arbitrarily small $\varepsilon > 0$, consider two infinitesimal time intervals $[\tilde{t}-\varepsilon, \tilde{t})$ and $(\tilde{t}, \tilde{t}+\varepsilon]$. The normal vectors on time-varying boundary Γ at three locations which vary according to moment are given by ${}^{\tilde{t}-\varepsilon}\mathbf{n}_\Gamma$, ${}^{\tilde{t}}\mathbf{n}_\Gamma$ and ${}^{\tilde{t}+\varepsilon}\mathbf{n}_\Gamma$.

(i) For time interval $[\tilde{t}-\varepsilon, \tilde{t})$, we say flow $\mathbf{x}_t$ is *approaching* the reference surface Γ at time $\tilde{t}$, if

$$ {}^{\tilde{t}}\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x}_{\tilde{t}} - \overline{\mathbf{x}}_{\tilde{t}}) - {}^{\tilde{t}-\varepsilon}\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x}_{\tilde{t}-\varepsilon} - \overline{\mathbf{x}}_{\tilde{t}-\varepsilon}) > 0 \quad \text{for} \quad \mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}}) < 0; $$

or

$$ {}^{\tilde{t}}\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x}_{\tilde{t}} - \overline{\mathbf{x}}_{\tilde{t}}) - {}^{\tilde{t}-\varepsilon}\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x}_{\tilde{t}-\varepsilon} - \overline{\mathbf{x}}_{\tilde{t}-\varepsilon}) < 0 \quad \text{for} \quad \mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}}) > 0; $$

(ii) For time interval $(\tilde{t}, \tilde{t}+\varepsilon]$, we say flow $\mathbf{x}_t$ is *leaving from* the reference surface Γ at time $\tilde{t}$, if

$$ {}^{\tilde{t}}\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x}_{\tilde{t}} - \overline{\mathbf{x}}_{\tilde{t}}) - {}^{\tilde{t}+\varepsilon}\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x}_{\tilde{t}+\varepsilon} - \overline{\mathbf{x}}_{\tilde{t}+\varepsilon}) > 0 \quad \text{for} \quad \mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}}) < 0; $$

or

$$ {}^{\tilde{t}}\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x}_{\tilde{t}} - \overline{\mathbf{x}}_{\tilde{t}}) - {}^{\tilde{t}+\varepsilon}\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x}_{\tilde{t}+\varepsilon} - \overline{\mathbf{x}}_{\tilde{t}+\varepsilon}) < 0 \quad \text{for} \quad \mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}}) > 0; $$

(iii) For time intervals $[\tilde{t}-\varepsilon, \tilde{t})$ and $(\tilde{t}, \tilde{t}+\varepsilon]$, we say flow $\mathbf{x}_t$ is *passing through* the reference surface Γ at time $\tilde{t}$, if

for $\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}}) < 0,\ t \in [\tilde{t}-\varepsilon, \tilde{t})$,

$$ {}^{\tilde{t}}\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x}_{\tilde{t}} - \overline{\mathbf{x}}_{\tilde{t}}) - {}^{\tilde{t}-\varepsilon}\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x}_{\tilde{t}-\varepsilon} - \overline{\mathbf{x}}_{\tilde{t}-\varepsilon}) > 0, $$

and

$$ {}^{\tilde{t}}\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x}_{\tilde{t}} - \overline{\mathbf{x}}_{\tilde{t}}) - {}^{\tilde{t}+\varepsilon}\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x}_{\tilde{t}+\varepsilon} - \overline{\mathbf{x}}_{\tilde{t}+\varepsilon}) < 0, $$

or, for $\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}}) > 0,\ t \in [\tilde{t}-\varepsilon, \tilde{t})$,

$$ {}^{\tilde{t}}\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x}_{\tilde{t}} - \overline{\mathbf{x}}_{\tilde{t}}) - {}^{\tilde{t}-\varepsilon}\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x}_{\tilde{t}-\varepsilon} - \overline{\mathbf{x}}_{\tilde{t}-\varepsilon}) < 0, $$

and

$$ {}^{\tilde{t}}\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x}_{\tilde{t}} - \overline{\mathbf{x}}_{\tilde{t}}) - {}^{\tilde{t}+\varepsilon}\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x}_{\tilde{t}+\varepsilon} - \overline{\mathbf{x}}_{\tilde{t}+\varepsilon}) > 0. $$

For a small time interval, the displacement difference between interior flow in $N(t)$ and reference flow in $M(t)$ in normal direction(i.e., normal component) of the reference surface is used in the above conceptions. Utilizing the same idea, we introduce a new function describing the time-changing rate as follows.

