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SYNCHRONIZATION OF COUPLED SYSTEMS TO PERIODIC DIAGONAL SOLUTIONS WITH SYNCHRONIZED ASYMPTOTIC PHASES

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Abstract Using the change of coordinates, parameterization and characteristic multipliers, we prove the synchronization of a class of coupled nonlinear systems with nontrivial periodic solution. The periodic diagonal solution of the coupled system is asymptotically orbitally stable with asymptotic phase. Examples are given to illustrate the theorem.

Keywords Periodic solutions, coupled systems, periodic diagonal solutions, variational systems, characteristic multipliers, synchronized asymptotical phases.

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1. Introduction

Synchronization phenomena exist in almost all branches of sciences, engineering, and social life [7]. For the definition of the synchronization, see [1,4]. When identical dynamical systems are coupled, the diagonal in the phase space is invariant, and if the linear invariant manifold is stable, the identical synchronization occurs [2].

In this paper, we investigate the identical synchronization for a class of coupled nonlinear systems with asymptotically orbitally stable periodic orbit. We will show that under suitable conditions, synchronization occurs when the periodic diagonal solution is asymptotically orbitally stable. A solution near the periodic diagonal solution, with an initial condition, approaches the periodic diagonal solution with asymptotic phase. The asymptotic phase is determined by the initial condition of the solution. We have the following theorem:

Theorem 1.1. *Consider a system of coupled nonlinear equations*

$$\begin{aligned} \frac{dx}{dt} &= f(x) + K(x-y) + g(t, x-y), \\ \frac{dy}{dt} &= f(y) + K(y-x) + g(t, y-x). \end{aligned} \tag{1.1}$$

where $f \in C^1(\mathbf{R}^n, \mathbf{R}^n)$, $g \in C^1(\mathbf{R}^{n+1}, \mathbf{R}^n)$ *and* $g(t, -\zeta) = -g(t, \zeta)$ *with* $|g(t, \zeta)| = o(|\zeta|)$ *as* $|\zeta| \to 0$.

If the following conditions are satisfied by system (1.1)

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(a) $x = u(t)$ *is a* T*-periodic solution of the nonlinear system* $\frac{dx}{dt} = f(x)$*, and the characteristic multipliers of the variational system* $\frac{dw}{dt} = f'(u(t))w$ *satisfy*

$$\mu_1 = 1, \quad 0 < |\mu_j| < 1, \quad j = 2, ..., n,$$

(b) $f'(u(t))$ *is* s*-block diagonal,* $1 \le s \le n$*. Each block has size* $n_i \times n_i$*,* $1 \le i \le s$*.* $K = diag\{\alpha_1 I_{n_1}, ..., \alpha_s I_{n_s}\}$*, where*

$$\alpha_1 < 0 \quad and \quad \alpha_i < \min_{\substack{n_1 + \cdots + n_{(i-1)} + 1 \le \\ j \le n_1 + \cdots + n_{i-1} + n_i}} \left\{ -\frac{1}{2T}(ln|\mu_j|) \right\}, \quad 2 \le i \le s,$$

then

(i) the system (1.1) *is synchronized;*

(ii) for $(x(t_0), y(t_0)) = (x_0, y_0)$ *near the periodic diagonal solution, the solution of the coupled system approaches the periodic diagonal solution with synchronized asymptotic phase, i.e.* $(x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0)) \to (u(t + t_1), u(t + t_1))$ *as* $t \to \infty$ *for some constant* t_1 *depending on* (t_0, x_0, y_0)*.*

Note that in the theorem if $f'(u(t))$ is s-block diagonal with $s \ge 2$, then for $j \ge 2$, α_j can be chosen as any negative number, zero or a positive number restricted by (b). For the case $\alpha_j = 0$, a partially coupled system is formed.

The approach of proving this theorem is as follows. First, we decompose the space into the diagonal space and it's orthogonal complement. By using the new bases, we rewrite the solution $(x(t), y(t))$ into $(p(t), q(t))$, where $p(t)$ is in the diagonal space and $q(t)$ is in the orthogonal complement space. We then parameterize the periodic solution and use this parameterized function for the function $p(t)$ in a neighborhood of the periodic diagonal solution. Finally we use a new variational system and the characteristic multipliers to prove the result. In the last section, we present some examples to apply and illustrate the theorem.

2. Change of Coordinates

The diagonal space of the product space has an orthonormal base $\{b_j = (e_j, e_j)/\sqrt{2}, j = 1, 2, ..., n\}$, and its orthogonal complement has a base $\{c_j = (e_j, -e_j)/\sqrt{2}, j = 1, 2, ..., n\}$, where e_j is the j-th unit vector in $\mathbf{R}^n$. Therefore for any function $(x(t), y(t))$, we have

$$\begin{aligned}(x(t), y(t)) =& \sum_{j=1}^{n} [b_j \cdot (x(t), y(t))] b_j + \sum_{j=1}^{n} [c_j \cdot (x(t), y(t))] c_j \\ =& \sum_{j=1}^{n} [(x_j(t) + y_j(t))/\sqrt{2}] b_j + \sum_{j=1}^{n} [(x_j(t) - y_j(t))/\sqrt{2}] c_j \\ =&((x(t) + y(t))/2, (x(t) + y(t))/2) + ((x(t) - y(t))/2, (-x(t) + y(t))/2) \\ =&(p(t), p(t)) + (q(t), -q(t)), \qquad (2.1)\end{aligned}$$

where $p(t) = (x(t) + y(t))/2$ and $q(t) = (x(t) - y(t))/2$. Since we are interesting in the solution that is near the periodic orbit, therefore we can assume that each component of $p(t)$ is close to the periodic solution, and the absolute value of each component of $q(t)$ is small. Equation (2.1) contains formulas from $(x(t), y(t))$ to $(p(t), q(t))$ and vice versa.

Let $u(\theta)$ be the periodic solution as described in Theorem 1.1. Then there exists a local moving orthonormal system $[v(\theta), z_2(\theta), ..., z_n(\theta)]$ on this periodic solution, where $v(\theta) = f(u(\theta))/|f(u(\theta))|$ (see [3] for details). Denote the n by $n-1$ matrix whose columns are $z_2(\theta), ..., z_n(\theta)$ by $Z(\theta)$. For solutions near the periodic solution $u(\theta)$, there exists a parameterized function $\theta(t)$ such that

$$(p(t), q(t)) = (u(\theta(t)) + Z(\theta(t))\rho(t), q(t)). \tag{2.2}$$

It was mentioned in [3] that we can assume $\frac{d\theta}{dt} = 1$, if periodic orbit is under consideration. Here we will be using this. In the case $n = 2$, one can see [6] for more detail work.

Now we consider the system of ordinary differential equations for the vector function $(\theta(t), \rho(t), q(t)) \in \mathbf{R} \times \mathbf{R}^{n-1} \times \mathbf{R}^n$. Since $p(t)$ is close to the periodic orbit $u(\theta)$, thus we can assume the norms of $\rho(t)$ and $q(t)$ are small. It follows from equation (1.1) and (2.1), we have

$$\begin{aligned} \frac{dp}{dt} &= \frac{1}{2}[f(p+q) + f(p-q)], \\ \frac{dq}{dt} &= \frac{1}{2}[f(p+q) - f(p-q)] + 2Kq + g(t, 2q). \end{aligned} \tag{2.3}$$

Applying (2.2) to the first equation of (2.3), we have

$$\begin{aligned} &\left[\frac{du(\theta)}{d\theta} + \frac{dZ(\theta)}{d\theta}\rho\right]\frac{d\theta}{dt} + Z(\theta)\frac{d\rho}{dt} \\ =&\frac{1}{2}[f(u(\theta) + Z(\theta)\rho + q) + f(u(\theta) + Z(\theta)\rho - q)]. \end{aligned} \tag{2.4}$$