Definition 3.2. Consider dynamical systems (2.6) and (3.1) with corresponding flows $\mathbf{x}_t$ and $\overline{\mathbf{x}}_t$. Suppose that for any $(t, \overline{\mathbf{x}}_t) \in \Gamma$, corresponding normal vector on the separation boundary is denoted as $\mathbf{n}_\Gamma(t, \overline{\mathbf{x}}_t)$. Hence, define G-function as

$$ G_\Gamma(t) = \mathbf{n}_\Gamma^{\mathrm{T}}(t, \overline{\mathbf{x}}_t) \cdot \dot{\mathbf{x}}_t. $$

Remark 3.1. The G-function is the normal component of relative vector field function between $M(t)$ and $N(t)$, which means the relative speed of the flow in the normal direction of Γ. Once flow $\mathbf{x}_t$ experiences a vibration with boundary Γ at instant t^*(i.e. $\mathbf{x}_{t^*} = \overline{\mathbf{x}}_{t^*}$), the foregoing definitions can be simplified by deleting corresponding factors. Especially, for a flow *passing through* the surface Γ, since flow $\overline{\mathbf{x}}_t$ always stays in $M(t)$, the flow $\mathbf{x}_t$ is called *transversal* to such surface at instant t^*. This case is flexibly related to the definition of vibration point.

4. Chatter Criterion

In this section, we will utilize the method of flow theory by considering the state-dependent impulse surface as a time-varying boundary which obstructs the flow. As mentioned above, in this paper, we focus on looking for conditions to guarantee the nonoccurrence of chatter from impulsive point of view. Here we present a general result of the nonoccurrence of regenerative chatter phenomena.

Theorem 4.1. *Consider dynamical systems* (2.6) *and* (3.1) *with corresponding flows* $\boldsymbol{x}_t$ *and* $\overline{\boldsymbol{x}}_t$. *Suppose that for the separation boundary* Γ, *corresponding normal vector is* $\boldsymbol{n}_\Gamma$. *If*

$$\textit{for} \quad \boldsymbol{n}_\Gamma^{\mathrm{T}} \cdot (\boldsymbol{x} - \overline{\boldsymbol{x}}) < 0, \qquad \boldsymbol{n}_\Gamma^{\mathrm{T}} \cdot F(t, \boldsymbol{x}) < 0,$$

or

$$\textit{for} \quad \boldsymbol{n}_\Gamma^{\mathrm{T}} \cdot (\boldsymbol{x} - \overline{\boldsymbol{x}}) > 0, \qquad \boldsymbol{n}_\Gamma^{\mathrm{T}} \cdot F(t, \boldsymbol{x}) > 0.$$

Then for any solution of equation (2.6) *starting from* $(t_0, \boldsymbol{x}_0)$ *in* $N(t)$, *there is at most only one vibration point for the impulse surface* Γ, *i.e., chatter phenomena would not be regenerated.*

Proof. First we consider any solution of equation (2.6) starting from $(t_0, \mathbf{x}_0)$ in $N(t)$, denoted by $\mathbf{x}(t) = \mathbf{x}(t, t_0, \mathbf{x}_0)$. According to the definition of $N(t)$, we have $\Gamma(t_0, \mathbf{x}_0) > 0$, or $\Gamma(t_0, \mathbf{x}_0) < 0$. Before discussing the motion of the solutions, we can describe the derivative of the foregoing threshold function $\Gamma(t, \mathbf{x})$ along equation (2.6) as

$$D\Gamma(t, \mathbf{x})\Big|_{(2.6)} = \frac{\partial \Gamma(t, \mathbf{x})}{\partial \mathbf{x}} \cdot \mathbf{x}'\Big|_{(2.6)},$$

where the Hamiltonian operator denotes $\frac{\partial \Gamma(t,\mathbf{x})}{\partial \mathbf{x}} = (\nabla \Gamma(t, \mathbf{x}))^{\mathrm{T}} = (\frac{\partial \Gamma(t,\mathbf{x})}{\partial x_1}, \frac{\partial \Gamma(t,\mathbf{x})}{\partial x_2})$.