Multiplying both sides of (2.4) by $v^T(\theta)$, one has

$$\begin{aligned} &\left[\left|\frac{du(\theta)}{d\theta}\right| + v^T(\theta)\frac{dZ(\theta)}{d\theta}\rho\right]\frac{d\theta}{dt} \\ =&\frac{1}{2}v^T(\theta)[f(u(\theta) + Z(\theta)\rho + q) + f(u(\theta) + Z(\theta)\rho - q)]. \end{aligned}$$

Let

$$h(\theta, \rho) = \left[\left|\frac{du(\theta)}{d\theta}\right| + v^T(\theta)\frac{dZ(\theta)}{d\theta}\rho\right]^{-1} v(\theta), \tag{2.5}$$

then equation (2.4) is changed into

$$\begin{aligned} \frac{d\theta}{dt} =&\frac{1}{2}h^T(\theta, \rho)[f(u(\theta) + Z(\theta)\rho + q) + f(u(\theta) + Z(\theta)\rho - q)] \\ =&1 + h^T(\theta, \rho)\left\{-\frac{dZ(\theta)}{d\theta}\rho\right. \\ &\left.+\frac{1}{2}[f(u(\theta) + Z(\theta)\rho + q) + f(u(\theta) + Z(\theta)\rho - q) - 2f(u(\theta))]\right\}. \end{aligned} \tag{2.6}$$

Multiplying both sides of (2.4) by $Z^T(\theta)$, one has

$$\frac{d\rho}{dt} = Z^T(\theta)\left\{-\frac{dZ(\theta)}{d\theta}\rho\frac{d\theta}{dt} + \frac{1}{2}[f(u(\theta)+Z(\theta)\rho+q) + f(u(\theta)+Z(\theta)\rho-q)]\right\}. \tag{2.7}$$

Therefore the system (2.3) becomes

$$\begin{aligned}\frac{d\theta}{dt} &= 1 + S_1(\theta,\rho) + S_2(\theta,\rho,q),\\ \frac{d\rho}{dt} &= R_1(\theta,\rho) + R_2(\theta,\rho,q),\\ \frac{dq}{dt} &= Q_1(\theta,q) + Q_2(\theta,\rho,q,t),\end{aligned} \tag{2.8}$$

where

$$\begin{aligned} S_1(\theta,\rho) &= h^T(\theta,\rho)\left(-\frac{dZ(\theta)}{d\theta} + f'(u(\theta))Z(\theta)\right)\rho = O(|\rho|),\\ S_2(\theta,\rho,q) &= \frac{1}{2}h^T(\theta,\rho)\{f(u(\theta)+Z(\theta)\rho+q) + f(u(\theta)+Z(\theta)\rho-q)\\ &\quad - 2f(u(\theta)) - 2f'(u(\theta))Z(\theta)\rho\} = o(|(\rho,q)|),\\ R_1(\theta,\rho) &= Z^T(\theta)\left(-\frac{dZ(\theta)}{d\theta} + f'(u(\theta))Z(\theta)\right)\rho = O(|\rho|),\\ R_2(\theta,\rho,q) &= \frac{1}{2}Z^T(\theta)\{f(u(\theta)+Z(\theta)\rho+q) + f(u(\theta)+Z(\theta)\rho-q)\\ &\quad - 2f(u(\theta)) - 2f'(u(\theta))Z(\theta)\rho\}\\ &\quad - Z^T(\theta)\frac{dZ(\theta)}{d\theta}\rho(S_1(\theta,\rho)+S_2(\theta,\rho,q)) = o(|(\rho,q)|),\\ Q_1(\theta,q) &= [f'(u(\theta)) + 2K]q = O(|q|),\\ Q_2(\theta,\rho,q,t) &= \frac{1}{2}[f(u(\theta)+Z(\theta)\rho+q) - f(u(\theta)+Z(\theta)\rho-q)\\ &\quad - 2f'(u(\theta))q + 2g(t,2q)] = o(|(\rho,q)|).\end{aligned} \tag{2.9}$$

The orders here refer to those when the limits go to zero.

Since S_1 is $O(|\rho|)$ and S_2 is $o(|(\rho,q)|)$, thus one can choose $|(\rho,q)|$ to be small enough so that

$$\frac{1}{2} \le \frac{d\theta}{dt} \le 2. \tag{2.10}$$

This condition allows us to rewrite the last two equations of the system (2.8) into a system of the variable θ, which is as follows.

$$\begin{aligned}\frac{d\rho}{d\theta} &= R_1(\theta,\rho) + \tilde{R}_2(\theta,\rho,q),\\ \frac{dq}{d\theta} &= Q_1(\theta,q) + \tilde{Q}_2(\theta,\rho,q,t),\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}\tilde{R}_2(\theta,\rho,q) &= \frac{R_2(\theta,\rho,q) - R_1(\theta,\rho)(S_1(\theta,\rho)+S_2(\theta,\rho,q))}{1+S_1(\theta,\rho)+S_2(\theta,\rho,q)} = o(|(\rho,q)|),\\ \tilde{Q}_2(\theta,\rho,q,t) &= \frac{Q_2(\theta,\rho,q,t) - Q_1(\theta,q)(S_1(\theta,\rho)+S_2(\theta,\rho,q))}{1+S_1(\theta,\rho)+S_2(\theta,\rho,q)} = o(|(\rho,q)|).\end{aligned} \tag{2.12}$$

Now, one can see that the linear system of (2.11) can be decoupled for the variables ρ and q in the spaces $\mathbf{R}^{n-1}$ and $\mathbf{R}^n$, we have the following

$$\begin{aligned} \frac{d\rho}{d\theta} =& R_1(\theta,\rho) = Z^T(\theta,\rho)\left(-\frac{dZ(\theta)}{d\theta} + f'(u(\theta))Z(\theta)\right)\rho, \\ \frac{dq}{d\theta} =& Q_1(\theta,q) = [f'(u(\theta)) + 2K]q. \end{aligned} \tag{2.13}$$

If the characteristic multipliers of linear system (2.13) is to satisfy the conditions (a) and (b) of Theorem 1.1, then one can use it to prove the synchronization theorem of system (1.1). We describe these results in the next section. For simplicity, we use the same (ρ, q) to denote the functions of t and of θ.

3. Synchronization

Let A be an $n \times n$ monodromy matrix of the linear system $\frac{dw}{dt} = f'(u(t))w$. Since $\frac{du}{dt}$ is a solution of the system, thus A can be chosen in the following form

$$A = \begin{bmatrix} 1 & B_2 \\ 0 & B_1 \end{bmatrix},$$

where B_1 is an $(n-1)\times(n-1)$ matrix whose eigenvalues are $\mu_2, ..., \mu_n$. If $f'(u(t))$ is block diagonal, then A and a fundamental matrix $\Phi(t)$ can be chosen as block diagonal with same sizes for their related blocks. We start with the following lemma:

Lemma 3.1. *Suppose $f \in C^1(\mathbf{R}^n, \mathbf{R}^n)$ and the conditions (a) and (b) of Theorem 1.1 are satisfied. Suppose A is to be chosen as block diagonal matrix as the same size of $f'(u(t))$, ie $A = diag\{A_1, ..., A_s\}$. Let M be a monodromy matrix of (2.13), then*

(i) $M = diag\{B_1, e^{2\alpha_1 T}A_1, ..., e^{2\alpha_s T}A_s\}$;

(ii) $2n-1$ *characteristic multipliers of M are*

$$\begin{aligned} &\mu_2, ..., \mu_n, \quad e^{2\alpha_1 T}, \quad e^{2\alpha_1 T}\mu_2, \quad ..., \quad e^{2\alpha_1 T}\mu_{n_1}, \quad ..., \\ &e^{2\alpha_i T}\mu_{n_1+\cdots+n_{(i-1)}+1}, \quad ..., \quad e^{2\alpha_i T}\mu_{n_1+\cdots+n_i}, \quad ... \quad \textit{for} \quad 2 \le i \le s. \end{aligned}$$