Then for the first case, let us suppose $\Gamma(t_0, \mathbf{x}_0) < 0$. When the solution stays in $N(t)$,

(i) for $\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}}) < 0$, according to equation (2.6) and the definition of the normal vector of the separation boundary, it is obvious that

$$\begin{aligned} \mathbf{n}_\Gamma^{\mathrm{T}} \cdot \mathbf{F}(t, \mathbf{x}) &= (\nabla \Gamma(t, \mathbf{x}))^{\mathrm{T}} \cdot \mathbf{F}(t, \mathbf{x}) \\ &= (\frac{\partial \Gamma(t, \mathbf{x})}{\partial x_1}, \frac{\partial \Gamma(t, \mathbf{x})}{\partial x_2}) \cdot (x_1', x_2')^{\mathrm{T}} \\ &= D\Gamma(t, \mathbf{x})\Big|_{(2.6)}. \end{aligned} \tag{4.1}$$

Therefore, together with the negative initial value and the condition $\mathbf{n}_\Gamma^{\mathrm{T}} \cdot \mathbf{F}(t, \mathbf{x}) <$ 0, we know that, for any solution $\mathbf{x}(t)$ starting from $N(t)$, once the solution stays in it, we have $\Gamma(t, \mathbf{x}(t)) < \Gamma(t_0, \mathbf{x}_0) < 0$, for $\forall t > t_0$ in terms of the derivative of

$\Gamma(t, \mathbf{x})$. That is, the solution would not get to a vibration point with the impulse surface Γ, so that chatter phenomena would not be regenerated.

(ii) for $\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}}) > 0$, since that $\Gamma(t_0, \mathbf{x}_0) < 0$, while the derivative of $\Gamma(t, \mathbf{x})$ is positive referred to the condition $\mathbf{n}_\Gamma^{\mathrm{T}} \cdot \mathbf{F}(t, \mathbf{x}) > 0$ with the similar discussing process as stage (i). Hence, $\Gamma(t, \mathbf{x}(t))$ would increase until there is a moment $t_m > t_0$, such that $\Gamma(t_m, \mathbf{x}(t_m)) = 0$, i.e., when at the instant moment t_m, flow $\mathbf{x}_t$ would encounter the vibratory condition set $M(t)$ with $\mathbf{x}_{t_m} = \overline{\mathbf{x}}_{t_m}$, which lies on the reference surface Γ. That is, the solution experiences a vibration point with the impulse surface with $\mathbf{x}(t_m) = \overline{\mathbf{x}}(t_m)$.

Next, we would show that the state $\mathbf{x}(t_m)$ is a fixed point of the operator Φ.

Actually, in virtue of the property of increase for $\Gamma(t, \mathbf{x}(t))$ just before the solution encounters the boundary such that $\Gamma(t_m, \mathbf{x}(t_m)) = 0$, we know the flow is approaching the reference surface Γ for a small time interval $[t_m - \varepsilon, t_m)$, when $\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}}) > 0$,

$$\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}})\Big|_{t_m} - \mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}})\Big|_{t_m - \varepsilon} < 0. \tag{4.2}$$

With the Taylor series expansions of $\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}})$ at $t_m - \varepsilon$ with t_m up to the ε-term, and because in the small interval the reference surface is time-independent, it gives

$$\begin{aligned}
\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}})\Big|_{t_m - \varepsilon} &= \mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}})\Big|_{t_m} - \varepsilon D(\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}}))\Big|_{t_m} + o(\varepsilon) \\
&= \mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}})\Big|_{t_m} - \varepsilon[(D\mathbf{n}_\Gamma)^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}}) \\
&\quad + \mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\dot{\mathbf{x}} - D\overline{\mathbf{x}})]\Big|_{t_m} + o(\varepsilon) \\
&= \mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}})\Big|_{t_m} - \varepsilon[(D\mathbf{n}_\Gamma)^{\mathrm{T}}\Big|_{t_m} \cdot (\mathbf{x} - \overline{\mathbf{x}}) \\
&\quad + \mathbf{n}_\Gamma^{\mathrm{T}} \cdot \dot{\mathbf{x}}\Big|_{t_m} - \mathbf{n}_\Gamma^{\mathrm{T}} \cdot \dot{\overline{\mathbf{x}}}\Big|_{t_m}] + o(\varepsilon) \\
&= -\varepsilon G_\Gamma(t_m) + o(\varepsilon),
\end{aligned}$$

where $D(\cdot) = d(\cdot)/dt$.