Proof. Since $w = \frac{du(\theta)}{d\theta}$ is a solution of $\frac{dw}{d\theta} = f'(u(\theta))w$, one can choose a fundamental matrix solution in the form of $[\frac{du(\theta)}{d\theta}, \xi_2(\theta), ..., \xi_n(\theta)]$, where $\xi_j(\theta) = a_j(\theta)\frac{du}{d\theta} + Z(\theta)\rho_j(\theta),\ 2 \le j \le n$. Here $Z(\theta)$ is the n by $n-1$ matrix whose columns are $z_2(\theta), ..., z_n(\theta)$. By using the fact that each $\xi_j(\theta)$ is a solution of $\frac{dw}{d\theta} = f'(u(\theta))w$, one has

$$\begin{aligned} \frac{d\xi_j(\theta)}{d\theta} =& a_j'(\theta)\frac{du(\theta)}{d\theta} + a_j(\theta)\frac{d^2u(\theta)}{d\theta^2} + \frac{dZ(\theta)}{d\theta}\rho_j(\theta) + Z(\theta)\frac{d\rho_j(\theta)}{d\theta} \\ =& f'(u(\theta))[a_j(\theta)\frac{du(\theta)}{d\theta} + Z(\theta)\rho_j(\theta)]. \end{aligned}$$

Multiply both sides of previous equation by $Z^T(\theta)$ and use $\frac{d^2u(\theta)}{d\theta^2} = f'(u(\theta))\frac{du}{d\theta}$, we have

$$\frac{d\rho_j(\theta)}{d\theta} = Z^T(\theta)\left(-\frac{dZ(\theta)}{d\theta} + f'(u(\theta))Z(\theta)\right)\rho_j(\theta). \tag{3.1}$$

Therefor $\rho_j(\theta)$ satisfies the first equation of (2.13) for each j. This implies that $\Gamma(\theta) \equiv [\rho_2(\theta), ..., \rho_n(\theta)]$ is a fundamental solution of the first equation of (2.13). By using the definition of the monodromy matrix and both $\frac{du}{d\theta}$ and $Z(\theta)$ are T-periodic, we have

$$\left[\frac{du(T)}{d\theta}, Z(T)\right]\left[\begin{array}{cc} 1 & C(T) \\ 0 & \Gamma(T) \end{array}\right] = \left[\frac{du(0)}{d\theta}, Z(0)\right]\left[\begin{array}{cc} 1 & C(0) \\ 0 & \Gamma(0) \end{array}\right]\left[\begin{array}{cc} 1 & B_2 \\ 0 & B_1 \end{array}\right], \quad (3.2)$$

where $C(\theta) \equiv [c_2(\theta), \cdots, c_n(\theta)]$ is a $1\times(n-1)$ matrix. Thus we have $\Gamma(T) = \Gamma(0)B_1$. Hence B_1 is a monodromy matrix of the first equation of (2.13).

Let $A = diag\{A_1, ..., A_s\}$ and $\Phi(\theta) = diag\{\Phi_1(\theta), ..., \Phi_s(\theta)\}$ be as defined in the beginning of this section and have the same block diagonal structure of $f'(u(\theta))$. Since each block of $e^{2K\theta}$ is diagonal with constant entries $e^{2\alpha_j\theta}$ and

$$\frac{d}{d\theta}(e^{2K\theta}\Phi(\theta)) = e^{2K\theta}f'(u(\theta))\Phi(\theta) + 2Ke^{2K\theta}\Phi(\theta) = (f'(u(\theta)) + 2K)e^{2K\theta}\Phi(\theta).$$

This show that $e^{2K\theta}\Phi(\theta)$ is a fundamental matrix solution of $\frac{dw}{d\theta} = [f'(u(\theta))+2K]w$. Therefore, we obtain

$$\begin{aligned} e^{2KT}\Phi(T) =& \Phi(T)e^{2KT} = \Phi(0)Ae^{2KT} \\ =& \Phi(0)diag\{e^{2\alpha_1 T}A_1, ..., e^{2\alpha_s T}A_s\}. \end{aligned} \quad (3.3)$$

Equation (3.3) implies $diag\{e^{2\alpha_1 T}A_1, ..., e^{2\alpha_s T}A_s\}$ is a monodromy matrix of the second equation of (2.13). This concludes (i). (ii) follows directly from (i). □

Now we are ready to prove the Theorem 1.1.

Proof of Theorem 1.1. Let $\Psi(t)$ be a fundamental matrix solution of (2.13). Suppose that the conditions (a) and (b) are satisfied, then it follows from Lemma 3.1 that all the characteristic multipliers are less than 1. Therefore there exist positive constants γ_1 and β_1 such that

$$|\Psi(\theta)\Psi^{-1}(\theta_0)| \leq \gamma_1 e^{-\beta_1(\theta-\theta_0)}. \quad (3.4)$$

By applying the variation of constant formula to equation (2.11), we obtain

$$\left[\begin{array}{c} \rho(\theta) \\ q(\theta) \end{array}\right] = \Psi(\theta)\Psi^{-1}(\theta_0)\left[\begin{array}{c} \rho_0 \\ q_0 \end{array}\right] + \int_{\theta_0}^{\theta} \Psi(\theta)\Psi^{-1}(\eta)\left[\begin{array}{c} \tilde{R}_2(\eta, \rho(\eta), q(\eta)) \\ \tilde{Q}_2(\eta, \rho(\eta), q(\rho), t(\eta)) \end{array}\right] d\eta.$$

Now take the norms on both sides of previous equation and use the facts that $|(\rho(\theta), q(\theta))|$ is small and (2.12), inequality (3.4) leads to

$$e^{\beta_1\theta}|(\rho(\theta), q(\theta))| \leq \gamma_1 e^{\beta_1\theta_0}|(\rho_0, q_0)| + \int_{\theta_0}^{\theta} \gamma_1\delta e^{\beta_1\eta}|(\rho(\eta), q(\eta))|d\eta,$$

for some small positive value δ. By applying Gronwall's inequality to the previous inequality, one has

$$e^{\beta_1\theta}|(\rho(\theta), q(\theta))| \leq \gamma_1 e^{\beta_1\theta_0}|(\rho_0, q_0)|e^{\gamma_1\delta(\theta-\theta_0)}.$$

These two inequalities lead to

$$|(\rho(\theta), q(\theta))| \leq \gamma_1|(\rho_0, q_0)|e^{-(\beta_1-\gamma_1\delta)(\theta-\theta_0)} \equiv \gamma e^{-\beta(\theta-\theta_0)}, \quad (3.5)$$

for some positive constants γ, β. Letting θ go to infinity on equation (3.5), we obtain the synchronization of $x(t)$ and $y(t)$. To show they approach to the periodic orbit with asymptotic phase, we use inequality (3.5) on the first equation of (2.8) to obtain

$$\begin{aligned}&\int_{t_0}^{t} \Big| [S_1(\theta(s), \rho(\theta(s))) + S_2(\theta(s), \rho(\theta(s)), q(\theta(s)))] \Big| ds \\ \leq &\int_{\theta_0}^{\theta} 2L(\gamma e^{-\beta(\theta-\theta_0)}) d\theta < \infty\end{aligned}$$

for some positive constant L. Now, apply the variation of constant formula on the first equation of (2.8), one has

$$\theta(t) - t = \theta_0 - t_0 + \int_{t_0}^{t} [S_1(\theta(s), \rho(\theta(s))) + S_2(\theta(s), \rho(\theta(s)), q(\theta(s)))] ds \to t_1$$

for some constant t_1 as $t \to \infty$. Combine all the results, we have

$$\begin{aligned}(x(t), y(t)) =&(u(\theta(t)) + Z(\theta(t))\rho(t) + q(t), u(\theta(t)) + Z(\theta(t))\rho(t) - q(t)) \\ &\to (u(t + t_1), u(t + t_1))\end{aligned}$$

as $t \to \infty$. This completes the proof. □

In the case that K is a simple diagonal matrix, we have

Corollary 3.1. *If $K = \alpha I$ for any constant $\alpha < 0$ in condition (b) of Theorem 1.1, then the same conclusions of the theorem hold.*

4. Examples

In this section, we will give two examples to apply and illustrate the theorem. Example 4.1 is for one block ($s = 1$) and Example 4.2 is for two blocks ($s = 2$). For numerical computation of the systems of ordinary differential equations, we use the Runge-Kutta method of order four with a step size $h = 10^{-3}$.