As the positive $\varepsilon \to 0$, together with the former equation (4.2),it gives that

$$-\varepsilon G_\Gamma(t_m) > \ \mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}})\Big|_{t_m} = 0,$$

which implies $G_\Gamma(t_m) < 0$. Refer to the definition of G-function in 3.2, it gives

$$\mathbf{n}_\Gamma^{\mathrm{T}}(t_m, \overline{\mathbf{x}}_{t_m}) \cdot \mathbf{x}'_{t_m} = \mathbf{n}_\Gamma^{\mathrm{T}}(t_m, \overline{\mathbf{x}}_{t_m}) \cdot \mathbf{F}(t_m, \mathbf{x}(t_m)) < 0.$$

Similarly discussing as the former equation (4.1) in stage (i), the derivative of $\Gamma(t, \mathbf{x}(t))$ in any infinitesimal time interval $(t_m, t_m + \varepsilon)$ $(\varepsilon > 0)$ would be negative. Since that $\Gamma(t_m, \mathbf{x}(t_m)) = 0$, it gives

$$\Gamma(t, \mathbf{x}(t)) < 0, \quad \text{for } t \in (t_m, t_m + \varepsilon).$$

That is, for those moments after the vibration, corresponding states of the cutting displacement all among belong to $N(t)$, in which the certain small interval keeping the states in the kernel of $\Gamma(t, \mathbf{x}(t))$ could not be found out, i.e., the state $\mathbf{x}(t_m)$ is a fixed point of the operator Φ. Therefore, by repeating the process of (i) and (ii), the phenomena of chatter would not be regenerated.

On the other hand, for the second case, if $\Gamma(t_0, \mathbf{x}_0) > 0$, the proceeding of the proof is just like the first case. When the solution stays in $N(t)$,

(i$'$) for $\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}}) > 0$, according to the positive initial value and the condition, we know that, once the solution stays in the state of $N(t)$, we have $\Gamma(t, \mathbf{x}(t)) > \Gamma(t_0, \mathbf{x}_0) > 0$, for $\forall t > t_0$ in terms of the derivative of $\Gamma(t, \mathbf{x})$ similarly as equation (4.1) demonstrated. Hence, the solution would not get to a vibration point with the impulse surface Γ, that is chatter phenomena would not be regenerated.

(ii$'$) for $\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}}) < 0$, we could conclude from the process of (ii) and reveal the fact that once we find out the vibration point $(t_m', \mathbf{x}(t_m'))$, it is the only vibration point for Γ. That is, we will manifest that at the instant moment t_m', flow $\mathbf{x}_t$ is *transversal* to the boundary Γ, which means the state $\mathbf{x}(t_m')$ is a fixed point of the operator Φ. Actually, the similar equation like equation (4.2) could also be obtained as

$$\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}})\Big|_{t_m'} - \mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}})\Big|_{t_m' - \varepsilon} > 0,$$

due to the monotonicity property of $\Gamma(t, \mathbf{x}(t))$ among the same domain and the flow's approaching to the reference surface for a small time interval $[t_m' - \varepsilon, t_m')$. Also with the Taylor series expansions of $\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}})$ up to the ε-term, it gives

$$\mathbf{n}_\Gamma^{\mathrm{T}} \cdot (\mathbf{x} - \overline{\mathbf{x}})\Big|_{t_m' - \varepsilon} = -\varepsilon G_\Gamma(t_m') < 0,$$

which implies $G_\Gamma(t_m') > 0$. Similarly discussing as the former analysis, for any infinitesimal time interval $(t_m', t_m' + \varepsilon')$,

$$\Gamma(t, \mathbf{x}(t)) > 0, \quad \text{for } t \in (t_m', t_m' + \varepsilon'),$$

which is against the definition of the phenomena of regenerative chatter.

Therefore, together the two cases, we could obtain the nonoccurrence of the regenerative chatter phenomena. □

Remark 4.1. From the condition of the above result, we can obviously find the convenience of flow theory, which comes from the idea of the specific discontinuous dynamical system, i.e., impulsive differential system. From the introductory of the instantaneous impact, i.e., impulse, we can get a more general mathematical model for the process of machining, and obtain a more concise result avoiding the complex testing on the parameters. Besides, we would modify the conditions and improve the results involving more cases further.

5. Applications

In this section, we will apply our analysis and criterion to a specific planing model of coal plough.

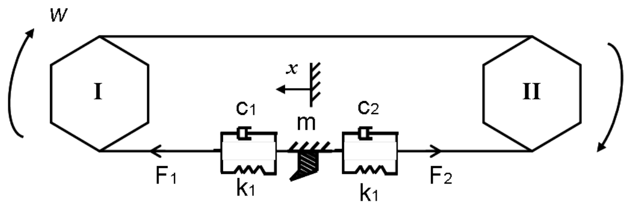

Figure 3. An orthogonal ploughing process with a coal planer tool with double-wheel driving.