Example 4.1. In this example, we consider the coupling of two identical Van der Pol equations with nonlinear function $g(t, q_1, q_2) = -\begin{bmatrix} \sin(q_1) - q_1 \\ \sin(q_2) - q_2 \end{bmatrix}$. It is well known that the variational system at the periodic orbit of the Van der Pol equation is one-block and its characteristic multipliers are $\mu_1 = 1, \mu_2 < 1$. For more detail, see [5]. Here we consider the following coupled system

$$\begin{aligned}\frac{dx_1}{dt} =&x_2 + \alpha_1(x_1 - y_1) - (\sin(x_1 - y_1) - (x_1 - y_1)), \\ \frac{dx_2}{dt} =& - x_1 - (x_1^2 - 1)x_2 + \alpha_1(x_2 - y_2) - (\sin(x_2 - y_2) - (x_2 - y_2)), \\ \frac{dy_1}{dt} =&y_2 + \alpha_1(y_1 - x_1) - (\sin(y_1 - x_1) - (y_1 - x_1)), \\ \frac{dy_2}{dt} =& - y_1 - (y_1^2 - 1)y_2 + \alpha_1(y_2 - x_2) - (\sin(y_2 - x_2) - (y_2 - x_2)).\end{aligned} \tag{4.1}$$

By letting $\alpha_1 = -1$ and initial data be $(-1, 3, 0, 2)$, we observe synchronization. The result of this data is given in Figure 1 for $0 \leq t \leq 10$, where the $x-$ and $y-$ planes are drawn separately in the same space.

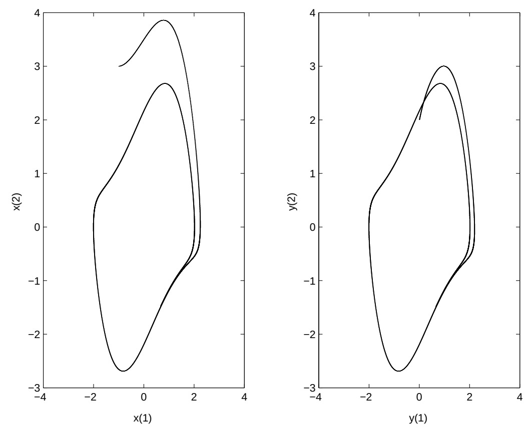

Figure 1. x(t) and y(t) for Example 4.1.

Example 4.2. Let

$$f(x) = \begin{bmatrix} x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ x_1 + x_2 - x_2(x_1^2 + x_2^2) \\ -3x_3 \end{bmatrix}, \quad g(t, q_1, q_2, q_3) = \sin(t) \begin{bmatrix} q_1^3 \\ q_2^3 \\ q_3^3 \end{bmatrix}.$$

A periodic solution of $\frac{dx}{dt} = f(x)$ is $u(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ 0 \end{bmatrix}$ and its variational equation is

$$\frac{dx}{dt} = f'(u(t))x = \begin{bmatrix} -2\cos^2(t) & -1 - 2\cos(t)\sin(t) & 0 \\ 1 - 2\cos(t)\sin(t) & -2\sin^2(t) & 0 \\ 0 & 0 & -3 \end{bmatrix} x, \tag{4.2}$$

which has fundamental solution

$$\begin{aligned} &\begin{bmatrix} -\sin(t) & e^{-2t}\cos(t) & 0 \\ \cos(t) & e^{-2t}\sin(t) & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix} \\ =& \begin{bmatrix} -\sin(t) & \cos(t) & 0 \\ \cos(t) & \sin(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix}. \end{aligned} \tag{4.3}$$

It is clear from equation (4.2) that the variational equation at the periodic solution is two-block, therefore in this example we consider the following coupled system

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 - x_2 - x_1(x_1^2 + x_2^2) + \alpha_1(x_1 - y_1) + \sin(t)(x_1 - y_1)^3, \\ \frac{dx_2}{dt} &= x_1 + x_2 - x_2(x_1^2 + x_2^2) + \alpha_1(x_2 - y_2) + \sin(t)(x_2 - y_2)^3, \\ \frac{dx_3}{dt} &= -3x_3 + \alpha_2(x_3 - y_3) + \sin(t)(x_3 - y_3)^3, \\ \frac{dy_1}{dt} &= y_1 - y_2 - y_1(y_1^2 + y_2^2) + \alpha_1(y_1 - x_1) + \sin(t)(y_1 - x_1)^3, \end{aligned} \tag{4.4}$$

$$\frac{dy_2}{dt} = y_1 + y_2 - y_2(y_1^2 + y_2^2) + \alpha_1(y_2 - x_2) + \sin(t)(y_2 - x_2)^3,$$
$$\frac{dy_3}{dt} = -3y_3 + \alpha_2(y_3 - x_3) + \sin(t)(y_3 - x_3)^3.$$

It follows from equation (4.3) and the definition that the characteristic multipliers are $\mu_1 = 1, \mu_2 = e^{-4\pi} < 1, \mu_3 = e^{-6\pi} < 1$. Therefore according to Theorem 1.1, the requirements for the conditions are $\alpha_1 < 0, \alpha_2 < -\frac{1}{4\pi}(ln|\mu_3|)\} = 1.5$. Here we choose $\alpha_1 = -0.1$, $\alpha_2 = 1.4$ and initial data $(1.2, 1.3, 0.2, 1.1, 1.4, 0.1)$. By using the inverse sine function or the inverse cosine function, we can compute $t_1 \approx 0.86511377$. In Figure 2, $x(t)$ and $y(t)$ of the solution curve are give separatively for $0 \le t \le 10$. It also shows that both $x(t)$ and $y(t)$ go to $u(t+t_1) = \begin{bmatrix} \cos(t+t_1) \\ \sin(t+t_1) \\ 0 \end{bmatrix}$ as $t \to \infty$.

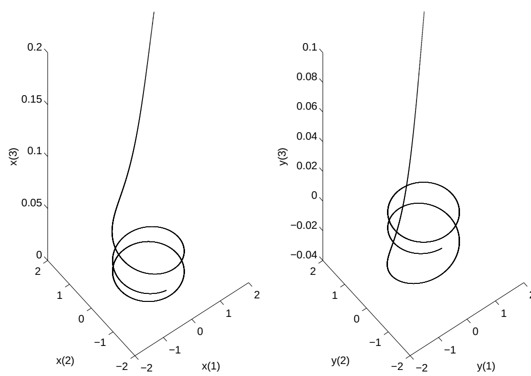

Figure 2. x(t) and y(t) for Example 4.2.

We list in Table 1 the asymptotic behavior of the curves for $x(t)$ and $y(t)$, where

$$d_j(x) = x_j(t) - u_j(t+t_1), \quad d_j(y) = y_j(t) - u_j(t+t_1), \quad 1 \le j \le 3.$$

From the table, we can see the behavior of the synchronized asymptotic phase described in Theorem 1.1.

Table 1. Data for Example 4.2

t	$d_1(x)$	$d_2(x)$	$d_3(x)$	$d_1(y)$	$d_2(y)$	$d_3(y)$
0	0.5514	0.5388	0.2000	0.4514	0.6388	0.1000
5	-0.0061	-0.0134	0.0187	0.0059	0.0135	-0.0187
10	-0.0054	0.0007	0.0069	0.0054	-0.0007	-0.0069
15	-0.0003	0.0020	0.0025	0.0003	-0.0020	-0.0025
20	0.0007	0.0003	0.0009	-0.0007	-0.0003	-0.0009
25	0.0002	-0.0002	0.0003	-0.0002	0.0002	-0.0003
30	-0.0001	-0.0001	0.0001	0.0001	0.0001	-0.0001
35	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

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