Example 5.1. Figure 3 shows a double-wheel driving coal plough system(driving wheel I and driven wheel II), with a simplified form of traction $F_i(i = 1, 2)$ describing by two spring damping systems, where c_i, $k_i(i = 1, 2)$ are corresponding damping and stiffness coefficients. Firstly, we give traction of this double-wheel driving system

$$\begin{cases} F_1 = k_1(x - \omega Rt) + c_1(x' - \omega R), \\ F_2 = k_2(\omega Rt - x) + c_2(\omega R - x'), \end{cases} \tag{5.1}$$

where F_1 represents the chain traction between driving wheel I and the plough tool, F_2 represents another traction with a reverse direction in virtue of the driven wheel II. And ω is the angular velocity of the driving wheel, R is the radius of the sprocket pitch wheel.

Besides, for simplicity, denote $F_b(t)$ as the cutting force in the feed direction and friction $F_\mu(t)$ is

$$F_\mu = \mu mg, \tag{5.2}$$

where g is the acceleration of gravity.

Therefore, according to equations (2.1) and (2.2), the force analysis in the feed direction x is

$$mx^{''} = F_1 - F_2 - F_\mu - F_b(t), \tag{5.3}$$

in virtue of equations (5.1) and (5.2).

In Figure 3, the planing blade would run along the coal seam surface back and forth, in the traction of the plough chain, making such dynamical system a discontinuous one, including the free-vibration area and the impact area. Especially in this model, the plough would vibrate more intensely in a transverse, which is the velocity direction but not the feed, so we consider the variable speed boundary on which the displacement of feed shows a linear correlation with the threshold function.

To apply the analysis and criterion in Section 4, we consider following parameters for numerical illustrations of periodic motions,

$$\mu = 0.3,\ R = 0.243m,\ \omega = 1.5/R\ rad/s,\ c_1 = c_2 = 500Ns/m,\ m = 3400kg.$$

Besides, we take $k_1 = K/(L_0 + x)$, $k_2 = K/(L - L_0 - x)$ as the time-varying stiffness coefficients in virtue of Hooke's law with $K = 6.95 \times 10^7 N$, and $L = 200m$ represents the length of the chain, $L_0 = 2m$ represents the remaining length of the chain between the plough and the driven wheel II when the planing blade is excited.

Also we take the traction difference between the plough chain as a simplified stationary stochastic process with a fixed average according to equation (5.1). Then the following figures illustrate that how different types of cutting force and its alternation in feed direction would influence the transverse vibration.

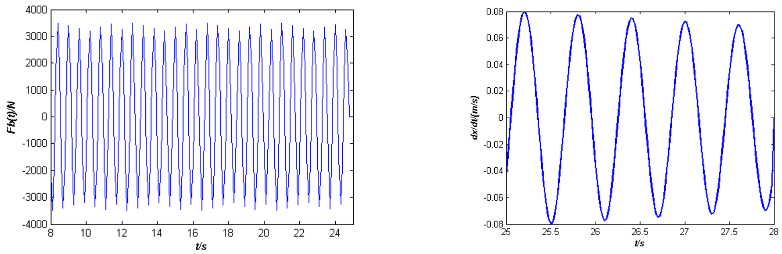

Figure 4. A ploughing process without the phenomena of chatter.

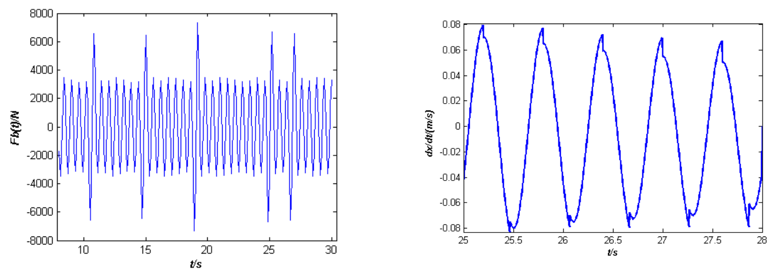

Figure 5. A ploughing process which would lead to chatter phenomena.

When we take the stationary stochastic ploughing resistance average of $2\times 10^3 N$ to ignore the perturbation of state x, which means the normal vector of the variable speed separation boundary is irrelevant with the state. And in equation (5.3), the G-function reduces only the second item which just offsets to zero according to equation (5.2). Therefore, Figure 4 shows an almost smooth free motion of the ploughing process as transverse vibration.

When the ploughing resistance shows a periodic motion but an abrupt increase when the oscillation state reaches the separation boundary, making the G-function zero. Hence in Figure 5, we can see some complex perturbation as impact in the velocity direction which would lead to unsteadiness after certain time scale, even more complicated phenomena such as chatter.

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